APPENDIX A1

BASIC CONCEPTS OF VECTOR ALGEBRA

In this Appendix, we state the basic results of vector algebra that you need to know while studying this course. You have studied these concepts in Units 1 and 2 of the course BPHCT-131 entitled Mechanics.

Recall that in its geometric representation, the vector is described by a directed line segment. In its algebraic representation the vector is described by its components in a specific coordinate system like the Cartesian coordinate system. The properties of vectors do not depend on the chosen representation.

A vector is represented geometrically or graphically by a **directed line segment** or an arrow, that is, a straight line with an arrowhead. The length of the arrow represents the **magnitude** of the vector quantity, which is a positive scalar quantity and the **arrowhead points along the direction of the vector**.

In the Cartesian coordinate system, a vector $\vec{\mathbf{a}}$ in **two-dimensional space** with tail at the point (x_1, y_1) and head at the point (x_2, y_2) can be represented algebraically in terms of its *x* and *y* components as

$$a = a_x\,\hat{\mathbf{i}} + a_y\,\hat{\mathbf{j}} \qquad \text{(A1.1a)}$$

where

$$a_x = x_2 - x_1 = a\cos\theta, \quad a_y = y_2 - y_1 = a\sin\theta \qquad \text{(A1.1b)}$$

The **magnitude** of the vector is given by $\left|\vec{\mathbf{a}}\right| = a = \sqrt{a_x^2 + a_y^2}$ (A1.1c)

and its **direction** is given by the angle θ that the vector makes with the positive *x*-axis: $\theta = \tan^{-1}\left(\dfrac{a_y}{a_x}\right)$ (A1.1d)

A vector $\vec{\mathbf{a}}$ in **three-dimensional space** with tail at the point (x_1, y_1, z_1) and head at the point (x_2, y_2, z_2) can be represented algebraically in terms of its *x*, *y* and *z* components as

$$\vec{\mathbf{a}} = a_x\,\hat{\mathbf{i}} + a_y\,\hat{\mathbf{j}} + a_z\,\hat{\mathbf{k}} \qquad \text{(A1.2a)}$$

where $a_x = x_2 - x_1, \quad a_y = y_2 - y_1, \quad a_z = z_2 - z_1$ (A1.2b)

The **magnitude** of $\vec{\mathbf{a}}$ is given by $a = \sqrt{(a_x^2 + a_y^2 + a_z^2)}$ (A1.2c)

The **direction** of the vector $\vec{\mathbf{a}}$ is given by the direction cosines cos α, cos β and cos γ where α, β and γ are the angles that the vector $\vec{\mathbf{a}}$ makes with the *x*, *y* and z-axes. Thus,

$$a_x = (\vec{\mathbf{a}}.\hat{\mathbf{i}}) = a\cos\alpha \qquad \text{(A1.2d)}$$

$$a_y = (\vec{\mathbf{a}}.\hat{\mathbf{j}}) = a\cos\beta \qquad \text{(A1.2e)}$$

$$a_z = (\vec{\mathbf{a}}.\hat{\mathbf{k}}) = a\cos\gamma \qquad \text{(A1.2f)}$$

We now state the properties of vectors.

Equality of Vectors

Two free vectors are equal if they have the same magnitude and direction, regardless of the position of the tail of the vector or their respective components are equal:

$$\vec{\mathbf{a}} = \vec{\mathbf{b}} \text{ iff } a_x = b_x,$$
$$a_y = b_y,$$
$$a_z = b_z$$

If a vector $\vec{\mathbf{b}}$ has the same magnitude but the opposite direction as any other vector $\vec{\mathbf{a}}$, then we can write

$$\vec{\mathbf{b}} = -\vec{\mathbf{a}}$$

Unit Vector

A vector of length or magnitude 1 is called a **unit vector**. By convention, unit vectors are taken to be dimensionless. A unit vector is used to denote a direction in space. Any vector $\vec{\mathbf{a}}$ can be represented as the product of its magnitude (a) and a unit vector along its direction denoted by $\hat{\mathbf{a}}$. Then we have:

$$\vec{\mathbf{a}} = a\,\hat{\mathbf{a}} \qquad \text{or} \qquad \hat{\mathbf{a}} = \frac{\vec{\mathbf{a}}}{a} = \frac{\vec{\mathbf{a}}}{|\vec{\mathbf{a}}|} \tag{A1.3}$$

Addition and Subtraction of Vectors

Triangle Law of Vector Addition: If two vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ to be added are represented in magnitude and direction by the two sides of a triangle taken in order (which means that the tail of $\vec{\mathbf{b}}$ is at the head of the vector $\vec{\mathbf{a}}$), then their sum or resultant is given in magnitude and direction by the third side of the triangle taken in the opposite order, that is from the tail of the first vector to the head of the second vector (Fig. A1.1).

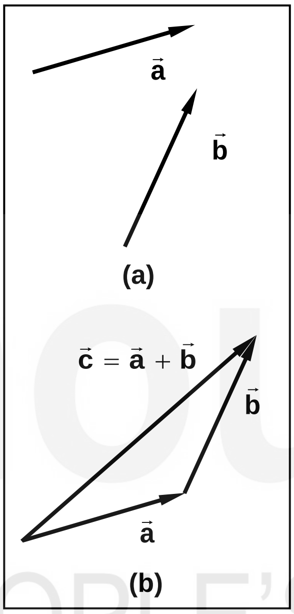

Fig. A1.1: **The triangle law of vector addition.**

Parallelogram Law of Vector Addition: If the two vectors to be added are represented in magnitude and direction by the adjacent sides of a parallelogram, then their resultant is given in magnitude and direction by the diagonal of the parallelogram drawn through the point of intersection of the two given vectors (see Fig. A1.2).

The expressions for the magnitude and direction of the resultant $\vec{\mathbf{c}}$ for two vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ having the angle θ between them are given as follows:

$$c = \sqrt{b^2 + 2ab\cos\theta + a^2} \tag{A1.4a}$$

$$\alpha = \tan^{-1}\left[\frac{a\sin\theta}{b + a\cos\theta}\right] \tag{A1.4b}$$

Here a, b and c are the magnitudes of the vectors $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$ and $\vec{\mathbf{c}}$, respectively, and the angle α between the vectors $\vec{\mathbf{b}}$ and $\vec{\mathbf{c}}$ gives the direction of the vector $\vec{\mathbf{c}}$ (see Fig. A1.2).

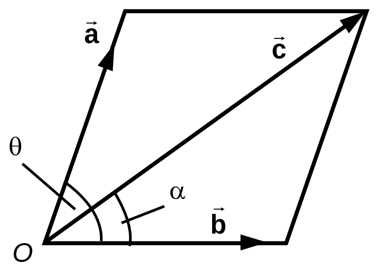

Fig A1.2: **The parallelogram law of vector addition.**

In terms of components in the Cartesian system, the sum

$$\vec{\mathbf{c}} = (c_x \hat{\mathbf{i}} + c_y \hat{\mathbf{j}} + c_z \hat{\mathbf{k}})$$

of the vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ is given as

$$\vec{\mathbf{c}} = \vec{\mathbf{a}} + \vec{\mathbf{b}} \Rightarrow (a_x + b_x)\hat{\mathbf{i}} + (a_y + b_y)\hat{\mathbf{j}} + (a_z + b_z)\hat{\mathbf{k}}$$

It is possible to add any number of vectors by the repeated application of the triangle law of vector addition. We also use the polygon law of vector addition.

Polygon Law of Vector Addition: If a number of vectors are represented in magnitude and direction, by the sides of a polygon, taken in order, then the resultant vector is represented in magnitude and direction by the closing side of the polygon taken in the opposite order, that is from the tail of the first vector to the head of the last vector (see Fig. A1.3).

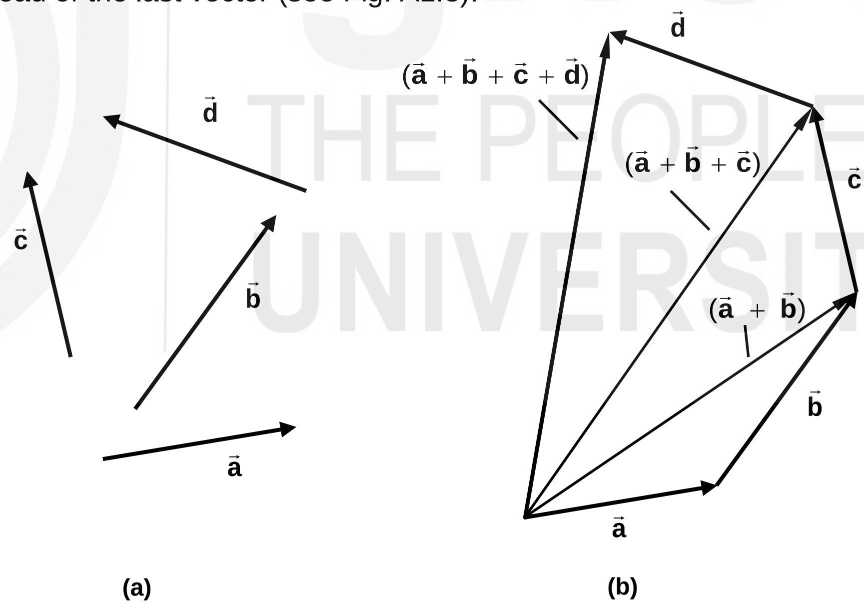

Fig. A1.3: **Polygon law of vector addition applied for determining the resultant $(\vec{\mathbf{a}} + \vec{\mathbf{b}} + \vec{\mathbf{c}} + \vec{\mathbf{d}})$ of four vectors $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$, $\vec{\mathbf{c}}$ and $\vec{\mathbf{d}}$.**

Vector addition is commutative and associative:

$$\vec{\mathbf{a}} + \vec{\mathbf{b}} = \vec{\mathbf{b}} + \vec{\mathbf{a}} \quad \text{and} \quad (\vec{\mathbf{a}} + \vec{\mathbf{b}}) + \vec{\mathbf{c}} = \vec{\mathbf{a}} + (\vec{\mathbf{b}} + \vec{\mathbf{c}}) \qquad \text{(A1.5)}$$

Subtraction of a vector $\vec{\mathbf{b}}$ from a vector $\vec{\mathbf{a}}$ denoted by $\vec{\mathbf{a}} - \vec{\mathbf{b}}$ is just the sum of the vectors $\vec{\mathbf{a}}$ and $(-\vec{\mathbf{b}})$:

$$\vec{\mathbf{a}} - \vec{\mathbf{b}} = \vec{\mathbf{a}} + (-\vec{\mathbf{b}}) \qquad \text{(A1.6)}$$

Multiplication of a vector by a scalar

A vector $\vec{\mathbf{a}}$ when multiplied by a scalar quantity m, is equal to the vector $m\vec{\mathbf{a}}$, in the same direction as $\vec{\mathbf{a}}$ and having the magnitude $|m||\vec{\mathbf{a}}|$. The following is true for the multiplication of a vector by a scalar:

$$m(n)\vec{\mathbf{a}} = (m)n\vec{\mathbf{a}} = mn\vec{\mathbf{a}} \qquad \textbf{Associative Law} \qquad \text{(A1.7a)}$$

$$(m+n)\vec{\mathbf{a}} = m\vec{\mathbf{a}} + n\vec{\mathbf{a}} \quad \text{and} \quad m(\vec{\mathbf{a}} + \vec{\mathbf{b}}) = m\vec{\mathbf{a}} + m\vec{\mathbf{b}} \qquad \textbf{Distributive Laws} \qquad \text{(A1.7b)}$$

If $m = 0$, then $m\vec{\mathbf{a}}$ is a **null** or **zero** vector, which has zero magnitude but no definite direction.

Components of a vector in a given direction

A vector can be resolved into its **component vectors** along any arbitrary direction. The components of a vector $\vec{\mathbf{a}}$ parallel and perpendicular to any other vector $\vec{\mathbf{b}}$ which makes an angle θ with the vector $\vec{\mathbf{a}}$ are given as (see also Fig. A1.4):

The component of $\vec{\mathbf{a}}$ parallel to $\vec{\mathbf{b}} = \dfrac{\vec{\mathbf{a}}.\vec{\mathbf{b}}}{|\vec{\mathbf{b}}|} = |\vec{\mathbf{a}}|\cos\theta$ (A1.8a)

The component of $|\vec{\mathbf{a}}|$ perpendicular to $\vec{\mathbf{b}} = |\vec{\mathbf{a}}|\sin\theta$ (A1.8b)

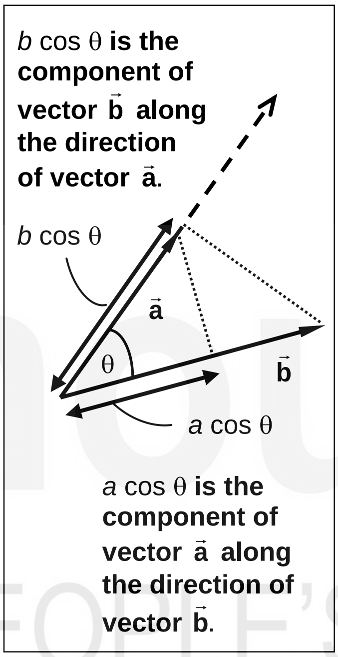

Fig. A1.4: **Projection or the component of a vector along the direction of another vector**.

Scalar product

The **scalar product** of two vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ called "a dot b" and denoted by $\vec{\mathbf{a}}.\vec{\mathbf{b}}$ is a scalar quantity defined as

$$\vec{\mathbf{a}}.\vec{\mathbf{b}} = |\vec{\mathbf{a}}||\vec{\mathbf{b}}|\cos\theta = ab\cos\theta \qquad \text{(A1.9a)}$$

In component form $\vec{\mathbf{a}}.\vec{\mathbf{b}} = a_x b_x + a_y b_y + a_z b_z$ (A1.9b)

Vector product

The **vector product** of two vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ called "a cross b" and denoted by $\vec{\mathbf{a}} \times \vec{\mathbf{b}}$ is a vector quantity defined as

$$\vec{\mathbf{c}} = \vec{\mathbf{a}} \times \vec{\mathbf{b}} = ab\sin\theta\,\vec{\mathbf{c}} \text{ with magnitude } c = ab\sin\theta \qquad \text{(A1.10a)}$$

The direction of $\vec{\mathbf{c}}$ is determined by the right hand rule. The vector $\vec{\mathbf{c}}$ is perpendicular to the plane containing the vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$.

In the component form,

$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = (a_y b_z - a_z b_y)\hat{\mathbf{i}} + (a_z b_x - a_x b_z)\hat{\mathbf{j}} + (a_x b_y - a_y b_x)\hat{\mathbf{k}}$$

(A1.10b)