APPENDIX A2

DOUBLE INTEGRALS

In Unit 3, you have learnt how to integrate vector functions of single variables. It is possible to extend the idea of a definite integral to calculate double and triple integrals which are integrals of functions of two and three variable respectively. Double and triple integrals have many applications in physics. For example, we use these integrals to determine the volume of an object bound by an arbitrary surface, its mass, its centre of mass or its moment of inertia. In this appendix, we explain in brief how to evaluate a double integral, which is an integral of a function of two variables.

A2.1 DOUBLE INTEGRALS

We first develop the geometrical concept of the double integral. Before we do this, however, you should revise the concept of the definite integral of a function.

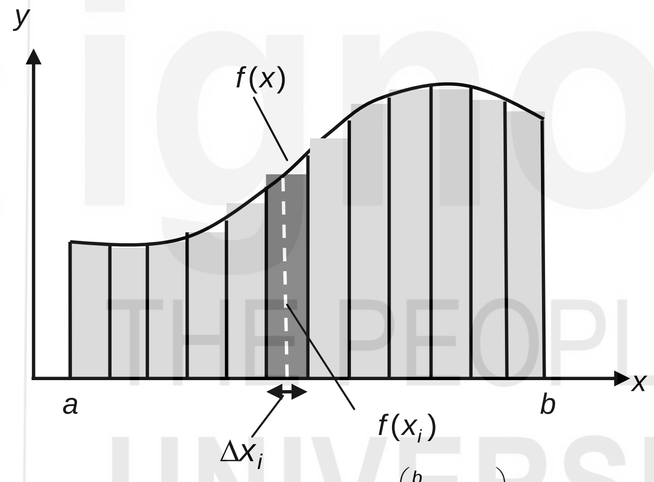

Fig. A2.1: **Definite integral of the function** $f(x)$: $\left(\int_a^b f(x)dx\right)$ **as area under a curve.**

We define the definite integral of a function *f* (*x*) over the interval [*a*, *b*] on the *x*-axis, denoted by $\int_a^b f(x)dx$ as the limit of a sum. We start by dividing the interval [*a*, *b*] into *n* sub-intervals, the *i*th sub-interval having a width Δx_i, as shown in Fig. A2.1. The sum $\sum f(x_i)\Delta x_i$ is the total area of the *n* rectangles we see in the figure. Then the sum of the areas of the rectangles is approximately the area under the curve. It is also clear that if we increase the number of sub-intervals, i.e., increase the value of *n*, the rectangles become narrower, and the total area of the rectangles comes closer and closer to the area under the curve. The exact area is then given by the limit of the sum as *n* goes to infinity:

$$\text{Area under the curve } f(x) = \lim_{n\to\infty}\sum_{i=1}^{i=n} f(x_i)\Delta x_i \qquad \text{(A2.1a)}$$

The expression on the right hand side of Eq. (A2.1a) is called the definite integral of *f*(*x*) from *a* to *b* and denoted as follows:

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{i=n} f(x_i)\Delta x_i \qquad \text{(A2.1b)}$$

This definition of the definite integral holds even if $f\,(x)$ has both positive and negative values in the interval [a, b]. The integral exists if the function f is continuous on [a, b] or has only a finite number of jump discontinuities.

Let us now explain the concept of **double Integral** of the function $f\,(x,y)$ over a bounded region R on the xy-plane denoted by

$$\iint_R f(x,y)\,dx\,dy$$

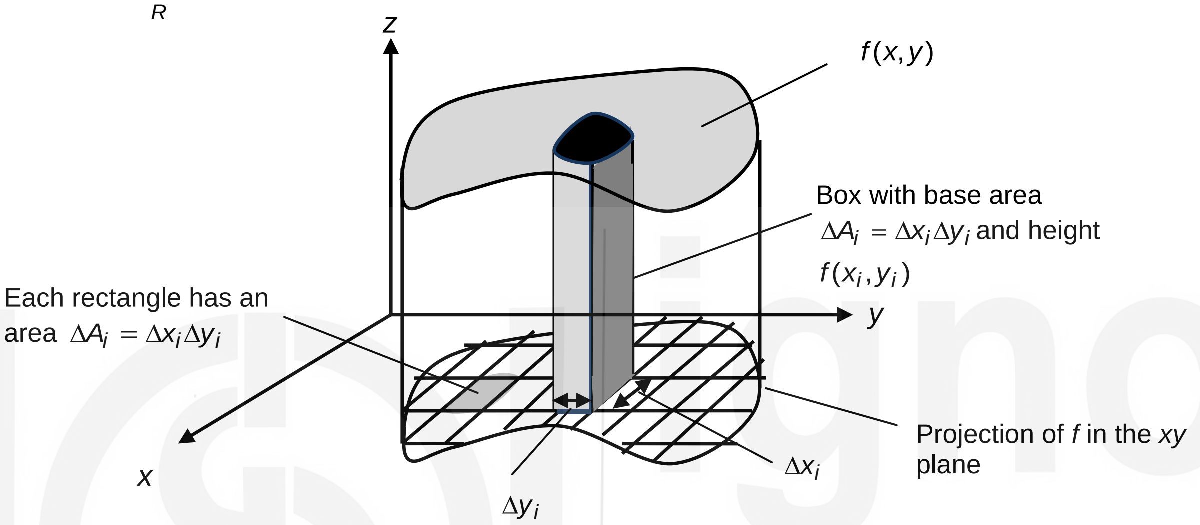

Fig. A2.2: **Double Integral** $\iint_R f(x,y)dxdy$ **as the volume under a surface** $f(x,y)$ **and above the region** R **in the** xy **plane.**

The definition of the double integral is similar to that of a definite integral. We divide the region R in the xy plane into n tiny rectangles by drawing lines parallel to the x and y axes. Each rectangle has an area ΔA_i (see Fig. A2.2).

We number the rectangles within R from i = 1 to i = n and choose a point $(x_i,\ y_i)$ in each rectangle. Now consider the sum:

$$S_n = \sum_{i=1}^{n} f(x_i,\ y_i)\Delta A_i = \sum_{i=1}^{n} f(x_i,\ y_i)\Delta x_i \Delta y_i \qquad \text{(A2.2)}$$

We can evaluate this sum for increasing values of n such that the maximum diagonal of the rectangles goes to zero as the number of rectangles goes to infinity. If $f(x,y)$ is a continuous function in R, these sums (also called the Reimann sums) converge to a limiting value which does not depend on either the values of (x_i, y_i) or the choice of subdivision. This limit is the double integral of the function $f(x,y)$ over the region R.

The double integral of a function $f(x,\ y)$, which is defined for all ($x,\ y$) in a closed, bounded region R in the xy plane, is written as the limit of a sum as follows:

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i, y_i)\Delta A_i = \iint_R f(x,y)dA = \iint_R f(x,y)dxdy \qquad \text{(A2.3)}$$

Just as the area under the curve $f(x)$ in Fig. A2.1b is the area under the curve above the x-axis.

For $f(x,y) \geq 0$, the double integral gives us the volume of the solid that lies below the surface $f(x,y)$ and above the region R in the xy plane (read the margin remark).

PROPERTIES OF THE DOUBLE INTEGRALS

For two functions $f(x,y)$ and $g(x,y)$, which are defined and continuous in a region R:

1. $$\iint_R c\,f(x,y)dxdy = c\iint_R f(x,y)dxdy \tag{A2.4}$$

   where c is a constant.

2. **Linearity**

   $$\iint_R [\alpha f(x,y) + \beta g(x,y)]dxdy = \alpha\iint_R f(x,y)dxdy + \beta\iint_R f(x,y)dxdy \tag{A2.5}$$

   where α and β are constants.

3. **Additivity**

   If the region of integration R can be broken up into a finite number of non overlapping regions R_1, R_2 ....R_n, (Fig. A2.3), then we can write:

   $$\iint_R f(x,y)dxdy = \iint_{R_1} f(x,y)dxdy + \iint_{R_2} f(x,y)dxdy + ... + \iint_{R_n} f(x,y)dxdy \tag{A2.6}$$

4. **Area Property**

   If the function being integrated is $f(x,y) = 1$, then

   $$\iint_R [1]\,dxdy = \text{Area of the region } R \tag{A2.7}$$

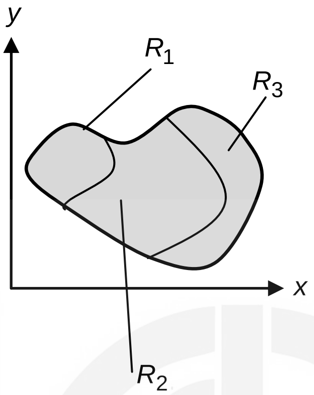

Fig. A2.3: **The region R is broken up into three overlapping regions R_1, R_2 and R_3.**

We briefly explain how to evaluate double integrals

A2.2 EVALUATION OF DOUBLE INTEGRALS

To evaluate a double integral $\iint_R f(x,y)dxdy$ over a region R, we have to carry out two successive integrations over the variables x and y. How is this done? You will see that there are actually two ways of doing this. Let us see what these are.

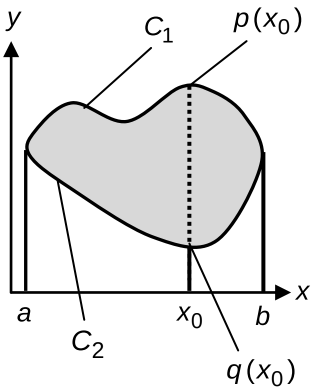

Fig. A2.4: **Evaluating the double integral as an iterated integral (Eq. A2.9).**

First let us define the region of integration R shown in Fig. A2.4 as:

$$a \leq x \leq b;\; q(x) \leq y \leq p(x) \tag{A2.8}$$

As you can see from Fig. A2.4, the values of the x coordinate at the two extremities of the region are $x = a$ and $x = b$. Now in between these two values of x, the region R is bound by the two curves C_1 and C_2. These two curves are given by the equations $y = p(x)$ and $y = q(x)$, respectively. What does this tell us?

It tells us that for each value of x in the interval [a, b], the value of y ranges between $q(x)$ and $p(x)$ which are the points on the lower and upper curves bounding the region R. So for any value of x, for example $x = x_0$ in [a, b], the values of y range from $q(x_0)$ to $p(x_0)$. Now if we were to first integrate the function $f(x, y)$ over the variable y, holding x as a constant, the limits in y would be from $y = q(x)$ to $y = p(x)$. The result would be a function of only x. Next we integrate this function of x with respect to x from $x = a$ to $x = b$. Thus, we cover the entire region of R while integrating over the two variables.

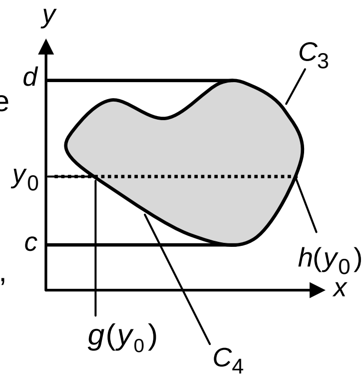

Fig. A2.5: **Evaluating the double integral as an iterated integral (Eq. A2.11).**

Therefore:

$$\iint_R f(x,y)dxdy = \int_{x=a}^{b}\left[\int_{y=q(x)}^{p(x)} f(x,y)dy\right]dx \qquad \text{(A2.9)}$$

The quantity in the brackets, which is evaluated first is the integral of $f(x, y)$ over y alone, with the limits as specified. The result of this integral is a function of x alone which is then integrated over x, over the limits shown.

We could have chosen to carry out this integration in another way. Refer to Fig. A2.5. We can write down the limits on x and y for the same region R in a different way as we describe below:

$$c \le y \le d;\ g(y) \le x \le h(y) \qquad \text{(A2.10)}$$

Both these iterated integrals defined in Eqs. (A2.9) and (A2.11) are equal if $p(x)$, $q(x)$, $g(y)$, $h(y)$ are continuous functions, for the limits defined in Eqs. (A2.8) and (A2.10). This is the consequence of a theorem in multivariable calculus called the Fubini's theorem, which is beyond the scope of this course.

Now for any value of y in the interval [c, d] the value of x is decided by the function $h(y)$ (curve C_3) and $g(y)$ (curve C_4), which respectively now defined the upper and lower boundaries of R. Now we can integrate the function $f(x, y)$ over the variable x, holding y as a constant, the limits of the integral would be from $x = g(y)$ to $x = h(y)$ and the result would be a function of only y. We then integrate this function of y over x from $y = c$ to $y = d$. So we get an alternative expression for the evaluation of the double integral:

$$\iint_R f(x,y)dxdy = \int_{y=c}^{d}\left[\int_{x=g(y)}^{h(y)} f(x,y)dx\right]dy \qquad \text{(A2.11)}$$

As before, the integral within the brackets is carried out first. Both Eqs. (A2.9) and (A2.11) are equivalent methods of determining the double integral. In Eq. (A2.9), the integral over the variable y is carried out first. In Eq. (A2.11), the integral over x is carried out first.

Suppose R cannot be represented by the inequalities shown in Eq. (A2.8) or Eq. (A2.10), but can be subdivided into many parts that can be represented by inequalities, then we evaluate the double integral over each part and sum up to get the result as the double integral over R.

The integrals of the form $\int_{y=c}^{d}\left[\int_{x=g(y)}^{h(y)} f(x,y)dx\right]dy$ and $\int_{x=a}^{b}\left[\int_{y=q(x)}^{p(x)} f(x,y)dy\right]dx$ are called **iterated** (repeated) **integrals** because they are evaluated first by integrating with respect to one variable, either x or y, as the case may be and then integrating the result with respect to the second variable. Multiple integrals are usually integrated as iterated integrals. We shall use the same technique to evaluate triple integrals as you will see in Unit 4. Let us summarise these results.

Recap

EVALUATION OF A DOUBLE INTEGRAL

Suppose that $f(x, y)$ is a continuous function on the region R. If R is described by the inequalities $a \le x \le b, q(x) \le y \le p(x)$, then

$$\iint_R f(x,y)dxdy = \int_{x=a}^{b}\left[\int_{y=q(x)}^{p(x)} f(x,y)dy\right]dx \qquad \text{(A2.9)}$$

If R is described by the inequalities $c \le y \le d, g(y) \le x \le h(y)$, then

$$\iint_R f(x,y)dxdy = \int_{y=c}^{d}\left[\int_{x=g(y)}^{h(y)} f(x,y)dx\right]dy \qquad \text{(A2.11)}$$

In some textbooks that the iterated integrals are sometimes written without the bracket as follows:

$$\int_a^b\left[\int_c^d f(x,y)dy\right]dx = \int_a^b\int_c^d f(x,y)dydx \qquad \text{(A2.12a)}$$

and

$$\int_c^d\left[\int_a^b f(x,y)dx\right]dy = \int_c^d\int_a^b f(x,y)dxdy \qquad \text{(A2.12b)}$$

By convention, the limits of integration on the variable over which the integration is carried out first, appears on the inner integral sign.

Note: The order of integration in Eqs. (A2.12a and b) is different. It is as shown in the left hand side of each of these equations.

In Eq. (A2.12a), we write *dy dx* in the integrand. This means that we first integrate with respect to y over the interval $[c,d]$ and then with respect to x over $[a,b]$. In Eq. (A2.12b), we write *dx dy* in the integrand. So, the integration is first with respect to x and then with respect to y.

A Special Case

An important special case is when we evaluate double integrals for which the following is true:

i) the region R is a rectangle defined by the limits, say $a \le x \le b, c \le y \le d$ (Fig. A2.6).

ii) The function $f(x, y) = h(x)\, g(y)$, that is $f(x, y)$ is a product of two functions, one of which is a function of only x, $h(x)$ and the other a function of only y, that is $g(y)$.

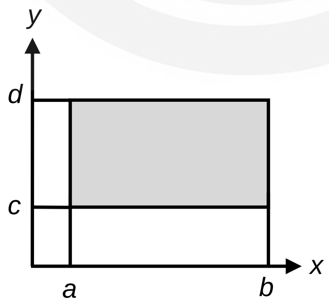

Fig. A2.6: **A rectangular region of integration** $a \le x \le b, c \le y \le d$.

Then the double integral can be evaluated as:

$$\iint_R f(x,y)dxdy = \left[\int_{x=a}^{b} h(x)dx\right]\left[\int_{y=c}^{d} g(y)dy\right] \qquad \text{(A2.13)}$$

As you can see, here we can integrate with respect to each variable separately. We now evaluate some double integrals to illustrate these methods.

EXAMPLE A2.1: DOUBLE INTEGRAL OVER A RECTANGULAR REGION

Evaluate the integral $\iint_R \sin x \cos y \, dx \, dy$ where R is a square on the xy plane defined by $0 \le x \le \pi/2, 0 \le y \le \pi/2$.

SOLUTION ■ Using (Eq. A2.13), we solve the integral as:

$$\iint_R \sin x \cos y dx dy = \left[\int_{x=0}^{\pi/2} \sin x \, dx\right]\left[\int_{y=0}^{\pi/2} \cos y \, dy\right] = [-\cos x]_0^{\frac{\pi}{2}} \times [\sin y]_0^{\frac{\pi}{2}} = 1$$

Note that if the region R is rectangular, but $f(x,y)$ cannot be written as the product of two functions i.e., $f(x,y) \neq h(x)g(y)$, we shall have to carry out an iterated integral.

Let us now work out an example of an iterated integral.

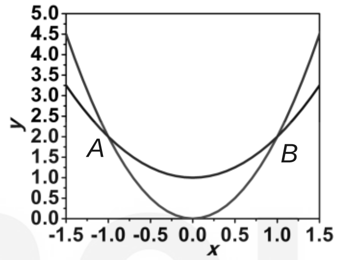

Fig. A2.7: Region of integration for Example A2.2. The two curves intersect at $x = -1$ and $x = 1$.

EXAMPLE A2.2: DOUBLE INTEGRAL

Evaluate the integral $\iint_R (x+4y) dx \, dy$ where R is the region bounded by the curves $y = 2x^2$ and $y = 1 + x^2$.

SOLUTION ■ In Fig. A2.7, we plot the two curves $y = 2x^2$ and $y = x^2 + 1$ which define R. Now, as you can see from the figure, the two curves intersect at the points A and B. At the points of intersection of the two curves, we have $2x^2 = x^2 + 1$. Solving for x we have:

$$2x^2 = x^2 + 1 \Rightarrow x^2 = 1 \quad \Rightarrow x = -1, 1$$

So the points of intersection are $x = 1$ and $x = -1$. This marks the limits of x for the region of integration. As you can see, for each value of x in the range $-1 \le x \le 1$, the value of y will vary in the range $2x^2 \le y \le 1 + x^2$. Now let us use Eq. (A2.9) to evaluate the integral with $q(x) = 2x^2$ and $p(x) = x^2 + 1$. Then we write:

$$\iint_R (x+4y) \, dx \, dy = \int_{x=-1}^{1} \left[\int_{y=2x^2}^{1+x^2} (x+4y) dy\right] dx \qquad \text{(i)}$$

You can see that we have used $a = -1, b = 1$. Now we first carry out the integration within the bracket, integrating over y and taking x as a constant.

$$\int_{y=2x^2}^{x^2+1} (x+4y) dy = \left[xy + 2y^2\Big|_{2x^2}^{1+x^2}\right] = x(1 + x^2 - 2x^2) + 2(1+x^2)^2 - 2(2x^2)^2$$

$$= 2 + x + 4x^2 - x^3 - 6x^4 \qquad \text{(ii)}$$

Substituting the quantity in the bracket in Eq. (i) by the expression in Eq. (ii) we get,

$$\iint_R (x+4y)\,dx\,dy = \int_{x=-1}^{1}\left[2+x+4x^2-x^3-6x^4\right]dx$$

$$= \left[2x+\frac{x^2}{2}+4\frac{x^3}{3}-\frac{x^4}{4}-6\frac{x^5}{5}\right]_{-1}^{1} = \frac{64}{15} \quad \text{(iii)}$$

SAQ - Double integrals over a rectangular region

1. Evaluate the following integrals:

   a) $\int_0^2\int_0^2 xye^{(x+y)}dydx$

   b) $\int_{-1}^{0}\int_0^2 y\sin\frac{\pi x}{4}dxdy$

2. Evaluate $\iint_R (x^4-2y)dx\,dy,$ where R is the region defined by the equations $-1\le x\le 1$ and $-x^2\le y\le x^2$.

A2.3 SOLUTIONS AND ANSWERS

Self-Assessment Questions

When we integrate over y, x is a constant.

1. a) Using Eq. (A2.13), we write the integral:

$$I = \int_{x=0}^{2}\int_{y=0}^{2} x\,y\,e^{(x+y)}\,dy\,dx = \left[\int_0^2 x\,e^{x}dx\right]\left[\int_0^2 y\,e^{y}dy\right]$$

$$= \left[x\,e^x - e^x\right]_0^2 \times \left[y\,e^y - e^y\right]_0^2 = \left(e^2+1\right)^2$$

b) Using Eq. (A2.13), we write

$$I = \int_{y=-1}^{0}\int_{x=0}^{2} y\sin\left(\frac{\pi x}{4}\right)dx\,dy = \left[\int_{-1}^{0} y\,dy\right]\left[\int_0^2 \sin\left(\frac{\pi x}{4}\right)dx\right]$$

$$= \left[\frac{y^2}{2}\right]_{-1}^{0}\left[-\left(\frac{4}{\pi}\right)\cos\left(\frac{\pi x}{4}\right)\right]_0^2 = \frac{2}{\pi}$$

2. We use Eq. (A2.9) to write:

$$I = \int\int_R \left(x^4-2y\right)dx\,dy = \int_{x=-1}^{1}\left[\int_{y=-x^2}^{x^2}\left(x^4-2y\right)dy\right]dx$$

Carrying out the integral over y first and applying the limits of integration, we get:

$$I = \int_{-1}^{1}\left[x^4y-y^2\right]_{-x^2}^{x^2}dx = \int_{-1}^{1}\left[2x^6\right]dx$$

We now integrate over x to get:

$$I = \left[\frac{2x^7}{7}\right]_{-1}^{1} = \frac{4}{7}$$

TABLES OF DERIVATIVES AND INTEGRALS

Table A1.1: Derivatives of simple functions

S. No.	df/dx	S. No.	df/dx
1.	$\frac{d}{dx}(c) = 0, c$ constant	10.	$\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}, \; x \neq 0$
2.	$\frac{d}{dx}(x) = 1$	11.	$\frac{d}{dx}(x^{-n}) = -\frac{n}{x^{n+1}}$
3.	$\frac{d}{dx}(x^n) = nx^{n-1}$	12.	$\frac{d}{dx}[g(x) + h(x)] = \left[\frac{d}{dx}g(x)\right] + \left[\frac{d}{dx}h(x)\right]$
4.	$\frac{d}{dx}(\sin x) = \cos x$	13.	$\frac{d}{dx}[f(x)\, g(x)] = \left[\frac{df}{dx}\right]g + f\left[\frac{dg}{dx}\right]$
5.	$\frac{d}{dx}(\cos x) = -\sin x$	14.	$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g\frac{df}{dx} - f\frac{dg}{dx}}{g^2}, \; g \neq 0$
6.	$\frac{d}{dx}(\tan x) = \sec^2 x$	15.	$\frac{d}{dx}(e^x) = e^x$
7.	$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}}$	16.	$\frac{d}{dx}\ln x = \frac{1}{x}$
8.	$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1 - x^2}}$	17.	$\frac{d}{dx}(c^x) = c^x \ln c, \; c > 0$
9.	$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1 + x^2}$	18.	$\frac{d}{dx}\log_c x = \frac{1}{x \ln c} \quad c \neq 1, \; c > 0$

Table A1.2: Integrals of simple functions

S. No.	Integral	S. No.	Integral
1.	$\int a\, dx = ax + c$, a and c constants	5.	$\int \sin x\, dx = -\cos x + c$, c constant
2.	$\int x^n\, dx = \dfrac{x^{n+1}}{n+1} + c$, c constant	6.	$\int \cos x\, dx = \sin x + c$, c constant
3.	$\int \dfrac{1}{x}\, dx = \ln\lvert x\rvert + c$, c constant	7.	$\int \tan x\, dx = \ln\lvert \sec x\rvert + c$, c constant
4.	$\int e^x\, dx = e^x + c$, c constant	8.	$\int e^{ax}\, dx = ae^{ax} + c$, a and c constants