Indira Gandhi National
Open University
School of Sciences

BPHCT-133
ELECTRICITY AND MAGNETISM

Block

2

ELECTROSTATICS

Lightning in clouds is the most powerful display of strong electrostatic forces and electric fields in nature!

UNIT 5

ELECTROSTATIC FORCE AND ELECTRIC FIELD

Structure

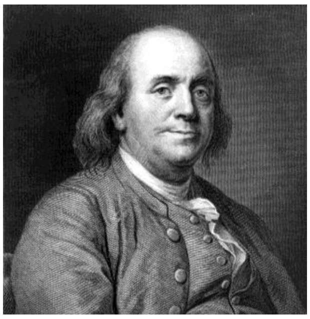
STUDY GUIDE

We hope that you have studied thoroughly the concepts of vector algebra given in Block 1 of the course BPHCT-131 on Mechanics and the concepts of vector calculus presented in Block 1 of this course. You can revise the basic concepts of vector algebra from the Appendix given in Block 1 of this course. You have to make sure that you know all these concepts very well and only then you should study this block and the remaining blocks of this course. In this unit, you will learn about the basic concept of electrostatic force between charges, its quantitative definition given by Coulomb's law, which you have learnt in school physics. You will also learn the concept of electric field and its relation with the electrostatic force. The presentation of these concepts may be new to you. To help you learn the concepts and their application better, we have given many Examples and SAQs within the unit and Terminal Questions at its end. Most of these should take you at most 5 to 10 minutes to solve. You should study all sections thoroughly and make sure that you can solve the SAQs and Terminal Questions on your own before studying the next unit.

"Science is beautiful when it makes simple explanations of phenomena or connections between different observations."

Stephen Hawking

5.1 INTRODUCTION



Benjamin Franklin (1706- 1790), an American polymath (meaning expert in many subjects), was one of the founding fathers of the United States of America. In physics, he is well known for his pioneering work on electricity. He was also a great inventor. The lightning rod, bifocal glasses and urinary catheter are some of his well known inventions in use today. Franklin coined several terms in electricity which we use today: battery, charge, conductor, plus, minus, positively, negatively, condenser (≡ capacitor).

In your school physics, you have studied about electrostatic force between electric charges and Coulomb's law. How are these concepts related to your direct experiences?

During the rainy season, you must have seen flashes of lightning in dark clouds lighting them up. You may have wondered what causes lightning. Do you know that it was Benjamin Franklin who first proved the electric nature of lightning through his experiment with the flying kite? He also gave the idea that clouds possess electric charges, which when discharged in the atmosphere, give rise to a giant spark of lightning.

Actually, human beings have known about the effect of electric charges for thousands of years – the Greeks knew that rubbing amber on a piece of fur made it attract light objects such as feathers. It was later found that many materials such as silk, wax, precious stones, flannel, etc., when rubbed with other materials developed the ability to attract light objects. For example, rubbing glass with silk made it attract pieces of paper. Such materials were called 'electrics'. It was said that the materials became 'electrified' or 'acquired vitreous or resinous electricity'. You may have observed this effect yourself. If you run a comb through your dry hair or rub any dry synthetic fabric, you will notice that small bits of paper or hair cling to them.

The concept of 'positive' and 'negative' charges was developed by Benjamin Franklin and other scientists in the eighteenth century to explain a large number of such observations (as above) made in many experiments. A notable thing about electric charges is that the force between them is extremely large. This force is now known as the **electrostatic force**. As you may recall from Sec. 6.2.5 of Unit 6 of the course Mechanics (BPHCT-131), the electrostatic force is a fundamental force in nature that controls everyday phenomena such as friction, tension, normal force, etc. It helps form electrically neutral stable atoms, molecules, solids and liquids.

'Electrica' is a Latin word coined from elektron, the Greek word for amber. Electrica was translated as electrics in English and later the two words electrical and electricity were coined. All electrical effects due to rubbing together of various materials were ascribed to two forms of electricity – 'vitreous' electricity and 'resinous' electricity. Franklin identified the term 'positive' with vitreous electricity and 'negative' with resinous electricity.

So in Sec. 5.2, we explain the concept of **electrostatic force** between positive and negative charges. To do so, we revise the concept of **electric charge.** Then we give the mathematical expression of the force law known as **Coulomb's law** and use it to calculate the electrostatic force between two charges. We then discuss the concept of **electric field** in Sec. 5.3. You have been introduced to vector fields in the first block of this course. You have learnt that the electric field is a vector field, which is set up due to a charge or distribution of charges in the region surrounding it. You will learn how to calculate the electric field due to different simple charge distributions. In the next unit, you will study the concept of **electric flux**. You will use it to learn the easier and more elegant Gauss's law for determining the electric field due to various charge distributions.

Expected Learning Outcomes

After studying this unit, you should be able to:

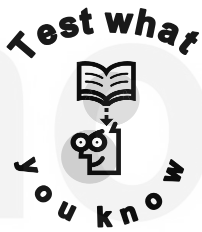
- ❖ use Coulomb's law to calculate the electrostatic force between two given charges at rest;

- apply the principle of superposition of forces to calculate the resultant force due to a system of more than two charges;
- define electric field due to multiple discrete charges and continuous distribution of charges; and
- calculate the net electric field due to a distribution of multiple discrete charges and infinite uniform line charge.

5.2 ELECTROSTATIC FORCE

Do you recall the concepts of charge and electrostatic forces between like and unlike charges and Coulomb's law from school physics? Do you remember studying that like charges repel each other and unlike charges attract each other? You have studied about positive and negative charges and the forces between them in your school physics. You may like to revise the concepts by solving the problems in the pre-test given below. Otherwise, study this section and then try to solve these problems again.

PRE-TEST



1. A glass rod rubbed with silk is said to be 'positively' charged and amber or plastic rubbed with fur, 'negatively' charged. Select the correct conclusion for each observation given below:

   Observation 1: An object is repelled by a piece of glass that has been rubbed with silk.

   a) The object is positively charged.

   b) The object is negatively charged.

   Observation 2: Two objects are both attracted to a piece of amber that has been rubbed with fur.

   a) Both objects are positively charged.

   b) Both objects are negatively charged.

2. State whether the following statements are true or false:

   a) The charge on free particles has also been measured to be a fraction of the charge on the electron, 1.6×10^{-19} C.

   b) Objects are electrically neutral because they have equal numbers of positive protons and negative electrons.

   c) The total charge in the universe is conserved.

   d) The force between two charges at rest is independent of their magnitude.

   e) The force between two charged particles at rest is proportional to the product of the magnitudes of the charge on them.

   f) The force between two charged particles at rest is an inverse square force.

Actually, electric charge could have been given any other name by scientists. How it came to be used is interesting. In older English language, the word charge was used for a *load* carried by anything, such as a cannon or a horse. Since the property/ substance/'fluid' was 'carried' by matter, it was called 'electric charge'.

If you have solved these problems correctly, you know the basic concepts about charges and the force between them. You may like to quickly go through the remaining part of Sec. 5.2 and solve the SAQs given in it. Otherwise, study it thoroughly and try the pre-test and SAQs again.

5.2.1 Electric Charge

In this section, we will quickly revise what you have learnt about electric charges in your school physics, viz., the types of charge, the unit of charge, quantisation of charge and charge conservation.

Types of Charge and the Unit of Charge

You have learnt in school physics that **charge** is a scalar quantity and **is of two types**: **positive** and **negative**. Electrons and protons are the most familiar examples of negative and positive charges having the same magnitude of charge, i.e., 1.6×10^{-19} C. As you can see, the SI unit of charge is **coulomb** (denoted by C) named after the French physicist Charles-Augustin de Coulomb (1736 – 1806).

Coulomb, the unit of charge is defined in terms of magnetic forces and you will learn about them in Block 3.

Atoms and molecules are electrically neutral because they are made up of an equal number of electrons and protons. You may also have read an explanation of how two materials when rubbed together become electrically charged. On rubbing, electrons flow from one material (which becomes positively charged) to another (which is then negatively charged). This way of charge transfer is called **charging by friction** (because you are rubbing one material with another). There are other ways of charging an object about which you have studied in your school physics and we will not go into those details here.

Quantisation of Charge

In the eighteenth century, scientists (including Benjamin Franklin) thought that electric charge was a continuous invisible fluid present in all matter and could flow in and out of objects to charge them positively or negatively. Later experiments about the nature of matter revealed that it was made up of atoms, and molecules and atoms were made up of electrons, protons and neutrons. Today we know that the **smallest free charge** that is possible to obtain is that of an electron or proton. The magnitude of this charge is denoted by *e*.

Electric charge was first measured in 1909 by an American Nobel Laureate physicist Robert Millikan (1868 – 1953). His famous experiment known as the oil-drop experiment is now performed in school and college laboratories. In this experiment, you can observe the motion of a charged oil drop falling between two electrified metallic plates under the influence of two forces: the force of gravitation and an electric force being exerted on it in a direction opposite to the gravitational force. Millikan made observations on a large number of drops and found that charges on different drops were integral multiples of an elementary charge 1.6×10^{-19} C. This is not only true for negative charges but also for positive charges.

Mathematically, any positive or negative charge on a free particle is written as

$$q = ne, \qquad n = \pm 1, \pm 2, \pm 3, \ldots.. \qquad (5.1a)$$

where

$$e = 1.6 \times 10^{-19}\,\text{C} \qquad (5.1b)$$

You may know that when a physical quantity can have only discrete values rather than any arbitrary continuous value, we say that it is **quantised**. We do not know why electric charge is quantised. But it is an experimental observation that has had no exception so far. Thus, we say that

Charge is quantised; it takes discrete values that are integral multiples of e.

For example, we can find a free particle (such as positron, α-particle) or charged object (say, a charged sphere or a charged drop) that has a charge equal to an integral multiple of e, i.e., $+4e$ or $-4e$, but never a free particle having a charge of, say, $+0.77e$ or $-2.55e$. You may know that protons and neutrons are made up of tightly bound quarks having charges $-e/3$ and $+2e/3$. However, quarks are yet to be detected as free particles. So on the basis of experimental evidence so far, we can say that

> **Charge is quantised, i.e., charges on free particles have always been measured to be integral multiples of 1.6×10^{-19} C, never a fraction.**

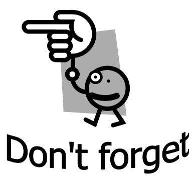
Conservation of Charge

Experiments on electric charges also show that whenever any two objects are in contact (e.g., due to rubbing, touching, etc.) and there is an excess charge on any one of these two objects after contact, then there is an excess charge on the other object too. These excess charges on the two objects in contact are equal in amount but **opposite** in sign. This means that when electric charge (electrons) is transferred from one object to another, no electrons are destroyed or created. Thus, the amount of charge contained in the two objects is a conserved quantity. This is true for all **isolated** systems in nature.

Actually, based on his experiments Benjamin Franklin was the first scientist to propose the hypothesis of conservation of charge. No violations of this law have been found in countless experiments done on microscopic particles such as elementary particles, nuclei, atoms and molecules as well as large charged objects. So, we can add electric charge to the list of conserved quantities such as linear momentum, energy and angular momentum and state **the law of conservation of charge**. Experimental evidence shows that

> Conservation of total electric charge in the universe also points to the existence of anti-particles.

> **In an isolated system, the total amount of electric charge (that is, the algebraic sum of the positive and negative charge present in the system at any time) never changes. We say that it is conserved. Charge-carrying particles can be transferred from one object to another, but the charge associated with those particles cannot be created or destroyed. It follows that the total electric charge in the universe is conserved.**



You may like to go back to the pre-test and attempt questions 1 and 2a to c before studying further. We now revise Coulomb's law which tells us how much force is exerted by one charged object on another.

Charles-Augustin de Coulomb (1736 – 1806) was a French physicist who is best known for his law describing the electrostatic forces between charged particles. Coulomb's law has been firmly established after countless experiments. It applies to all electrical charges whether free or between the positively charged nucleus and electrons bound within an atom. It accounts for the forces that bind atoms to form molecules, and atoms and molecules to form all types of matter. Thus, it accounts for the stability of matter.

5.2.2 Coulomb's Law

The force law for charged particles at rest was arrived at after a series of careful experiments by Coulomb. He discovered that the magnitude of the force (electric, Coulomb or **electrostatic** force as we know it today) between two charged particles q_1 and q_2 **at rest** is given by

$$F = k\,\frac{|q_1||q_2|}{r^2} \tag{5.2}$$

where r is the distance between the charged particles and k is the constant of proportionality. The force is directed along the line joining the two particles. The force on either particle is directed toward the other particle if the two have opposite (unlike) charges and away if the two have similar (like) charges. So we say that like charges repel and unlike charges attract each other. Since force is a vector quantity, let us write down Eq. (5.2) in vector form for both like and unlike charges in one place.

COULOMB'S LAW

The **electrostatic force on a particle carrying a charge** q_1 **by a particle carrying a charge** q_2 **situated at a distance** r **from it is given by**

$$\vec{\mathbf{F}}_{21} = k\,\frac{q_1 q_2}{|\vec{\mathbf{r}}_{21}|^2}\,\hat{\mathbf{r}}_{21} \tag{5.3a}$$

where $\hat{\mathbf{r}}_{21}$ is the unit vector along the line joining the particles and directed from q_2 to q_1 (see Fig. 5.1) and k is called the **Coulomb constant**. Note that $\vec{\mathbf{r}}_{21} = \vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2$ and $|\vec{\mathbf{r}}_{21}| = r$. Here $\vec{\mathbf{r}}_1$ and $\vec{\mathbf{r}}_2$ are the position vectors of q_1 and q_2, respectively. Note that the particles are at rest. In SI units, Coulomb's law is written as

$$\vec{\mathbf{F}}_{21} = \frac{1}{4\pi\varepsilon_0}\,\frac{q_1 q_2}{|\vec{\mathbf{r}}_{21}|^2}\,\hat{\mathbf{r}}_{21} \tag{5.3b}$$

where the units of q_1 and q_2 are coulomb, those of $\vec{\mathbf{r}}_{21}$ and $\vec{\mathbf{F}}_{21}$ are metre and newton, respectively. Here ε_0 is the permittivity of free space.

Coulomb constant $\dfrac{1}{4\pi\varepsilon_0} = 8.99\times10^9\,\mathrm{N\,m^2\,C^{-2}}$.

Note that Eqs. (5.3a and b) account for the attractive and repulsive nature of the electrostatic force if q_1 and q_2 include the sign of the charge. So, **if the charges are like**, that is, both charges are either positive or negative, **the force** $\vec{\mathbf{F}}_{21}$ on q_1 points away from q_2, along $\vec{\mathbf{r}}_{21}$, i.e., it is **repulsive**. **If the charges are unlike**, that is, one of them is positive and the other negative, **the force** $\vec{\mathbf{F}}_{21}$ on q_1 is towards q_2, in the direction opposite to $\vec{\mathbf{r}}_{21}$, i.e., it is **attractive**.

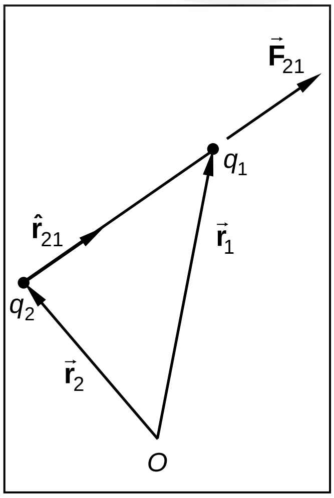
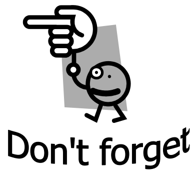


Fig. 5.1: **The electrostatic force between two electric charges at rest.**

Did you notice that the expression for the attractive Coulomb force between unlike charges is similar to the expression of the gravitational force you have studied in Unit 7 of Block 2 of the course on Mechanics (BPHCT-131)?

We have used the same sign convention here. The force of repulsion differs only in sign. So, the mathematical expression of Coulomb's law given by Eq. (5.3a or b) sums up four experimental observations:

1. **Unlike charges attract and like charges repel;**
2. **The force between two charged particles is exerted along the line joining them;**
3. **The force between any two charged particles is proportional to the magnitude of charge on each particle; and**
4. **It is an inverse square force, i.e., it is inversely proportional to the square of the distance between the particles.**



Let us now take up an example to show you how to apply Coulomb's law.

EXAMPLE 5.1 : APPLYING COULOMB'S LAW

Determine the magnitudes and directions of the electrostatic force on the following charged particles at rest and show them on a diagram:

$q_1 = 5.0$ C, $q_2 = -12$ C at a distance of 30 m.

SOLUTION ■ The electrostatic force on each charge is given by Coulomb's law, i.e., Eq. (5.3b).

We substitute the values of q_1, q_2 and r in each case and take

$$\frac{1}{4\pi\varepsilon_0} = 8.99 \times 10^9 \text{ N m}^2 \text{ C}^{-2}.$$

The magnitude of the force on each particle is

$$F = (8.99 \times 10^9 \text{ N m}^2 \text{ C}^{-2}) \times \frac{5.0 \text{ C} \times 12 \text{ C}}{(30 \text{ m})^2} = 6.0 \times 10^8 \text{ N}$$

Since the charges on the particles are unlike, they will attract each other. The force on each particle will be directed toward the other particle. Mathematically, we write the forces as:

Force on q_1 by q_2 is $\vec{\mathbf{F}}_{21} = -6.0 \times 10^8 \text{ N } \hat{\mathbf{r}}_{21}$ and

Force on q_2 by q_1 is $\vec{\mathbf{F}}_{12} = -6.0 \times 10^8 \text{ N } \hat{\mathbf{r}}_{12} = +6.0 \times 10^8 \text{ N } \hat{\mathbf{r}}_{21}$

since $\hat{\mathbf{r}}_{12} = -\hat{\mathbf{r}}_{21}$. Both forces are shown in Fig. 5.2.

You can see that the force is very large.

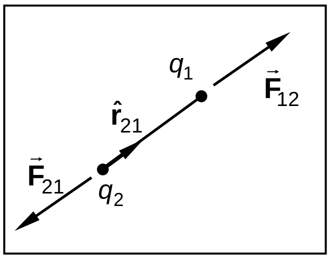


Fig. 5.2: The electrostatic forces for Example 5.1.

Let us take up another example of applying Coulomb's law and then you can test yourself by solving an SAQ.

EXAMPLE 5.2 : APPLYING COULOMB'S LAW

Two point charges Q_1 and Q_2 are 3.0 m apart and their combined charge is 20 μC. If one charge repels the other with a force of 0.075 N, what are the magnitudes of the two charges?

SOLUTION ■ Once again we use Coulomb's law given by Eq. (5.3b).

We are given that the charges repel each other. Therefore, they are like charges. Let Q_1 and Q_2 represent their magnitudes.

Substituting the values of the distance and the force in the scalar form of Eq. (5.3b), we get

$$0.075\text{ N} = (8.99 \times 10^9 \text{ Nm}^2\text{C}^{-2}) \times \frac{Q_1 Q_2}{(3.0\text{m})^2}$$

or $\quad Q_1 Q_2 = 75 \times 10^{-12}\text{ C}^2 = 75 \times (10^{-6}\text{ C})^2 = 75(\mu\text{C})^2 \qquad$ (i)

Also $\quad Q_1 + Q_2 = 20\mu\text{C} \quad \Rightarrow \quad Q_2 = 20\mu\text{C} - Q_1 \qquad$ (ii)

Substituting Q_2 from Eq. (ii) in Eq. (i), we get a quadratic equation in Q_1:

$$75(\mu\text{C})^2 = Q_1(20\mu\text{C} - Q_1) \quad \Rightarrow \quad Q_1^2 - 20Q_1 + 75 = 0$$

where Q_1 is in μC. Solving the equation gives the magnitudes of the charges

$Q_1 = 5.0\mu\text{C}$ and $Q_2 = 15\mu\text{C}$ or $Q_1 = 15\mu\text{C}$ and $Q_2 = 5.0\mu\text{C}$

SAQ 1 - Coulomb's law

a) Determine the electrostatic force on q_1 due to q_2 for :

i) $q_1 = 8.0\ \mu\text{C},\ q_2 = 8.0\ \mu\text{C}$ at a distance of 0.04 m.

ii) $q_1 = 15\text{ mC},\ q_2 = -10\text{ mC}$ at a distance of 3.0 m.

b) The hydrogen atom consists of an electron and a proton separated by an average distance of 5.3×10^{-11}m. Calculate the magnitude of the electrostatic force between the electron and proton taking them to be at rest. Compare it with the magnitude of the gravitational force between them. It is given that the mass of the electron is 9.1×10^{-31} kg, mass of the proton is 1.7×10^{-27} kg and $G = 6.7 \times 10^{-11}\text{ Nm}^2\text{ kg}^{-2}$.

In Example 5.2, we have used the term point charge. What does it mean? A **point charge** is a hypothetical **charge** located at a single **point** in space. In that sense, it has no size: it is dimensionless. It is a purely abstract mathematical concept used in electrostatics. For many purposes, we consider the electron to be a **point charge**. However, its size can be characterized by a length scale known as the electron radius. We often use the term **point charge in electrostatics** when we do not wish to take the size (dimensions) of the particle into consideration.

So far, you have learnt how to determine the electrostatic force between two static charged particles. How do we calculate the electrostatic force on a charge in a system having more than two charges at rest? We use the principle of superposition. Recall that you have studied this principle for the force of gravitation in Sec. 7.2.1 of Unit 7 of the course BPHCT-131 entitled Mechanics. Let us now explain it for electrostatic forces.

5.2.3 The Principle of Superposition

The first thing to understand is that electrostatic forces are two-body forces. This means that the electrostatic force between any pair of charged objects does not change if other charged objects are present in their surroundings. In a system having more than two charged objects, the electrostatic force between each pair of objects is given by Coulomb's law.

To determine the **net electrostatic force** on any given charged particle in a system of charged particles, exerted by the other charged particles in the system, we simply take the vector sum of the forces being exerted on it by the other charged particles in the system.

Suppose there are three charges q_1, q_2 and q_3 at rest in the system. Then the net electrostatic force $\vec{\mathbf{F}}_1$ exerted on q_1 by q_2 and q_3 is the vector sum of the electrostatic force $\vec{\mathbf{F}}_{21}$ exerted on q_1 by q_2 and the electrostatic force $\vec{\mathbf{F}}_{31}$ exerted on q_1 by q_3, i.e.,

$$\vec{\mathbf{F}}_1 = \vec{\mathbf{F}}_{21} + \vec{\mathbf{F}}_{31} \tag{5.4a}$$

or

$$\vec{\mathbf{F}}_1 = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r_{21}^2}\hat{\mathbf{r}}_{21} + \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_3}{r_{31}^2}\hat{\mathbf{r}}_{31} \tag{5.4b}$$

Note that while applying Eqs. (5.4b and 5.4c), you have to take into account the sign of the charges as shown in Example 5.3.

In general, the electrostatic force $\vec{\mathbf{F}}_i$ on the *i*th charge q_i due to all other charges $q_1, q_2, ..., q_j, ...$ in a many-particle system of charged particles is given by

$$\vec{\mathbf{F}}_i = \sum_{j \neq i} \vec{\mathbf{F}}_{ji} = \frac{1}{4\pi\varepsilon_0}\sum_{j \neq i}\frac{q_i q_j}{r_{ji}^2}\hat{\mathbf{r}}_{ji} \tag{5.4c}$$

Note that the summation in Eq. (5.4c) does not include the *i*th charge. This is indicated by putting $j \neq i$ under the summation signs.

PRINCIPLE OF SUPERPOSITION

Recap

According to the principle of superposition, in a many-particle system of charged particles, the resultant electrostatic force on any charged particle is the vector sum of the electrostatic forces exerted by all other charged particles on it [as given by Eq. (5.4c)].

You may like to work through an example to apply the principle of superposition.

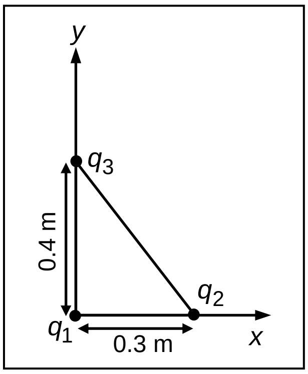


Fig. 5.3: **Diagram for Example 5.3.**

EXAMPLE 5.3 : PRINCIPLE OF SUPERPOSITION

Three charges $q_1 = -2.0\,\mu C$, $q_2 = 9.0\,\mu C$ and $q_3 = 16.0\,\mu C$ are situated at the corners of a right-angled triangle as shown in Fig. 5.3. Calculate the electrostatic force exerted on q_1 by q_2 and q_3.

SOLUTION ■ We use the principle of superposition given by Eq. (5.4b) for a system of three charges. Since $\hat{\mathbf{r}}_{21} = -\hat{\mathbf{i}}$ and $\hat{\mathbf{r}}_{31} = -\hat{\mathbf{j}}$, we have

$$\vec{\mathbf{F}}_1 = \frac{1}{4\pi\varepsilon_0}\left[\frac{q_1 q_2}{(r_{21})^2}(-\hat{\mathbf{i}}) + \frac{q_1 q_3}{(r_{31})^2}(-\hat{\mathbf{j}})\right] \qquad \text{(i)}$$

where $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ are unit vectors along the x and y-axes (Fig. 5.3). Substituting all numerical values (with the sign of the charges) in Eq. (5.4b), we get

$$\vec{\mathbf{F}}_1 = (8.99\times10^9\,\mathrm{N m^2 C^{-2}})\left[\frac{(-2.0\times10^{-6}\,\mathrm{C})\times(9.0\times10^{-6}\,\mathrm{C})}{(0.3\,\mathrm{m})^2}(-\hat{\mathbf{i}}) + \frac{(-2.0\times10^{-6}\,\mathrm{C})\times(16.0\times10^{-6}\,\mathrm{C})}{(0.4\,\mathrm{m})^2}(-\hat{\mathbf{j}})\right] \qquad \text{(ii)}$$

or $\vec{\mathbf{F}}_1 = \left(1.8\,\hat{\mathbf{i}} + 1.8\,\hat{\mathbf{j}}\right)\mathrm{N}$

The magnitude of the force is $\sqrt{(1.8)^2 + (1.8)^2}\ \mathrm{N} = 2.5\,\mathrm{N}$

The direction of the force is given by the angle θ it makes with the positive x-axis: $\theta = \tan^{-1}\left(\frac{1.8}{1.8}\right) = \tan^{-1}(1) = 45^\circ$

So far you have revised the concepts of charge and electrostatic force between charged particles/objects at rest. You have also revised Coulomb's law and the superposition principle, and learnt how to determine the magnitude and direction of electrostatic forces between like and unlike charges. We now discuss the concept of electric field that you have also learnt in school physics.

5.3 ELECTRIC FIELD

Although the notion of electric field first figured in the work of British physicist Michael Faraday (1791 – 1867) on electromagnetic induction, he did not develop its concept. This was done by James Clerk Maxwell (1831 – 1879), a Scottish physicist. You will read more about the work of these two physicists in Block 4 of this course. You are familiar with the concept of vector fields from

Block 1. You have studied about the gravitational field in Unit 7 of the course on Mechanics. You know that the concept of electric field is a very powerful concept that gives us a simple tool for determining the electrostatic force on any charge due to another charge.

The advantage of this concept is that to calculate the net electrostatic force on a given charge due to other charges, we need not follow the lengthy process of Coulomb's law (where we need to know the relative positions of these charges) and vector addition. You will appreciate this point better as you study this section further and learn the concept of electric field. You may ask: **How do we define electric field?** We begin with the simplest case of a point charge.

5.3.1 Electric Field due to a Point Charge

Let us define the electric field due to a point charge.

ELECTRIC FIELD

A point charge Q sets up an electric field in the region surrounding it. If another charge, say q, is placed in this region, it experiences the electrostatic force in accordance with Coulomb's law. The **electric field** generated by an electric charge or a group of charges **is a vector field defined as follows**:

Suppose a positive charge q of an infinitesimal (negligibly small) magnitude, called a **test charge**, is placed at a position $\vec{\mathbf{r}}$ relative to a point charge Q (Fig. 5.4). According to Coulomb's law, at that point, the test charge q will experience the electrostatic force

$$\vec{\mathbf{F}}(\vec{\mathbf{r}}) = \frac{1}{4\pi\varepsilon_0}\frac{Qq}{r^2}\,\hat{\mathbf{r}} \qquad (5.5)$$

where $\hat{\mathbf{r}}$ is the unit vector along $\vec{\mathbf{r}}$. Then the electric field of the point charge Q at a point having position vector $\vec{\mathbf{r}}$ is defined as the electrostatic force on a test charge at that point **divided by** the magnitude of the test charge. It is denoted by $\vec{\mathbf{E}}(\vec{\mathbf{r}})$. Mathematically, it is given by

$$\vec{\mathbf{E}}(\vec{\mathbf{r}}) = \frac{\vec{\mathbf{F}}(\vec{\mathbf{r}})}{q} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\,\hat{\mathbf{r}} \qquad (5.6a)$$

Its magnitude is given by

$$E = \frac{1}{4\pi\varepsilon_0}\frac{|Q|}{r^2} \qquad (5.6b)$$

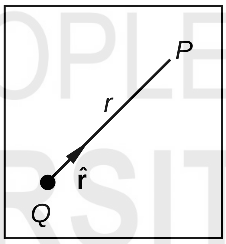
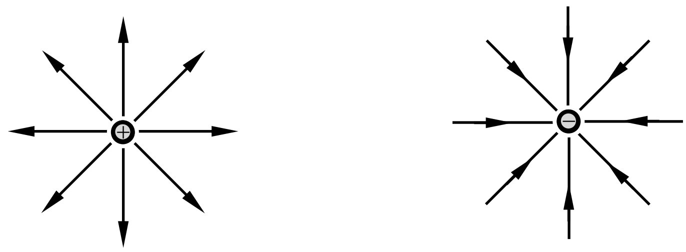


Fig. 5.4: **Unit vector for electric field at point *P* due to a point charge** *Q*.

You have learnt how to visualise electric fields due to a point charge Q defined by Eqs. (5.6a and b) in Sec. 2.2.2 of Unit 2. The representations of these electric fields are shown in Figs. 5.5a and b for positive and negative charges.

Note that the magnitude of the electric field is the same for both positive and negative electric charge ($+Q$ or $-Q$). However, the directions are different as these are given by the direction of the electrostatic force experienced by the respective test charges. The electric field due to a positive point charge is directed away from the charge (Fig. 5.5a). For a negative point charge, it points towards the charge (Fig. 5.5b). The arrows in both Figs. 5.5a and b indicate the direction of the electric field. The continuous lines are called **field lines** (or the lines of force).



(a) (b)

Fig. 5.5: **Electric field lines around a) positive electric charge**; **b) negative electric charge**.

So, to draw electric field lines, you should always remember that

> **Electric field lines (or lines of force) begin at positive charges and end at negative charges. Electric field lines may also go to infinity without terminating. These lines do not intersect.**
>
> **These are close together near the point charges where the electric field is strong and far apart at large distances from the charges where the electric field is weak.**

From Eq. (5.6a), you should also note that the electrostatic force on the charge q when it is placed in the electric field of charge Q is given by

$$\vec{\mathbf{F}} = q\vec{\mathbf{E}} \tag{5.7}$$

So, if you know the electric field in a region of space (could be due to a charge or system of charges), you can determine the electrostatic force on any charge placed in that electric field using Eq. (5.7).

Before studying further, you may like to calculate the electric field due to a few point charges. Work out SAQ 2.

SAQ 2 – Electric field due to point charge

a) Determine the electric field due to the point charges (i) $+5\ \mu C$ at a point 30 cm from it and (ii) $-10\ \mu C$ at a point 1 m from it. Show them in properly labelled diagrams.

b) If a point charge $+6\mu C$ is placed in the electric fields at the respective points given in part (a), what electrostatic force would be exerted on it in both cases?

You may now like to ask: **How is the electric field due to a group of charges defined**? This is what you will now learn.

5.3.2 Electric Field due to Multiple Discrete Charges

Consider a group of point charges q_j, having position vectors $\vec{\mathbf{r}}_j$. Let us place a test charge q_i having position vector $\vec{\mathbf{r}}_i$ in the electric field of these charges. From the principle of superposition for electrostatic forces, the net electrostatic force on the test charge q_i due to this group of charges is given by

$$\vec{\mathbf{F}} = \frac{1}{4\pi\varepsilon_0} \sum_{j \neq i} \frac{q_i\, q_j}{\left|\vec{\mathbf{r}}_i - \vec{\mathbf{r}}_j\right|^2} \hat{\mathbf{r}}_{ji} \tag{5.8}$$

The electric field due to the group of charges at the point with position vector $\vec{\mathbf{r}}_i$ is defined as

$$\vec{\mathbf{E}} = \frac{\vec{\mathbf{F}}}{q_i} = \frac{1}{4\pi\varepsilon_0} \sum_{j \neq i} \frac{q_j}{\left|\vec{\mathbf{r}}_i - \vec{\mathbf{r}}_j\right|^2} \hat{\mathbf{r}}_{ji} \tag{5.9}$$

Eq. (5.9) defines the electric field at a point in space due to a group of point charges. Now, in Eq. (5.9), each charge appears only once. So if only one charge, say q_j, were present, we could write the electric field due to it as

$$\vec{\mathbf{E}}_j = \frac{1}{4\pi\varepsilon_0} \frac{q_j}{\left|\vec{\mathbf{r}}_i - \vec{\mathbf{r}}_j\right|^2} \hat{\mathbf{r}}_{ji} \tag{5.10}$$

So, Eq. (5.9) becomes

$$\vec{\mathbf{E}} = \sum_j \vec{\mathbf{E}}_j \tag{5.11}$$

In other words, **the total electric field due to a group of charges is the vector sum of the individual electric fields of the charges**. This is just the principle of superposition at work. You may like to study Fig. 5.6 to get a sense of the vectors involved in Eq. (5.10) before reading further.

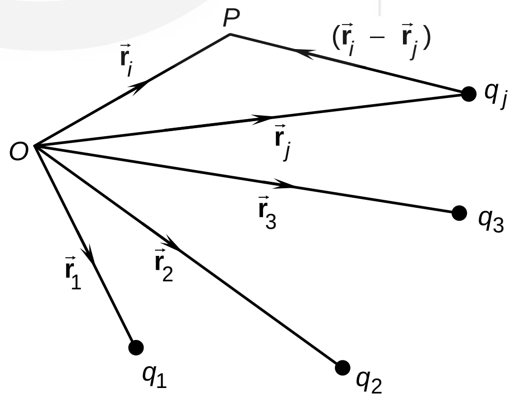


Fig. 5.6: **The vectors involved in defining the electric field due to a group of charges. The vector $\vec{\mathbf{r}}_{ji} = (\vec{\mathbf{r}}_i - \vec{\mathbf{r}}_j)$ represents the vector joining q_j to the point *P* having position vector $\vec{\mathbf{r}}_i$. The vector $\hat{\mathbf{r}}_{ji}$ is the unit vector along $\vec{\mathbf{r}}_{ji}$.**

Once again, if a charge q is placed in the electric field given by Eq. (5.9), the electrostatic force exerted on it will be given by Eq. (5.7). This makes the calculation of electrostatic force on a charge due to a group of charges much easier than using Coulomb's law. Let us now consider an example to calculate the electric field due to a special arrangement of two charges called the **electric dipole**.

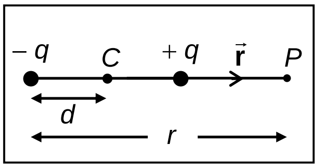


Fig. 5.7: **An electric dipole made up of equal and opposite charges, $\pm q$, separated by distance $2d$. The vector $2\vec{\mathbf{d}}$ along the axis of the dipole is drawn from the negative to the positive charge. The point P lies on the dipole axis at a distance r from the midpoint C.**

EXAMPLE 5.4 : ELECTRIC FIELD OF AN ELECTRIC DIPOLE

Two point charges $-q$ and $+q$ are separated by distance $2d$ (see Fig. 5.7). Such an arrangement of equal and opposite charges placed at some distance from each other is called an **electric dipole**. Determine the net electric field due to the charges at the point P located on the **dipole axis** (i.e., the line joining the charges) at a distance r from the midpoint C of the dipole axis.

SOLUTION ■ From Eq. (5.10), we determine the electric field due to each charge at the point P and then use Eq. (5.11).

From Eq. (5.9), the electric fields due to both charges at the point P are, respectively,

$$\vec{\mathbf{E}}_{+q} = \frac{q}{4\pi\varepsilon_0}\frac{\hat{\mathbf{r}}}{(r-d)^2} \quad \text{and} \quad \vec{\mathbf{E}}_{-q} = \frac{(-q)}{4\pi\varepsilon_0}\frac{\hat{\mathbf{r}}}{(r+d)^2}$$

Here $\hat{\mathbf{r}}$ is the unit vector pointing from the charge $-q$ to the charge $+q$ along the line joining them and d is the distance of the midpoint from each charge (see Fig. 5.7). From Eq. (5.11), the resultant or net electric field at the point P due to the two charges is:

$$\vec{\mathbf{E}} = \vec{\mathbf{E}}_{+q} + \vec{\mathbf{E}}_{-q} = \frac{q\hat{\mathbf{r}}}{4\pi\varepsilon_0}\left(\frac{4rd}{(r^2-d^2)^2}\right)$$

If we assume that the point P lies far away from the dipole so that $r >> d$, we can neglect the term d^2 in comparison to r^2 in the denominator of the expression for $\vec{\mathbf{E}}$. Under this assumption, the net electric field at P is

$$\vec{\mathbf{E}} = \frac{1}{4\pi\varepsilon_0}\times\frac{q\hat{\mathbf{r}}(4rd)}{r^4} = \frac{1}{4\pi\varepsilon_0}\frac{2\vec{\mathbf{p}}}{r^3} \qquad \text{(i)}$$

where $\vec{\mathbf{p}} = 2qd\hat{\mathbf{r}}\ (= 2q\vec{\mathbf{d}})$ is a vector quantity called **dipole moment.**

You may like to solve an SAQ to determine the electric field of a dipole.

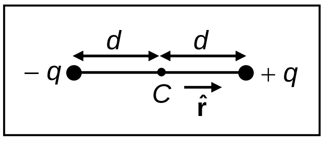


Fig. 5.8: **Diagram for SAQ 3.**

SAQ 3 – Electric field due to an electric dipole

Determine the electric field due to an electric dipole at the midpoint of its axis.

Let us now calculate the electric field of an electric dipole at a point off its axis.

EXAMPLE 5.5 : ELECTRIC FIELD OF AN ELECTRIC DIPOLE

Determine the net electric field due to the electric dipole of Example 5.4 at a point P situated on the perpendicular bisector at a distance r from the midpoint C of the dipole axis.

SOLUTION ■ As in Example 5.4, we use Eq. (5.10) to determine the electric field due to each charge at point P and then apply Eq. (5.11).

The distance of the point P from both the charges $+q$ and $-q$ is $\sqrt{(d^2 + r^2}$ and therefore, from Eq. (5.10), the magnitudes of the electric fields at P due to these charges are equal and, respectively, given by:

$$E_{+q} = \frac{1}{4\pi\varepsilon_0}\frac{q}{d^2 + r^2} \quad \text{and} \quad E_{-q} = \frac{1}{4\pi\varepsilon_0}\frac{q}{d^2 + r^2}$$

From Fig. 5.9a, you can see that the direction of the field is away from the charge $+q$ and towards the charge $-q$. To obtain the expression for the resultant field at P, we take the vector sum of the two electric fields using the **parallelogram law of vector addition**. From Fig. 5.9a, note that the angle between the two electric field vectors is 2θ. So we obtain the magnitude and direction of the resultant electric field as follows [Eqs. (A1.3a and b) in the Appendix A1 of Block 1]:

$$E = \sqrt{E_{+q}^2 + E_{-q}^2 + 2E_{+q}E_{-q}\cos 2\theta} = \frac{1}{4\pi\varepsilon_0}\frac{2q}{d^2 + r^2}\cos\theta$$

or $$E = \frac{1}{4\pi\varepsilon_0}\frac{2qd}{(d^2 + r^2)^{3/2}} \qquad \text{since } \cos\theta = \frac{d}{\sqrt{d^2 + r^2}}$$

The direction of the resultant electric field is given by the angle α it makes with $\vec{\mathbf{E}}_{-q}$ (Fig. 5.9b):

$$\alpha = \tan^{-1}\left[\frac{E_{+q}\sin 2\theta}{E_{-q} + E_{+q}\cos 2\theta}\right] = \tan^{-1}[\tan\theta] = \theta$$

Note that $\vec{\mathbf{E}}$ is anti-parallel to $\vec{\mathbf{p}}$. So, we can express $\vec{\mathbf{E}}$ at point P as

$$\vec{\mathbf{E}} = -\frac{\vec{\mathbf{p}}}{4\pi\varepsilon_0(r^2 + d^2)^{3/2}}$$

If the point P is located far away from the dipole so that $r >> d$, we can express the electric field due to the electric dipole at the point as

$$\vec{\mathbf{E}} = -\frac{\vec{\mathbf{p}}}{4\pi\varepsilon_0 r^3} \qquad \text{(i)}$$

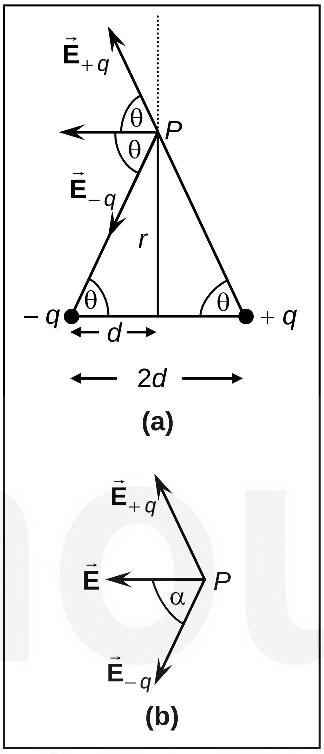


Fig. 5.9: **Diagram for Example 5.5.**

You may now like to learn how to determine the electric field due to a system of more than two charges. Consider the following example.

EXAMPLE 5.6: ELECTRIC FIELD OF MANY CHARGES

Three charges $+q$, $+2q$ and $-q$ are kept in the xy plane at three vertices of a square $ABCD$ of side a (as shown in Fig. 5.10). Determine the net electric field due to these charges at the point B.

SOLUTION ■ We use Eq. (5.10) to determine the electric field at point B due to each charge. Then we apply Eq. (5.11) to obtain the net electric field.

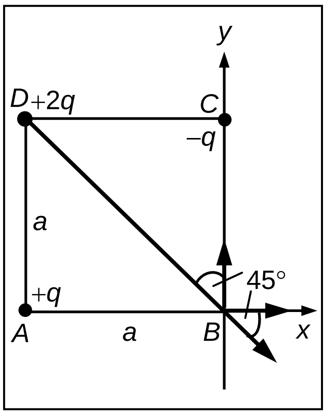


Fig. 5.10: **Diagram for Example 5.6.**

The electric field at B due to charge $+q$ is $\vec{\mathbf{E}}_{+q} = \frac{1}{4\pi\varepsilon_0}\frac{|q|}{a^2}\hat{\mathbf{i}}$ where $\hat{\mathbf{i}}$ is the unit vector along the x-axis. To simplify the algebra, we write

$$E_0 = \frac{1}{4\pi\varepsilon_0}\frac{|q|}{a^2} \quad \text{so that} \quad \vec{\mathbf{E}}_{+q} = E_0\,\hat{\mathbf{i}}$$

The electric field at B due to charge $-q$ is $\vec{\mathbf{E}}_{-q} = \frac{1}{4\pi\varepsilon_0}\frac{|q|}{a^2}(\hat{\mathbf{j}}) = E_0\,\hat{\mathbf{j}}$

Using the geometry of Fig. 5.10, we resolve the electric field at B due to charge $+2q$ along the x and y-axes to get

$$\vec{\mathbf{E}}_{+2q} = E_{+2q}\cos 45^\circ\,\hat{\mathbf{i}} - E_{+2q}\sin 45^\circ\,\hat{\mathbf{j}} \quad \text{where} \quad E_{+2q} = \frac{1}{4\pi\varepsilon_0}\frac{2q}{(\sqrt{2}a)^2} = E_0$$

The net electric field at B is, therefore,

$$\vec{\mathbf{E}} = \vec{\mathbf{E}}_{+q} + \vec{\mathbf{E}}_{-q} + \vec{\mathbf{E}}_{+2q} = E_0\,\hat{\mathbf{i}} + E_0\,\hat{\mathbf{j}} + \frac{E_0}{\sqrt{2}}(\hat{\mathbf{i}} - \hat{\mathbf{j}})$$

or $$\vec{\mathbf{E}} = E_0\left(1+\frac{1}{\sqrt{2}}\right)\hat{\mathbf{i}} + E_0\left(1-\frac{1}{\sqrt{2}}\right)\hat{\mathbf{j}}$$

The magnitude of the resultant electric field is [Eq. (A1.3a), Appendix A1 of Block 1]:

$$E = E_0\sqrt{\left[\left(1+\frac{1}{\sqrt{2}}\right)^2 + \left(1-\frac{1}{\sqrt{2}}\right)^2\right]} = \sqrt{3}\,E_0$$

The direction of the resultant electric field is given by the angle

$$\theta = \tan^{-1}\left[\frac{\left(1-\frac{1}{\sqrt{2}}\right)}{\left(1+\frac{1}{\sqrt{2}}\right)}\right] = \tan^{-1}[0.17] \Rightarrow \theta = 9.6^\circ$$

The resultant electric field has magnitude $\frac{\sqrt{3}}{4\pi\varepsilon_0}\frac{|q|}{a^2}$.

It makes an angle of 9.6° with the x-axis.

Before studying further, you may like to practice how to calculate the electric field due to many charges.

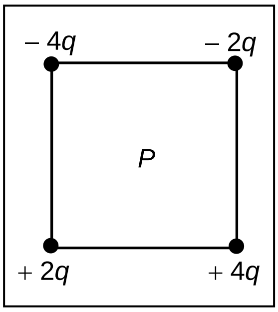


Fig. 5.11: **Diagram for SAQ 4.**

SAQ 4 – Electric field due to many charges

Four charges $+2q$, $-2q$, $+4q$ and $-4q$ are placed at the vertices of a square of side a (Fig. 5.11). Determine the net electric field due to the charges at the centre P of the square given that $q = 1.0\times10^{-9}$ C and $a = 6.0$ cm.

So far, we have defined the electric field and calculated its value for an isolated point charge or an arrangement of two or more point charges. You may like to ask: **What is the electric field of a continuous charge distribution, for example, charge distribution on a wire, lamina or sphere**? Let us find out.

5.3.3 Electric Field due to Continuous Charge Distributions

Let us calculate the electric field at point P with position vector $\vec{\mathbf{r}}_i$ due to any continuous distribution of charge (like the one shown in Fig. 5.12). Let us take the continuous charge distribution to be made up of infinitesimal charges dq_j. Then from Eq. (5.10), the electric field $d\vec{\mathbf{E}}_j$ due to the infinitesimal charge dq_j (having position vector $\vec{\mathbf{r}}_j$) at the point P is given by

$$d\vec{\mathbf{E}}_j = \frac{1}{4\pi\varepsilon_0}\frac{dq_j}{\left|\vec{\mathbf{r}}_i - \vec{\mathbf{r}}_j\right|^2}\hat{\mathbf{r}}_{ji} \qquad (5.12)$$

From the principle of superposition [Eqs. (5.11 and 5.9)], the net electric field $\vec{\mathbf{E}}$ at point P due to the charge distribution will be just the **vector sum of electric fields due to all such infinitesimal charges comprising the distribution**:

$$\vec{\mathbf{E}} = \sum_j d\vec{\mathbf{E}}_j = \frac{1}{4\pi\varepsilon_0}\sum_j \frac{dq_j}{\left|\vec{\mathbf{r}}_i - \vec{\mathbf{r}}_j\right|^2}\hat{\mathbf{r}}_{ji} \qquad (5.13)$$

But in the limit as the charges are infinitesimally small and tend to zero, the sum in Eq. (5.13) can be written as the following integral:

$$\vec{\mathbf{E}} = \frac{1}{4\pi\varepsilon_0}\int \frac{dq}{r^2}\hat{\mathbf{r}} \qquad (5.14)$$

The limits of the integral are defined so that the entire region over which charge is distributed is included. Remember that in Eq. (5.14), $\hat{\mathbf{r}}$ is the unit vector from the charge dq to the point P (having position vector $\vec{\mathbf{r}}$) at which the electric field is being determined (see Fig. 5.12).

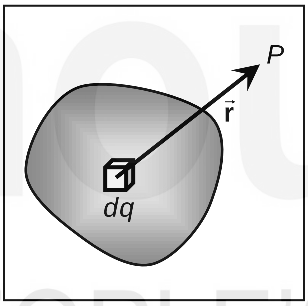


Fig. 5.12: **Determining electric field at a point due to a continuous charge distribution.**

Now, the charge may be continuously distributed over a line, a surface or a volume as shown in Figs. 5.13a, b and c. In such distributions, instead of charges, we speak of the density of charges. The charge density (line, surface or volume) will, in general, be a function of the coordinates. However, in this course, we will consider only those charge distributions that have constant charge density.

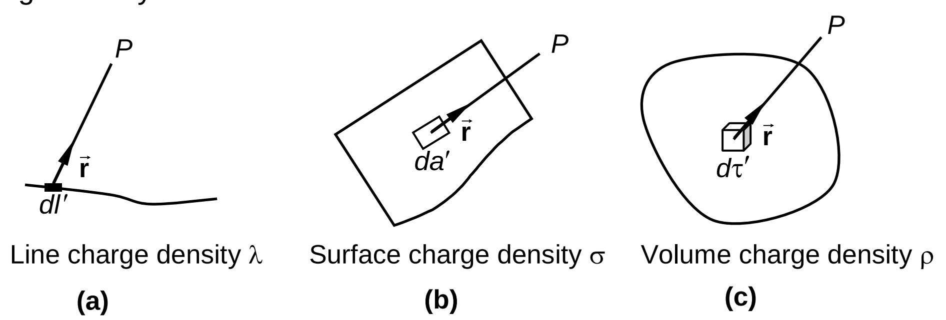


Fig. 5.13: **Determining electric field due to a) line charge distribution; b) surface charge distribution; c) volume charge distribution.**

If the charge is distributed over a line, as in a wire, (Fig. 5.13a), then we speak of the **line charge density**, i.e., charge per unit length and usually denote it by λ. The SI unit of λ is $\mathrm{C\,m^{-1}}$.

In general, when the line charge density is not constant, we have

$dq = \lambda(\vec{\mathbf{r}}')dL'$

and $q = \int_C \lambda(\vec{\mathbf{r}}')dL'$ (i)

Suppose we use the Cartesian coordinates to solve these integrals. Then in Eq. (i), we will integrate with respect to only one variable *x*, *y* or *z* depending on whether the line charge is distributed along the *x*, *y* or *z*-axis.

For a non-uniform surface charge distribution, σ is not constant, and we have

$dq = \sigma(\vec{\mathbf{r}}')dS'$

$q = \iint_S \sigma(\vec{\mathbf{r}}')dS'$ (ii)

Since an area is defined in two dimensions, we will integrate Eq. (ii) with respect to any two variables *x* and *y*, *y* and *z* or *z* and *x*.

For a non-uniform volume charge distribution, ρ is not constant, and we have

$dq = \rho(\vec{\mathbf{r}}')dV'$

and

$q = \iiint_V \rho(\vec{\mathbf{r}}')dV'$ (iii)

Now we will have to integrate Eq. (iii) with respect to the variables *x*, *y* and *z* since volume is defined in three dimensions. These calculations are beyond the scope of this course.

The line charge is, in general, a function of the position **along** the line. Its expression is given in the margin. If the line charge is distributed uniformly, i.e., the line charge density λ is constant, then we have

$$dq = \lambda dL \qquad (5.15a)$$

So, the electric field due to a **uniformly distributed line charge** is defined by

$$\vec{\mathbf{E}} = \frac{1}{4\pi\varepsilon_0}\int_C \frac{\lambda dL}{r^2}\hat{\mathbf{r}} \quad \text{or} \quad \vec{\mathbf{E}} = \frac{\lambda}{4\pi\varepsilon_0}\int_C \frac{dL}{r^2}\hat{\mathbf{r}} \qquad (5.15b)$$

For continuous charge distribution over a surface (Fig. 5.13b), we define the **surface charge density** σ as the **charge per unit area**. Its SI unit is $\text{C}\,\text{m}^{-2}$. It is constant for a uniformly distributed charge on any surface. In this case,

$$dq = \sigma dS \qquad (5.16a)$$

and the electric field due to a **uniformly distributed surface charge** is defined as

$$\vec{\mathbf{E}} = \frac{\sigma}{4\pi\varepsilon_0}\iint_S \frac{dS}{r^2}\hat{\mathbf{r}} \qquad (5.16b)$$

Eq. (5.16b) is a surface integral about which you have studied in Unit 4.

If the continuous charge distribution is spread over a volume (Fig. 5.13c), then we use the **volume charge density** ρ, which is the **charge per unit volume**. Its SI unit is Cm^{-3}. For a uniformly distributed charge over any volume, ρ is constant and

$$dq = \rho dV \qquad (5.17a)$$

The electric field due to a **uniformly distributed volume charge** is defined as

$$\vec{\mathbf{E}} = \frac{\rho}{4\pi\varepsilon_0}\iiint_V \frac{dV}{r^2}\hat{\mathbf{r}} \qquad (5.17b)$$

Let us take up an example to apply the simplest of these equations, Eq. (5.15b), to calculate the electric field of a uniform line charge.

EXAMPLE 5.7 : ELECTRIC FIELD OF INFINITE LINE CHARGE

A straight line of infinite length carries a uniform charge with line charge density λ. Determine the electric field at a distance *y* above the midpoint of the line.

SOLUTION ■ We apply Eq. (5.15b) to determine the electric field due to a uniformly distributed infinite line charge.

Study Fig. 5.14, which shows the charge distribution in the given geometry. Let us choose the *xy* coordinate system to solve this problem with its origin at the midpoint.

Here $dq = \lambda dx$ (i)

By definition, the **magnitude** of the electric field due to dq at the point P directly above the origin is given by

$$d\vec{\mathbf{E}} = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^2}\hat{\mathbf{r}} \qquad \text{(ii)}$$

where r is the distance of dq from P and $\hat{\mathbf{r}}$, the unit vector from dq to P. Note that the direction of $\hat{\mathbf{r}}$ will be different for different elements of charge. Now to determine the net electric field, we write the electric field $d\vec{\mathbf{E}}$ in terms of its x and y-components and then integrate each component over the respective variable x or y.

Our choice of the coordinate system simplifies the calculation. Note that for each infinitesimal charge dq placed at the point $+x$ to the right of the origin, we can place a corresponding infinitesimal charge dq at the point $-x$ to the left of the origin. So these form a pair. Now the x-components of the electric fields due to this pair cancel out as shown in Fig. 5.14. This will be the case for each pair of points $\pm x$ on the x-axis. Therefore, the x-component of the electric field $d\vec{\mathbf{E}}$ will be zero. The y-component of the electric field due to the element of charge dq is given by

$$dE = dE_y = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^2}\cos\theta = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^2}\frac{y}{r} \quad (\because \cos\theta = \frac{y}{r}) \qquad \text{(iii)}$$

where θ is the angle between $\hat{\mathbf{r}}$ and the y-axis. We add the y-components of the electric fields of the two elements at the points $\pm x$ that will be in the same direction to get the net electric field due to them as,

$$d\vec{\mathbf{E}}_{net} = \frac{1}{4\pi\varepsilon_0}\frac{2dq}{r^2}\frac{y}{r}\hat{\mathbf{j}} = \frac{1}{4\pi\varepsilon_0}\frac{2y\lambda dx}{(x^2+y^2)^{3/2}}\hat{\mathbf{j}} \quad (\because dq = \lambda dx)$$

The net electric field due to the infinite line charge is obtained by integrating $d\vec{\mathbf{E}}_{net}$ with respect to x with the limits from 0 to ∞.

Although the line extends from $-\infty$ to $+\infty$, we integrate over only half the line because the expression we are integrating is already the electric field of a charge pair dq.

Thus, $$\vec{\mathbf{E}}_{net} = \int d\vec{\mathbf{E}}_{net} = \int_{x=0}^{x=\infty}\frac{1}{4\pi\varepsilon_0}\frac{2y\lambda dx}{(x^2+y^2)^{3/2}}\hat{\mathbf{j}} \qquad \text{(iv)}$$

Integrating the right hand side of Eq. (iv) gives (read the margin remark):

$$\vec{\mathbf{E}} = \frac{1}{4\pi\varepsilon_0}\frac{2\lambda}{y}\hat{\mathbf{j}}$$

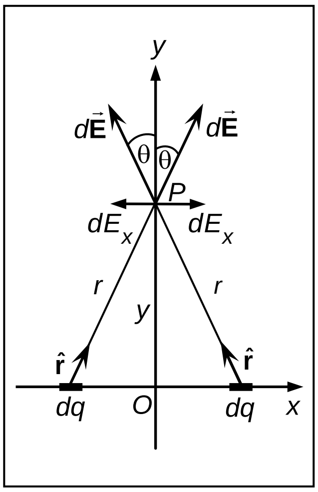


Fig. 5.14: Electric field of a uniform infinite line charge.

Let $x = y\tan\theta$.

Then $dx = y\, d\theta \sec^2\theta$ with the limits from 0 to $\frac{\pi}{2}$. The electric field is then given by

$$\vec{\mathbf{E}}_{net} = \frac{\lambda y}{2\pi\varepsilon_0}\int_0^{\pi/2}\frac{y\sec^2\theta\, d\theta}{y^3\sec^3\theta}\hat{\mathbf{j}}$$

$$= \frac{\lambda}{2\pi\varepsilon_0 y}\int_0^{\pi/2}\cos\theta\, d\theta\,\hat{\mathbf{j}}$$

$$= \frac{\lambda}{2\pi\varepsilon_0 y}\sin\theta\Big|_0^{\pi/2}\hat{\mathbf{j}}$$

$$= \frac{\lambda}{2\pi\varepsilon_0 y}\hat{\mathbf{j}}$$

You will agree that this way of calculating the electric field is quite lengthy as it involves solving complicated integrals. You will learn a much simpler way of determining the electric field of such continuous charge distributions that have some symmetry of this kind in the next unit.

Let us now stop and review what you have learnt in this section. To sum up, you have learnt the definition of the electric field and calculated it for a point charge, arrangements of discrete point charges and a continuous line charge. But while going through this section, this question may still have puzzled you: **What exactly *is* an electric field**?

You should think of the electric field as a **real** physical entity which exists in the space in the neighbourhood of any charge, groups of charges or continuous charge distributions, which set up the electric field. Any charge kept in the electric field experiences the electrostatic force given by Eq. (5.7). The concept of electric field is abstract and it is difficult to imagine it concretely. But you have learnt how to calculate the electric field and also the electrostatic force experienced by a charge kept in the electric field.

This is actually all that we are supposed to do in electrostatics: Determine the electrostatic forces and electric fields due to a given charge distribution. However, as you may have felt while working through Example 5.7, the integrals involved in calculating electric fields can be quite complicated even for simple charge distributions. So, much of electrostatics is about learning the tools and methods that simplify these calculations so that we have no need to solve such complicated integrals. This is what you will be learning in the remaining units of this block and Units 10 and 11 of the next block.

We now summarise the concepts you have studied in this unit.

5.4 SUMMARY

Concept	Description
Electric charge	■ From a large number of observations and experiments, it has been deduced that there exist two types of electric charges in nature, which are arbitrarily called positive and negative charges. In SI system, the unit of electric charge is coulomb denoted by C. Like charges repel and unlike charges attract. In an isolated system, electric charge is always conserved. Thus, the total positive charge is equal to the total negative charge in an isolated system. Free electric charge is quantised and can take only discrete values that are integer multiples of the charge on the electron.
Electrostatic force and Coulomb's law	■ The magnitude of the **electrostatic force** between two charged particles **at rest** is proportional to the product of the magnitudes of charges on them and inversely proportional to the square of the distance between them. The quantitative expression of the electrostatic force between two charges is given by **Coulomb's law**: The **electrostatic force** **on** a particle carrying a charge q_1 **by** a particle carrying a charge q_2 situated at a distance r from it is given by

$$\vec{\mathbf{F}}_{21} = k\,\frac{q_1 q_2}{\left|\vec{\mathbf{r}}_{21}\right|^2}\,\hat{\mathbf{r}}_{21}$$

where $\hat{\mathbf{r}}_{21}$ is the unit vector along the line joining the particles and is directed from q_2 to q_1. Note that $\left|\vec{\mathbf{r}}_{21}\right| = r$ and $\vec{\mathbf{r}}_{21} = \vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2$ where $\vec{\mathbf{r}}_1$ and $\vec{\mathbf{r}}_2$ are the position vectors of q_1 and q_2, respectively. In SI units, Coulomb's law is written as

$$\vec{\mathbf{F}}_{21} = \frac{1}{4\pi\varepsilon_0}\,\frac{q_1 q_2}{\left|\vec{\mathbf{r}}_{21}\right|^2}\,\hat{\mathbf{r}}_{21}$$

Principle of superposition

- According to the **principle of superposition**, in a many-particle system of charged particles at rest, the resultant electrostatic force on any charged particle is the vector sum of the electrostatic forces exerted by all other particles on it. In general, the electrostatic force $\vec{\mathbf{F}}_i$ on the *i*th charged particle due to all other charges $q_1, q_2, ..., q_j, ...$ in a many-particle system of charged particles at rest is given by

$$\vec{\mathbf{F}}_i = \sum_{j \neq i} \vec{\mathbf{F}}_{ji} = \frac{1}{4\pi\varepsilon_0} \sum_{j \neq i} \frac{q_i q_j}{r_{ji}^2}\,\hat{\mathbf{r}}_{ji}$$

Note that the above summation does not include the *i*th charge and $\hat{\mathbf{r}}_{ji}$ is the unit vector along the line joining the *i*th and *j*th particles and is directed from q_j to q_i.

Electric field due to a point charge

- The electric field due to a point charge or charge distribution at a point is defined in terms of the electrostatic force experienced by a test charge q placed at that point divided by the magnitude of the test charge:

$$\vec{\mathbf{E}} = \frac{\vec{\mathbf{F}}}{q}$$

The electric field due to a charge Q at a point having position vector $\vec{\mathbf{r}}$ is given by

$$\vec{\mathbf{E}} = \frac{\vec{\mathbf{F}}}{q} = \frac{1}{4\pi\varepsilon_0}\,\frac{Q}{r^2}\,\hat{\mathbf{r}}$$

where $\hat{\mathbf{r}}$ is the unit vector pointing from the charge to the point at which the electric field is being calculated.

Electric field due to multiple discrete charges

- The electric field due to a distribution of charges at the point with position vector $\vec{\mathbf{r}}_i$ is given from the principle of superposition as

$$\vec{\mathbf{E}} = \frac{\vec{\mathbf{F}}}{q_i} = \frac{1}{4\pi\varepsilon_0} \sum_j \frac{q_j}{\left|\vec{\mathbf{r}}_i - \vec{\mathbf{r}}_j\right|^2}\,\hat{\mathbf{r}}_{ji}$$

where $\hat{\mathbf{r}}_{ji}$ is the unit vector along the line joining the *i*th and *j*th particles and is directed from q_j to q_i. We can write this equation as

$$\vec{\mathbf{E}} = \sum_j \vec{\mathbf{E}}_j$$

where $$\vec{\mathbf{E}}_j = \frac{1}{4\pi\varepsilon_0}\frac{q_j}{\left|\vec{\mathbf{r}}_i - \vec{\mathbf{r}}_j\right|^2}\hat{\mathbf{r}}_{ji}$$

So, **the total electric field due to a group of charges is the vector sum of the electric fields due to individual charges of the distribution**.

Electric field due to continuous charge distributions

- The electric field due to a continuous distribution of charge is given by,

$$\vec{\mathbf{E}} = \frac{1}{4\pi\varepsilon_0}\int\frac{dq}{r^2}\hat{\mathbf{r}}$$

The electric field due to a **uniformly distributed line charge** with constant line charge density λ is given by

$$\vec{\mathbf{E}} = \frac{\lambda}{4\pi\varepsilon_0}\int_C\frac{dL}{r^2}\hat{\mathbf{r}}$$

The electric field due to a **uniformly distributed surface charge** with constant surface charge density σ is given by the surface integral

$$\vec{\mathbf{E}} = \frac{\sigma}{4\pi\varepsilon_0}\iint_S\frac{dS}{r^2}\hat{\mathbf{r}}$$

The electric field due to a **uniformly distributed volume charge** with constant volume charge density ρ is given by the volume integral

$$\vec{\mathbf{E}} = \frac{\rho}{4\pi\varepsilon_0}\iiint_V\frac{dV}{r^2}\hat{\mathbf{r}}$$

5.5 TERMINAL QUESTIONS

1. The electrostatic force exerted by two point charges on each other has magnitude 10 N when these are at rest and placed a distance r apart. What would the magnitude of the electrostatic force between them be if the distance between them is a) $4r$, b) $100r$, c) $\frac{r}{4}$ and d) $\frac{r}{100}$?

2. Two identical charged particles are placed at rest at a separation of 1 m. What is the charge on them if the magnitude of the electrostatic force exerted on each particle is 1 N?

3. Three charged particles A, B and C, each having a charge of 1.0 μC, are placed at rest on a straight line. The distance between A and B is 0.01 m. What is the net electrostatic force exerted on particle C if it is placed a) at a distance 0.01 m to the right of the particle B along the line AB, b) to the left of the particle B along the line AB, at the midpoint of AB?

4. Two point charges $+4q$ and $+q$ are placed at rest at a distance 'a' from each other. Determine the position of a charge $+q$ placed on a straight line joining these two charges, if it is in equilibrium.

5. What is the electric field of a particle having charge -9.0×10^{-9} C at a point 1.0 m away from it? Determine the electrostatic force exerted on a proton placed at that point.

6. The electric field due to a charged particle at a point 0.5 m away from it has magnitude $36\,\text{NC}^{-1}$. What is the magnitude of the electric charge on the particle?

7. When a particle having charge $-9\times10^{-9}\,\text{C}$ is placed at a certain point in an electric field, the electrostatic force exerted on it is of magnitude $3\times10^{-9}\,\text{N}$ and directed along the negative x-axis. What is the electric field at this point? What would the magnitude and direction of the electrostatic force acting on an electron placed at this point be?

8. Three particles each having charge $+q$ are placed at the vertices of an equilateral triangle with each side of length r. Calculate the magnitude of the net electric field at the midpoint of any side of the triangle.

9. Three particles having charge $+q$, $-q$ and $-2q$ are placed at the same distance a from the origin as shown in Fig. 5.15. Calculate the net electric field at the origin.

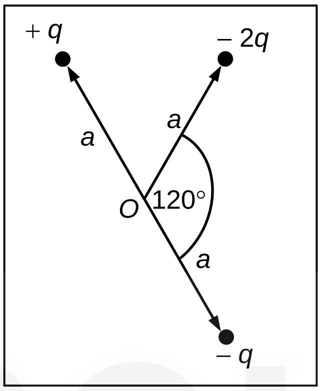


Fig. 5.15: **Diagram for TQ 9.**

10. Four charges $+2q$, $+2q$, $-2q$ and $-2q$ are placed at the vertices of a rectangle of sides 3.0 m and 4.0 m. What is the net electric field due to the charges at the point of intersection of the diagonals given that $q = 3.0\times10^{-9}\,\text{C}$?

5.6 SOLUTIONS AND ANSWERS

Pre-test

1. **Observation 1**: Correct answer is (a) since the glass rubbed with silk is positively charged. Since the object is repelled by the glass, it must have the same charge as the glass.

   Observation 2: Correct answer is (a) since the amber rubbed with fur is negatively charged. Since both objects are attracted to it, therefore, both must have the opposite charge to that of amber.

2. a) False. So far, no such measurements have been made for free particles.

   b) True.

   c) True.

   d) False. It depends on the product of their magnitudes.

   e) True.

   f) True.

Self-Assessment Questions

1. a) From Eq. (5.3b), the electrostatic force on charge q_1 due to charge q_2 at rest is given by Coulomb's law as $\vec{\mathbf{F}}_{21} = \dfrac{1}{4\pi\varepsilon_0}\dfrac{q_1q_2}{|\vec{\mathbf{r}}_{21}|^2}\,\hat{\mathbf{r}}_{21}$ where $\hat{\mathbf{r}}_{21}$ is the unit vector along the line joining the particles and directed from q_2 to q_1 and $|\vec{\mathbf{r}}_{21}| = r$. Also $\dfrac{1}{4\pi\varepsilon_0} = 8.99\times10^{9}\,\text{Nm}^2\,\text{C}^{-2}$.

Substituting the values of q_1, q_2 and r for both cases, we get

i) $$\vec{\mathbf{F}}_{21} = 8.99\times10^9\,\text{Nm}^2\,\text{C}^{-2} \times \frac{(8.0\,\mu\text{C})\,(8.0\,\mu\text{C})}{(0.04\,\text{m})^2}\,\hat{\mathbf{r}}_{21} = 360\,\text{N}\,\hat{\mathbf{r}}_{21}$$

ii) $$\vec{\mathbf{F}}_{21} = 8.99\times10^9\,\text{Nm}^2\,\text{C}^{-2} \times \frac{(15\,\text{mC})\,(-10\,\text{mC})}{(3.0\,\text{m})^2}\,\hat{\mathbf{r}}_{21}$$

$$= -1.5\times10^5\,\text{N}\,\hat{\mathbf{r}}_{21}$$

b) From Eq. (5.3b), the magnitude of the electrostatic force between the electron and the proton is given by

$$F_{elec} = \frac{1}{4\pi\varepsilon_0}\frac{|q_1||q_2|}{r^2} \quad \text{since} \quad |\vec{\mathbf{r}}_{21}| = r.$$

Substituting the values of the magnitudes of the charges of electron and proton, i.e., $|q_1| = |q_2| = 1.6\times10^{-19}$ C and $r = 5.3\times10^{-11}$m, we get

$$F_{elec} = 8.99\times10^9\,\text{Nm}^2\,\text{C}^{-2} \times \frac{(1.6\times10^{-19}\,\text{C})\times(1.6\times10^{-19}\,\text{C})}{(5.3\times10^{-11}\,\text{m})^2}$$

$$= 8.2\times10^{-8}\,\text{N}$$

The gravitational force between the electron and the proton is given by

$$F_{grav} = G\,\frac{m_1 m_2}{r^2}$$

Substituting the values of the masses of electron and proton, i.e., $m_1 = 9.1\times10^{-31}$kg, $m_2 = 1.7\times10^{-27}$ kg, $r = 5.3\times10^{-11}$m and $G = 6.7\times10^{-11}$ Nm^2 kg^{-2}, we get

$$F_{grav} = 6.7\times10^{-11}\,\text{Nm}^2\,\text{kg}^{-2} \times \frac{(9.1\times10^{-31}\,\text{kg})\times(1.7\times10^{-27}\,\text{kg})}{(5.3\times10^{-11}\,\text{m})^2}$$

$$= 3.7\times10^{-47}\,\text{N}$$

Hence, $$\frac{F_{elec}}{F_{grav}} = \frac{8.2\times10^{-8}}{3.7\times10^{-47}} = 2.2\times10^{39}$$

Thus, the electrostatic force is much stronger (~ 10^{39} times stronger) than the gravitational force.

2. a) Substituting for Q (with its sign) and r in Eq. (5.6a), we get

(i) For $Q = +5\,\mu\text{C}$ and $r = 0.30\,\text{m}$,

$$\vec{\mathbf{E}}\,(\vec{\mathbf{r}}) = 8.99\times10^9\,\text{Nm}^2\,\text{C}^{-2} \times \frac{(+5\times10^{-6}\,\text{C})}{(0.30\,\text{m})^2}\,\hat{\mathbf{r}} = 5\times10^5\,\text{NC}^{-1}\,\hat{\mathbf{r}}$$

(ii) For $Q = -10\,\mu\text{C}$ and $r = 1\text{m}$,

$$\vec{\mathbf{E}}\,(\vec{\mathbf{r}}) = 8.99\times10^9\,\text{Nm}^2\,\text{C}^{-2} \times \frac{(-10\times10^{-6}\,\text{C})}{(1\text{m})^2}\,\hat{\mathbf{r}} = -\,9\times10^4\,\text{NC}^{-1}\,\hat{\mathbf{r}}$$

Note that the electric field due to the negative charge is directed towards it. The electric fields at the points P and R are shown in Figs. 5.16a and b. Note that the tails of the electric fields are placed at P and R, respectively.

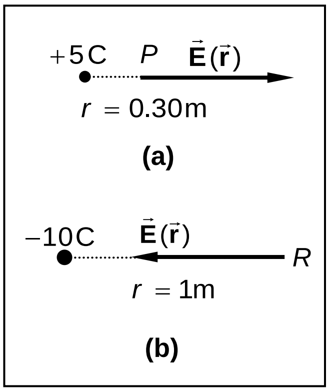


Fig. 5.16: **Diagrams for the answers of SAQ 2a (i) and (ii). The diagrams are not to scale.**

b) From Eq. (5.7), the electrostatic force is given by $\vec{\mathbf{F}}(\vec{\mathbf{r}}) = q\vec{\mathbf{E}}(\vec{\mathbf{r}})$. For $q = +6\,\mu\text{C}$, it is given as follows:

(i) $5 \times 10^5 \times 6 \times 10^{-6}\,\text{N}\,\hat{\mathbf{r}} = 3\text{N}\,\hat{\mathbf{r}}$

(ii) $-9\times10^4 \times 6\times10^{-6}\,\text{N}\,\hat{\mathbf{r}} = -0.5\text{N}\,\hat{\mathbf{r}}$ up to one significant digit.

3. See Fig. 5.17. The midpoint C of the dipole axis is at equal distance d from each charge. From Eq. (5.6b), the magnitudes of the electric fields of both charges at the midpoint are, respectively,

$$\left|\vec{\mathbf{E}}_{-q}\right| = \frac{1}{4\pi\varepsilon_0}\frac{|(-q)|}{d^2} = \frac{1}{4\pi\varepsilon_0}\frac{q}{d^2} \quad \text{and} \quad \left|\vec{\mathbf{E}}_{+q}\right| = \frac{1}{4\pi\varepsilon_0}\frac{q}{d^2}$$

The directions of the electric fields at the point C due to both charges are opposite to $\hat{\mathbf{r}}$, the unit vector along the line joining the two charges as shown in Fig. 5.17. From Eq. (5.11), the resultant or net electric field at the midpoint C due to the two charges is:

$$\vec{\mathbf{E}} = \vec{\mathbf{E}}_{+q} + \vec{\mathbf{E}}_{-q} = -\frac{1}{4\pi\varepsilon_0}\frac{2q}{d^2}\,\hat{\mathbf{r}}$$

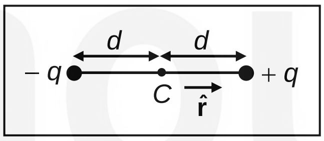


Fig. 5.17: **Diagram for the answer of SAQ 3.**

4. Let us choose the x and y-axes as shown in Fig. 5.18 by the dashed arrows.

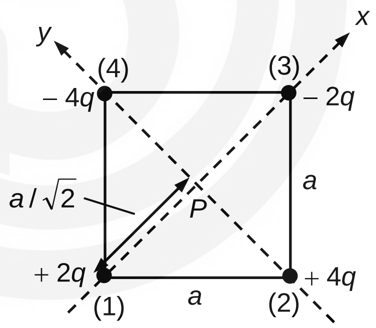


Fig. 5.18: **Diagram for answer to SAQ 4.**

Note from Fig. 5.18 that the distance of the point P from any of the four charges is $\frac{a\sqrt{2}}{2} = \frac{a}{\sqrt{2}}$, where a is the side of the square. Now the net electric field at the point P is the vector sum of the electric fields due to all charges at that point:

$$\vec{\mathbf{E}} = \vec{\mathbf{E}}_1 + \vec{\mathbf{E}}_2 + \vec{\mathbf{E}}_3 + \vec{\mathbf{E}}_4 \qquad \text{(i)}$$

where $\vec{\mathbf{E}}_1$ is the electric field due to the charge $+2q$, $\vec{\mathbf{E}}_2$, the electric field due to the charge $+4q$, $\vec{\mathbf{E}}_3$, the electric field due to the charge $-2q$ and $\vec{\mathbf{E}}_4$, the electric field due to the charge $-4q$. We use Eq. (5.6a) to determine each one of these electric fields and then take their vector sum. Note that with the choice of axes in Fig. 5.18, the direction of the position vector $\hat{\mathbf{r}}$ of the point P with respect to each charge can be expressed in terms of $\hat{\mathbf{i}}$ and

$\hat{\mathbf{j}}$ or their combinations. Also $r = \dfrac{a}{\sqrt{2}}$. So, from Fig. 5.18, we can write the electric field at P due to the charge 1 as

$$\vec{\mathbf{E}}_1 = \frac{1}{4\pi\varepsilon_0}\frac{(+2q)}{(a/\sqrt{2})^2}\hat{\mathbf{i}} = \frac{1}{4\pi\varepsilon_0}\frac{2\times 2q}{a^2}\hat{\mathbf{i}} = \frac{1}{4\pi\varepsilon_0}\frac{4q}{a^2}\hat{\mathbf{i}} \; (\because \hat{\mathbf{r}} = \hat{\mathbf{i}}) \quad \text{(ii)}$$

We write $E_0 = \dfrac{1}{4\pi\varepsilon_0}\dfrac{4q}{a^2}$ so that the expressions become simpler to write. The electric fields at P due to the charges 2, 3, 4 are:

$$\vec{\mathbf{E}}_2 = \frac{1}{4\pi\varepsilon_0}\frac{2\times 4q}{a^2}\hat{\mathbf{j}} = 2E_0\,\hat{\mathbf{j}}, \qquad (\because \hat{\mathbf{r}} = \hat{\mathbf{j}}) \quad \text{(iii)}$$

$$\vec{\mathbf{E}}_3 = \frac{1}{4\pi\varepsilon_0}\frac{2\times(-2q)}{a^2}(-\hat{\mathbf{i}}) = E_0\,\hat{\mathbf{i}} \qquad (\because \hat{\mathbf{r}} = -\hat{\mathbf{i}}) \quad \text{(iv)}$$

and
$$\vec{\mathbf{E}}_4 = \frac{1}{4\pi\varepsilon_0}\frac{2\times(-4q)}{a^2}(-\hat{\mathbf{j}}) = 2E_0\,\hat{\mathbf{j}} \qquad (\because \hat{\mathbf{r}} = -\hat{\mathbf{j}}) \quad \text{(v)}$$

Substituting Eqs. (ii) to (v) in Eq. (i), we can write

$$\vec{\mathbf{E}} = \vec{\mathbf{E}}_1 + \vec{\mathbf{E}}_2 + \vec{\mathbf{E}}_3 + \vec{\mathbf{E}}_4 = 2E_0\,\hat{\mathbf{i}} + 4E_0\,\hat{\mathbf{j}} = 2E_0(\hat{\mathbf{i}} + 2\hat{\mathbf{j}}) \quad \text{(vi)}$$

Now for $q = 1.0\times 10^{-9}\,\text{C}$ and $a = 0.06\,\text{m}$,

$$E_0 = \frac{1}{4\pi\varepsilon_0}\frac{4q}{a^2} = 8.99\times 10^9\,\text{Nm}^2\,\text{C}^{-2} \times \frac{4\times 1.0\times 10^{-9}\,\text{C}}{(0.06\,\text{m})^2}$$

$$\therefore \quad E_0 = 1.0\times 10^4\,\text{NC}^{-1} \quad \text{and} \quad \vec{\mathbf{E}} = 2.0\times 10^4\,\text{NC}^{-1}(\hat{\mathbf{i}} + 2\hat{\mathbf{j}})$$

Terminal Questions

1. We use Eq. (5.2) for the magnitude of the electrostatic force with $k = \dfrac{1}{4\pi\varepsilon_0}$. It is given that $F_1 = \dfrac{1}{4\pi\varepsilon_0}\dfrac{|q_1||q_2|}{r^2} = 10\,\text{N}$ is the magnitude of the electrostatic force exerted by two point charges on each other when these are placed a distance r apart. The magnitudes of the electrostatic force between them for various distances will be, respectively,

   a) $F_2 = \dfrac{1}{4\pi\varepsilon_0}\dfrac{|q_1||q_2|}{(4r)^2} = \dfrac{1}{4\pi\varepsilon_0}\dfrac{|q_1||q_2|}{16(r)^2} = \dfrac{F_1}{16} = \dfrac{5}{8}\,\text{N}$ since

   $$F_1 = \frac{1}{4\pi\varepsilon_0}\frac{|q_1||q_2|}{r^2} = 10\,\text{N}$$

   b) $F_3 = \dfrac{1}{4\pi\varepsilon_0}\dfrac{|q_1||q_2|}{(100r)^2} = \dfrac{F_1}{10000} = \dfrac{10}{10000}\,\text{N} = 10^{-3}\,\text{N}$

   c) $F_4 = \dfrac{1}{4\pi\varepsilon_0}\dfrac{|q_1||q_2|}{(r/4)^2} = 16F_1 = 160\,\text{N}$ and

d) $F_5 = \dfrac{1}{4\pi\varepsilon_0}\dfrac{|q_1||q_2|}{(r/100)^2} = 10000F_1 = 10^5\,\text{N}$

2. Let the charge on the identical particles be q. We use Eq. (5.2) for the magnitude of the electrostatic force with $k = \dfrac{1}{4\pi\varepsilon_0}$. It is given that the charges are identical and the distance between them is 1m. Substituting these values in Eq. (5.2), we have

$$1\text{N} = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{(1\text{m})^2}$$

or $$q = \sqrt{\frac{1\text{N}\times(1\text{m})^2}{8.99\times10^9\,\text{Nm}^2\,\text{C}^{-2}}} = 0.33\,\text{mC}$$

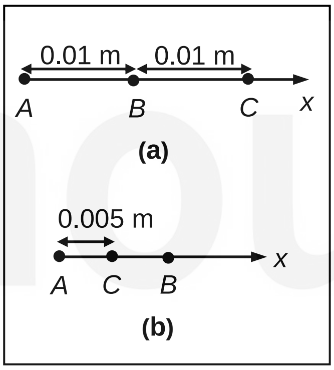


Fig. 5.19: **Diagram for the answer of TQ 3.**

3. a) Refer to Fig. 5.19a. The net electrostatic force exerted on particle C is the vector sum of the electrostatic forces exerted on it by the particles A and B as given by Eq. (5.4b). In terms of the unit vector $\hat{\mathbf{i}}$ along the x-axis, it is given by

$$\vec{\mathbf{F}} = \vec{\mathbf{F}}_{AC} + \vec{\mathbf{F}}_{BC} = \frac{1}{4\pi\varepsilon_0}\left(\frac{(1.0\,\mu\text{C})^2}{(0.02\,\text{m})^2} + \frac{(1.0\,\mu\text{C})^2}{(0.01\text{m})^2}\right)\hat{\mathbf{i}}$$

$$= (8.99\times10^9\,\text{Nm}^2\,\text{C}^{-2}\times10^{-12}\,\text{C}^2\times\frac{5}{4}\times10^4\,\text{m}^{-2})\,\hat{\mathbf{i}} = 1.1\times10^2\,\text{N}\,\hat{\mathbf{i}}$$

b) Refer to Fig. 5.19b. The net electrostatic force exerted on particle C is the vector sum of the electrostatic forces exerted on it by the particles A and B. In this case, the electric field due to B will be in the opposite direction to that of A since it points away from the positive charge. In terms of the unit vector $\hat{\mathbf{i}}$ along the x-axis, it is given by Eq. (5.4b):

$$\vec{\mathbf{F}} = \vec{\mathbf{F}}_{AC} + \vec{\mathbf{F}}_{BC} = \frac{1}{4\pi\varepsilon_0}\left(\frac{(1\,\mu\text{C})^2}{(0.005\,\text{m})^2}\,\hat{\mathbf{i}} - \frac{(1\,\mu\text{C})^2}{(0.005\,\text{m})^2}\,\hat{\mathbf{i}}\right) = \vec{\mathbf{0}}$$

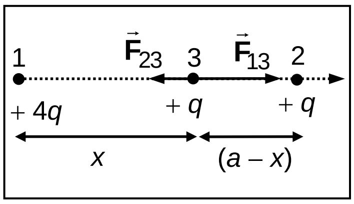


Fig. 5.20: **Diagram for the answer of TQ 4.**

4. Refer to Fig. 5.20. Let the charges lie along the x-axis. Let the position of the charge 3 ($= +q$) be at a distance x from the charge 1 ($= +4q$) such that $x < a$. At this point the charge 2 ($= +q$) is at a distance $(a - x)$ from the charge 3. Therefore, the net electrostatic force exerted on the charge 3 due to the charges 1 and 2 is given by Eq. (5.4b) as

$$\vec{\mathbf{F}}_3 = \vec{\mathbf{F}}_{13} + \vec{\mathbf{F}}_{23} = \frac{1}{4\pi\varepsilon_0}\left(\frac{4q\times q}{x^2}\,\hat{\mathbf{i}} + \frac{q\times q}{(a-x)^2}(-\hat{\mathbf{i}})\right)$$

When the charge 3 is in equilibrium, the net force on it is zero. Thus,

$$\vec{\mathbf{F}}_3 = \frac{1}{4\pi\varepsilon_0}\left(\frac{4q\times q}{x^2}\,\hat{\mathbf{i}} + \frac{q\times q}{(a-x)^2}(-\hat{\mathbf{i}})\right) = \vec{\mathbf{0}}$$

or $\quad \dfrac{4}{x^2} = \dfrac{1}{(a-x)^2} \quad \Rightarrow \quad 4(a-x)^2 = x^2 \Rightarrow 2(a-x) = \pm x$

For the positive sign of x, $x = \dfrac{2a}{3}$ and for the negative sign of x, $x = 2a$.

Since $x < a$, $x = \dfrac{2a}{3}$ is the only possible value of x. Therefore, for the charge 3 $(+q)$ to be in equilibrium, it should be placed at a distance $\dfrac{2a}{3}$ from the charge $+4q$.

5. From Eq. (5.6a), the electric field of a particle having charge $Q = -9\times10^{-9}\,\text{C}$ at a point 1 m away from it is given by

$$\vec{\mathbf{E}}(\vec{\mathbf{r}}) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{\mathbf{r}} = 8.99\times10^9\,\text{N m}^2\,\text{C}^{-2}\,\frac{(-9\times10^{-9}\,\text{C})}{(1\text{m})^2}\,\hat{\mathbf{r}} = -81\,\text{NC}^{-1}\hat{\mathbf{r}}$$

It is directed towards the negatively charged particle. The electrostatic force experienced by a proton placed at that point is an attractive force directed towards the charge Q and is given by

$$\vec{\mathbf{F}}(\vec{\mathbf{r}}) = e\vec{\mathbf{E}}(\vec{\mathbf{r}}) = (1.6\times10^{-19})\,\text{C}\times(-81\,\text{NC}^{-1})\hat{\mathbf{r}} = -1.3\times10^{-17}\,\text{N}\,\hat{\mathbf{r}}$$

up to 2 significant digits.

6. Substituting $E = 36\,\text{NC}^{-1}$ and $r = 0.5\,\text{m}$ in Eq. (5.6a), we have

$$E(\vec{\mathbf{r}}) = \frac{1}{4\pi\varepsilon_0}\frac{|Q|}{r^2} = 8.99\times10^9\,\text{N m}^2\,\text{C}^{-2}\,\frac{|Q|}{(0.5\,\text{m})^2} = 36\,\text{NC}^{-1}$$

$$\therefore \quad |Q| = 1\times10^{-9}\,\text{C} = 1\text{nC}$$

7. From Eq. (5.7), we have

$$\vec{\mathbf{F}}(\vec{\mathbf{r}}) = Q\vec{\mathbf{E}}(\vec{\mathbf{r}}) \qquad \text{(i)}$$

where $Q = -9\times10^{-9}\,\text{C}$ and $\vec{\mathbf{F}} = -3\times10^{-9}\,\text{N}\,\hat{\mathbf{i}}$. Substituting these values in Eq. (i), we get

$$\vec{\mathbf{E}}(\vec{\mathbf{r}}) = \frac{\vec{\mathbf{F}}(\vec{\mathbf{r}})}{Q} = \frac{-3\times10^{-9}\,\text{N}}{-9\times10^{-9}\,\text{C}}\,\hat{\mathbf{i}} = 0.3\,\text{NC}^{-1}\,\hat{\mathbf{i}}$$

It is directed along the positive x-axis. The electrostatic force exerted on an electron placed at this point is given by

$$\vec{\mathbf{F}}(\vec{\mathbf{r}}) = e\vec{\mathbf{E}}(\vec{\mathbf{r}}) = (-1.6\times10^{-19}\,\text{C})\times(0.3\,\text{NC}^{-1})\,\hat{\mathbf{i}}$$

$$= -0.48\times10^{-19}\,\hat{\mathbf{i}}\,\text{N} = -5\times10^{-20}\,\hat{\mathbf{i}}\,\text{N}$$

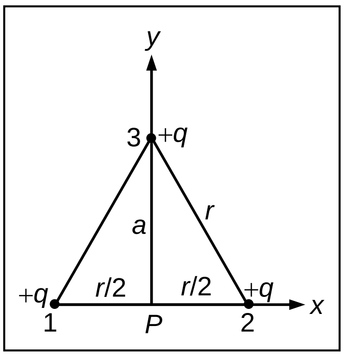


Fig. 5.21: Diagram for the answer to TQ 8.

8. Refer to Fig. 5.21. Let us take the x and y-axes as shown in the figure. Then the electric fields at the midpoint P due to two charges 1 and 2 of

magnitude $+q$ along the x-axis will be equal in magnitude and opposite in direction to each other:

$$\vec{\mathbf{E}}_1(\vec{\mathbf{r}}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{(r/2)^2}\hat{\mathbf{i}} \quad \text{and} \quad \vec{\mathbf{E}}_2(\vec{\mathbf{r}}) = -\frac{1}{4\pi\varepsilon_0}\frac{q}{(r/2)^2}\hat{\mathbf{i}}$$

The magnitude of the net electric field will be just the magnitude of the electric field due to charge 3 on the y-axis. The distance of the charge 3 from the midpoint of the side of the triangle along the x-axis is given by

$$a = \sqrt{r^2 - \left(\frac{r}{2}\right)^2} = \frac{\sqrt{3}}{2}r$$

Therefore, the magnitude of the net electric field at the midpoint of the base of this equilateral triangle is given by

$$E_3(\vec{\mathbf{r}}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{a^2} = \frac{1}{4\pi\varepsilon_0}\frac{4q}{3r^2} = \frac{q}{3\pi\varepsilon_0 r^2}$$

This result holds for any side of the equilateral triangle.

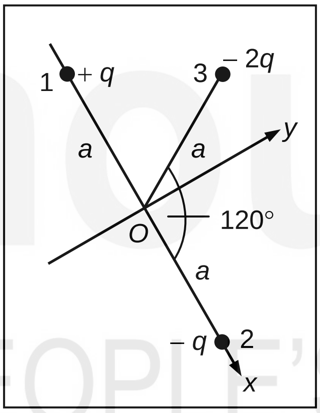


Fig. 5.22: **Diagram for the answer of TQ 9.**

9. Refer to Fig. 5.22. Let us choose the coordinate axes so that the problem becomes simplified. We choose the x-axis to be along the line joining the charges 1 and 2 as shown in the figure. The net electric field at the origin is the vector sum of the electric fields due to the charges 1, 2 and 3:

$$\vec{\mathbf{E}}(\vec{\mathbf{r}}) = \vec{\mathbf{E}}_1(\vec{\mathbf{r}}) + \vec{\mathbf{E}}_2(\vec{\mathbf{r}}) + \vec{\mathbf{E}}_3(\vec{\mathbf{r}}) \qquad \text{(i)}$$

Let us determine the electric fields due to the three charges at the origin. You can see that the charges $+q$ and $-q$ are at the same distance (a) from the origin. So, the origin is at the midpoint of the line joining them. Therefore, for our choice of the x-axis, we get

$$\vec{\mathbf{E}}_1(\vec{\mathbf{r}}) + \vec{\mathbf{E}}_2(\vec{\mathbf{r}}) = \frac{1}{4\pi\varepsilon_0}\frac{2q}{a^2}\hat{\mathbf{i}} \qquad \text{(ii)}$$

The magnitude of the electric field due to the third charge $-2q$ is

$$E_3(\vec{\mathbf{r}}) = \frac{1}{4\pi\varepsilon_0}\frac{2q}{a^2}$$

Since the charge 3 is a negative charge, the direction of the electric field due to it is towards the charge. The net electric field $\vec{\mathbf{E}}'$ due to the charges 1 and 2 and the electric field $\vec{\mathbf{E}}_3$ due to charge 3 are shown in Fig. 5.23. Note that the tails of the vectors are placed at the point where the net electric field is to be determined. To determine the net electric field at the origin, we resolve the electric field $\vec{\mathbf{E}}_3(\vec{\mathbf{r}})$ along the x and y-axes:

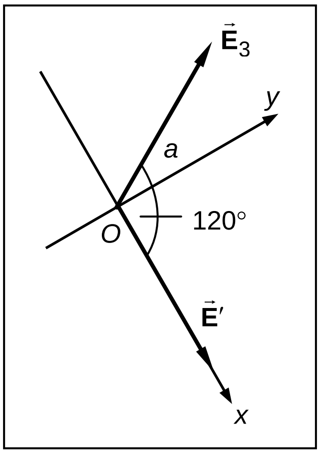


Fig. 5.23: **Electric fields for the answer of TQ 9.**

$$E_{3x} = E_3\cos 120^\circ = -\frac{E_3}{2} \quad \text{and} \quad E_{3y} = E_3\sin 120^\circ = \frac{\sqrt{3}}{2}E_3 \qquad \text{(iii)}$$

Therefore, combining the results of Eqs. (ii) and (iii) with Eq. (i), the net electric field at O is given as:

$$\vec{\mathbf{E}}(\vec{\mathbf{r}}) = \frac{1}{4\pi\varepsilon_0}\frac{2q}{a^2}\hat{\mathbf{i}} - \frac{1}{2}\frac{1}{4\pi\varepsilon_0}\frac{2q}{a^2}\hat{\mathbf{i}} + \frac{\sqrt{3}}{2}\frac{1}{4\pi\varepsilon_0}\frac{2q}{a^2}\hat{\mathbf{j}} = \frac{1}{4\pi\varepsilon_0}\frac{q}{a^2}\left(\hat{\mathbf{i}} + \sqrt{3}\,\hat{\mathbf{j}}\right)$$

The magnitude of the net electric field is given by

$$E(\vec{\mathbf{r}}) = \sqrt{E_x^2 + E_y^2} = \frac{1}{4\pi\varepsilon_0}\frac{q}{a^2}\sqrt{1+3} = \frac{1}{2\pi\varepsilon_0}\frac{q}{a^2}$$

10. Refer to Fig. 5.24 showing the four charges A, B, C, D, viz. $+2q$, $+2q$, $-2q$ and $-2q$ placed at the vertices of a rectangle of sides $AB = 3.0\,\text{m}$ and $BC = 4.0\,\text{m}$. The net electric field due to the charges at the point of intersection of the diagonals is the vector sum of the electric fields of the respective charges at that point. Let us choose the x and y-axes as shown in the figure. The length of the diagonal of the rectangle is $\sqrt{(3.0)^2 + (4.0)^2}\,\text{m} = 5.0\,\text{m}$. Note from Fig. 5.24 that the electric fields due to the charges placed at the vertices A and C point in the same direction since the charges are unlike. So is the case for the charges placed at the vertices B and D. The magnitudes of the electric fields due to all four charges are the same since the magnitudes of the charges are equal and their distances from the point P are equal. Thus, the magnitude of the electric field due to each charge is given by

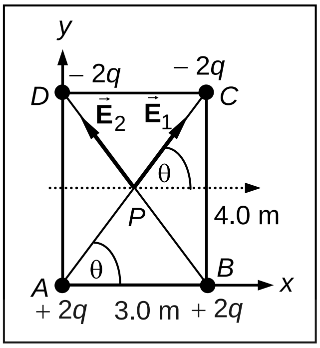


Fig. 5.24: **Diagram for the answer of TQ 10.**

$$E = \frac{1}{4\pi\varepsilon_0}\frac{|2q|}{r^2} = 8.99\times10^9\,\text{Nm}^2\,\text{C}^{-2}\,\frac{6.0\times10^{-9}\,\text{C}}{(2.5\,\text{m})^2} = 8.6\,\text{NC}^{-1}$$

The net electric field is the resultant of the electric fields $\vec{\mathbf{E}}_1$ and $\vec{\mathbf{E}}_2$ shown in Fig. 5.24 with their tails at the point P. Note that their magnitudes are:

$$E_1 = E_2 = 2E$$

Note also from Fig. 5.24 that the x-components of these electric fields are equal and opposite so they cancel out. Their y-components are equal in magnitude and in the same direction and are given by:

$$E_{1y} = E_{2y} = E_1 \sin\theta = 2E\frac{BC}{AC} = 2\times8.6\,\text{NC}^{-1}\,\frac{4}{5} = 13.8\,\text{NC}^{-1}$$

So, the magnitude of the net electric field is

$$E = E_{1y} + E_{2y} = 13.8\,\text{NC}^{-1} + 13.8\,\text{NC}^{-1} = 28\,\text{NC}^{-1}$$

up to 2 significant digits. It is directed along the y-axis.

Gauss's law is used to find the electric fields in symmetrical capacitors. The Earth is a huge spherical capacitor that we use all the time. How do we do so? You will learn the answer in this unit!

UNIT 6

GAUSS'S LAW AND APPLICATIONS

Structure

STUDY GUIDE

In Unit 5, you have studied the concepts of charge, electrostatic force, Coulomb's law, electric field and calculated the electrostatic force on charges, electric field of point charges and continuous line charge.

In this unit, you will study Gauss's law that simplifies the calculation of electric fields and electrostatic forces for distributions of discrete point charges and symmetric continuous charge distributions. You will learn how to apply Gauss's law to a point charge and spherically symmetric systems like uniformly charged sphere and spherical shell for which the electric field has spherical symmetry. You have learnt about the divergence theorem in Unit 4, which you will also use in this unit. You should revise Units 1 to 4 of this course as you will be using them all the time to learn the concepts of this unit. Of course, you should also know the concepts of vector algebra thoroughly. We advise you to solve the SAQs and Terminal Questions given in this unit. You should study all sections of this unit thoroughly and make sure you can solve the SAQs and Terminal Questions on your own.

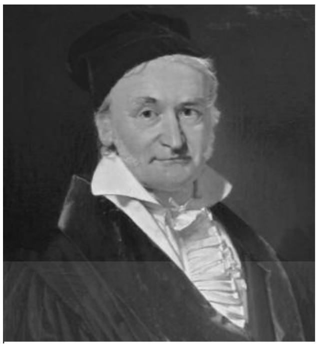
"All the measurements in the world do not balance one theorem by which the science of eternal truths is actually advanced."

Carl F. Gauss

6.1 INTRODUCTION

In Unit 5, you have revised the concept of charge and Coulomb's law. You have learnt the concept of electric field and calculated the electric field due to point charges and continuous line charge. You have also learnt how to calculate the electrostatic force on a charge kept in any given electric field.

This is what electrostatics is about: Calculating electric fields due to charges and electrostatic forces on a charge or distribution of charges placed in an electric field. You also saw how involved the calculation of the electric field of a line charge was. Would you not like to learn simpler methods for doing these calculations? This is what we do in the rest of this block. Most of this block involves **learning the tools** that simplify the calculation of electric fields and electrostatic forces.



Carl Friedrich Gauss (1777 – 1855), a German mathematician and physicist, is referred to as the 'greatest mathematician since antiquity'. He made exceptional contributions in the areas of mathematics such as algebra, number theory, analysis, differential geometry, and physics such as mechanics, electrostatics, magnetic fields, optics, etc. He is known as one of history's most influential mathematicians with equally significant contributions in physics.

In this unit, we describe an alternative to Coulomb's law and the principle of superposition to help us determine electric fields of discrete charges and charge distributions. This is the **Gauss's law** which relates electric charge distributions and electric fields. It gives us a simpler method to determine electric fields associated with symmetric charge distributions. If we know the electric fields in any region, we can also use the law to determine the net charge of the charge distributions that give rise to them.

We begin our study of Gauss's law by defining a new quantity called **electric flux** (Sec. 6.2). We then present the law in Sec. 6.3. You will learn that Gauss's law is particularly useful when applied to systems that possess some symmetry, a concept that you may know but will learn again in this unit. In Sec. 6.4, we apply the law to **spherically symmetric** systems and determine the electric fields due to a point charge, a uniform spherical charge distribution and a uniformly charged spherical shell.

You may ask: Why is it important for you to learn these applications of Gauss's law? One of the most important uses of these applications is in calculating the electric fields in capacitors and consequently their capacitances. You would know from your school physics that capacitors are important devices used to store electric charge and electrical energy. You will learn in detail about them in Unit 11 of Block 3. The Earth is one huge spherical capacitor that we use all the time as you will learn in Sec. 6.5.

In the next unit, we continue the discussion on Gauss's law for systems having cylindrical and planar symmetry such as a uniform line charge, a uniformly charged cylinder and a plane sheet of charge. You will learn some more applications of the law and then you will be able to appreciate the power of this law.

Expected Learning Outcomes

After studying this unit, you should be able to:

- define electric flux and calculate the electric flux due to an arbitrary distribution of charges;
- state Gauss's law;

- apply Gauss's law to calculate the electric field due to a point charge;
- apply Gauss's law to calculate the electric field due to a uniformly charged sphere; and
- using Gauss's law, determine the electric field due to a uniformly charged spherical shell.

6.2 ELECTRIC FLUX

You have learnt the concept of flux of a vector field in Sec. 4.2.1 of Unit 4 of this course. Here we briefly explain the concept again so that you can understand the concept of electric flux. You know that flux is defined for any vector field but is most easily pictured for the flow of fluids. So, we begin the discussion with a brief revision of the concept of flux for fluid flow.

The word flux has its origins in the old French word 'flus' and the Latin word 'fluxus' both meaning 'flowing' or 'to flow'. When we say that something is in the state of flux, we mean that it is changing.

Imagine that a stream of water or some fluid is flowing and the velocity of the particles in it is described by the velocity vector field. We now place a very small flat wire loop of area dS in the stream so that it is normal (perpendicular) to the direction of the flow (Fig. 6.1a). We choose this flat element of area to be small enough so that the velocity of all fluid particles flowing through it is constant. The **volume flux** of the fluid through the loop is defined as the **rate of flow** of the fluid through the area (of the loop). Let us determine its value.

Suppose ΔV is the volume of the fluid that passes through the small loop of area dS in time Δt. Since its area is flat and very small, we can take the speed v of the small amount of fluid flowing through it to be constant. So, during the time interval Δt, the fluid moves a length $\Delta x = v\Delta t$. The volume of fluid that flows through the loop during that time interval is then given by

$$\Delta V = dS\Delta x = dS\, v\, \Delta t \qquad (6.1)$$

So, the rate of flow of fluid through the very small area dS is given by

$$\frac{\Delta V}{\Delta t} = v\, dS \qquad (6.2a)$$

This is just the **volume flux** of the fluid when the small area chosen is **normal** (**perpendicular**) to the direction of its flow.

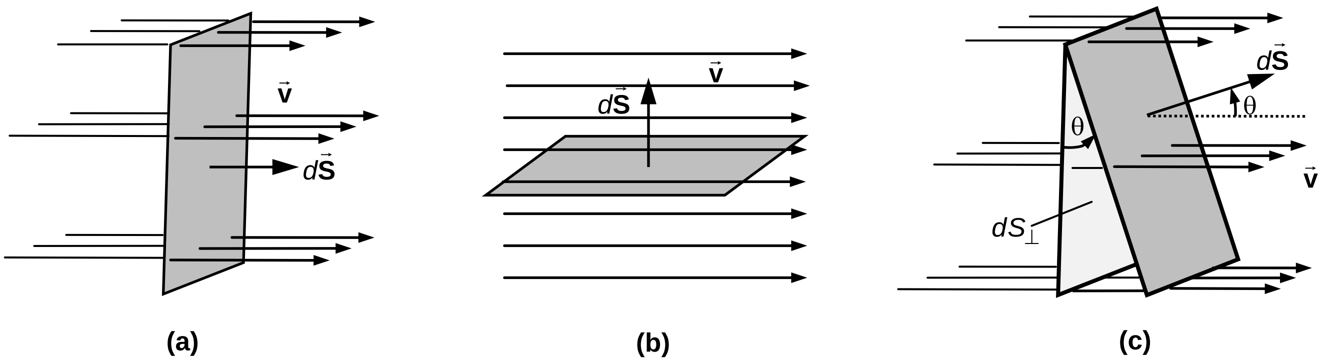


Fig. 6.1: **A wire loop placed in a stream a) normal and b) parallel to the direction of the flow or the velocity field $\vec{v}$; c) the same loop placed at an angle θ to the direction of fluid flow. In parts (a) and (c) of this figure, we have shown only a few lines for the fluid flow but the loop is immersed in the stream.**

What would the flux be if we kept the loop parallel to the direction of fluid flow as shown in Fig. 6.1b? You can see that **no fluid will now flow across the wire loop** or across the area dS. So, the volume flux will be zero in this case.

What would the flux be if we kept the loop at some angle θ to the direction of fluid flow as shown in Fig. 6.1c?

In this case, the fluid will pass through only that component of the area, which is perpendicular to the direction of fluid flow. This is just $dS_{\perp} = dS\cos\theta$.

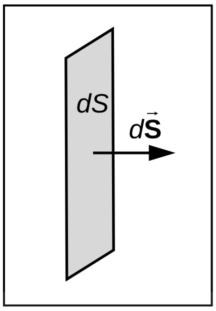


Fig. 6.2: Area vector $d\vec{\mathbf{S}}$ for any surface of area dS is directed normal to the surface (refer to Sec. 4.3 of Unit 4 for the sense of the normal vector to the surface).

Therefore, substituting $dS\cos\theta$ for dS in Eq. (6.2a), the **volume flux** through the loop kept at an angle θ to the direction of the fluid flow will be

$$v\,dS_{\perp} = v\,dS\cos\theta \tag{6.2b}$$

Now, we use the definition of the scalar product to express the volume flux given by Eqs. (6.2a and b) as

$$\Phi_{dS} = \vec{\mathbf{v}}.d\vec{\mathbf{S}} \tag{6.3}$$

where $\vec{\mathbf{v}}$ is the velocity field and $d\vec{\mathbf{S}}$, the area vector corresponding to the area dS of the loop (see Fig. 6.2). The area vector gives the magnitude of the area and its direction gives the sense of the flux through the area. In our example (Figs. 6.1a and c), the sense of the flux is from left-hand side of the loop to its right-hand side. If we choose the direction of the area vector to be opposite to this, i.e., from right to left, the sense of the flux would also be from the right-hand side of the loop to its left-hand side. We can choose either direction for the area vector but once chosen, it should remain the same and be specified.

Note that the scalar product of Eq. (6.3) reflects all three situations we have considered: When the loop is normal to the flow, $\theta = 90^{\circ}$ and Eq. (6.3) gives the volume flux as $v\,dS$, which is just Eq. (6.2a). If the loop is parallel to the flow, $\theta = 0^{\circ}$ and the flux through the loop is zero. For any other value of θ, Eq. (6.3) gives the volume flux as $v\,dS\cos\theta$, which is just Eq. (6.2b).

The definition of volume flux can be extended to the flux of any vector field including the electric field. **In an electrostatic field, nothing is flowing** but we define the flux of the electric field in analogy to Eq. (6.3).

By definition, the **electric flux** $d\Phi_E$ of an electric field $\vec{\mathbf{E}}$ through a small flat surface of area dS is defined as

$$d\Phi_E = \vec{\mathbf{E}}.\,d\vec{\mathbf{S}} \tag{6.4}$$

where $d\vec{\mathbf{S}}$ is the area vector of magnitude dS **directed normal to the surface. Its orientation is defined to be outward to the surface**. Note that electric flux is a scalar quantity.

In Eq. (6.4), we have considered a small flat surface of area dS to define electric flux. You may ask: **What is the electric flux through a surface of any arbitrary shape**?

In that case, we divide the surface into a large number (say n) of small flat surfaces represented by area vectors $d\vec{\mathbf{S}}_i$, all pointing outwards from the same side of the surface. Let $\vec{\mathbf{E}}_i$ be the electric field through the element of surface area $d\vec{\mathbf{S}}_i$. Since flux is a scalar quantity, the electric flux through the surface S is just the sum of the electric flux through all such flat surfaces:

$$\Phi_E = \sum_{i=1}^{n} \vec{\mathbf{E}}_i \,.\, d\vec{\mathbf{S}}_i \tag{6.5}$$

We then make the sizes of the flat surfaces smaller and smaller so that $n \to \infty$ and collectively these surface elements approach the surface S. Then as you have learnt in Unit 4, the sum given in Eq. (6.5) approaches a limiting value which is equal to the electric flux through the surface S. In that limit, we can write the sum as a two-dimensional surface integral and the electric flux is given by

$$\Phi_E = \lim_{n \to \infty} \sum_i \vec{\mathbf{E}}_i \,.\, d\vec{\mathbf{S}}_i = \iint_S \vec{\mathbf{E}}.d\vec{\mathbf{S}} \tag{6.6a}$$

As you have learnt in Unit 4, the subscript S under the integral sign tells us that the area of integration is the entire surface S. If the surface is closed, we write the surface integral and Eq. (6.6a) as follows:

$$\Phi_E = \oiint_S \vec{\mathbf{E}}.d\vec{\mathbf{S}} \tag{6.6b}$$

In Unit 4, you have learnt how to determine surface integrals for different cases. From Eqs. (6.6a and b), you can see that electric flux is expressed as a surface integral. You may now like to determine the electric flux of an electric field through a surface using Eq. (6.6b). We take up the example of calculating the electric flux of a point charge through a closed surface. In the process, we shall arrive at Gauss's law.

EXAMPLE 6.1 : ELECTRIC FLUX OF A POINT CHARGE

Determine the electric flux for the electric field generated by a point charge q through a closed surface S of a sphere of radius R enclosing the charge such that the charge is placed at the centre of the sphere.

SOLUTION ■ We use Eq. (6.6b) to determine the electric flux through the surface of a sphere (of radius R) enclosing the charge q. From Eq. (6.6b), the electric flux through a closed surface is given by

$$\Phi_E = \oiint_S \vec{\mathbf{E}}.d\vec{\mathbf{S}}$$

where S is the surface of a sphere of radius R enclosing the charge q, which is kept at its centre. The electric field of the charge q at a point on the surface of the sphere is given from Eq. (5.6a) as

$$\vec{\mathbf{E}} = \frac{1}{4\pi\varepsilon_0} \frac{q}{R^2} \hat{\mathbf{r}}$$

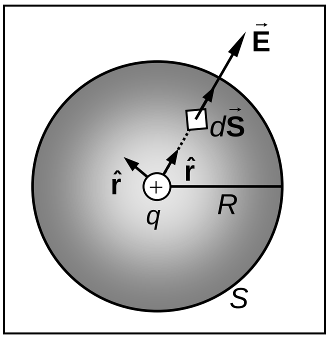


Fig. 6.3: **Calculation of the electric flux through a spherical surface enclosing charge q.**

where $\hat{\mathbf{r}}$ is the unit vector along the radial direction. Now, for a sphere, the direction of the area vector $d\vec{\mathbf{S}}$ is along the outward normal to its surface at all points on the surface. From Fig. 6.3 (showing one such point), you can see that it is along the vector $\hat{\mathbf{r}}$. Thus, we have

$$d\vec{\mathbf{S}} = dS\,\hat{\mathbf{r}} \qquad \text{and} \qquad \vec{\mathbf{E}}.d\vec{\mathbf{S}} = E\,dS\,\hat{\mathbf{r}}.\hat{\mathbf{r}} = E\,dS$$

The electric flux of the point charge through the sphere's surface is then

$$\oiint_S \vec{\mathbf{E}}.d\vec{\mathbf{S}} = \frac{q}{4\pi\varepsilon_0 R^2}\iint dS = \left(\frac{q}{4\pi\varepsilon_0 R^2}\right)\times 4\pi R^2 = \frac{q}{\varepsilon_0} \qquad (6.7)$$

Did you note in Example 6.1 that the radius of the sphere cancels out? This is because while the field **decreases** as $\dfrac{1}{r^2}$, the surface area **increases** as r^2. So, their product is constant. **REMEMBER**: This result arises because of the **inverse square nature of the electrostatic force field and the electric field**.

Also note that we have obtained Eq. (6.7) in Example 6.1 for the electric flux of a point charge across a **spherical surface** enclosing the charge. However, it is true for a surface of any shape enclosing a charge. This is what Gauss's law is about. So you will study it in greater detail in the next section. But before that, you may like to attempt an SAQ to determine electric flux for another simple situation.

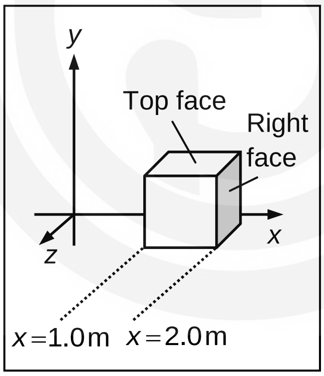
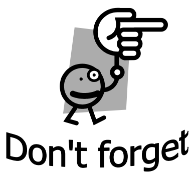


Fig. 6.4: **Diagram for SAQ 1.**

SAQ 1 - Electric flux

A cube of side 1.0 m is kept in an electric field (in units of NC^{-1}) given by $\vec{\mathbf{E}} = 8.0x\,\hat{\mathbf{i}} + 5.0\,\hat{\mathbf{j}}$ as shown in Fig. 6.4. Determine the electric flux through the right and top faces of the cube.

You should always remember the following about electric flux.



- **Electric flux through a surface (of area S) represents the summation of electric flux elements $(\vec{\mathbf{E}}.d\vec{\mathbf{S}})$ over the entire surface.**
- **Each electric flux element represents the product of a small flat element of area on the surface with the <u>component of the electric field along the normal</u> to that area element.**
- **This product is nothing but the scalar product of the electric field vector and the area element vector.**
- **Electric flux does not represent flow or change the way volume flux does.**

Let us now study Gauss's law.

6.3 GAUSS'S LAW

In Example 6.1, we have enclosed a point charge in a spherical surface and arrived at Eq. (6.7), which relates the electric flux through a spherical surface to the point charge q enclosed by it. This is just Gauss's law for a point charge. However, we have enclosed the point charge in a spherical surface, which is a special case. **Gauss's law applies to any arbitrary surface enclosing a charge or charge distribution**. Any imaginary surface enclosing a charge or a charge distribution is called a **Gaussian surface**. We usually choose the Gaussian surface so that our calculations become easier.

Therefore, in this section, we first generalise Eq. (6.7) for any arbitrary surface enclosing the point charge and arrive at a formal statement of Gauss's law. So, let us find out whether the same equation [Eq. (6.7)] applies to **any arbitrary surface** enclosing a point charge.

Consider the electric field of a positive point charge in free space. Imagine that the charge is enclosed in a **closed Gaussian surface** S of an **arbitrary shape** (Fig. 6.5).

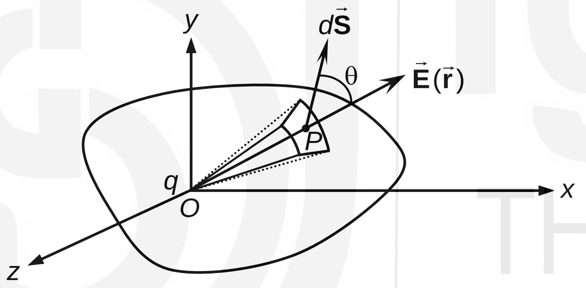


Fig. 6.5: **Gauss's law for a point charge enclosed by an arbitrary surface.**

Note from Fig. 6.5 that we have chosen the origin of the coordinate system to be at the location of the charge. Let P be a point on the Gaussian surface, having position vector $\vec{\mathbf{r}} = r\,\hat{\mathbf{r}}$. We choose a small element of area $d\vec{\mathbf{S}}$ centred at the point P on the Gaussian surface. As you know from Eq. (5.6a), the electric field at the point P is given by

$$\vec{\mathbf{E}} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{\mathbf{r}} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\frac{\vec{\mathbf{r}}}{r} \qquad (6.8)$$

Then from Eq. (6.4), the element of electric flux passing through $d\vec{\mathbf{S}}$ is given by

$$d\Phi_E = \vec{\mathbf{E}}.d\vec{\mathbf{S}} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^3}\vec{\mathbf{r}}.d\vec{\mathbf{S}} \qquad (6.9)$$

Now, you know that if θ is the angle between $\vec{\mathbf{r}}$ and $d\vec{\mathbf{S}}$, then

$$\vec{\mathbf{r}}.\,d\vec{\mathbf{S}} = r\,dS\cos\theta \qquad (6.10a)$$

You also know from vector algebra that $dS\cos\theta$ is the projection of $d\vec{\mathbf{S}}$ along $\vec{\mathbf{r}}$. From Sec. 4.3.5 of Unit 4, you know that the quantity

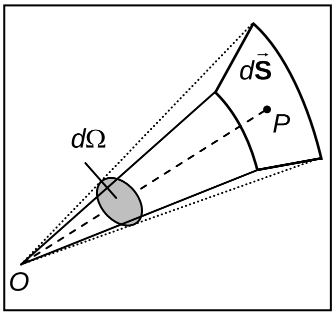


Fig. 6.6: **The solid angle $d\Omega$ subtended by an area element $d\vec{\mathbf{S}}$ at a point O. Recall Sec. 4.3.5 of Unit 4 for the definition of solid angle.**

$$\left(\frac{dS\cos\theta}{r^2}\right) = d\Omega \tag{6.10b}$$

is defined as the solid angle ($d\Omega$) subtended by the area $d\vec{\mathbf{S}}$ at O, the location of the charge (Fig. 6.6). Then using Eq. (6.10b), we can write Eq. (6.9) as

$$d\Phi_E = \vec{\mathbf{E}}.d\vec{\mathbf{S}} = \frac{q}{4\pi\varepsilon_0}\frac{r\,dS\cos\theta}{r^3} = \frac{q}{4\pi\varepsilon_0}d\Omega \tag{6.11a}$$

The total electric flux through the surface S is determined by integrating over the entire closed surface as follows:

$$\Phi_E = \oiint_S \vec{\mathbf{E}}.d\vec{\mathbf{S}} = \frac{q}{4\pi\varepsilon_0}\oiint_S d\Omega \tag{6.11b}$$

Now since the surface S surrounds the point O and the total solid angle around any point is 4π (see Sec. 4.3.5 of Unit 4), we have

$$\oiint_S d\Omega = 4\pi \tag{6.11c}$$

So, we can write Eq. (6.11b) as

$$\Phi_E = \oiint_S \vec{\mathbf{E}}.d\vec{\mathbf{S}} = \frac{q}{\varepsilon_0} \tag{6.12}$$

Eq. (6.12) is the same as Eq. (6.7) for a spherical surface. Let us see whether we can extend Eq. (6.12) to a distribution of charges. Suppose that instead of a single charge at the centre of a sphere, many charges are situated in some region of space. From the principle of superposition [Eq. (5.11)], you know that the net electric field is the vector sum of all individual electric fields:

$$\vec{\mathbf{E}} = \sum_j \vec{\mathbf{E}}_j \tag{6.13}$$

By definition [Eq. (6.6b)], the electric flux through a closed surface that encloses all these charges is given by

$$\Phi_E = \oiint_S \vec{\mathbf{E}}.d\vec{\mathbf{S}} = \sum_j\left(\oiint_S \vec{\mathbf{E}}_j.d\vec{\mathbf{S}}\right) = \frac{1}{\varepsilon_0}\sum_j\left(q_j\right) \tag{6.14}$$

where we have substituted $\vec{\mathbf{E}}$ from Eq. (6.13) and used Eq. (6.12) for individual charges, i.e., we have written

$$\oiint_S \vec{\mathbf{E}}_j.d\vec{\mathbf{S}} = \frac{q_j}{\varepsilon_0} \tag{6.15a}$$

Let us write the sum of all charges enclosed by the surface as Q_{encl}, i.e., Q_{encl} is the total or net charge enclosed by the surface S:

$$Q_{encl} = \sum_j\left(q_j\right) \tag{6.15b}$$

Then, we can write Eq. (6.14) as follows:

$$\oint\!\!\oint_S \vec{\mathbf{E}}.d\vec{\mathbf{S}} = \frac{Q_{encl}}{\varepsilon_0} \qquad (6.16)$$

Eq. (6.16) is the quantitative statement of **Gauss's law**. Let us now give a formal statement of Gauss's law.

GAUSS'S LAW

Gauss's law states that the net electric flux through any imaginary closed surface *S* (called the **Gaussian surface**) is directly proportional to the net charge (Q_{encl}) enclosed by the surface. In SI units, it is equal to $\frac{Q_{encl}}{\varepsilon_0}$.

The net charge is the algebraic sum (sum with sign of the charge included) *of all charges enclosed within the Gaussian surface*.

Mathematically, we write the law as

$$\oint\!\!\oint_S \vec{\mathbf{E}}.d\vec{\mathbf{S}} = \frac{Q_{encl}}{\varepsilon_0} \qquad (6.16)$$

What Eq. (6.16) tells us is that the flux of the electric field through any surface would be the same regardless of its shape. It is proportional to the charge enclosed by it. This point is easier to visualise for a point charge if you picture its electric field in terms of the field lines passing through a surface. A surface of any shape enclosing the charge would have the same number of field lines passing through as that of the sphere's surface (Fig. 6.7). So the electric flux through any surface enclosing charge *q* is $\frac{q}{\varepsilon_0}$.

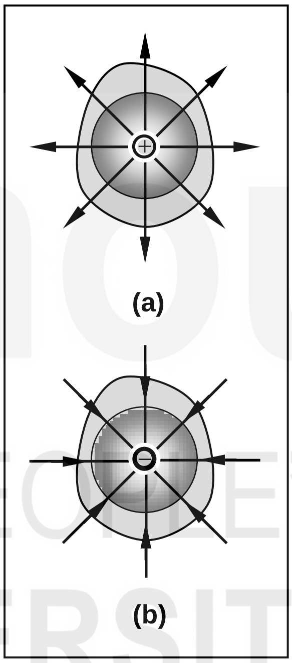


Fig. 6.7: **The same number of electric field lines will pass through surfaces of different shapes. Two Gaussian surfaces, one spherical and the other of arbitrary shape, are shown here for positive and negative charges**.

Eq. (6.16) is the integral form of Gauss's law. We can write **Gauss's law in the <u>differential</u> form** using the divergence theorem, which you have studied in Unit 4. For this, we write the charge enclosed by a surface in terms of the volume charge density ρ and substitute it in Eq. (6.16). Then we get

$$Q_{encl} = \iiint_V \rho\, dV \qquad (6.17a)$$

And

$$\oint\!\!\oint_S \vec{\mathbf{E}}.d\vec{\mathbf{S}} = \frac{1}{\varepsilon_0}\iiint_V \rho\, dV \qquad (6.17b)$$

Now you may recall the divergence theorem from Unit 4 given as

$$\oint\!\!\oint_S \vec{\mathbf{E}}.d\vec{\mathbf{S}} = \iiint_V \vec{\nabla}.\vec{\mathbf{E}}\, dV \qquad (6.17c)$$

We substitute the value of $\oint\!\!\oint_S \vec{\mathbf{E}}.d\vec{\mathbf{S}}$ from Eq. (6.17c) in the left hand side of Eq. (6.17b).

Then Eq. (6.17b) becomes

$$\iiint_V \vec{\nabla}.\vec{\mathbf{E}}\, dV = \frac{1}{\varepsilon_0} \iiint_V \rho\, dV \qquad (6.17d)$$

Since Eq. (6.17d) holds for any volume, the integrands must be equal and we have:

$$\vec{\nabla}.\vec{\mathbf{E}} = \frac{1}{\varepsilon_0} \rho \qquad (6.18)$$

Eq. (6.18) gives **Gauss's law in its differential form**.

It is easier to apply Gauss's law in its differential form. **However, note that we have expressed it only for volume charge density. Since the integral form of Gauss's law can be applied to point, line, surface and volume charges, it has wider use.**

In the next section, we consider some applications of Gauss's law to spherically symmetric systems. But before that you may like to remember the following aspects of Gauss's law and then try an SAQ to check your understanding.

- **In Eq. (6.16),** Q_{encl} is the **net charge** enclosed by the surface **taking into account the algebraic sign of the charges** (in case of many charges)**. So, if a surface encloses equal and opposite charges, the net electric flux through it is zero.**
- From the statement of Gauss's law, it is clear that the **charges lying outside the closed surface are not included in** Q_{encl}. **If the closed surface does not enclose any net charge, or if all charges lie outside the closed surface, then the electric flux through the surface is zero**. This implies that the electric field through such a surface is zero.
- We can **calculate the net charge enclosed inside any closed surface** using this law if we know the net electric flux through the surface enclosing the charges.
- **The form and location of the charges inside the closed surface do not matter in the calculations**. **What matters is the total charge enclosed by the closed surface and its sign**. This very fact makes the calculation of electric fields using the Gauss's law far easier in comparison with Coulomb's law.
- Gauss's law essentially follows from Coulomb's law and the principle of superposition. It contains no additional information that was not already present in Coulomb's law. The law follows from the inverse square nature of the electrostatic force. Without that, the cancellation of r^2 would not take place. Then the total flux would also depend on the surface chosen and not only upon the charge enclosed.

SAQ 2 - Gauss's law

a) Can we apply Gauss's law to the surfaces shown in Figs. 6.1a, b and c?

b) A point charge is enclosed by a spherical Gaussian surface. Would the electric flux through the surface change

   i) If the Gaussian surface is chosen to be a closed cylinder or a cube?

   ii) If the sphere is replaced by a cube that has one-tenth of its volume?

   iii) If the charge is located at some other point within the sphere instead of its centre?

   iv) If the charge is moved outside the Gaussian surface?

   v) If another charge is placed inside the Gaussian surface?

   vi) If another charge is placed outside the Gaussian surface?

c) The electric flux through a closed spherical Gaussian surface of radius 0.5 m surrounding a charged particle is equal to $500\,\text{Nm}^2\,\text{C}^{-1}$. Determine the value of the charge on the particle. If the radius of the surface were to be halved, what would the value of the electric flux through it be?

d) Determine the net electric flux through two overlapping closed surfaces S_1 and S_2 shown in Fig. 6.8, given that the values of the charges on the three particles are $q_1 = +3.1\,\text{nC}$, $q_2 = -5.9\,\text{nC}$ and $q_3 = -3.1\,\text{nC}$. The particle P enclosed by the surface S_1 carries no charge.

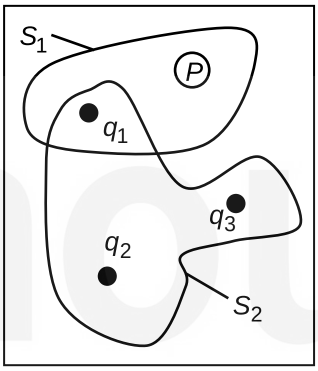


Fig. 6.8: **Diagram for SAQ 2d.**

You may be wondering: **Why do we need another method for calculating electric fields when we already have Coulomb's law**? This is because we can use Gauss's law to calculate the electric fields due to symmetric charge distributions in a much simpler way. You will discover in the next section and the next unit that Gauss's law is a powerful tool for determining electric fields of symmetric continuous charge distributions. Let us explain this point further.

6.3.1 Gauss's Law and Symmetric Charge Distributions

Let us first explain: **What are symmetric charge distributions**?

Symmetric charge distributions are arrangements of charges that **remain unchanged (or invariant) or look the same after a transformation**.

These charge distributions could be translated along some axis, reflected or rotated about some axis and would still appear the same.

Symmetry in physics essentially means that a system or an object remains unchanged (or invariant) under some transformation. You may already know of several examples of symmetric objects, e.g., a straight line, square, plane, sphere, cylinder, etc.

Due to the symmetries of charge distributions, the calculations of electric flux and electric fields due to them become far easier.

We will be dealing with three kinds of symmetry while applying Gauss's law:

1. Spherical symmetry
2. Cylindrical symmetry
3. Planar symmetry

We will talk about each of these symmetries when we apply Gauss's law to symmetric charge distributions in this unit and the next unit.

In the next three sections of this unit, you will learn how to apply Gauss's law. We will determine the electric field due to a point charge. We will also determine the electric fields due to **spherically symmetric** charge distributions such as a uniformly charged sphere and a spherical shell carrying uniform charge using Gauss's law. In the next unit, we will apply Gauss's law to infinitely long line of uniform charge, which has cylindrical symmetry and a plane sheet of charge having planar symmetry. So we will explain both these symmetries in the next unit.

Here we answer the question: **What is a spherically symmetric charge distribution**?

A charge distribution is said to be **spherically symmetric** if it remains **invariant** (**the same**)

- when it is **rotated around any axis passing through its centre**. It is said to possess **rotational symmetry** about that axis.
- when it is **reflected across any plane passing through its centre**. This is the **reflection symmetry**.

For such spherically symmetric charge distributions, we choose a spherical Gaussian surface. For a point charge, the centre of the Gaussian surface lies at the position of the charge. For a spherical charge distribution or a spherical shell, the Gaussian surfaces are concentric with them.

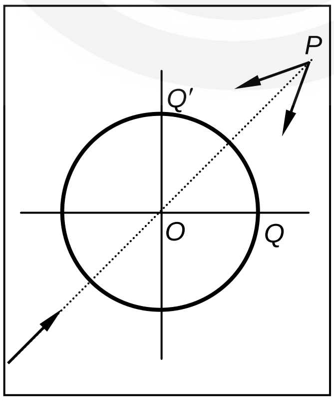


Fig. 6.9: **If the electric field is not radially directed, it will not remain the same under rotation or any other symmetry transformation of the sphere.**

The electric field of a spherically symmetric charge distribution is in the radial direction. It points outward from the centre of the sphere for positive charge and inward for negative charge. The magnitude of the electric field depends only on the distance r from the centre of the sphere. You may ask: Why is it so? Let us answer this question for both the direction and the magnitude of the electric field due to a spherically symmetric charge distribution.

Let us first answer the question: **Why is the electric field due to a spherically symmetric charge distribution directed radially i.e., it either points outward from the centre of the sphere, or inward along the radius of the sphere**?

Suppose the electric field at some point P outside the sphere **is not directed radially**, i.e., along the radius of the sphere. Suppose it points in some other direction, say in the direction of a point Q on the sphere's surface along the line PQ (see Fig. 6.9). Now suppose we rotate the sphere around the sphere's axis that passes through point P by 180°. The point Q shifts to position Q' on the sphere. Note that the sphere remains exactly the same and the point P

would also be in the same place. But the electric field would now point in a different direction – in the direction of Q' along the line PQ'.

This is a contradiction because you know that the electric field **at the same point due to the same charge distribution has to be in the same direction**; it cannot be in two different directions. When will the electric field **at any point** be in the same direction under any symmetry operation performed on the spherical charge distribution? This will happen only **if the electric field is directed along the axis of rotation of the sphere passing through that point**. This means that it must point along the axis of rotation (or the radius) of the sphere, i.e., in the **radial direction**.

Let us now answer the question: **Why does the magnitude of the electric field due to a symmetric charge distribution at any point depend only on its distance *r* from the centre of symmetry**?

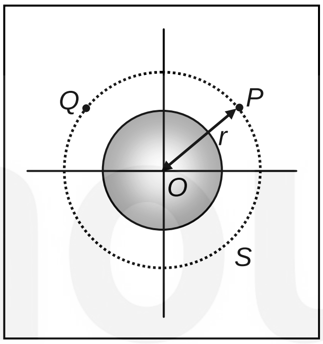


Fig. 6.10: **The magnitude of the electric field at any point *P* on the spherical surface *S* depends only on the radius *r* of the surface, i.e., the radial coordinate of *P*. Due to spherical symmetry, it is independent of the angular coordinates of the point.**

Study Fig. 6.10. Suppose we have to determine the electric field at a point *P* at a distance *r* from the sphere. Consider a spherical surface *S* of radius *r* passing through that point, concentric with the spherical charge distribution. Now, consider any two points *P* and *Q* on the surface *S*. Note that these two points have the same radial coordinate but different angular coordinates.

Let us now ask: **What would happen if the magnitude of the electric field depended on the angular coordinates of the points** *P* and *Q*? If this were so, the magnitude of the electric field due to the spherical charge distribution would be different at these two points.

But this is a contradiction **because due to spherical symmetry, the spherical charge distribution looks the same for all points on** *S* and hence for **both these points.** Therefore, **for the same charge distribution**, **the magnitude of the electric field cannot be different for different points on** *S*. It has to be the same for **all points** on the spherical surface *S*, i.e., all points at the same distance *r* from the centre of the charge distribution.

Hence, the **magnitude of the electric field at any point** on the spherical surface *S* (of a fixed radius) **cannot depend on the angular coordinates of that point**. It will only depend on the radius of the spherical surface, i.e., the radial coordinate of the point, which is just the distance of the point from the centre of the charge distribution. Therefore, we have

$$E(\vec{\mathbf{r}}) = E(r) \text{ for a spherically symmetric charge distribution}$$

So, all points on the spherical surface *S* of radius *r* are equivalent as far as the magnitude of the electric field is concerned. You must always remember the following for any spherically symmetric charge distribution.

- **The electric field due to the spherical charge distribution is directed radially.**
- **The magnitude of the electric field at any point depends only on the distance *r* of the point from the centre of the charge distribution.**

Let us now apply Gauss's law to determine the electric field due to a point charge.

6.4 ELECTRIC FIELD DUE TO A POINT CHARGE

Using Gauss's law, let us determine the electric field due to a positive point charge q at point P situated at a distance r from the charge.

We use Gauss's law given by Eq. (6.16) taking $Q_{encl} = q$.

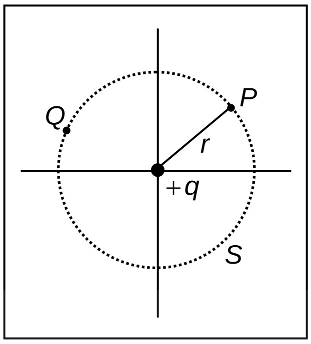


Fig. 6.11: **Spherical Gaussian surface *S* for determining electric field due to a positive point charge.**

We draw a spherical Gaussian surface of radius r passing through the point P with the charge at the centre of the sphere (Fig. 6.11). Now, you have learnt in Sec. 6.3.1 that for spherical symmetry, the electric field points radially outwards for a positive charge, i.e., the direction of the electric field is normal to the sphere's surface. The area vector $d\vec{\mathbf{S}}$ for any surface area element of the sphere is also normal to its surface. So, it is parallel to the electric field $\vec{\mathbf{E}}$ and $\vec{\mathbf{E}}.d\vec{\mathbf{S}} = E\,dS.$ Then Gauss's law becomes

$$\oiint_S \vec{\mathbf{E}}.d\vec{\mathbf{S}} = \oiint_S E\,dS = \frac{q}{\varepsilon_0}$$

Due to spherical symmetry, the magnitude of the electric field due to the charge would be the same for all points on the spherical surface and we can take it to be constant for S. So, we can take E out of the integral and write

$$\oiint_S E\,dS = E\oiint_S dS = \frac{q}{\varepsilon_0}$$

So, the integral is just the area of the spherical surface, i.e., it is $4\pi r^2$. Thus,

$$E\,4\pi r^2 = \frac{q}{\varepsilon_0}$$

or $$E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}$$

and $$\vec{\mathbf{E}} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{\mathbf{r}} \tag{6.19}$$

Did you notice that Eq. (6.19) is the same as Eq. (5.6a) of Unit 5 that was obtained from Coulomb's law? This means that Gauss's law and Coulomb's law give us the same result for the electric field due to a point charge. Gauss's law is equally true for a distribution of charges. You have seen it in arriving at Eq. (6.16).

The result for the electric field due to a charge distribution will be the same whether we use Gauss's law or Coulomb's law to calculate it. The only difference between the two laws is this: It is easier to use Coulomb's law for a charge distribution having many discrete point charges. But it is far easier to use Gauss's law if the charge distributions are continuous and symmetric. You have learnt this in this section for a point charge and will learn in the next two sections and next unit for other charge distributions. Otherwise, these two laws are not independent laws but the same law expressed in different ways.

6.5 ELECTRIC FIELD DUE TO A UNIFORMLY CHARGED SPHERE

Let us now apply Gauss's law to a spherical charge distribution having uniform volume charge density. You can verify that a charged sphere possesses spherical symmetry. It remains **invariant** (**the same**)

- when it is **rotated around any axis passing through its centre**; and
- when it is **reflected across any plane passing through its centre**.

The **volume charge density** (charge per unit volume) of a spherically symmetric charge distribution such as the charged sphere is the same at all points situated at the distance *r* from its centre. At any point, it depends only on the distance of that point from the centre of the sphere and not on the direction. Thus, the volume charge density ρ of a spherically symmetric charge distribution is a function of only *r*.

You have learnt in Sec. 6.3.1 that the **magnitude of the electric field** due to a spherically symmetric charge distribution at any point depends only on *r*. The direction of the electric field is radially outward for positive charge distribution and radially inward for a negative charge distribution. Let us now apply Gauss's law to determine the electric field due to a uniformly charged sphere.

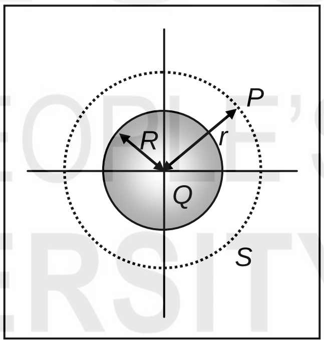


Fig. 6.12: **Determining the electric field due to a uniformly charged sphere of radius *R* carrying net charge *Q* at a point *P* outside the sphere.**

Consider a non-conducting charged sphere of radius *R* carrying total positive charge *Q* (Fig. 6.12). It is uniformly charged, which means that its volume charge density ρ is constant. Let us determine the electric field due to this charge distribution at a point *P* outside it, at a distance *r* from the centre of the sphere.

We draw a spherical Gaussian surface *S* of radius *r* through the point *P*. Since the point *P* lies outside the sphere, $r > R$ and $Q_{encl} = Q.$ From Gauss's law [Eq. (6.16)], we have

$$\oint\!\!\!\oint_S \vec{\mathbf{E}}.d\vec{\mathbf{S}} = \frac{Q}{\varepsilon_0} \tag{6.20}$$

Due to spherical symmetry, the magnitude of the electric field is the same on all points on the Gaussian surface. So we can take it to be constant for this Gaussian surface. The direction of the electric field is radially outwards for the positive charge, i.e., in the same direction as $d\vec{\mathbf{S}}$. So, $\vec{\mathbf{E}}$ and $d\vec{\mathbf{S}}$ are parallel and

$$\vec{\mathbf{E}}.d\vec{\mathbf{S}} = E\,dS \tag{6.21a}$$

Since *E* (the magnitude of the electric field on the Gaussian surface) is constant, we can pull it out of the surface integral.

Therefore, Eq. (6.21a) becomes

$$\oiint_S \vec{\mathbf{E}}.d\vec{\mathbf{S}} = E\oiint_S dS = E\,4\pi r^2 = \frac{Q}{\varepsilon_0} \tag{6.21b}$$

or

$$E = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2} \quad \text{for } r \geq R \tag{6.21c}$$

The electric field is given by

$$\vec{\mathbf{E}} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{\mathbf{r}} \quad \text{for } r \geq R \tag{6.22}$$

Notice that we have included the points lying on the surface of the spherical charge distribution in the result because the Gaussian sphere of radius R would enclose the entire charge. Did you also notice that the electric field given by Eq. (6.22) is the same as that due to a point charge [given by Eq. (6.19)]? It is as if the entire charge within the spherical surface is concentrated at the centre of the sphere. **Note that this result is a consequence of spherical symmetry**. So, a uniformly charged sphere would exert the same force on a charge placed anywhere **outside** it as an equivalent single charge would.

The electric field due to a uniformly charged sphere and the electrostatic force exerted by it on a charge situated outside the sphere are the same as the electric field and electrostatic force due to a point charge (equal to the charge of the sphere) situated at its centre.

Let us now determine the electric field at a point **inside** a spherical charge distribution carrying net charge Q , i.e., at points for which $r < R$ (see Fig. 6.13).

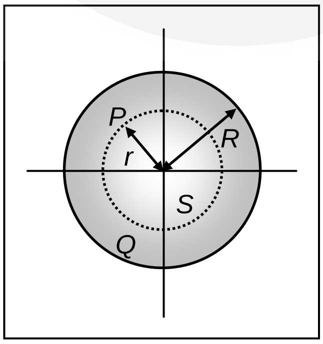


Fig. 6.13: **Determining the electric field of a uniformly charged sphere of radius** R **carrying net charge** Q **at a point** P **inside the sphere.**

For this, we draw a spherical Gaussian surface of radius $r < R$. We apply Eq. (6.20), in which Q has now to be replaced by the charge (q) enclosed by the Gaussian sphere of radius r.

What is the value of the charge enclosed by the Gaussian sphere of radius r?

You know that the volume charge density is uniform for the charged sphere of radius R, i.e., ρ is constant. The volume of the spherical charge distribution is $\frac{4\pi}{3}R^3$. Since the volume charge density ρ (charge per unit volume) is constant, for the sphere of volume $\frac{4\pi}{3}R^3$ carrying charge Q, it is given by

$$\rho = \frac{Q}{\frac{4\pi}{3}R^3} \tag{6.23a}$$

Therefore, the charge enclosed by the Gaussian sphere of volume $\frac{4\pi}{3}r^3$ will be the product of its volume with the volume charge density:

$$q = \rho \frac{4\pi}{3} r^3 = \frac{Q}{\frac{4\pi}{3} R^3}\left[\frac{4\pi}{3} r^3\right] = Q \frac{r^3}{R^3} \qquad (6.23b)$$

Using Eq. (6.23b) for q and the result $\oiint_S \vec{\mathbf{E}}.d\vec{\mathbf{S}} = E \oiint_S dS = E\, 4\pi r^2$ from Eq. (6.21b) in Eq. (6.16), we have

$$E\, 4\pi r^2 = \frac{q}{\varepsilon_0} = \frac{Q}{\varepsilon_0} \frac{r^3}{R^3}$$

or $$E = \frac{Q}{4\pi\varepsilon_0} \frac{r}{R^3} \quad \text{for} \quad r < R \qquad (6.24a)$$

The electric field at a point inside the uniformly charged sphere is given by

$$\vec{\mathbf{E}} = \frac{Q}{4\pi\varepsilon_0} \frac{r}{R^3} \hat{\mathbf{r}} \quad \text{for} \quad r < R \qquad (6.24b)$$

Note from Eqs. (6.24a) and (6.22) that the electric field inside the spherical charge distribution increases linearly with distance from its centre ($E \propto r$). However, for points outside the sphere, the electric field falls off as $\frac{1}{r^2}$. We show this behaviour of the electric field in Fig. 6.14.

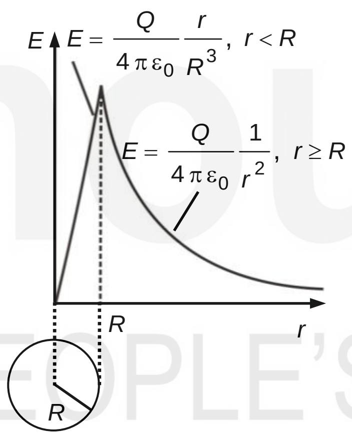


Fig. 6.14: **The behaviour of the electric field due to a uniformly charged sphere of radius *R*.**

We have said in the introduction that these results for the electric field due to a spherical charge distribution will be of use when you determine the capacitance of a spherical capacitor. As we have said on the first page of this unit, the Earth is one huge spherical capacitor that we use all the time. The Earth's capacitance is so large (~ 0.0007 F) that we can dump charge in it or take it out without changing its electric field much. That is why, we 'ground' or 'earth' the electrical circuits in our homes and all electrical appliances and instruments. That is also why we connect the lightning rods in buildings to the Earth so that most excess charge flows into it without hurting people.

Another example of spherical charge distributions is an isolated atom of inert gases. Since the atom is neutral, it carries no net charge and from Gauss's law, the electric field outside it is zero. Even when the atoms of an inert gas are in the neighbourhood of other atoms in it, they depart only slightly from spherical symmetry, and the electric fields near them remain small. So, we can say that the feeble chemical activity of inert gases is related to their spherically symmetric charge distributions. In the next section, you will learn how to apply Gauss's law to determine the electric field due to a spherical shell. Before that, you should solve an SAQ.

SAQ 3 - Applying Gauss's law to charged sphere

The electric field due to a uniformly charged sphere of radius 0.1 m has the magnitude 9.0 NC^{-1} at a distance of 0.3 m from the centre. What is the net charge on the sphere? What is the volume charge density of the charge distribution?

6.6 ELECTRIC FIELD DUE TO A UNIFORMLY CHARGED THIN SPHERICAL SHELL

As a first step, do convince yourself that a thin spherical shell possesses spherical symmetry, i.e., it remains the same under any rotation about its axis and any reflection about a plane passing through its centre and axis of rotation. You can rotate or reflect a hollow sphere with a thin surface (such as a hollow ball) to verify the spherical symmetry of a spherical shell.

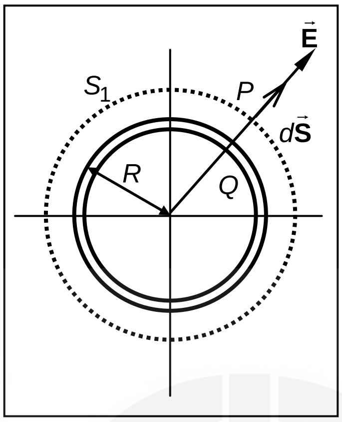


Fig. 6.15: **A thin uniformly charged spherical shell of radius *R* carrying a net charge *Q*. The cross-section of the Gaussian surface S_1 is shown for a point lying outside the shell. It is concentric with the shell.**

Now, consider a non-conducting thin spherical shell of radius *R* carrying a total positive charge *Q* that is distributed uniformly over its surface (Fig. 6.15). Let us determine the electric field due to this shell at a point lying outside it.

For a point *P* lying outside the shell, we draw a spherical Gaussian surface S_1 through the point and concentric with the spherical shell. You can see that the Gaussian surface lies outside the shell. Let us determine the electric field at the point *P* (see Fig. 6.15).

Due to the spherical symmetry of the charged spherical shell, its electric field has the same magnitude at every point on any spherical Gaussian surface and is directed radially. We apply Gauss's law [Eq. (6.16)] with $Q_{encl} = Q$ to the spherical surface S_1 and note that the electric field $\vec{\mathbf{E}}$ is in the same direction as $d\vec{\mathbf{S}}$ for S_1 so that $\vec{\mathbf{E}}$ and $d\vec{\mathbf{S}}$ are parallel. Therefore,

$$\vec{\mathbf{E}}.d\vec{\mathbf{S}} = E\,dS \tag{6.25a}$$

and since *E* (the magnitude of the electric field on the Gaussian surface) is constant, we can pull it out of the surface integral. Therefore, Eq. (6.16) becomes

$$\oint_S \vec{\mathbf{E}}.d\vec{\mathbf{S}} = E\oint_S dS = E\,4\pi r^2 = \frac{Q}{\varepsilon_0} \tag{6.25b}$$

or

$$E = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2} \quad \text{for} \quad r \geq R \tag{6.25c}$$

The electric field at any point lying outside the spherical shell of radius *R* is given by

$$\vec{\mathbf{E}} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{\mathbf{r}} \quad \text{(spherical shell, for } r \geq R\text{)} \tag{6.26}$$

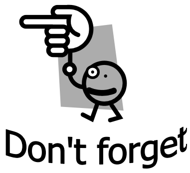
Note that the electric field given by Eq. (6.26) is the same as that due to a point charge [given by Eq. (6.19)].

For the electric field at a point lying outside the spherical shell, it is as if the entire charge *Q* of the spherical shell were replaced by a single equal charge placed at the centre of the shell.

Thus, a uniformly charged spherical shell would exert the same force on a charge placed anywhere **outside** the shell as a single equal charge would.

So always remember,

The electric field due to a spherical shell with a uniform charge distribution and the electrostatic force exerted by it on a charge situated outside the shell are the same as due to a single charge (equal to the charge of the shell) situated at its centre.



What is the electric field at a point **inside** the shell, i.e., at a point lying anywhere in the empty interior part of the shell?

For a point lying inside the shell, we draw a spherical Gaussian surface S_2 concentric with the spherical shell, lying in the empty interior of the shell (see Fig. 6.16). Since this Gaussian surface encloses no net charge, from Gauss's law, the electric field is zero at all points inside the shell:

$$\vec{\mathbf{E}} = \vec{\mathbf{0}} \qquad \text{(spherical shell, for } r < R\text{)} \qquad (6.27)$$

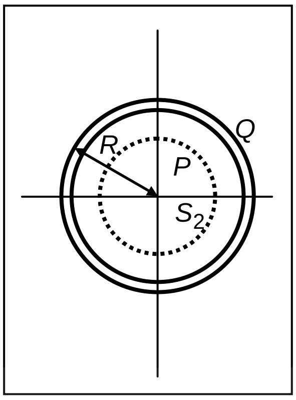


Fig. 6.16: **The cross-section of a Gaussian surface S_2 enclosing the empty interior of the thin uniformly charged spherical shell of radius R carrying a net charge Q.**

So, **always remember, when a charge is enclosed by a uniformly charged spherical shell so that the charge lies inside the shell, no electrostatic force is exerted on the charge by the shell**.

Let us apply what you have learnt in this section to an example of two concentric thin spherical shells.

EXAMPLE 6.2: TWO CONCENTRIC THIN SPHERICAL SHELLS

Two concentric thin spherical shells of radii R_1 and R_2 (with $R_2 > R_1$) carry uniformly distributed charges q_1 and q_2, respectively (Fig. 6.17). Use Gauss's law to determine the electric fields at the points

a) $r < R_1$,

b) $R_2 < r < R_1$ and

c) $r \geq R_2$.

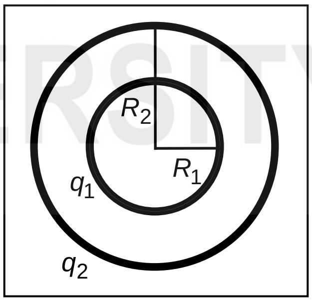


Fig. 6.17: **Diagram for Example 6.2.**

SOLUTION ■ We use Gauss's law along with the results obtained for a thin spherical shell.

a) For the point $r < R_1$, that is, any point A lying inside the inner spherical shell, we can draw the spherical Gaussian surface through it (Fig. 6.18).

You can see that the **charge enclosed by that Gaussian surface is zero**. From Eq. (6.27) obtained using Gauss's law for a point inside the thin spherical shell, we get the result that the electric field for $r < R_1$ is zero:

$$\vec{\mathbf{E}} = \vec{\mathbf{0}} \qquad \text{(inside the inner spherical shell, for } r < R_1\text{)}$$

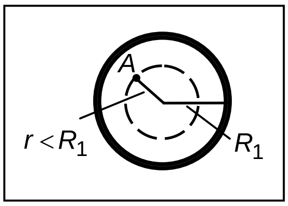


Fig. 6.18: **The electric field at a point inside the inner shell is zero since the charge enclosed by it is zero.**

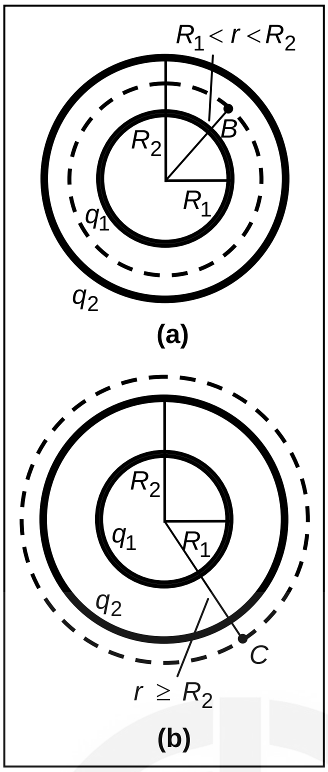


Fig. 6.19: **Diagram for parts (b) and (c) of Example 6.2.**

b) For the point $R_1 < r < R_2$, that is, the point lying between the two concentric shells, the net charge enclosed by the Gaussian surface of radius r is just the charge q_1 on the inner spherical shell (Fig. 6.19a). Therefore, from Eq. (6.26), the electric field at any point between the two thin concentric shells is

$$\vec{\mathbf{E}} = \frac{1}{4\pi\varepsilon_0}\frac{q_1}{r^2}\hat{\mathbf{r}} \qquad (\text{for } R_1 < r < R_2)$$

c) For the point $r \geq R_2$, that is, the point lying outside the outer spherical shell (Fig. 6.19b), the net charge enclosed by the Gaussian surface of radius r is the sum of the charges q_1 and q_2. Therefore, from Eq. (6.26), the electric field at any point outside the outer spherical shell is

$$\vec{\mathbf{E}} = \frac{1}{4\pi\varepsilon_0}\frac{(q_1 + q_2)}{r^2}\hat{\mathbf{r}} \qquad (\text{for } r \geq R_2)$$

What would your answers be if the charges on the inner shell and outer shell were equal to $+q$? To know this, answer the following SAQ!

SAQ 4 – Uniformly charged thin spherical shell

Each of two concentric thin spherical shells of radii R_1 and R_2 (with $R_2 > R_1$) carries uniformly distributed charge $+q$. Use Gauss's law to determine the electric fields due to the shells at the points a) $r < R_1$, b) $R_2 < r < R_1$ and c) $r \geq R_2$.

With this discussion on the applications of Gauss's law to spherically symmetric charge distributions, we end this unit. In the next unit, we continue our study of the applications of Gauss's law to charge distributions possessing cylindrical and planar symmetry. Let us now summarise what you have learnt in this unit.

6.7 SUMMARY

Concept	Description
Electric flux	■ The **electric flux** through a surface (of area S) represents the sum of electric flux elements $(\vec{\mathbf{E}}.d\vec{\mathbf{S}}_i)$ over the entire surface. Each flux element represents the product of a small flat element of area on the surface and the **component of the electric field along the normal** to that area element. This product is nothing but the scalar product of the electric field vector and the area element vector. Mathematically, electric flux or the flux of an electric field $\vec{\mathbf{E}}$ through a surface of area S is defined as $$\Phi_E = \iint_S \vec{\mathbf{E}}.d\vec{\mathbf{S}}$$

Remember, electric flux does not represent flow or change of any entity.

Gauss's law

- **Gauss's law** states that the **net electric flux** through any imaginary **closed** surface *S* of arbitrary shape (called the Gaussian surface) is directly proportional to the net charge (Q_{encl}) enclosed by the surface. In SI units, it is equal to $\frac{Q_{encl}}{\varepsilon_0}$. *The net charge is the algebraic sum* (sum with sign of the charge included) *of all charges enclosed within the Gaussian surface*.

  Mathematically, the law in its integral form is $\oint\!\!\oint_S \vec{\mathbf{E}}.d\vec{\mathbf{S}} = \frac{Q_{encl}}{\varepsilon_0}$

  The differential form of Gauss's law is $\vec{\nabla}.\vec{\mathbf{E}} = \frac{\rho}{\varepsilon_0}$

  where ρ is the volume charge density of the charge distribution.

Applications of Gauss's law to spherically symmetric systems

- Using Gauss's law, we can determine the electric field due to a point charge, distribution of discrete charges and continuous charge distributions enclosed by arbitrary surfaces. In this unit, we have considered spherically symmetric charge distributions.

  A charge distribution is said to be **spherically symmetric** if it remains **invariant** (**the same**)

  - when it is **rotated around any axis passing through its centre**. It is said to possess **rotational symmetry** about that axis.
  - when it is **reflected across any plane passing through its centre**. This is the **reflection symmetry**.

  Examples are a point charge, a uniformly charged sphere and a uniformly charged spherical shell.

  The **magnitude** of the electric field of a spherically symmetric charge distribution at any point depends only on *r*, the distance of the point from the centre of symmetry. The **direction** of the electric field is radially outward for positive charge distribution and radially inward for a negative charge distribution.

Point charge

- The **electric field of a point charge** *q* at a distance *r* from it is given by

$$\vec{\mathbf{E}} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}$$

Uniformly charged sphere

- The **electric field due to a uniformly charged sphere of radius** *R* **carrying charge** *Q* at a point located outside the sphere at a distance *r* is given by

$$\vec{\mathbf{E}} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{\mathbf{r}} \qquad \text{for } r \geq R$$

For a point inside the sphere, it is given by

$$\vec{\mathbf{E}} = \frac{Q}{4\pi\varepsilon_0}\frac{r}{R^3}\hat{\mathbf{r}} \qquad \text{for} \quad r < R$$

Uniformly charged thin spherical shell

- The **electric field due to a uniformly charged thin spherical shell of radius** R **carrying charge** Q at any point lying outside the shell at a distance r from its centre is given by

$$\vec{\mathbf{E}} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{\mathbf{r}} \qquad \text{(spherical shell, for } r \geq R\text{)}$$

At all points lying anywhere in the empty interior part of the shell, the electric field is zero:

$$\vec{\mathbf{E}} = \vec{\mathbf{0}} \qquad \text{(spherical shell, for } r < R\text{)}$$

6.8 TERMINAL QUESTIONS

1. Calculate the flux of the electric field $\vec{\mathbf{E}} = 100\,\text{NC}^{-1}\,\hat{\mathbf{i}}$ through the surfaces of area $1.0\,\text{m}^2$ situated in the *xy*, *xz* and *yz* planes, respectively.

2. A particle carrying a charge of $2.7\times10^{-9}\,\text{C}$ is enclosed in a cubical Gaussian surface of side 0.5 m. Calculate the electric flux through the surface of the cube and any one of its faces.

3. Consider a system of four charges: $3q$, q, $-3q$ and $-q$. Draw a Gaussian surface enclosing at least two charges of the system so that the net electric flux through it is a) zero, b) $+\left(\frac{4q}{\varepsilon_0}\right)$, c) $+\left(\frac{2q}{\varepsilon_0}\right)$ and d) $-\left(\frac{2q}{\varepsilon_0}\right)$.

4. The electric field in some region of space is given by $\vec{\mathbf{E}} = cr\,\hat{\mathbf{r}}$, where c is a constant. Use the differential form of Gauss's law to calculate the volume charge density, which gives rise to this electric field. Obtain the total charge contained in a sphere of radius R, centred at the origin in this region of space.

5. Suppose that a Gaussian surface encloses zero net charge. (a) Does Gauss's law require that the electric field be zero for all points on the surface? (b) If the electric field is zero everywhere on the Gaussian surface, does Gauss's law require that the net charge inside the surface be zero?

6. Is Gauss's law useful in calculating the electric field due to three equal charges placed at the corners of an equilateral triangle? Explain.

7. A charge q is placed at a corner of a cube as shown in Fig. 6.20. Determine the flux of the electric field of the charge through the right face (*ABCD*) of the cube? (Hint: Solving this problem requires a clever choice of the Gaussian surface.)

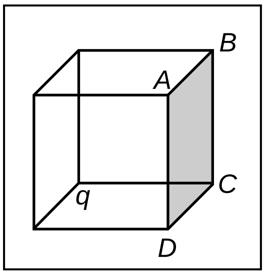


Fig. 6.20: **Diagram for TQ 7.**

8. a) The electric flux due to a point charge passing through a spherical Gaussian surface of radius 0.10 m centred on the charge is

$-900\,\text{Nm}^2\,\text{C}^{-1}$. What is the value of the point charge? What is the electric field due to the point charge at a point on the Gaussian surface? What would the electric flux through the Gaussian surface be if its radius were increased to 0.30 m?

b) The magnitude of the electric field due to a non-conducting charged sphere of radius 0.30 m at a distance of 0.10 m from its centre is $3.0 \times 10^3\,\text{NC}^{-1}$. What is the net charge on the sphere?

9. A non-conducting sphere of radius *R* carrying net positive charge *Q* is enclosed by a concentric non-conducting thin spherical shell of radius *r* carrying net negative charge *q*. Determine the electric field (a) inside the sphere, (b) between the sphere and the shell, and (c) outside the shell.

10. A charged non-conducting spherical shell having inner radius 3.0 m and outer radius10 m carries a charge of magnitude 9.0 nC distributed uniformly over its volume. Determine the electric field due to it at a distance of 6.0 m from its centre.

6.9 SOLUTIONS AND ANSWERS

Self-Assessment Questions

1. We can determine the electric flux through the faces of the cube by using Eq. (6.6a), i.e., by integrating the scalar product $\vec{\mathbf{E}}.d\vec{\mathbf{S}}$ over the right and top faces of the cube. Refer to Fig. 6.21. For the choice of the coordinate axes, the area vector for the right face is $d\vec{\mathbf{S}} = dS\,\hat{\mathbf{i}}$ and for the top face, it is $d\vec{\mathbf{S}} = dS\,\hat{\mathbf{j}}$. So, the electric flux through the right face of the cube is given as:

$$\Phi_E = \iint_S \vec{\mathbf{E}}.d\vec{\mathbf{S}} = \iint_S (8.0\,x\,\hat{\mathbf{i}} + 5.0\,\hat{\mathbf{j}}).dS\,\hat{\mathbf{i}}$$

$$= \iint_S (8.0x)\hat{\mathbf{i}}.\,\hat{\mathbf{i}}\,dS = (8.0)\iint_S (x)dS \quad (\because\ \hat{\mathbf{i}}.\,\hat{\mathbf{i}} = 1,\ \hat{\mathbf{i}}.\,\hat{\mathbf{j}} = 0)$$

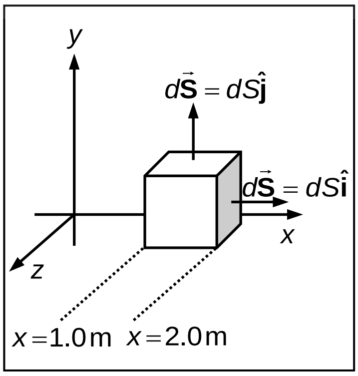


Fig. 6.21: **Diagram for the answer of SAQ 1.**

Now, on the right face of the cube, *x* is constant and has the value *x* = 2.0 m. Therefore, for the right face, we get

$$\Phi_E = (8.0)\iint_S (2.0)dS = (16.0)\iint_S dS$$

Now the integral $\iint_S dS$ is equal to the area of the right face of the cube, which is just $1.0\,\text{m}^2$. Therefore, the electric flux through the right face of the cube is

$$\Phi_E = (16.0)\,\text{NC}^{-1}\text{m}^2$$

Now, we follow the same steps for the top face of the cube as we followed for the right face of the cube. Since for the top face, $d\vec{\mathbf{S}} = dS\,\hat{\mathbf{j}}$ and $\hat{\mathbf{j}}.\,\hat{\mathbf{j}} = 1,\ \hat{\mathbf{i}}.\,\hat{\mathbf{j}} = 0,$ the electric flux through the top face of the cube is given as:

$$\Phi_E = \iint_S \vec{\mathbf{E}}.d\vec{\mathbf{S}} = \iint_S (8.0\,x\,\hat{\mathbf{i}} + 5.0\,\hat{\mathbf{j}}).dS\,\hat{\mathbf{j}} = \iint_S (5.0)dS = (5.0)\iint_S dS$$

Now the integral $\iint_S dS$ is equal to the area of the top face of the cube, which is just $1.0\,\text{m}^2$. Therefore, the electric flux through the top face of the cube is $\Phi_E = (5.0)\text{NC}^{-1}\text{m}^2$.

2. a) We cannot apply Gauss's law to the surfaces shown in Figs. 6.1a, b and c as these are open surfaces (these do not define an enclosed volume) and Gauss's law can be applied to only closed surfaces.

   b) i) No, the electric flux through the surface would not change as the Gaussian surface can be of any shape and the electric flux is equal to only the net charge enclosed by it.

      ii) No, since the net charge enclosed by the surface is the same.

      iii) No, because the location of the charge within the surface does not matter.

      iv) Yes, because the net charge enclosed by the surface would change.

      v) Yes, because the net charge enclosed by the surface would change.

      vi) No, since the net charge enclosed by the surface is the same.

   c) From Eq. (6.12), the value of the charge on the particle is given by

$$q_{encl} = \varepsilon_0\,\Phi_E = (8.85\times10^{-12}\,\text{C}^2\text{N}^{-1}\text{m}^{-2})\times 500\,\text{Nm}^2\,\text{C}^{-1} = 4.42\times10^{-9}\,\text{C}$$

The electric flux through the surface would not change since the net charge enclosed by it remains the same.

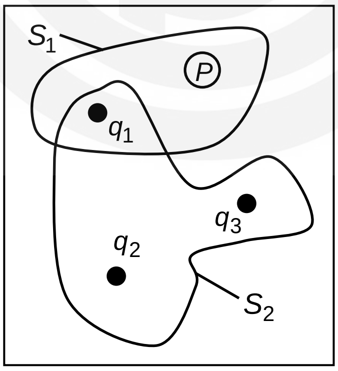


Fig. 6.22: **Diagram for the answer of SAQ 2d.**

   d) Refer to Fig. 6.22. The net charge enclosed by the surface S_1 is $q_1 = +3.1\,\text{nC}$. Since the particle P enclosed by the surface S_1 carries no charge, it makes no contribution to the electric flux. The remaining charges are outside the surface. Therefore, from Eq. (6.12),

$$\Phi_E = \frac{q_{encl}}{\varepsilon_0} = \frac{q_1}{\varepsilon_0} = \frac{3.1\times10^{-9}\,\text{C}}{8.85\times10^{-12}\,\text{C}^2\text{N}^{-1}\text{m}^{-2}} = 350\,\text{Nm}^2\,\text{C}^{-1}$$

The net charge enclosed by the surface S_2 is

$$q_1 + q_2 + q_3 = +3.1\,\text{nC} + (-5.9\,\text{nC}) + (-3.1\text{nC}) = -5.9\,\text{nC}$$

Therefore, from Eq. (6.12),

$$\Phi_E = \frac{q_{encl}}{\varepsilon_0} = \frac{-5.9\times10^{-9}\,\text{C}}{8.85\times10^{-12}\,\text{C}^2\text{N}^{-1}\text{m}^{-2}} = -6.7\times10^2\,\text{Nm}^2\,\text{C}^{-1}$$

3. We use Eq. (6.22) for the electric field of a uniformly charged sphere since the point lies outside the sphere and take the magnitude only. Therefore, the net charge on the sphere is

$$Q = E(4\pi\varepsilon_0 r^2) = \frac{(9.0\,\text{NC}^{-1}) \times (0.3\,\text{m})^2}{8.99 \times 10^9\,\text{C}^{-2}\text{Nm}^2} = 0.09\,\text{nC} = 0.1\,\text{nC}$$

Since the sphere is uniformly charged, its volume charge density is given by Eq. (6.23a):

$$\rho = \frac{Q}{\frac{4\pi}{3}R^3} = \frac{0.09 \times 10^{-9}\,\text{C}}{\frac{4\pi}{3}(0.1\,\text{m})^3} = 2.1 \times 10^{-8}\,\text{C}\,\text{m}^{-3}$$

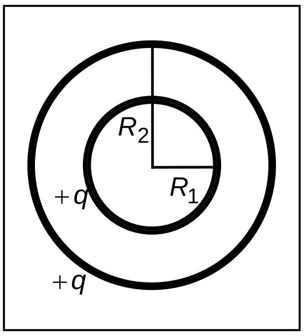


Fig. 6.23: **Diagram for the answer of SAQ 4.**

4. Refer to Fig. 6.23. We follow the steps in Example 6.2 with $q_1 = q_2 = +q$.

   a) For the point $r < R_1$, the net charge enclosed by a spherical Gaussian surface passing through r, (i.e., a point inside the inner shell) is zero. Hence, for $r < R_1$,

   $$\vec{\mathbf{E}} = \vec{\mathbf{0}}$$

   b) For $R_2 < r < R_1$, (i.e., a point lying between the two concentric shells), the net charge enclosed by a spherical Gaussian surface passing through r is just $+q$ and hence, for $R_2 < r < R_1$,

   $$\vec{\mathbf{E}} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}$$

   c) For $r \geq R_2$, (i.e., a point lying outside the outer shell), the net charge enclosed by a spherical Gaussian surface passing through r is $+q+q = +2q$ and hence, for $r \geq R_2$,

   $$\vec{\mathbf{E}} = \frac{1}{4\pi\varepsilon_0}\frac{2q}{r^2}\hat{\mathbf{r}}$$

Terminal Questions

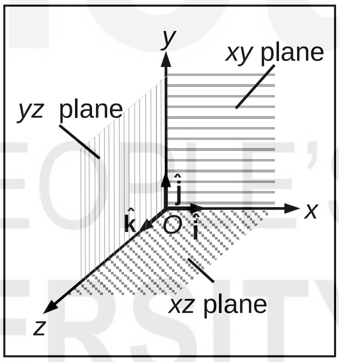


Fig. 6.24: **Area vectors for the answer of TQ 1.**

1. Refer to Fig. 6.24. A surface of area S in the xy plane is represented by the vector $S\hat{\mathbf{k}}$ since $\hat{\mathbf{k}}$ is the unit vector perpendicular to the xy plane. Therefore, the flux of the electric field $\vec{\mathbf{E}} = 100\,\text{NC}^{-1}\,\hat{\mathbf{i}}$ through a surface of area $S = 1.0\,\text{m}^2$ situated in the xy plane is

   $$\Phi_E = \vec{\mathbf{E}}.S\hat{\mathbf{k}} = 100\,\text{NC}^{-1}\,\hat{\mathbf{i}}.(1.0\,\text{m}^2)\hat{\mathbf{k}} = 100\,\text{NC}^{-1}\text{m}^2\,(\hat{\mathbf{i}}.\hat{\mathbf{k}}) = 0$$

   The area vector in the xz plane is given by $S\,\hat{\mathbf{j}}$ and for $S = 1.0\,\text{m}^2$, the flux of the electric field $\vec{\mathbf{E}} = 100\,\text{NC}^{-1}\,\hat{\mathbf{i}}$ through the xz plane is

   $$\Phi_E = \vec{\mathbf{E}}.S\,\hat{\mathbf{j}} = 100\,\text{NC}^{-1}\,\hat{\mathbf{i}}.(1.0\,\text{m}^2)\,\hat{\mathbf{j}} = 100\,\text{NC}^{-1}\text{m}^2\,(\hat{\mathbf{i}}.\hat{\mathbf{j}}) = 0$$

   In the yz plane, the area vector is given by $S\,\hat{\mathbf{i}}$ and for $S = 1.0\,\text{m}^2$, the flux of the electric field $\vec{\mathbf{E}} = 100\,\text{NC}^{-1}\,\hat{\mathbf{i}}$ through the yz plane is

   $$\Phi_E = \vec{\mathbf{E}}.S\hat{\mathbf{i}} = 100\,\text{NC}^{-1}\,\hat{\mathbf{i}}.(1.0\,\text{m}^2)\,\hat{\mathbf{i}} = 100\,\text{NC}^{-1}\text{m}^2\,(\hat{\mathbf{i}}.\hat{\mathbf{i}}) = 100\,\text{NC}^{-1}\text{m}^2$$

2. Let us choose the surface of the cube as the Gaussian surface. From Gauss's law [Eq. (6.16)], the electric flux through this surface is

   $$\Phi_E = \frac{q}{\varepsilon_0} = \frac{2.7 \times 10^{-9}\,\text{C}}{8.85 \times 10^{-12}\,\text{C}^2\text{N}^{-1}\text{m}^{-2}} = 3.0 \times 10^2\,\text{NC}^{-1}\text{m}^2$$

Since the cube has 6 faces, the electric flux through any one of the cube's faces is

$$\Phi_{E'} = \frac{\Phi_E}{6} = \frac{q}{6\varepsilon_0} = \frac{3.0\times10^2\,\text{NC}^{-1}\text{m}^2}{6} = 50\,\text{NC}^{-1}\text{m}^2$$

3. Refer to Figs. 6.25a to e. We use Eq. (6.16): $\Phi_E = \frac{q}{\varepsilon_0}$

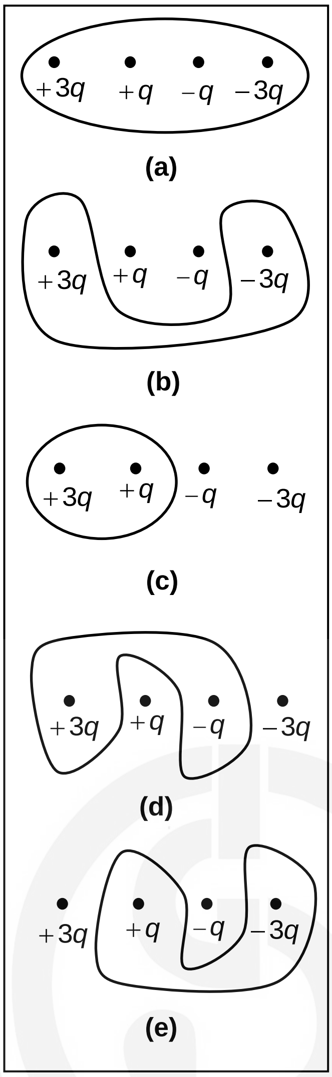


Fig. 6.25: **Diagram for the answer of TQ 3.**

a) For the net electric flux through the Gaussian surface to be zero, the net electric charge enclosed by it should be zero. In Figs. 6.24a and b, the two Gaussian surfaces shown enclose the charges so that the net charge within each one of them and hence the electric flux through them is zero. You can draw a third one too enclosing only the charges $+q$ and $-q$.

b) For the net electric flux through the Gaussian surface to be $(+\,4q/\varepsilon_0)$, the net electric charge enclosed by it should be $+4q$. In Fig. 6.24c, the Gaussian surface encloses the charges $+3q$ and $+q$ so that the net charge within it is $+4q$ and the net electric flux through it is $(+\,4q/\varepsilon_0)$.

c) For the net electric flux through the Gaussian surface to be $(+\,2q/\varepsilon_0)$, the net electric charge enclosed by it should be $+2q$. In Fig. 6.24d, the Gaussian surface encloses the charges $+3q$ and $-q$ so that the net charge within it is $+2q$ and the net electric flux through it is $(+\,2q/\varepsilon_0)$.

d) For the net electric flux through the Gaussian surface to be $(-\,2q/\varepsilon_0)$, the net electric charge enclosed by it should be $-2q$. In Fig. 6.24e, the Gaussian surface encloses the charges $-3q$ and $+q$ so that the net charge within it is $-2q$ and the net electric flux through it is $(-\,2q/\varepsilon_0)$.

4. We use Eq. (6.18) to calculate the volume charge density ρ and write

$$\rho = \varepsilon_0\,\vec{\nabla}.\vec{\mathbf{E}} = \varepsilon_0\,\vec{\nabla}.(cr\,\hat{\mathbf{r}}) = \varepsilon_0\,\vec{\nabla}.(c\,\vec{\mathbf{r}}) \qquad \left(\because\ \hat{\mathbf{r}} = \frac{\vec{\mathbf{r}}}{r}\right)$$

In Unit 2, you have learnt how to calculate the divergence of a vector field.

$$\therefore\ \rho = \varepsilon_0\,\vec{\nabla}.(c\,\vec{\mathbf{r}}) = \varepsilon_0\,c\,\vec{\nabla}.\vec{\mathbf{r}} = \varepsilon_0\,c\,(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}).(x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}})$$

$$= \varepsilon_0\,c\,(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}) = 3\,\varepsilon_0\,c$$

In this region of space, the total charge contained in a sphere of radius R, centred at the origin is just the volume integral $Q = \iiint_V \rho\,dV$. Since ρ is constant and the volume integral equals the volume of the sphere of radius R, we have

$$Q = 3\varepsilon_0\,c\int_V dV = 3\varepsilon_0\,c\,\frac{4\pi}{3}R^3 = 4\pi\varepsilon_0\,c\,R^3$$

5. a) When the Gaussian surface encloses zero net charge, Gauss's law yields $\vec{\mathbf{E}}.d\vec{\mathbf{S}} = 0$. However, this does not mean that the electric field is zero for all points on the surface. $\vec{\mathbf{E}}.d\vec{\mathbf{S}}$ can be zero even when $\vec{\mathbf{E}}$ and $d\vec{\mathbf{S}}$ are perpendicular to each other.

b) If the electric field is zero everywhere on the Gaussian surface, Gauss's law requires that there should be no net charge inside the surface, i.e., the net charge should be zero.

6. Gauss's law is not useful in calculating the electric field due to three equal charges placed at the corners of an equilateral triangle because it is not possible to find a closed surface of appropriate symmetry over which the electric field can be taken to be constant and its direction can be taken to be either parallel or normal to the surface to evaluate the surface integral.

7. The electric flux through the shaded right face (*ABCD*) of the cube having area, say S', is

$$\Phi_{S'} = \iint_{S'} \vec{\mathbf{E}}.d\vec{\mathbf{S}}$$

To determine $\Phi_{S'}$, the trick is to choose an appropriate Gaussian surface that encloses the charge q. We can put together 8 cubes of the same size as the original cube in the problem to construct the Gaussian surface as shown in Fig. 6.26. It includes the right face *ABCD* of the original cube and encloses the charge q. Note that the area of the Gaussian surface is 24 times the area of the right face *ABCD*. So, now we can apply Gauss's law to this problem.

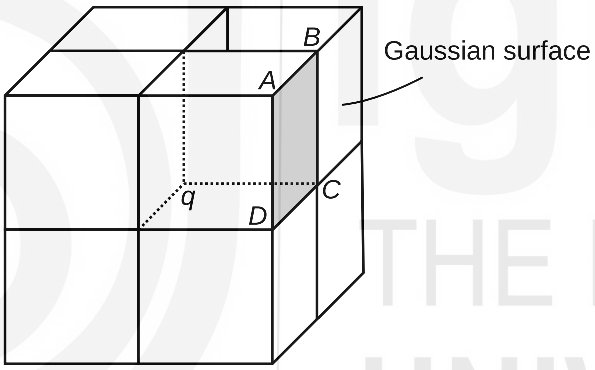


Fig. 6.26: **Diagram for answer to TQ 7.**

From Gauss's law, we have $\oiint_S \vec{\mathbf{E}}.d\vec{\mathbf{S}} = \dfrac{Q_{encl}}{\varepsilon_0}$ where S is the surface area of the Gaussian surface enclosing the charge. Since the area of the Gaussian surface is 24 times the area S' of *ABCD*, we have

$$\oiint_S \vec{\mathbf{E}}.d\vec{\mathbf{S}} = 24 \times \iint_{S'} \vec{\mathbf{E}}.d\vec{\mathbf{S}} = \frac{q}{\varepsilon_0} \quad \text{or} \quad \iint_{S'} \vec{\mathbf{E}}.d\vec{\mathbf{S}} = \frac{q}{24\,\varepsilon_0}$$

Thus, the electric flux through the right face (*ABCD*) of the cube is

$$\Phi_{S'} = \frac{q}{24\,\varepsilon_0}$$

8. a) We use Eq. (6.12) to determine the value of the point charge:

$$q = \varepsilon_0\,\Phi_E = (8.85\times10^{-12}\,\mathrm{C^2N^{-1}m^{-2}})\times(-900\,\mathrm{Nm^2C^{-1}}) = -7.96\,\mathrm{nC}$$

From Eq. (6.19), the electric field of q at a distance of 0.10 m from it is

$$\vec{\mathbf{E}} = (8.99\times10^9\,\mathrm{C^{-2}Nm^2})\times\frac{-7.96\times10^{-9}\,\mathrm{C}}{(0.10\,\mathrm{m})^2}\hat{\mathbf{r}} = (-7.2\times10^3\,\mathrm{NC^{-1}})\hat{\mathbf{r}}$$

The electric flux through the Gaussian sphere of radius 0.30 m would remain the same as the charge enclosed by it is the same. It will be $-900\,\mathrm{Nm^2C^{-1}}$.

b) We use Eq. (6.24b) for the magnitude E to determine the net charge on the sphere since the point at which the electric field is given lies inside the sphere. From Eq. (6.24b) for E, we have $Q = 4\pi\varepsilon_0 \dfrac{R^3}{r} E$.

Upon substituting the numerical values given in the problem, we get

$$Q = \frac{1}{(8.99\times10^9\,\text{C}^{-2}\text{Nm}^2)} \times \frac{(0.30\,\text{m})^3}{(0.10\,\text{m})} \times (3.0\times10^3\,\text{NC}^{-1}) = 90\,\text{nC}$$

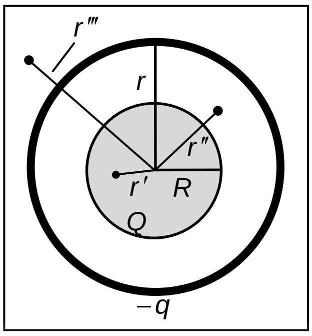


Fig. 6.27: **Diagram for the answer of TQ 9.**

9. a) Since the electric field inside the shell is zero, from Eq. (6.24b), the net electric field at a point r' inside the sphere (see Fig. 6.27) is given as:

$$\vec{\mathbf{E}} = \frac{Q}{4\pi\varepsilon_0}\frac{r'}{R^3}\hat{\mathbf{r}}$$

b) At a point r'' between the sphere and the shell, the total charge enclosed by a spherical Gaussian surface passing through r'' is the charge Q on the sphere and the electric field is given by Eq. (6.22):

$$\vec{\mathbf{E}} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r''^2}\hat{\mathbf{r}}$$

c) At a point r''' outside the shell, the total charge enclosed by a spherical Gaussian surface passing through the point is the charge Q on the sphere and the charge $-q$ on the spherical shell. The electric field is given by Eq. (6.22) or Eq. (6.26) where the net charge enclosed by the Gaussian surface passing through r''' is $(Q-q)$:

$$\vec{\mathbf{E}} = \frac{1}{4\pi\varepsilon_0}\frac{(Q-q)}{r'''^2}\hat{\mathbf{r}}$$

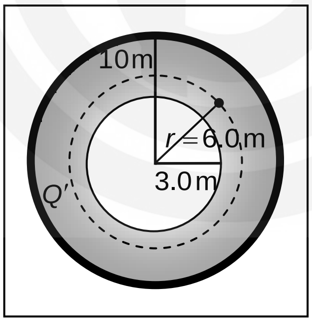


Fig. 6.28: **Diagram for the answer of TQ 10 (not to scale).**

10. We have to first determine the volume charge density of the spherical shell: $\rho = \dfrac{Q}{V}$. For this, we need to calculate the volume of the spherical shell, which is $V = \dfrac{4\pi}{3}\left[(10\,\text{m})^3 - (3.0\,\text{m})^3\right] = 4077\,\text{m}^3$

$$\therefore \quad \rho = \frac{Q}{V} = \frac{9.0\,\text{nC}}{4077\,\text{m}^3} = 2.2\times10^{-12}\,\text{Cm}^{-3}$$

To determine the electric field at the point 6.0 m away from the centre, we draw a spherical Gaussian surface of radius 6.0 m passing through the point (Fig. 6.28). Let us first calculate the total charge Q' enclosed by the Gaussian surface of radius 6.0 m. The volume of the part of the spherical shell that contains the charge Q' is

$$V' = \frac{4\pi}{3}\left[(6.0\,\text{m})^3 - (3.0\,\text{m})^3\right] = 792\,\text{m}^3$$

$$\Rightarrow Q' = \rho V' = 2.2\times10^{-12}\,\text{Cm}^{-3} \times 792\,\text{m}^3 = 1.7\,\text{nC}$$

From Gauss's law, we have

$$\Phi_E = E(4\pi R^2) = \frac{Q'}{\varepsilon_0} \qquad \Rightarrow \qquad E = \frac{Q'}{4\pi\varepsilon_0 R^2} \text{ or}$$

$$\vec{\mathbf{E}} = (8.99\times10^9\,\text{C}^{-2}\text{Nm}^2)\left(\frac{1.7\,\text{nC}}{(6.0\,\text{m})^2}\right)\hat{\mathbf{r}} = 0.42\,\text{NC}^{-1}\hat{\mathbf{r}}$$

How can sitting inside a closed conducting surface such as a car prevent you from being struck by lightning? Find the answer in this unit!

UNIT 7

APPLICATIONS OF GAUSS'S LAW

Structure

STUDY GUIDE

In Unit 6, you have studied the concept of electric flux and Gauss's law. You have learnt how to apply Gauss's law to discrete point charges and continuous charge distributions that are spherically symmetric such as a uniformly charged sphere and thin spherical shell. In this unit, you will learn applications of Gauss's law to some more continuous charge distributions having cylindrical and planar symmetry such as a uniform infinite line charge and a plane sheet of charge. You will determine the electric fields due to a uniformly charged infinite wire, a uniform cylindrical charge distribution and an infinite sheet of charge. You will also learn of its application to an isolated charged conductor.

You will learn how to choose appropriate Gaussian surfaces to solve the surface integrals involved in each case. Revise the divergence theorem that you have learnt in Unit 4. You should also revise the methods of solving surface and volume integrals to be able to master the concepts of this unit. Try to solve the Examples, SAQs and Terminal questions given in this unit on your own.

"It is not knowledge, but the act of learning, not possession but the act of getting there, which grants the greatest enjoyment."

Carl F. Gauss

7.1 INTRODUCTION

In Unit 6, you have learnt the concept of electric flux and studied Gauss's law. You have also learnt how to apply Gauss's law to obtain the electric flux and electric field due to discrete charges. You have applied the law to continuous charge distributions that are spherically symmetric and uniformly charged such as a uniformly charged sphere and thin spherical shell. You have learnt that the Gaussian surface for such charge distributions is spherical and concentric with them. It also passes through the point at which the electric field is to be determined.

In this unit, you will first learn how to apply Gauss's law to charge distributions having cylindrical symmetry such as uniform line charge and uniformly charged cylinder (Sec. 7.2). You will begin by learning the concept of cylindrical symmetry. Then we will explain why Gauss's law is useful for determining the electric fields due to cylindrically symmetric charge distributions. With this understanding, you can learn how to apply Gauss's law to determine the electric fields due to an infinite uniform line charge and infinite uniformly charged cylinder.

In Sec. 7.3, you will learn how to apply Gauss's law to calculate the electric field due to an infinite uniformly charged sheet that possesses planar symmetry. Once again, we will explain what planar symmetry is and how Gauss's law is useful for determining the electric fields due to charge distributions having planar symmetry.

The applications of Gauss's law described in Secs. 7.2 and 7.3 find use in computing the capacitance of coaxial capacitors and parallel plate capacitors as you will learn in Unit 11 of the next block. As you may know, such capacitors are very commonly used around us, for example, in electronic appliances like the TV and computers, and power storage systems, etc. Finally, in Sec. 7.4, we apply Gauss's law to an isolated charged solid conductor and a conductor with a cavity. This too has many interesting applications in real life. One of these is shown in the picture on the first page of this unit.

In the next two units, we introduce the concept of electric potential and its relation with the electric field. You will learn another way of calculating electric fields and electrostatic forces using the concept of electric potential.

Expected Learning Outcomes

After studying this unit, you should be able to:

- apply Gauss's law to determine the electric field due to cylindrically symmetric charge distributions such as a uniform infinite line charge and an infinite uniformly charged cylinder;
- determine the electric field due to an infinite uniform plane sheet of charge using Gauss's law; and
- use Gauss's law to explain why the electric field inside an isolated charged conductor is zero and why the charge on it is distributed entirely on its surface.

7.2 ELECTRIC FIELD DUE TO CYLINDRICALLY SYMMETRIC CHARGE DISTRIBUTIONS

We have said in the introduction of this unit that in this section we will determine the electric field due to charge distributions that possess cylindrical symmetry such as a line charge and a charged cylinder. You may like to ask:

- **What is cylindrical symmetry**?
- **How is Gauss's law useful for a charge distribution that possesses cylindrical symmetry**?

Let us begin our discussion by answering these two questions.

7.2.1 Gauss's Law and Cylindrically Symmetric Charge Distributions

Let us answer the first question and define **cylindrical symmetry**.

A charge distribution (or any object) is said to possess **cylindrical symmetry** if it remains **unchanged** (or is **invariant**) when it is

- moved along its axis (*AB* in Fig. 7.1a and *CD* in Fig. 7.1b), that is, the line running through its core (translational symmetry);
- rotated around its axis (rotational symmetry);
- rotated by 180° around any axis perpendicular to its axis, (*PQ* in Fig. 7.1a and *RS* in Fig. 7.1b), (180° rotational symmetry);
- reflected across any plane passing through its axis (reflection symmetry);
- reflected across any plane perpendicular to its axis (reflection symmetry).

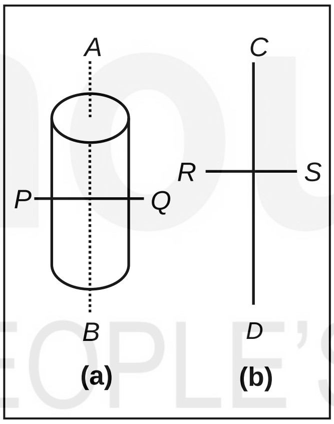


Fig. 7.1: **a) The axis *AB* (dotted line) of a cylinder; b) for a line or a wire, the axis *CD* lies on the line/wire itself.**

Try to apply the above transformations to any cylindrical object around you such as a can or a water pipe. Verify that it possesses cylindrical symmetry before studying further. An infinite line or wire (like the axis of an infinite cylinder) *also* possesses cylindrical symmetry (Fig. 7.1b).

Let us now answer the second question and explain **how Gauss's law is useful** for determining the electric field due to a cylindrically symmetric charge distribution.

While studying Secs. 6.4 to 6.6 of Unit 6, you would have noted that due to the choice of the spherical Gaussian surface enclosing the charge distribution, the calculations became very simple for two reasons:

- the electric field was directed parallel to the area vector for a surface element on the Gaussian surface so that $\vec{\mathbf{E}}.d\vec{\mathbf{S}} = E\,dS$; and
- the magnitude E of the electric field was the same at all points on the Gaussian surface so that it could be treated as constant and taken out of the surface integral.

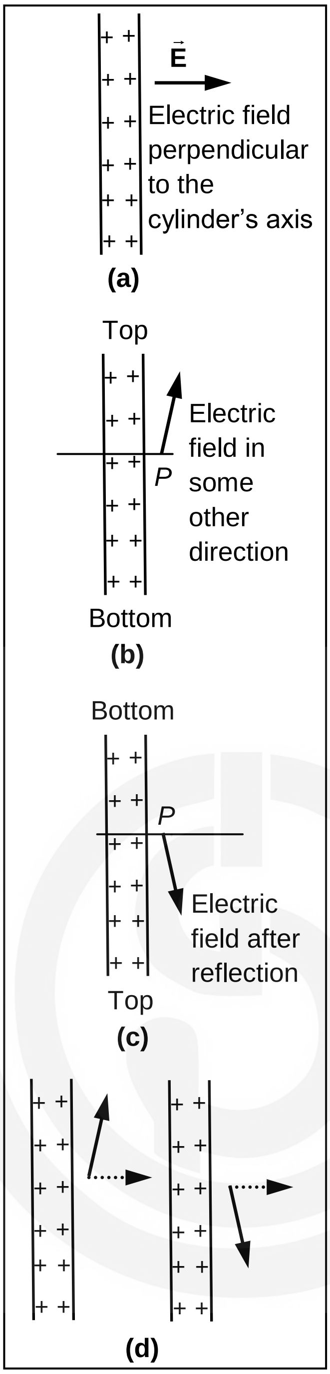


Fig. 7.2: **a) The direction of electric field $\vec{E}$ due to a section of an infinite charged cylinder is perpendicular to its axis; b) electric field in some other direction; c) reflected electric field; d) the directions of electric fields are different at the same point for the same charge distribution, which is not possible. So, $\vec{E}$ can only be in the direction shown by dotted arrows.**

Let us now ask: What is the **direction of the electric field** at any point due to a cylindrical charge distribution?

Consider Fig. 7.2a showing a small section of an infinite cylinder carrying positive charge (it could also be a charged infinite wire). The direction of the electric field of the charge distribution is perpendicular to its axis, and the electric field is directed radially outward from the axis for positively charged cylinder (Fig. 7.2a). For a negative cylindrical charge distribution, it will be directed radially inward and perpendicular to its axis. You may ask: **Why**?

To answer this question, suppose that the electric field due to the cylindrical charge distribution at some point *P* is directed in some other direction as shown in Fig. 7.2b. Note that we have arbitrarily labelled one end of this section of the cylinder as 'top' and the other one as 'bottom' just to show what happens when it is reflected.

Now let us reflect this cylindrical charge distribution about a horizontal line perpendicular to its axis and passing through *P*. So the 'top' of the section is now its 'bottom' and the 'bottom', its 'top' (Fig. 7.2c).

What is the direction of the electric field after the cylindrical charge distribution is reflected? After reflection, the direction of the electric field becomes as shown in Fig. 7.2c because the electric field is also reflected in the same manner.

Now compare Figs. 7.2b and 7.2c by putting them alongside each other as in Fig. 7.2d. What do you find? You can see that the charge distribution remains the same after reflection but the electric fields are different. (The labels 'top' and 'bottom' were only for our convenience. Otherwise, we cannot tell the difference.)

This is a contradiction: **How can there be different electric fields at the same point for the same charge distribution**? If the charge distribution remains unchanged, the electric field also has to be the same. If it is not so, there must be some mistake.

Now we ask: **What is the direction of the electric field that will <u>not</u> lead to such a contradiction**? From Fig. 7.2d, you can see that if the electric field (shown by dotted arrows) were in the direction perpendicular to the cylindrical axis, it would remain the same under this symmetry operation.

You can check it for all other symmetry operations on the cylinder. This is how we conclude from symmetry considerations that **the direction of the electric field due to a cylindrical charge distribution at a point can only be perpendicular to its axis**.

It points outward, for a positive charge distribution and inward, for a negative charge distribution.

Now you may like to know: **What does the magnitude of the electric field of a charge distribution having cylindrical symmetry depend on**?

The answer is: It depends only on the perpendicular distance, say *r*, of the point from the cylinder's axis. **Why is it so**?

Suppose the **magnitude of the electric field due to the cylindrical charge distribution at any point varied with the angular coordinates of the point**. Then it would have different values at different points, say, P and Q situated at the **same** perpendicular distance from the axis (i.e., on the dotted cylindrical surface of the same radius r in Fig. 7.3). But this is a contradiction. This is because due to cylindrical symmetry, the charged cylinder will look the same from all points on the cylindrical surface of radius r (Fig. 7.3). So,

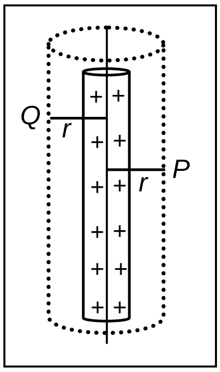


Fig. 7.3: **The magnitude of the electric field due to a cylindrical charge distribution at any point depends on its perpendicular distance *r* from the axis of symmetry. If it were not so, the magnitude of the electric field would be different at different points, (e.g., *P* and *Q*) on the same surface for the same charge distribution, which is incorrect.**

The **magnitude of the electric field cannot have different values at different points on a given cylindrical surface for the same cylindrical charge distribution**.

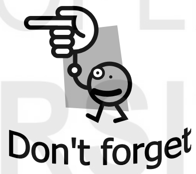
Therefore, at any point, it will depend only on the perpendicular distance of the point from the axis of the cylindrical charge distribution.

To conclude, due to cylindrical symmetry, the magnitude of the electric field due to a cylindrical charge distribution at any point depends only on the perpendicular distance of the point from the cylindrical axis. So, all points on the cylindrical surface of a given radius are equivalent as far as the magnitude of the electric field of any cylindrical charge distribution is concerned: it could be a line charge, charged wire or charged solid/hollow cylinder.

Then we can treat the magnitude of the electric field of such systems at a given cylindrical surface as constant and take it out of the surface integral. You will appreciate this point better in the next section.

To sum up, you must always remember the following for **any charge distribution having cylindrical symmetry**:

- **The electric field due to a charge distribution having cylindrical symmetry is directed perpendicular to its axis of symmetry.**
- **The magnitude of the electric field at any point depends only on its perpendicular distance from the axis of symmetry.**



So now can you quickly say what kind of Gaussian surface we should choose for a cylindrically symmetric charge distribution such as a line charge? The Gaussian surface should indeed be **cylindrical**. Why so?

As you have learnt just now, for a cylindrical Gaussian surface coaxial with the cylindrical charge distribution (charged line or cylinder), the electric field is normal to the surface at all points on it. You know that for any area element centred at a point on the Gaussian surface, the area vector $d\vec{\mathbf{S}}$ is directed normal to the surface (Fig. 7.4). So, the electric field $\vec{\mathbf{E}}$ at any point due to the cylindrically symmetric charge distribution will be parallel to the area vector $d\vec{\mathbf{S}}$ and, therefore

$$\vec{\mathbf{E}}.d\vec{\mathbf{S}} = E\,dS \tag{7.1}$$

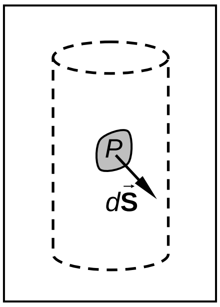


Fig. 7.4: **Area vector $d\vec{\mathbf{S}}$ for an element of area centred at any point *P* on a cylindrical Gaussian surface is normal to the surface.**

You have also learnt that the magnitude of the electric field at any point is the same everywhere on the cylindrical Gaussian surface passing through that point. So we can treat it as constant for that surface and take it out of the surface integral.

With this understanding of cylindrical symmetry of charge distributions, we can apply Gauss's law to a uniform infinite line charge.

7.2.2 Infinite Uniform Line Charge

Recall that you have calculated the electric field for an infinite line charge using Coulomb's law in Example 5.7 of Unit 5. You have learnt how to solve the lengthy integral involved in the calculation. Let us now apply Gauss's law to a similar problem. Consider an infinitely long wire carrying uniform linear charge density λ. Let us determine the electric field at a distance r from the wire using Gauss's law.

Before studying further, you may like to quickly verify that the infinite line charge distribution has cylindrical symmetry by carrying out the symmetry operations on a wire. Let us now draw a Gaussian surface, i.e., the surface of a right circular cylinder of radius r and length L coaxial with the wire (Fig. 7.5).

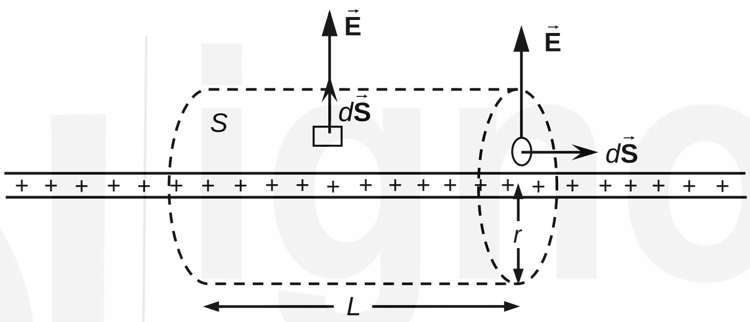


Fig. 7.5: **Applying Gauss's law to an infinite uniformly charged wire carrying positive charge. The Gaussian surface is cylindrical having length *L* and radius *r*. It encloses a section of the charged wire.**

What is the **magnitude** of the electric field at any point on the cylindrical Gaussian surface? You have learnt in Sec. 7.2.1 that due to cylindrical symmetry, it would be the same everywhere on the surface of the cylinder as it depends only on the perpendicular distance of the point from the wire's axis. As you can see in Fig. 7.5, this distance is just the radius (r) of the cylinder. So, it is the same for all points on the cylindrical surface of radius r and can be treated as constant for that particular surface.

For a cylindrical surface, the direction of the electric field is normal to the surface at all points as shown in Fig. 7.5. For positively charged wire, the electric field is directed radially outwards from the wire's axis. If the charge on the wire were negative, the electric field would point inwards towards the wire's axis. Thus, $\vec{\mathbf{E}}$ and $d\vec{\mathbf{S}}$ are parallel to each other for each area element on the curved part of the cylinder's surface:

$$\vec{\mathbf{E}}.d\vec{\mathbf{S}} = E\,dS \tag{7.2a}$$

The electric flux at all points through both circular ends of the cylinder is zero because $\vec{\mathbf{E}}$ and $d\vec{\mathbf{S}}$ are perpendicular to each other on these ends (Fig. 7.5). Therefore, the product $\vec{\mathbf{E}}.d\vec{\mathbf{S}}$ is finite only for the curved part of the cylindrical surface.

Thus, from Gauss's law, we have

$$\oiint_S \vec{\mathbf{E}}.d\vec{\mathbf{S}} = \oiint_S E\,dS = E\oiint_S dS \quad dS = E\,2\pi rL = \frac{Q_{encl}}{\varepsilon_0} \qquad (7.2b)$$

where we have taken E out of the integral as it is constant on this Gaussian surface S. In Eq. (7.2b), we have also used the result that the area of the curved surface of a cylinder of radius r and length L is $2\pi rL$. So from Eq. (7.2b), we have

$$E\,2\pi rL = \frac{Q_{encl}}{\varepsilon_0}$$

or

$$E = \frac{Q_{encl}}{2\pi\varepsilon_0\, r\, L} \qquad (7.2c)$$

For the uniform line charge density λ, the charge enclosed by the cylinder of length L is given by

$$Q_{encl} = \int_0^L \lambda\, dl' = \lambda\int_0^L dl' = \lambda L, \text{ since } \lambda \text{ is constant} \qquad (7.2d)$$

Substituting Eq. (7.2d) in Eq. (7.2c), we get

$$E = \frac{Q_{encl}}{2\pi\varepsilon_0\, r\, L} = \frac{\lambda L}{2\pi\varepsilon_0\, r\, L}$$

or

$$\boxed{E = \frac{\lambda}{2\pi\varepsilon_0\, r}} \qquad (7.3)$$

Note that the electric field in Eq. (7.3) does not depend on the length of the cylindrical Gaussian surface.

The electric field is directed perpendicular to the line charge or charged wire. This is the same result as the one we got in Example 5.7 after a very lengthy calculation! So, you see that for a symmetrical distribution of charges, the calculation of electric field becomes quite simple if we use Gauss's law.

You should, however, note that **Gauss's law is always true**, no matter what the distribution of charges. **But it is very useful for symmetric charge distributions** since its application makes the calculation much simpler.

In applying Gauss's law, the choice of the Gaussian surface is very important for simplifying calculations. This is especially true for symmetric charge distributions as you have learnt in Unit 6. You will appreciate this point time and again in this unit.

You may like to know: **Why do charge distributions have to be symmetric for Gauss's law to be applied to determine electric fields**?

Recall what you have learnt so far and you will be able to arrive at the answer: The symmetry of the distribution helps us determine the surfaces over which the magnitude of the electric field is constant (i.e., the distance r is constant). Also we know the direction of the electric field for a given type of symmetry.

Then the trick is to **choose the Gaussian surface to be the surface over which the magnitude of the electric field is constant**. Also, **the direction of the electric field should be parallel/perpendicular to the area vector at all points on the surface**.

You must note that this is true for all applications of Gauss's law that you have studied so far, such as the charged sphere and the spherical shell in Unit 6 and the infinite charged wire in this section. For example, in this unit, for the infinite line charge, you have seen that the magnitude of the electric field is the same at all points of the curved part of the cylindrical surface as its radius is constant. The direction of $\vec{\mathbf{E}}$ is normal to the curved part of the surface and therefore, in the same direction as the area element $d\vec{\mathbf{S}}$. Since $\vec{\mathbf{E}}$ is perpendicular to $d\vec{\mathbf{S}}$ on the cylinder's ends, for all points on the circular ends of the cylinder, $\vec{\mathbf{E}}.d\vec{\mathbf{S}} = 0.$ This has made the calculation of electric field quite simple. Of course, it is also simple because the line charge density is uniform, i.e., constant.

Suppose, we had chosen some other shape for the Gaussian surface, then Gauss's law would still apply but $\vec{\mathbf{E}}$ may not have been in the same direction as $d\vec{\mathbf{S}}$ and its magnitude may not have been constant over the surface. Then we could not have taken E out of the integral. That would have made the calculation difficult. So, **symmetry is important for such applications of Gauss's law**. You must have appreciated this point by now having studied charge distributions possessing spherical and cylindrical symmetry. We end this section with an SAQ for you.

SAQ 1 – Applying Gauss's law to line charge

The electric field due to an infinite line charge has magnitude 9.0×10^3 NC^{-1} at a distance of 1.0 m. Calculate the linear charge density.

Let us now determine the electric field due to an infinite uniformly charged cylinder using the same symmetry considerations as for the wire at points both outside and inside the cylinder. Such calculations of electric fields for a cylindrical charge distribution are required for determining the capacitance of capacitors having cylindrical geometry.

7.2.3 Uniformly Charged Infinite Cylinder

Consider an infinitely long charged solid cylinder of radius R, which has uniform volume charge density ρ. Let us determine the electric field due to this charge distribution at a point outside the cylinder.

We use Gauss's law to obtain the electric field for the uniformly charged infinite cylinder at a point P lying outside it at a distance r from its axis.

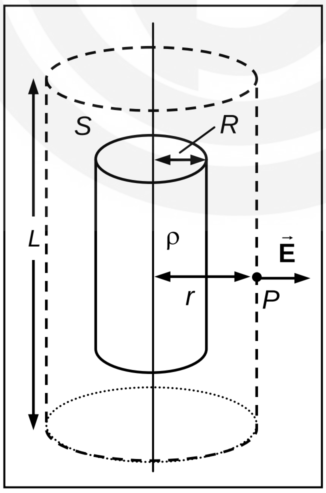


Fig. 7.6: **Electric field at a point *P* lying outside a uniformly charged infinite cylinder. The cylindrical Gaussian surface is of length *L* and radius $r > R$.**

Study Fig. 7.6 showing a section of the infinite cylinder by a solid line. You can verify that the charge distribution is cylindrically symmetric. For a point P outside the cylinder, we draw a cylindrical Gaussian surface of length L and radius r, passing through P. Recall that we have drawn a similar surface for the infinitely long wire in Fig. 7.5. We now follow the same steps and argument as in Sec. 7.2.1 to determine the electric field due to a uniformly charged infinitely long cylinder for $r \geq R$.

Once again we note that for the curved part of the cylindrical Gaussian surface, the direction of the electric field is normal to the surface at all points. Also, the electric field is directed radially outwards from the positively charged cylinder's axis. Therefore, $\vec{\mathbf{E}}$ and $d\vec{\mathbf{S}}$ are parallel to each other for each area element on the curved part of the Gaussian surface and $\vec{\mathbf{E}}.d\vec{\mathbf{S}} = E\,dS.$ As in Sec. 7.2.2, you can see that the electric flux through both circular ends of the cylindrical Gaussian surface is zero because $\vec{\mathbf{E}}$ and $d\vec{\mathbf{S}}$ are perpendicular to each other at all points on these ends. Therefore, from Gauss's law, we have

$$\oiint_S \vec{\mathbf{E}}.d\vec{\mathbf{S}} = \oiint_S E\,dS = E\oiint_S dS \quad dS = E\,2\pi rL = \frac{Q_{encl}}{\varepsilon_0} \qquad (7.4a)$$

Here, since E is the same on all points of the Gaussian surface S, we have taken it to be constant for the surface and have taken it out of the integral. In Eq. (7.4a), we have also used the result that the total surface area of a cylinder of radius r and length L is $2\pi rL.$ So from Eq. (7.4a), we have

$$E = \frac{Q_{encl}}{2\pi\varepsilon_0\, r\, L} \quad \text{for } r \geq R \qquad (7.4b)$$

We now have to determine Q_{encl} in Eq. (7.4b), which is the net charge enclosed by the cylindrical Gaussian surface, given that ρ is constant. It is just the charge on the cylinder of length L and radius R (because the charge distribution of the infinite cylinder is zero beyond its radius R). By definition, it is given by the following volume integral:

$$Q_{encl} = \iiint_V \rho\, dV \qquad (7.5a)$$

Since ρ is uniform (constant), we can take it out of the integral and write

$$Q_{encl} = \rho \iiint_V dV = \rho\,\pi R^2 L \qquad (7.5b)$$

where the volume integral is just the volume of the cylinder of length L and radius R. Therefore,

$$E = \frac{\rho\,\pi R^2 L}{2\pi\varepsilon_0\, r\, L} \quad \text{for } r \geq R \qquad (7.5c)$$

or

$$\vec{\mathbf{E}} = \frac{\rho R^2}{2\varepsilon_0\, r}\,\hat{\mathbf{r}} \quad \text{for } r \geq R \qquad (7.6)$$

where $\hat{\mathbf{r}}$ is the unit vector in the radial direction pointing outward from the cylinder's axis. Notice from Eq. (7.6) that the electric field of a cylindrical charge distribution at points lying outside it decreases as the distance from the axis increases.

Let us now ask: **What is the electric field of an infinite uniformly charged cylinder at a point inside it**?

You will learn the answer in Example 7.1.

EXAMPLE 7.1 : ELECTRIC FIELD INSIDE A CYLINDER

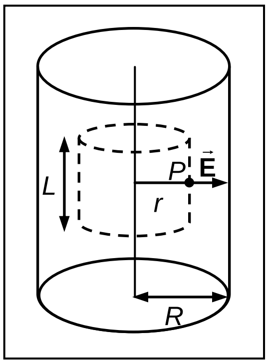


Fig. 7.7: **Electric field inside an infinite uniformly charged cylinder. The Gaussian surface is a cylindrical surface of length** *L* **and radius** $r < R$**.**

An infinitely long uniformly charged cylinder of radius R has positive volume charge density ρ. Determine the electric field at a point inside the cylinder.

SOLUTION ■ We use Gauss's law to obtain the electric field at a point P inside the cylinder at a distance r from its axis.

Since the charge distribution is cylindrically symmetric, we draw a **cylindrical** Gaussian surface of length L and radius r passing through P (Fig. 7.7). For any point inside the cylinder, $r < R$ and the Gaussian surface lies inside the cylinder. From symmetry considerations that you have learnt in Sec. 7.2.1 for cylindrical charge distributions, you know that the electric flux has contribution only from the curved surface of the Gaussian cylinder and not its ends. Hence, from Gauss's law, we have

$$\oiint_S \vec{\mathbf{E}}.d\vec{\mathbf{S}} = E\,2\pi r\,L = \frac{Q_{encl}}{\varepsilon_0} \quad \text{for } r < R \qquad \text{(i)}$$

The charge enclosed by this Gaussian surface is

$$Q_{encl} = \rho \iiint_V dV$$

where the volume is just the volume of the cylinder of length L and radius r. Therefore,

$$Q_{encl} = \rho\pi r^2 L$$

and from Eq. (i), $$E\,2\pi r\,L = \frac{Q_{encl}}{\varepsilon_0} = \frac{\rho\pi r^2 L}{\varepsilon_0}$$

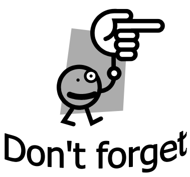
$$\therefore \qquad E = \frac{\rho r}{2\varepsilon_0} \quad \text{for } r < R$$

and $$\vec{\mathbf{E}} = \frac{\rho r}{2\varepsilon_0}\hat{\mathbf{r}} \quad \text{for } r < R \qquad (7.7)$$

where $\hat{\mathbf{r}}$ is the unit vector in the radial direction pointing outward from the cylinder's axis.

So, inside the cylindrical charge distribution, the electric field increases linearly with an increase in the distance from the axis.

For an infinite uniformly charged cylinder, always remember the following:



- **The electric field of an infinite uniformly charged cylinder at points outside it decreases with an increase in distance from its axis.**
- **The electric field inside an infinite uniformly charged cylinder increases linearly with an increase in distance from its axis.**

In the next section, we will apply Gauss's law to a charge distribution having planar symmetry. Examples of such charge distributions are uniform two-dimensional sheets of charge, thin plate carrying charge or uniform slabs of charge as well as combinations of such sheets or slabs like the ones used in parallel plate capacitors. But before you study further, try an SAQ to revise what you have learnt in this section.

SAQ 2 – **Applying Gauss's law to a solid charged cylinder**

A long non-conducting solid cylinder of radius 0.60 m carries a uniform volume charge density $+4.8\mu\text{Cm}^{-3}$. Calculate the magnitude of the electric field at a distance of (a) 0.40 m and (b) 1.0 m from the axis of the cylinder.

7.3 ELECTRIC FIELD DUE TO AN INFINITE UNIFORMLY CHARGED PLANE SHEET

In this section, we apply Gauss's law to an infinite uniformly charged plane sheet carrying a constant surface charge density σ. A large plastic sheet uniformly charged on one side is an example of a non-conducting sheet of charge. An aluminium foil is an example of a conducting sheet.

What kind of symmetry does an infinite sheet (planar charge distribution) possess? It remains the same if it is

- translated parallel to itself,
- rotated about any axis perpendicular to its plane, and
- reflected about any axis lying in its plane or perpendicular to its plane.

It follows from the symmetry considerations for a sheet of charge that the electric field due to it is everywhere perpendicular to the plane of the sheet. It is directed outward from the sheet, if positively charged and inward, if negatively charged. You may like to know: **Why is the electric field perpendicular to the plane everywhere**?

To answer this question, we follow the same line of argument as we have done for all symmetric charge distributions so far.

Refer to Fig. 7.8a, which shows the side view of a small section of the infinite sheet of charge. Suppose that the electric field of the sheet at some point *P* were directed in some other direction as shown in Fig. 7.8a. Note that we have arbitrarily labelled one end of this section of the infinite sheet as 'top' and the other one as 'bottom' just to show what happens when the sheet is reflected.

Let us now reflect this sheet of charge about a horizontal line perpendicular to its plane and passing through *P*. So the 'top' of the sheet is now the 'bottom' of the sheet and the 'bottom', its 'top' (Fig. 7.8b). What is the direction of the electric field after the sheet is reflected? After reflection, the direction of the electric field at *P* becomes as shown in Fig. 7.8b because the electric field is also reflected in the same manner.

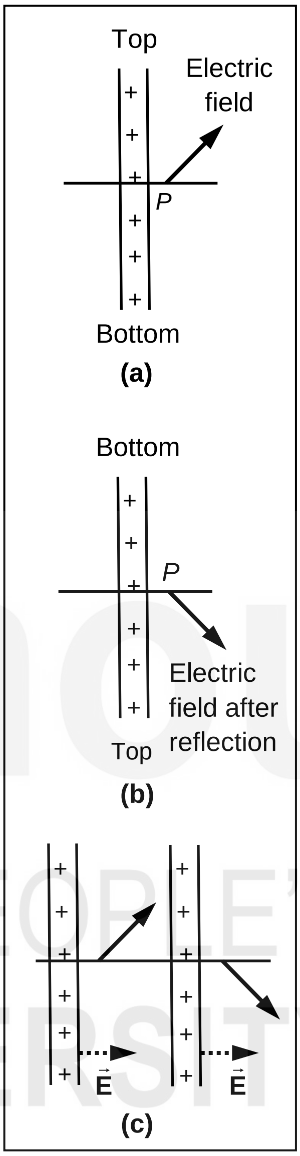


Fig. 7.8: **a) and b) If the electric field at any point due to a sheet of charge is not perpendicular to its plane everywhere, it will have different values at the same point for the same charge distribution under reflection, which is a contradiction; c) this contradiction does not exist if the electric field (shown by the dotted arrows) at any point due to the sheet of charge is perpendicular to its plane everywhere.**

But you can compare Figs. 7.8a and b by putting them alongside each other as in Fig. 7.8c. What do you find? You can see that the charge distribution remains the same after reflection. As before, we have labelled 'top' and 'bottom' on the sheet for our convenience. Otherwise, we cannot tell the difference. In this case, we find again that the electric fields before and after reflection are different at the same point. This is a contradiction: **How can there be different electric fields at a given point for the same charge distribution**? **If the charge distribution remains unchanged, the electric field at the point *P* cannot be different; it has to be the same**. Since it is not so, the direction of the electric field in Fig. 7.8a is incorrect.

Again we ask: **What is the direction of the electric field that does <u>not</u> lead to such a contradiction**? From Fig. 7.8c, you can see that the electric field remains the same under reflection only if it is directed perpendicular to the sheet of charge. It is shown by dotted arrows in Fig. 7.8c. You can verify that this is indeed the electric field direction for all other symmetry operations on the sheet. This is how we conclude that from symmetry of the sheet of charge, the direction of the electric field can only be perpendicular to its plane.

Let us now determine the electric field due to the infinite uniformly charged sheet at a distance r from it. Let its surface charge density be σ. Here we assume that the thickness of the sheet is much less than r. Now to use Gauss's law meaningfully, we need to choose a Gaussian surface that exploits the fact that the electric field is directed normal to the charged sheet. **What is that Gaussian surface**? We choose a closed cylindrical Gaussian surface perpendicular to the sheet with each end of the cylinder located at an equal distance (r) from the sheet. So, the length of the Gaussian cylindrical surface is $2r$ (see Fig. 7.9a). Such a Gaussian surface is also called the **Gaussian 'pillbox'**. In Fig. 7.9b, we show the side view of the sheet and the pillbox. Let the area of cross-section of the Gaussian pillbox (i.e., the area of its ends) be S.

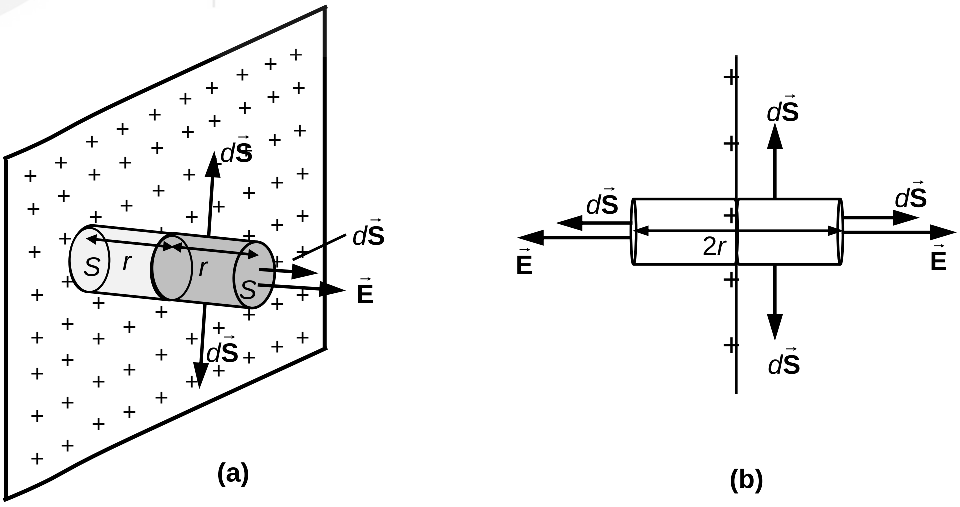


Fig. 7.9: **a) A sheet of positive charge and the Gaussian pillbox for which the electric field $\vec{E}$ and area vector $d\vec{S}$ are parallel at the ends and perpendicular to each other on the curved part of the surface; b) the sheet in its side view showing the electric field vectors and area vectors for the pillbox.**

Since the charge is positive, the electric field is directed away from the sheet and is perpendicular to the sheet. This means that for the curved part of the cylindrical Gaussian surface, the electric field vector is perpendicular to the area vector at all points (see Fig. 7.9b). Thus,

$$\vec{\mathbf{E}}.d\vec{\mathbf{S}} = 0 \quad \text{for all points on the curved part of the cylindrical surface}$$

The electric field vectors point in an outward direction from the two ends of the Gaussian pillbox, i.e., in the same direction as the area vectors for the ends. So, the contribution to the electric flux is only from the ends of the Gaussian pillbox and

$$\vec{\mathbf{E}}.d\vec{\mathbf{S}} = E\,dS \quad \text{for all points on one end of the cylindrical surface}$$

Since there are two ends on the Gaussian pillbox, we need to consider the surfaces of both ends while applying Gauss's law and divide the surface integral into three parts corresponding to the two ends and the curved part. Then Gauss's law gives us

$$\oiint_S \vec{\mathbf{E}}.d\vec{\mathbf{S}} = \iint_{\substack{\text{Curved}\\ \text{part}}} \vec{\mathbf{E}}.d\vec{\mathbf{S}} + \iint_{\text{Both ends}} \vec{\mathbf{E}}.d\vec{\mathbf{S}} = 0 + ES + ES = \frac{Q_{encl}}{\varepsilon_0} \qquad (7.8a)$$

or
$$E = \frac{Q_{encl}}{2\varepsilon_0 S} \qquad (7.8b)$$

Now, we need to express the charge on the sheet enclosed by the Gaussian cylinder in terms of the uniform surface charge density σ. This is just the charge enclosed by the area of the sheet equal to the cylinder's cross-section, i.e., the area S. Since σ is uniform (i.e., constant), it is equal to the ratio of the charge on a given surface to its area. Therefore, for the charge Q_{encl} enclosed by the area S, it is

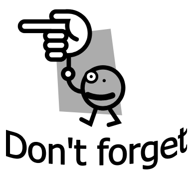
$$\sigma = \frac{Q_{encl}}{S} \quad \Rightarrow \quad Q_{encl} = \sigma S \qquad (7.9)$$

Substituting the value of Q_{encl} from Eq. (7.9) in Eq. (7.8b), we get

$$E = \frac{\sigma}{2\varepsilon_0} \qquad (7.10)$$

where the direction of the electric field is perpendicular to the sheet. Eq. (7.10) holds for both non-conducting and conducting sheets of charge provided the layer of charge on the sheet is very thin (or its thickness is very small compared to the distance at which the electric field is being calculated). It also holds for very large sheets of charge at points far from the edges of the sheet and at distances much larger than the thickness of the sheet or the layer of charge on the sheet. Eq. (7.10) tells us that

> **The electric field due to an infinite (or very large) uniformly charged sheet has the same value at all points lying outside it and points in a direction perpendicular to the sheet.**



Let us apply Gauss's law to two infinite or large sheets of charge in the following example.

EXAMPLE 7.2 : TWO INFINITE SHEETS OF CHARGE

Two thin infinite non-conducting charged sheets are kept parallel to each other as shown in Fig. 7.10a. The surface charge density of the negatively charged left sheet is σ_1 and that of the right sheet carrying a positive charge is σ_2. Determine the net electric field in the region (1) to the left of the sheets, (2) between the sheets and (3) to the right of the sheets.

SOLUTION ■ We apply Gauss's law to both sheets using the result obtained for an infinite uniformly charged sheet. We use the fact that the charges are fixed and obtain the electric field due to each sheet as if it were isolated. Then we apply the principle of superposition to obtain the net electric field.

Remember that from Eq. (7.10), the magnitude of the electric field at any point does not depend on the distance of the point from the sheet. It depends only on the surface charge density. The directions of the electric fields depend on the sign of the charge carried by them. The magnitudes of the electric field due to the negatively and positively charged sheets having surface charge densities σ_1 and σ_2, respectively, are given by

$$E_- = \frac{\sigma_1}{2\varepsilon_0} \quad \text{and} \quad E_+ = \frac{\sigma_2}{2\varepsilon_0}$$

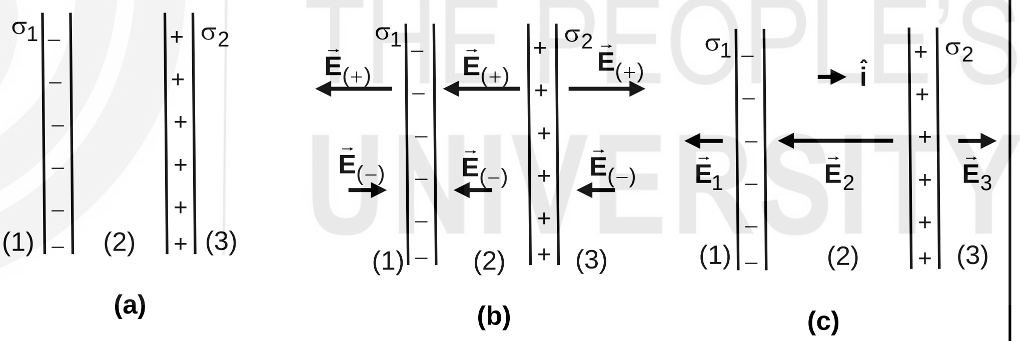


Fig. 7.10: **Diagram for Example 7.2.**

Fig. 7.10b shows the directions of the electric fields in each region. Note that the electric field due to the positively charged sheet points away from it in each of the three regions. The electric field due to the negatively charged sheet points towards it in each region. Let us denote the unit vector to the right of the sheets by $\hat{\mathbf{i}}$ (Fig. 7.10c). Then the resultant electric field in each of these regions (Fig. 7.10c) is given by

a) Region (1): $\vec{\mathbf{E}}_1 = \vec{\mathbf{E}}_+ + \vec{\mathbf{E}}_- = (E_+)(-\hat{\mathbf{i}}) + (E_-)\hat{\mathbf{i}} = \dfrac{1}{2\varepsilon_0}(\sigma_1 - \sigma_2)\hat{\mathbf{i}}$

b) Region (2): $\vec{\mathbf{E}}_2 = \vec{\mathbf{E}}_+ + \vec{\mathbf{E}}_- = (E_+ + E_-)(-\hat{\mathbf{i}}) = -\dfrac{1}{2\varepsilon_0}(\sigma_1 + \sigma_2)\hat{\mathbf{i}}$

c) Region (3): $\vec{\mathbf{E}}_3 = \vec{\mathbf{E}}_+ + \vec{\mathbf{E}}_- = (E_+)\,\hat{\mathbf{i}} + (E_-)(-\hat{\mathbf{i}}) = \dfrac{1}{2\varepsilon_0}(\sigma_2 - \sigma_1)\hat{\mathbf{i}}$

You will realise the importance of these calculations when you determine the electric fields of parallel plate capacitors in the next block and learn how useful capacitors are in our daily lives. You may now like to attempt an SAQ.

SAQ 3 – Uniformly charged thin sheets

Suppose in Example 7.2, the surface charge density of the negatively charged sheet is $\sigma_1 = 9.0 \times 10^{-9}\,\text{Cm}^{-2}$ and that of the positively charged sheet is $\sigma_2 = 6.0 \times 10^{-9}\,\text{Cm}^{-2}$. Determine the magnitudes and directions of the electric fields in the three regions. What would the net electric fields in the three regions be if the two sheets were interchanged?

While studying Unit 6 and Unit 7 so far, you must have realised that the symmetry of the charge distribution plays an important role in applications of Gauss's law. As you have learnt, the calculation of the surface integral in Gauss's law is greatly simplified for symmetric charge distributions. You have learnt about three kinds of symmetry for which application of Gauss's law is particularly useful. These are: **spherical symmetry**, **cylindrical symmetry** and **planar symmetry**. Let us revise the method of applying Gauss's law for each one of these.

APPLICATIONS OF GAUSS'S LAW

Recap

1. For a **spherically symmetric charge distribution, you should draw a concentric Gaussian sphere**. This means that the centre of the Gaussian sphere should be on the point charge or the centre of the charged sphere or spherical shell. Also the point on which the electric field is to be determined should be on the surface of the Gaussian sphere. Then the electric field is normal to the Gaussian spherical surface, $\vec{\mathbf{E}} \parallel d\vec{\mathbf{S}}$ so that $\vec{\mathbf{E}}.d\vec{\mathbf{S}} = E\,dS$ and E is constant on the surface.

2. For a **cylindrically symmetric charge distribution, you should draw a coaxial cylindrical Gaussian surface.** This means that the axis of the cylindrical Gaussian surface should be the same as that of the charge distribution (charged wire or charged cylinder). Also the point on which the electric field is to be determined should lie on the Gaussian surface. Then the electric field is normal to the curved part of the Gaussian cylindrical surface ($\vec{\mathbf{E}} \parallel d\vec{\mathbf{S}}$ so that $\vec{\mathbf{E}}.d\vec{\mathbf{S}} = E\,dS$) and parallel to the flat ends of the Gaussian cylinder ($\vec{\mathbf{E}} \perp d\vec{\mathbf{S}}$ so that $\vec{\mathbf{E}}.d\vec{\mathbf{S}} = 0$). Also, E is constant on the Gaussian surface.

3. For a **planar charge distribution, you should draw a Gaussian pillbox with its axis perpendicular to the plane of the charge distribution.** Then the electric field is perpendicular to the curved surface of the Gaussian pill box ($\vec{\mathbf{E}} \perp d\vec{\mathbf{S}}$ so that $\vec{\mathbf{E}}.d\vec{\mathbf{S}} = 0$) and parallel to the flat ends of the Gaussian pill box ($\vec{\mathbf{E}} \parallel d\vec{\mathbf{S}}$ so that $\vec{\mathbf{E}}.d\vec{\mathbf{S}} = E\,dS$). Also E is constant on the Gaussian surface.

So far, we have applied Gauss's law to non-conducting charged distributions. Does the law give different results for charged conductors? In the last section of this unit, we will apply Gauss's law to isolated charged conductors. This application of Gauss's law is quite important in our daily lives. This is especially so when we are caught in a thunderstorm. It will help you understand what you should do when you are travelling in a vehicle and are caught in a thunderstorm accompanied by lightning.

7.4 CHARGED ISOLATED CONDUCTOR

We can use Gauss's law to verify the following property of charged isolated conductors:

> "**If any excess unbalanced, static charges are placed on a conductor, they must reside on the surface of the conductor**. **The excess amount of charge moves to the surface of the conductor. When the charges stop moving, none of the charges will remain within the body of the conductor**."

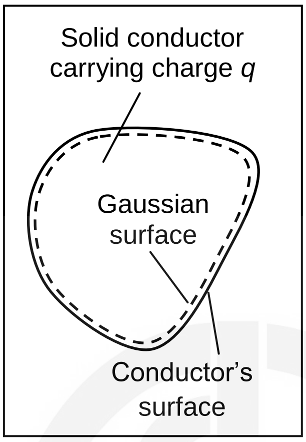


Fig. 7.11: **An isolated charged solid metallic conductor carrying excess charge *q* and the Gaussian surface just inside it**.

Let us use Gauss's law to explain how this is possible.

Consider the cross-section of an insulated solid metallic conductor such as the one shown in Fig. 7.11 carrying an excess charge q. We choose the Gaussian surface to lie just inside the actual surface of the conductor. The dashed line in Fig. 7.11 shows the Gaussian surface.

Once the excess charge stops moving, the electric field inside the charged conductor must become zero. **Why is this so**? We can see why this is so without a formal calculation. Suppose that this were not true and that there was an electric field inside conductor. Then a force would be exerted by the electric field on the charges inside the conductor that are always present in it and are free to move (e.g., electrons in this case).

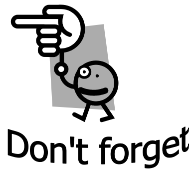
Thus, internal currents would be set up and would always exist within a conductor because charge would flow from one point to another under the action of this force. But no such perpetual currents are observed in any isolated charged conductor. So, the only conclusion is that the internal electric field of an isolated charged conductor is zero. Its interior is always free of electric fields.

For the time when the conductor is being charged, internal electric fields do exist inside it. But once the charging stops and the conductor is isolated, the excess charge is quickly distributed in a way that the **net electric field is zero everywhere inside the conductor**.

Now since the electric field is zero inside the conductor, it must be zero for all points on the Gaussian surface because that surface, though close to the surface of the conductor, lies **inside** it. This means that the electric flux through the Gaussian surface is zero. Then according to Gauss's law, the net charge enclosed by the Gaussian surface is also zero. So, if the excess charge is not inside the Gaussian surface, it must lie outside it. This means that it must lie on the actual surface of the isolated conductor.

So, always remember that

The net electric field is zero everywhere inside the conductor. If a net charge does reside on an isolated conducting body/object, it can be distributed only over the surface layer of that conductor.

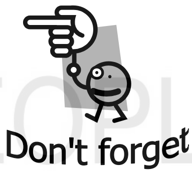


You may like to know: **What is the electric field at any point lying outside a conductor carrying a net charge on its outer surface**?

The results for **the electric fields for all <u>conducting</u> symmetric charge distributions at a point outside the conductor will be the same as the results obtained for the corresponding non-conducting charge distributions**. So, the electric fields due to various symmetric **conducting** and **non-conducting charge distributions** (**at any point lying outside them**) are same and are given in Table 7.1.

Table 7.1: Electric fields due to conducting and non-conducting charge distributions at points lying outside them.

Conducting and non-conducting charge distribution	Electric field at a point lying outside the charge distribution
Uniform spherical charge distribution of radius R carrying net positive charge Q	$\vec{\mathbf{E}} = \dfrac{1}{4\pi\varepsilon_0}\dfrac{Q}{r^2}\hat{\mathbf{r}} \qquad r \geq R$
Uniformly charged thin spherical shell of radius R carrying net charge Q	$\vec{\mathbf{E}} = \dfrac{1}{4\pi\varepsilon_0}\dfrac{Q}{r^2}\hat{\mathbf{r}} \qquad r \geq R$
Infinite line of charge having uniform line charge density λ	$E = \dfrac{\lambda}{2\pi\varepsilon_0 r}$ directed perpendicular to the line charge
Infinite cylindrical charge distribution of radius R having uniform volume charge density ρ	$\vec{\mathbf{E}} = \dfrac{\rho R^2}{2\varepsilon_0 r}\hat{\mathbf{r}} \qquad r \geq R$
Infinite thin sheet of charge having uniform surface charge density σ	$E = \dfrac{\sigma}{2\varepsilon_0}$ directed perpendicular to the sheet



SAQ 4 – Charged isolated conductor

An isolated conducting sphere of radius 1.0 m carries a uniform surface charge density $2.7\,\mu\text{C}\,\text{m}^{-2}$. What is the net charge on the sphere? Calculate the net electric flux leaving the surface of the sphere. What is the electric field due to the conductor at a point 3.0 m from its centre?

We now consider an example for determining the electric field due to two concentric conductors in different regions around them. Such problems are useful in determining the electric fields due to various geometries in capacitors.

EXAMPLE 7.3 : CONCENTRIC SPHERE AND SHELL

A solid conducting sphere is concentric with a thin conducting spherical shell as shown in Fig. 7.12a. The sphere of radius r_1 carries charge Q_1 and the spherical shell of radius r_2 carries charge Q_2 with $r_1 < r_2$. Determine the electric fields at a distance r from the centre of the sphere for (a) $r < r_1$, (b) $r_1 < r < r_2$ and (c) $r > r_2$. d) What will happen if the sphere and the shell are connected with a wire? e) What will the electric fields be for $r < r_2$ and $r > r_2$ after this?

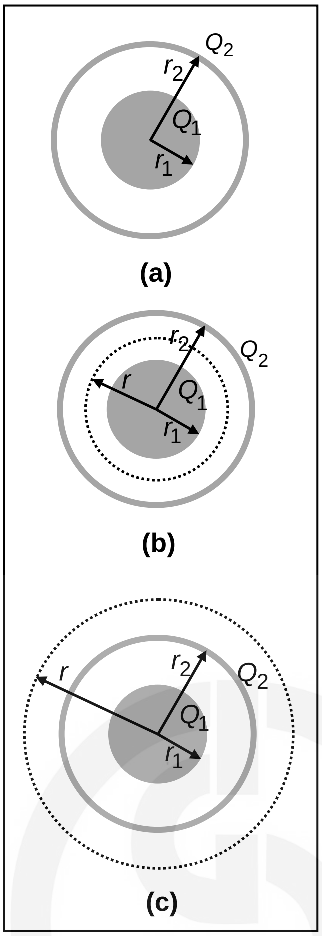


Fig. 7.12: **Diagram for Example 7.3.**

SOLUTION ■ We apply Gauss's law to both conducting sphere and conducting shell using the result obtained for a conductor in this section.

Remember that the electric field at any point inside a conductor is zero.

a) The points corresponding to $r < r_1$ lie **inside** the conducting sphere. Therefore, the electric field at all such points is zero.

b) For the points $r_1 < r < r_2$, we draw a spherical Gaussian surface of radius r at any point between the conducting sphere and the conducting shell (see Fig. 7.12b). The net charge enclosed by it is just the charge on the conducting sphere, i.e., Q_1. Therefore, from Eq. (6.22), the electric field is given as

$$\vec{\mathbf{E}} = \frac{1}{4\pi\varepsilon_0}\frac{Q_1}{r^2}\hat{\mathbf{r}} \qquad \text{for} \quad r_1 < r < r_2$$

c) For the points $r > r_2$, we draw a spherical Gaussian surface of radius $r\ (> r_2)$ lying outside the conducting shell (Fig. 7.12c). The surface encloses a net charge $(Q_1 + Q_2)$. Therefore, from Eq. (6.22), the electric field is given as

$$\vec{\mathbf{E}} = \frac{1}{4\pi\varepsilon_0}\frac{(Q_1 + Q_2)}{r^2}\hat{\mathbf{r}} \qquad \text{for} \quad r > r_2$$

Remember, you have to take the algebraic sum of charges to determine the net charge. So, while solving problems, always take into account the signs of the charges.

d) When the conducting sphere and the conducting shell are connected with a wire, charges flow in the system until equilibrium is reached. At equilibrium, there is no charge inside both the conductors and the system behaves like a single conductor. So there is no charge on either the inner sphere or the inner surface of the shell. The net charge $(Q_1 + Q_2)$ resides on the outer surface of the spherical shell.

e) The electric field for $r < r_2$ will be zero since the point lies inside a conductor.

If we draw a spherical Gaussian surface for $r > r_2$, it encloses the net charge $(Q_1 + Q_2)$. Therefore, we have

$$\vec{\mathbf{E}} = \frac{1}{4\pi\varepsilon_0}\frac{(Q_1 + Q_2)}{r^2}\hat{\mathbf{r}} \qquad \text{for} \quad r > r_2$$

You may quickly like to apply the results of Example 7.3 for practice. Attempt the following SAQ.

SAQ 5 – Charged conductors

Suppose in Example 7.3, $Q_1 = Q$ and $Q_2 = -2Q$. What will the electric fields be for (a) $r < r_1$, (b) $r_1 < r < r_2$ and (c) $r > r_2$, if all other parameters are the same?

Now suppose we create a cavity inside the conductor. Will the results for charged isolated conductors still hold? We explain what happens in this case in the following example.

EXAMPLE 7.4 : AN ISOLATED CONDUCTOR WITH A CAVITY

A cavity is created inside an isolated conductor. Explain why any excess charge placed on the conductor will reside on its outer surface.

SOLUTION ■ We use Gauss's law to give the explanation.

Consider Fig. 7.13, which shows an isolated conductor with a cavity inside it. Now, you have learnt that there are no unbalanced charges inside the solid conductor. Therefore, we can assume reasonably that when we scoop out some of the material, leaving a hollow cavity, we do not change the charge distribution or the electric fields that existed in the solid conductor.

Once again, we draw the Gaussian surface so that it is **inside the conductor** and surrounds the cavity wall very close to it as shown in Fig. 7.13. Since the net electric field inside the conductor is zero $(\vec{\mathbf{E}}_{net} = \vec{\mathbf{0}})$, the electric flux through this surface must also be zero. Therefore, from Gauss's law, this surface cannot enclose any net charge. Thus, we can say that there is no net charge on the cavity wall. All excess charge remains on the outer surface of the isolated conductor.

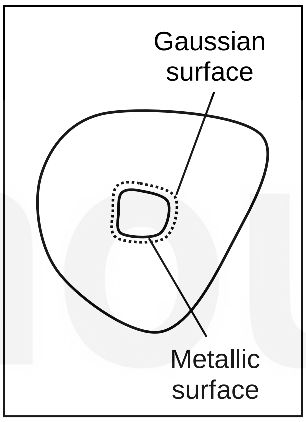


Fig. 7.13: **An isolated charged conductor having a cavity within it. The Gaussian surface lies within the conductor outside the cavity and very close to the cavity's surface.**

The results obtained in this section have many practical applications. We can now answer the question: What should we do when we get caught in a thunderstorm while travelling in a vehicle? From what you have studied in this section, you can answer the question as follows:

We should shut all windows and doors of the vehicle and keep ourselves insulated from all electronic gadgets present in it. If lightning strikes the vehicle, the entire charge will be distributed on its outer metallic surface. Its effects inside of the conductor (vehicle) will be substantially reduced: We will not be struck by lightning if we are sitting in a closed vehicle or any other closed space that is made of conducting material. On the other hand, if we were inside a non-conducting material like a wooden crate, lightning would pass right through it and we would be struck by it. The crate could also catch fire.

The fact that the electric field inside an isolated conductor with a cavity is zero has an interesting application in experimental physics called the **Faraday cage**. It is used in experiments which involve the measurement of very low power electrical signals generated, e.g., in computer chips or in neurons of animals. You can read about it at https://en.wikipedia.org/wiki/Faraday_cage. This is also the reason why your mobile phones, radio receivers, etc. will not work inside metal cages or metallic buildings.

With this we complete the discussion on Gauss's law and its applications. Let us now summarise the contents of this unit.

7.5 SUMMARY

Concept	Description
Infinite line charge	■ From Gauss's law, the electric field due to conducting and non-conducting **infinite line or wire of charge with uniform line charge density** λ is directed perpendicular to the line of charge and its magnitude is given by $$E = \frac{\lambda}{2\pi\varepsilon_0 r} \quad \text{at any point } r$$
Infinite non-conducting cylindrical charge distribution	■ The electric field due to a **non-conducting infinite solid cylinder of radius** R **with uniform volume charge density** ρ is given by $$\vec{\mathbf{E}} = \frac{\rho R^2}{2\varepsilon_0 r}\hat{\mathbf{r}} \qquad \text{for } r \geq R$$ $$\vec{\mathbf{E}} = \frac{\rho r}{2\varepsilon_0}\hat{\mathbf{r}} \qquad \text{for } r < R$$
Infinite non-conducting sheet of charge	■ The electric field due to a **non-conducting infinite sheet of charge** with **uniform surface charge density** σ at any point is given by $$E = \frac{\sigma}{2\varepsilon_0}$$ and points in a direction perpendicular to the sheet.
Charged isolated conductor without and with a cavity	■ If any excess unbalanced, static charges are placed on an isolated conductor, they must reside on the surface of the conductor. It follows that **if a net charge does reside on an isolated conducting body/object, it can be distributed only over the surface layer of that conductor**. In an isolated conductor having a cavity, all excess charge placed on the conductor will reside only on its outer surface.
Electric field due to charged isolated conductor at points lying inside and outside the conductor	■ The **electric field at points lying inside** an isolated charged conductor is zero. The electric field **at a point lying outside** an isolated charged conductor is the same as that of a non-conductor of the same geometry/symmetry (see Table 7.1).

Electric field due to charged isolated conductor at points lying inside and outside the conductor

- The electric field due to
  - a uniform **conducting spherical charge distribution** (of radius R and carrying charge Q) at a point at a distance r from its centre is

$$\vec{\mathbf{E}} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{\mathbf{r}} \qquad r \geq R$$

  - a thin uniform **conducting spherical shell** (of radius R and carrying charge Q) at a point at a distance r from its centre is

$$\vec{\mathbf{E}} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{\mathbf{r}} \qquad r \geq R$$

  - an **infinite conducting wire** carrying uniform linear charge density λ at r is

$$E = \frac{\lambda}{2\pi\varepsilon_0 r} \quad \text{directed perpendicular to the line of charge}$$

  - an **infinite conducting solid cylinder** having radius R and uniform volume charge density ρ at a point at a distance r is

$$E = \frac{\rho R^2}{2\varepsilon_0 r} \quad \text{in the radial direction for } r \geq R$$

  - an **infinite conducting thin sheet of charge** carrying uniform surface charge density σ is

$$E = \frac{\sigma}{2\varepsilon_0} \quad \text{directed perpendicular to the sheet.}$$

7.6 TERMINAL QUESTIONS

1. What is the magnitude of the electric field at a distance of 2.0 m from an infinitely charged wire given that the linear charge density is $3.6\,\mu\text{C m}^{-1}$?

2. A solid metal wire of length 1000 m and diameter 1.0 cm carries a net charge $q = 5.0\,\mu\text{C}$, which is distributed uniformly in it. Determine the electric field at a distance of a) 5.0 cm and b) 0.50 cm, respectively, from the wire's axis. Assume that the point where the electric field is to be determined is far from the ends of the wire.

3. A thin metal wire of length 30 m and diameter 0.04 cm carries a net charge $6.0\,\mu\text{C}$ distributed uniformly over its surface. Calculate the electric field at the points at the distances of (a) 0.01 cm and (b) 0.09 cm from its axis. Assume that these points lie far away from the ends of the wire.

4. A Gaussian surface of cylindrical shape (of radius 1.0 m and height 20 m) encloses a few positive charges. Assuming that the electric field due to these charges is normal to the Gaussian surface and has magnitude $900\,\text{NC}^{-1}$, calculate the volume charge density of the charge distribution.

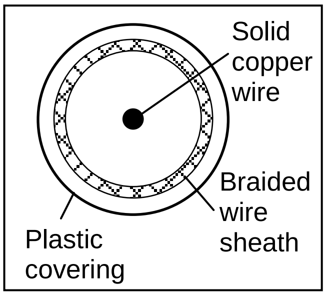


Fig. 7.14: Diagram for TQ 5.

5. A coaxial cable consists of a thin inner solid copper wire and an outer sheath of braided copper wire (see Fig. 7.14). The linear charge density of the inner wire is $-\lambda$ and that of the outer wire is λ. Determine the electric

fields at a point (a) in the region inside the inner wire, (b) in the region between the wires and (c) in the region outside the coaxial cable.

6. A flat sheet of charge of surface area *A* has uniform surface charge density σ. An electrostatic force of magnitude $3.6\times10^{-12}\,\text{N}$ pointing in a perpendicular direction away from the sheet, is exerted on an electron at a distance of 0.03 m from its centre. Calculate the net charge on the sheet for $A = 2.56\,\text{m}^2$.

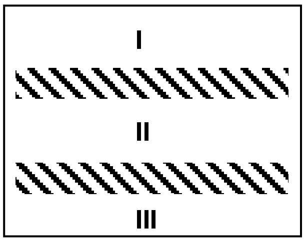


Fig. 7.15: **Diagram for TQ 7.**

7. Two identical infinite non-conducting sheets having equal positive surface charge densities σ are kept parallel to each other as shown in Fig. 7.15. Determine the electric field at a point in (a) region I above the sheets, (b) region II between the sheets and (c) region III below the sheets.

8. A very long conducting thin solid cylinder of length *L* carrying a net charge $+q$ is enclosed in a thin conducting cylindrical hollow tube of the same length. The tube carries a net charge $+2q$. Determine the electric fields at

   (a) a point lying outside the conducting tube; and

   (b) a point lying in the region between the solid cylinder and the tube.

   In both cases, the point lies far away from the edges of the conductors.

9. The net charge on an isolated conductor is $q_1 = 15\,\mu\text{C}$. A charge $q_2 = 5.0\,\mu\text{C}$ is later placed inside a cavity in the conductor. Determine the charge on the wall of the cavity. What is the charge on the outer surface of the conductor after q_2 is placed inside the cavity?

10. A concentric spherical cavity of radius 3.0 m is created in a conducting sphere of radius 6.0 m. A point charge *Q* is kept at the centre of the sphere/cavity. The net charge on the conducting sphere is $+9.0\,\text{nC}$. The electric field at a point 2.0 m away from the centre of the sphere is $7.2\,\text{NC}^{-1}$ and points radially inward.

    a) What is the value of the charge *Q*?

    b) What is the charge on the wall of the cavity, i.e., the inner surface of the sphere?

    c) Calculate the value of the charge on the sphere's outer surface.

    d) Determine the electric field at a point 4.0 m away from the centre of the sphere.

7.7 SOLUTIONS AND ANSWERS

Self-Assessment Questions

1. From Eq. (7.3), the linear charge density is $\lambda = 2\pi\varepsilon_0 r E$. Substituting the numerical values of *r* and *E* along with the constants, we get

   $$\lambda = 2\pi\times8.85\times10^{-12}\,\text{C}^2\text{N}^{-1}\text{m}^{-2}\times(1.0\,\text{m})\times9.0\times10^3\,\text{NC}^{-1} = 5.0\times10^{-7}\,\text{C}\,\text{m}^{-1}$$

2. a) The point at a distance of 0.40 m from the cylinder's axis lies inside it. Therefore, we use Eq. (7.7) to calculate the magnitude of the electric field:

$$E = \frac{\rho r}{2\varepsilon_0} = \frac{4.8\,\mu\text{Cm}^{-3} \times 0.40\,\text{m}}{2 \times 8.85 \times 10^{-12}\,\text{C}^2\text{N}^{-1}\text{m}^{-2}} = 1.1 \times 10^5\,\text{NC}^{-1}$$

b) The point at a distance of 1.0 m from the cylinder's axis lies outside it. Therefore, we use Eq. (7.6) to calculate the magnitude of the electric field:

$$E = \frac{\rho R^2}{2\varepsilon_0 r} = \frac{4.8\,\mu\text{Cm}^{-3} \times (0.60\,\text{m})^2}{2 \times 8.85 \times 10^{-12}\,\text{C}^2\text{N}^{-1}\text{m}^{-2} \times (1.0\,\text{m})} = 9.8 \times 10^4\,\text{NC}^{-1}$$

3. As explained in Example 7.2, for $\sigma_1 = 9.0 \times 10^{-9}\,\text{Cm}^{-2}$ and $\sigma_2 = 6.0 \times 10^{-9}\,\text{Cm}^{-2}$, the magnitudes and directions of the electric fields in the three regions are given by

Region (1): $$\vec{\mathbf{E}}_1 = \frac{1}{2\varepsilon_0}(\sigma_1 - \sigma_2)\hat{\mathbf{i}} = \frac{(9.0 - 6.0) \times 10^{-9}\,\text{Cm}^{-2}}{2 \times 8.85 \times 10^{-12}\,\text{C}^2\text{N}^{-1}\text{m}^{-2}}\,\hat{\mathbf{i}}$$

$$= 1.7 \times 10^2\,\text{NC}^{-1}\,\hat{\mathbf{i}}$$

Region (2): $$\vec{\mathbf{E}}_2 = -\frac{1}{2\varepsilon_0}(\sigma_1 + \sigma_2)\hat{\mathbf{i}} = -\frac{(9.0 + 6.0) \times 10^{-9}\,\text{Cm}^{-2}}{2 \times 8.85 \times 10^{-12}\,\text{C}^2\text{N}^{-1}\text{m}^{-2}}\,\hat{\mathbf{i}}$$

$$= -8.5 \times 10^2\,\text{NC}^{-1}\,\hat{\mathbf{i}}$$

Region (3): $$\vec{\mathbf{E}}_3 = \frac{1}{2\varepsilon_0}(\sigma_2 - \sigma_1)\hat{\mathbf{i}} = \frac{(6.0 - 9.0) \times 10^{-9}\,\text{Cm}^{-2}}{2 \times 8.85 \times 10^{-12}\,\text{C}^2\text{N}^{-1}\text{m}^{-2}}\,\hat{\mathbf{i}}$$

$$= -1.7 \times 10^2\,\text{NC}^{-1}\,\hat{\mathbf{i}}$$

Refer to Fig. 7.16. If the two sheets are interchanged, then we have negative $\sigma_1 = 9.0 \times 10^{-9}\,\text{Cm}^{-2}$ and positive $\sigma_2 = 6.0 \times 10^{-9}\,\text{Cm}^{-2}$.

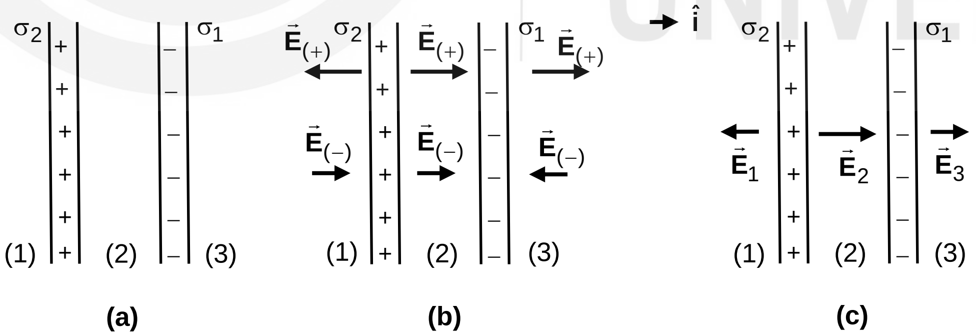


Fig. 7.16: **Diagram for answer of SAQ 3. Part (c) is not to scale.**

From Fig. 7.16b, the magnitudes and directions of the electric fields in the three regions are now given by

$$\vec{\mathbf{E}}_1 = \vec{\mathbf{E}}_+ + \vec{\mathbf{E}}_- = (E_+)(-\hat{\mathbf{i}}) + (E_-)\hat{\mathbf{i}} = \frac{1}{2\varepsilon_0}[\sigma_2(-\hat{\mathbf{i}}) + \sigma_1\hat{\mathbf{i}}] = \frac{1}{2\varepsilon_0}(\sigma_1 - \sigma_2)\hat{\mathbf{i}}$$

$$= 1.7 \times 10^2\,\text{NC}^{-1}\,\hat{\mathbf{i}}$$

$$\vec{\mathbf{E}}_2 = \vec{\mathbf{E}}_+ + \vec{\mathbf{E}}_- = E_+\hat{\mathbf{i}} + E_-\hat{\mathbf{i}} = \frac{1}{2\varepsilon_0}(\sigma_1 + \sigma_2)\hat{\mathbf{i}} = 8.5 \times 10^2\,\text{NC}^{-1}\,\hat{\mathbf{i}}$$

$$\vec{\mathbf{E}}_3 = \vec{\mathbf{E}}_+ + \vec{\mathbf{E}}_- = (E_+)\hat{\mathbf{i}} + (E_-)(-\hat{\mathbf{i}}) = \frac{1}{2\varepsilon_0}(\sigma_2 - \sigma_1)\hat{\mathbf{i}} = -1.7\times10^2\,\text{NC}^{-1}\hat{\mathbf{i}}$$

Of course, when you solve this problem, you have to start from the beginning and follow all steps given in Example 7.2.

4. The net charge on the sphere is $Q = \sigma S$, where $S = 4\pi R^2$ is the area of the surface of the sphere of radius R. Therefore,

$$Q = \sigma 4\pi R^2 = 4\pi\times 2.7\,\mu\text{Cm}^{-2}(1.0\,\text{m})^2 = 34\,\mu\text{C}$$

From Gauss's law [Eq. (7.4a)], the net electric flux leaving the surface of the sphere is

$$\Phi_E = \frac{Q}{\varepsilon_0} = \frac{34\,\mu\text{C}}{8.85\times10^{-12}\,\text{C}^2\text{N}^{-1}\text{m}^{-2}} = 3.8\times10^6\,\text{Nm}^2\text{C}^{-1}$$

Since the point lies outside the sphere, the electric field due to the conductor at a point 3.0 m from its centre is

$$\vec{\mathbf{E}} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{\mathbf{r}} = (8.99\times10^9\,\text{C}^{-2}\text{Nm}^2)\times\frac{34\,\mu\text{C}}{(3.0\,\text{m})^2}\hat{\mathbf{r}} = 3.4\times10^4\,\text{NC}^{-1}\hat{\mathbf{r}}$$

5. Substituting $Q_1 = Q$ and $Q_2 = -2Q$ in the results of Example 7.3, we get

a) For $r < r_1$, $\quad \vec{\mathbf{E}} = \vec{\mathbf{0}}$

b) For $r_1 < r < r_2$, $\quad \vec{\mathbf{E}} = \dfrac{1}{4\pi\varepsilon_0}\dfrac{Q_1}{r^2}\hat{\mathbf{r}} = \dfrac{1}{4\pi\varepsilon_0}\dfrac{Q}{r^2}\hat{\mathbf{r}}$, and

c) For $r > r_2$, $\quad \vec{\mathbf{E}} = \dfrac{1}{4\pi\varepsilon_0}\dfrac{(Q_1 + Q_2)}{r^2}\hat{\mathbf{r}} = \dfrac{1}{4\pi\varepsilon_0}\dfrac{(Q - 2Q)}{r^2}\hat{\mathbf{r}} = -\dfrac{1}{4\pi\varepsilon_0}\dfrac{Q}{r^2}\hat{\mathbf{r}}$

Terminal Questions

1. From Eq. (7.3), the magnitude of the electric field is

$$E = \frac{\lambda}{2\pi\varepsilon_0 r} = 2\times(8.99\times10^9\,\text{C}^{-2}\text{Nm}^2)\times\frac{3.6\,\mu\text{Cm}^{-1}}{2.0\,\text{m}} = 3.2\times10^4\,\text{NC}^{-1}$$

2. Although the wire is not infinite, for points close to it and sufficiently far from its ends, we can approximate it as one. This is because at such points we can neglect the contribution of the electric fields due to distant charges.

   a) We use Eq. (7.4b) to calculate the electric field at a point 5.0 cm from the wire's axis, since it lies outside the wire and get

$$E = \frac{Q_{encl}}{2\pi\varepsilon_0 r\,L} = 2\times(8.99\times10^9\,\text{C}^{-2}\text{Nm}^2)\times\frac{5.0\,\mu\text{C}}{5.0\times10^{-3}\,\text{m}\times1000\,\text{m}}$$

$$= 1.8\times10^4\,\text{NC}^{-1}$$

   b) The metal wire is a conductor. Therefore, the electric field at the point 0.50 cm from the wire's axis is zero, $\vec{\mathbf{E}} = \vec{\mathbf{0}}$ since the point lies inside the conductor.

3. The thin metal wire in the problem cannot strictly be taken as an infinite line charge. But for points close to the wire and sufficiently far from its ends, the contribution of the electric fields from distant charges can be taken to be negligible. Therefore, we can approximate the electric field of the wire to that of an infinite line charge. (a) Since the metal wire is a conductor, the electric field at the point 0.01 cm from the wire's axis, which lies inside it, will be zero: $\vec{\mathbf{E}} = \vec{\mathbf{0}}$. (b) The electric field at the point outside the wire at a distance of 0.09 cm from its axis is given by Eq. (7.2c) and we get

$$E = \frac{Q_{encl}}{2\pi\varepsilon_0 r\,L} = 2\times(8.99\times10^9\,\text{C}^{-2}\text{Nm}^2)\times\frac{6.0\,\mu\text{C}}{0.09\times10^{-2}\,\text{m}\times30\,\text{m}} = 4.0\times10^6\,\text{NC}^{-1}$$

4. We are given the electric field and the radius and height of the cylindrical Gaussian surface and we have to determine the volume charge density of the charge distribution enclosed by it. Since the surface area of the cylinder is $2\pi r\,h$, the electric flux through the Gaussian surface is

$$\Phi_E = E\,S = E\times(2\pi r\,h) = \frac{Q_{encl}}{\varepsilon_0}$$

or $$Q_{encl} = 2\pi\varepsilon_0 r\,hE$$

The volume charge density ρ of the charge distribution is the net charge enclosed per unit volume.

$$\therefore\ \rho = \frac{Q_{encl}}{V} = \frac{2\pi\varepsilon_0 r\,hE}{\pi r^2 h} = \frac{2\varepsilon_0 E}{r} = \frac{2\times8.85\times10^{-12}\,\text{C}^2\text{N}^{-1}\text{m}^{-2}\times900\,\text{NC}^{-1}}{1.0\,\text{m}}$$

$$= 1.6\times10^{-8}\,\text{Cm}^{-3}$$

5. a) The electric field at a point inside the inner copper wire (region I) is zero since it is a conductor: $\vec{\mathbf{E}} = \vec{\mathbf{0}}$.

b) Refer to Fig. 7.17. We take the Gaussian surface to be a coaxial cylindrical surface of radius r and length L lying in the region II between the wires. Note that the net charge enclosed by the Gaussian surface is $Q_{encl} = -\lambda L$, where $-\lambda$ is the linear charge density of the inner wire. From Gauss's law given by Eq. (7.4a),

$$\oint\!\!\oint_S \vec{\mathbf{E}}.d\vec{\mathbf{S}} = \oint\!\!\oint_S E\,dS = E\oint\!\!\oint_S dS = E\,2\pi r\,L = \frac{Q_{encl}}{\varepsilon_0} = -\frac{\lambda L}{\varepsilon_0}$$

So, we have $$\vec{\mathbf{E}} = -\frac{\lambda}{2\pi\varepsilon_0 r}\hat{\mathbf{r}}$$

where $\hat{\mathbf{r}}$ is the unit vector perpendicular to the cylindrical axis pointing away from the axis. So, the electric field in region II is directed radially inward.

c) For the point that lies outside the cable, the electric field is zero. This is because the two wires have equal and opposite linear charge densities and the net charge enclosed by a Gaussian surface outside both wires will be zero: $\vec{\mathbf{E}} = \vec{\mathbf{0}}$.

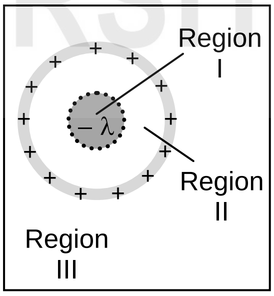


Fig. 7.17: **Diagram for the answer of TQ 5.**

6. The sheet is effectively infinite for the point at a distance of 0.03 m from its centre assuming that the point lies far from its edges. Let the net charge on the sheet be q. The electric field due to the sheet is given by Eq. (7.10):

$$E = \frac{\sigma}{2\varepsilon_0} \quad \text{directed perpendicular to the sheet, where} \quad \sigma = \frac{q}{A}$$

The magnitude of the electrostatic force on an electron is given by

$$F = -eE \qquad \text{or} \qquad F = -e\frac{\sigma}{2\varepsilon_0} = \frac{-eq}{2\varepsilon_0 A}$$

since the surface charge density is charge per unit area and A is the area of the sheet. Thus,

$$q = \frac{2\varepsilon_0 AF}{-e} = \frac{2\times 8.85\times 10^{-12}\,\mathrm{C^2N^{-1}m^{-2}}\times 2.56\,\mathrm{m^2}\times 3.6\times 10^{-12}\,\mathrm{N}}{-1.6\times 10^{-19}\,\mathrm{C}} = -1.0\,\mathrm{mC}$$

The negative sign shows that the charge on the sheet is negative. This is expected because the electrostatic force between the sheet and the electron is negative, i.e., the electron is repelled by the sheet.

7. Let $\vec{\mathbf{E}}_1$ be the electric field due to sheet 1 and $\vec{\mathbf{E}}_2$ be the electric field due to sheet 2 at some point in each of the three regions. The magnitudes of the electric fields due to the sheets will be equal since their surface charge densities are equal. Let us denote the magnitudes by E. Then from Eq. (7.10),

$$E = \frac{\sigma}{2\varepsilon_0}$$

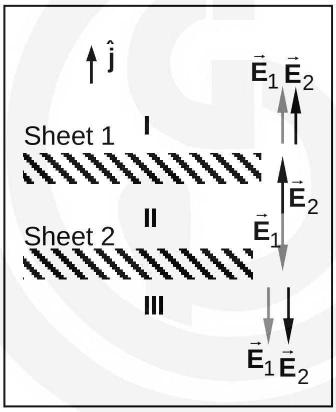


Fig. 7.18: **Diagram for answer of TQ 7.**

Since both sheets are charged positively, the electric fields due to them would be directed away from them in each region. The electric fields due to the sheets in the three regions are shown in Fig. 7.18. Now we can determine the net electric field at any given point in each region as follows:

a) Region I above the sheets: The electric fields due to the sheets are in the same direction, say, $\hat{\mathbf{j}}$, as both sheets are positively charged. Therefore, the net electric field at a point in region I is

$$\vec{\mathbf{E}} = \vec{\mathbf{E}}_1 + \vec{\mathbf{E}}_2 = 2\times\frac{\sigma}{2\varepsilon_0}\hat{\mathbf{j}} = \frac{\sigma}{\varepsilon_0}\hat{\mathbf{j}}$$

b) Region II between the sheets: The electric field due to sheet 1 is directed opposite to the electric field due to sheet 2. Therefore, the net electric field at a point in region II is

$$\vec{\mathbf{E}} = \vec{\mathbf{E}}_1 + \vec{\mathbf{E}}_2 = \frac{\sigma}{2\varepsilon_0}(-\hat{\mathbf{j}}) + \frac{\sigma}{2\varepsilon_0}\hat{\mathbf{j}} = \vec{\mathbf{0}}$$

c) Region III below the sheets: The electric fields are again in the same direction, but opposite to $\hat{\mathbf{j}}$. Therefore, the net electric field at a point in region III is

$$\vec{\mathbf{E}} = \vec{\mathbf{E}}_1 + \vec{\mathbf{E}}_2 = 2\times\frac{\sigma}{2\varepsilon_0}(-\hat{\mathbf{j}}) = -\frac{\sigma}{\varepsilon_0}\hat{\mathbf{j}}$$

8. We use Gauss's law given by Eq. (7.4a) to determine the electric fields in the two regions for conducting cylindrical charge distributions.

a) For a point lying outside the conducting tube, the net charge enclosed by a Gaussian cylindrical surface of radius r and length L passing through the point is the algebraic sum of the total charge on the solid cylinder and the cylindrical tube, i.e., $+3q$. Therefore, from Eq. (7.4a), we get

$$\oiint_S \vec{\mathbf{E}}.d\vec{\mathbf{S}} = \oiint_S E\,dS = E\oiint_S dS = E\,2\pi r\,L = \frac{Q_{encl}}{\varepsilon_0} = \frac{+3q}{\varepsilon_0}$$

or $\quad E = \dfrac{3q}{2\pi\varepsilon_0 r\,L}$ directed radially outward

b) For a point lying in the region between the solid cylinder and the tube, the net charge enclosed by a Gaussian cylindrical surface of radius r and length L passing through the point is just the charge on the solid cylinder, i.e., $+q$. Therefore, from Eq. (7.4a), we get

$$\oiint_S \vec{\mathbf{E}}.d\vec{\mathbf{S}} = \oiint_S E\,dS = E\oiint_S dS = E\,2\pi r\,L = \frac{Q_{encl}}{\varepsilon_0} = \frac{+q}{\varepsilon_0}$$

or $\quad E = \dfrac{q}{2\pi\varepsilon_0 r\,L}$ directed radially outward

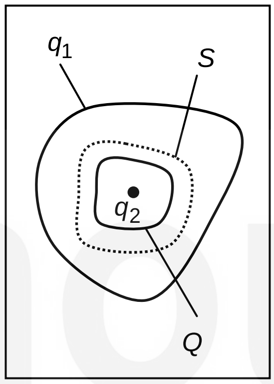


Fig. 7.19: **Diagram for the answer of TQ 9.**

9. Refer to Fig. 7.19. The net charge on the conductor is $q_1 = 15\,\mu\text{C}$. Suppose the charge on the wall of the cavity is Q. Let S be the Gaussian surface enclosing the cavity. The electric flux Φ_S through S is zero since the electric field inside the conductor is zero. Since the charge $q_2 = 5.0\,\mu\text{C}$ is placed inside the cavity in the conductor, the net charge enclosed by the Gaussian surface is the algebraic sum of the charge Q on the cavity wall (which is also the inner surface of the conductor) and q_2. So, from Gauss's law,

$$\Phi_S = \frac{Q_{encl}}{\varepsilon_0} = \frac{Q + q_2}{\varepsilon_0} = 0$$

$$\Rightarrow \quad Q + q_2 = 0 \;\Rightarrow\; Q = -q_2 = -5.0\,\mu\text{C}$$

Let the net charge on the outer surface of the conductor be q' after the charge $q_2 = 5.0\,\mu\text{C}$ is placed inside the cavity. From conservation of charge, the net charge on the conducting sphere is equal to the algebraic sum of the charge on its inner surface and the charge on its outer surface. Therefore, we have

$$q_1 = Q + q'$$

So, the charge on the outer surface of the conductor is

$$q' = 15\,\mu\text{C} - (-5.0\,\mu\text{C}) = 20\,\mu\text{C}$$

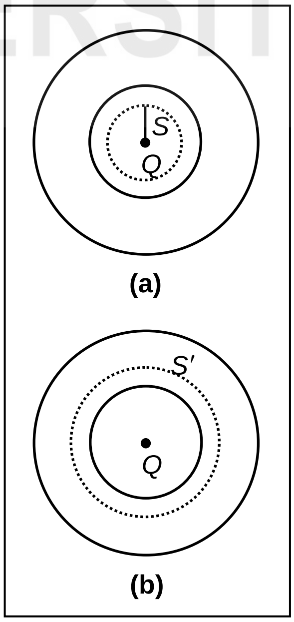


Fig. 7.20: **Diagram for the answer of TQ 10.**

10. a) It is given that the electric field at a point 2.0 m away from the centre of the sphere/cavity points inward. Refer to Fig. 7.20a. We draw a spherical Gaussian surface S of radius 2.0 m. So, its surface area is $4\pi(2.0\text{m})^2$ and the net charge enclosed by it is Q. Thus, from Gauss's law, we have

$$\Phi_S = -ES = \frac{Q_{encl}}{\varepsilon_0} = \frac{Q}{\varepsilon_0} \Rightarrow Q = -\varepsilon_0 ES = -4\pi\varepsilon_0 (2.0\,\text{m})^2 E$$

or

$$Q = -4\pi(8.85\times10^{-12}\,\text{C}^2\text{N}^{-1}\text{m}^{-2})\times(7.2\,\text{NC}^{-1})\,(2.0\,\text{m})^2 = -3.2\,\text{nC}$$

b) We follow the same steps as in the solution of TQ 9. The Gaussian surface S' lies inside the conductor and surrounds the cavity as shown in Fig. 7.20b. The net charge enclosed by S' is the algebraic sum of the charge Q and the charge on the wall of the cavity, say q. Since the electric field inside the conductor is zero, from Gauss's law, the net charge enclosed by the Gaussian surface is zero. Therefore,

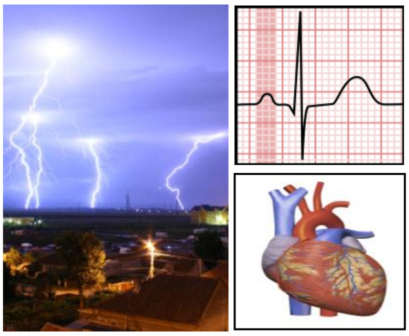
$$Q + q = 0 \;\Rightarrow\; q = -Q = 3.2\,\text{nC}$$

c) From conservation of charge, the total charge on the conducting sphere is equal to the algebraic sum of the charge q on its inner surface (i.e., the wall of the cavity) and the charge on its outer surface. Therefore, if the charge on the outer surface of the sphere is q', then we have

Net charge on the sphere $= +9.0\,\text{nC} = q + q'$

or $\quad q' = +9.0\,\text{nC} - 3.2\,\text{nC} = 5.8\,\text{nC}$

d) Since the point at a distance of 4.0 m from the centre lies inside the conducting sphere, the electric field at that point is zero.



Electric potential and potential differences abound in nature ranging from several hundred million volts in a typical lightning bolt to about 90 mV in heart cell membranes. (Picture source: Wikimedia Commons)

UNIT 8

ELECTRIC POTENTIAL

Structure

STUDY GUIDE

In this unit, you will study electric potential which is a concept closely related to electrostatic force and electric field. It is a very useful concept for studying the behaviour of charged objects in an electric field. You know that the electrostatic force and electric field are vector quantities. The electric potential, however, is a scalar quantity. Since electric potential is a scalar quantity, the calculation of electric potential at a point in space due to a charge or a system of charges is much easier than that of an electric field – a vector quantity. To understand the contents of this unit better, you should refresh vector algebra given in Block 1 and the concepts of conservative force and potential energy from Block 2 of the 1st semester course entitled Mechanics (BPHCT-131). You should also revise the vector calculus given in Block 1 of this course. In particular, you should refresh the concept of gradient of a scalar field, integration of a vector function, line integral of scalar and vector fields discussed in Block 1 of this course. We advise you to work through the steps of mathematical derivations as you study the unit. You should also try to solve SAQs and TQs yourself to check your understanding of the concepts discussed in the unit.

"You must be ready to give up even the most attractive ideas when experiment shows them to be wrong."

Alessandro Volta

8.1 INTRODUCTION

In Unit 5 of this block, you have learnt Coulomb's law which enables us to calculate the electrostatic force between any two charges. You have also learnt the concept of electric field which makes the computation of electrostatic force far easier and convenient than using Coulomb's law. In Units 6 and 7, you have learnt how to calculate electric field directly or by using Gauss's law.

In most problems in electrostatics, our aim is to calculate the electric field. Since electric field is a vector quantity, its determination requires calculation of each of its scalar components. Many a time, to make this calculation easier, we first calculate a scalar quantity known as the electric potential *V*, from which electric field can be determined using a simple relation. Since electric potential is a scalar quantity, its calculation in most cases is not as difficult as the calculation of electric field.

The concept of electric potential is also important because it is closely linked to the work done by the electrostatic force due to charged particles and their potential energies. To explain the concept of electric potential, we draw analogy from mechanics (Unit 10, BPHCT-131). In that unit, you studied gravitational potential energy, which arises from the work done in moving an object from one point to another against gravitational force. You have learnt in Unit 5 of this course that the gravitational force between charges is very small (compared to the electrostatic force). So, the gravitational potential energy of a charge is negligible. In the same way, when a charge is moved from one point to another against electrostatic force (or field), work needs to be done which is stored as electrostatic potential energy of the charge. And, the electric potential at a point in an electric field is defined as electrostatic potential energy per unit charge at that point.

We begin this unit by determining the work done in moving a charge from one point to another in an electric field in Sec. 8.2. In doing so, you will learn how to calculate the line integral of electric field $\vec{\mathbf{E}}$. In Sec. 8.3, we shall define a scalar quantity called **electric potential** in terms of the line integral of electric field and calculate its value at a point due to an isolated charge as well as due to a system of charges. In Sec. 8.4, you will learn how to calculate electric field at a point if the value of electric potential at that point is known.

You have learnt the concept of electric dipole in Unit 5. You know that it is a unique configuration of two charges which is of immense practical utility in physics. Therefore, in Sec. 8.5 of this unit, we shall explain how to determine electric potential due to electric dipole at a given point. In Sec. 8.6, we discuss the effect of electric field on an electric dipole and explain the conditions under which electrostatic potential energy can be stored in an electric dipole.

In the next unit, you will learn how to calculate electric potential due to continuous charge distributions.

Expected Learning Outcomes

After studying this unit, you should be able to:

- ❖ calculate the work done in moving a charge from one point to another in an electric field;
- ❖ define electric potential as line integral of an electric field;
- ❖ determine the electric potential at a point due to a single charge and a system of charges;
- ❖ establish the relation between electric potential and electric field;
- ❖ calculate electric field at a point knowing the electric potential;
- ❖ determine the electric potential due to an electric dipole at a given point; and
- ❖ determine the torque experienced by an electric dipole in a uniform electric field.

8.2 WORK DONE IN MOVING A CHARGE

The concept of electric potential is closely linked to (a) the work done by electrostatic force in moving a charge from one point to another in an electric field, and (b) the relation between work done and potential energy. For the gravitational force, you have learnt how to determine the work done in moving an object from one point to another in Example 9.8 of Unit 9 of the course on Mechanics (BPHCT-131). You have also learnt in Unit 10 that the gravitational force is a conservative force which enables us to define the gravitational potential energy. On similar lines, we shall determine the work done by the electrostatic force in moving a charge from one point to another in an electric field. We shall also show that the electrostatic force is conservative and thereby define electrostatic potential energy and electric potential.

From Sec. 5.3 of Unit 5, you know that a single charge, say Q, *sets up* an electric field in the region surrounding it. The electric field $\vec{\mathbf{E}}$ due to the charge at a point is defined as the electrostatic force experienced by a unit positive test charge placed at that point. If, instead of a unit positive charge, we place a charge q at that point, then electrostatic force $\vec{\mathbf{F}}$ experienced by the charge q in the electric field $\vec{\mathbf{E}}$ is given by

$$\vec{\mathbf{F}} = q\vec{\mathbf{E}} \tag{8.1a}$$

where the electric field $\vec{\mathbf{E}}$ is given by Eq. (5.6a) of Unit 5:

$$\vec{\mathbf{E}} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{\mathbf{r}} \tag{8.1b}$$

where $\hat{\mathbf{r}}$ is a unit vector in the radial direction away from charge Q.

Now, let us suppose that the charge *q* is moving from point *a* to *b* along an arbitrary path as shown in Fig. 8.1.

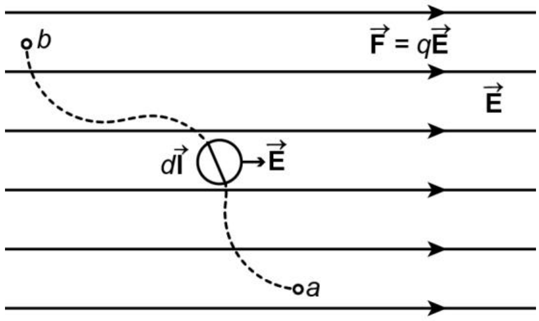


Fig. 8.1: **The charge *q* moves from point *a* to point *b* along an arbitrary path in electric field $\vec{E}$ of charge *Q* (not shown in the figure).**

From Sec. 3.3 of Unit 3, you may recall that the work *W*, done in moving the charge *q* from point *a* to *b* is given by the line integral [Eq. (3.18b)]:

$$W = \int_a^b \vec{\mathbf{F}}.d\vec{\mathbf{l}} = q\int_a^b \vec{\mathbf{E}}.d\vec{\mathbf{l}} \tag{8.2}$$

Now, what happens if instead of the charge *q*, we move only a unit positive charge between *a* and *b*? You can see that in this case, the work W', done is obtained simply by dividing *W* by *q*, i.e.,

$$W' = \frac{W}{q} = \int_a^b \vec{\mathbf{E}}.d\vec{\mathbf{l}} \tag{8.3}$$

We will be solving the line integrals of Eqs. (8.2) and (8.3) and obtain W' for a given charge. We will thus arrive at some interesting results. But, before proceeding further, we would like you to solve an SAQ.

SAQ 1 – **Work done in moving a charge**

Calculate the work done in moving a unit positive charge through a distance *l* in a uniform electric field parallel to the field direction.

Let us now evaluate the line integral of the electric field.

8.2.1 Line Integral of Electric Field

Let us consider the electric field due to a charge *Q* as shown in Fig. 8.2a. Let there be two points *a* and *b* at distances r_a and r_b from the charge *Q* as shown in Fig. 8.2a. Let us determine the line integral given by Eq. (8.3) for an arbitrary path between points *a* and *b*. Note that the path from *a* to *b* is a continuous curve. Let us evaluate Eq. (8.3), i.e., the work done in moving a unit positive charge from *a* to *b*. Suppose the unit charge moves from *a* to *a*′ (which is an arc of a circle) and then from *a*′ to *b* as shown in Fig. 8.2a. Then we can write [(recall Eq. (3.33), Unit 3, Block 1 of this course]:

$$W' = \int_a^b \vec{\mathbf{E}}.d\vec{\mathbf{l}} = \int_a^{a'} \vec{\mathbf{E}}.d\vec{\mathbf{l}} + \int_{a'}^b \vec{\mathbf{E}}.d\vec{\mathbf{l}} \tag{8.4}$$

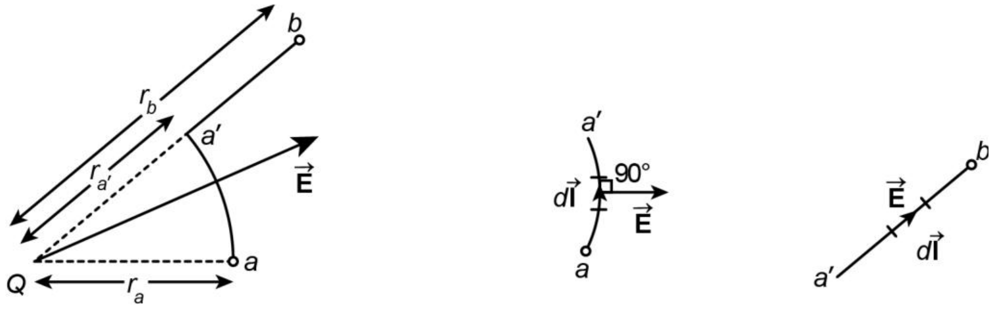


Fig. 8.2: **Work done in moving a unit positive charge from point** *a* **to point** *b* **along the path shown as continuous curve.**

The first line integral on the right-hand side of Eq. (8.4) represents the work done in moving the unit charge from *a* to a' along the arc of a circle of radius say, r_a. The second integral represents the work done in moving the same charge from a' to *b* along a straight line. The integrand $\vec{\mathbf{E}}.d\vec{\mathbf{l}}$ of the first line integral is equal to zero as $\vec{\mathbf{E}}$ and $d\vec{\mathbf{l}}$ are perpendicular to each other (see Fig. 8.2b). The integrand $\vec{\mathbf{E}}.d\vec{\mathbf{l}}$ of the second line integral in Eq. (8.4) is equal to $|\vec{\mathbf{E}}||d\vec{\mathbf{l}}|$ as both $\vec{\mathbf{E}}$ and $d\vec{\mathbf{l}}$ are parallel to each other along the path $a'b$ (see Fig. 8.2c). Can you tell why it is so? This is because, in the case of first integral, θ is 90° and hence $\cos\theta = 0$ and in the second integral, θ is zero and hence $\cos\theta = 1$ (see the margin remark).

The dot or scalar product of two vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ is defined as

$$\vec{\mathbf{a}}.\vec{\mathbf{b}} = |\vec{\mathbf{a}}||\vec{\mathbf{b}}|\cos\theta$$

where θ is the angle between vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$.

Let us now determine the second integral of Eq. (8.4). Using Eq. (8.1b) for $\vec{\mathbf{E}}$, replacing $d\vec{\mathbf{l}}$ by $d\vec{\mathbf{r}}$ (since the path from a' to *b* is radial), and writing $d\vec{\mathbf{r}} = \hat{\mathbf{r}}dr$ (where $\hat{\mathbf{r}}$ is a unit vector in the radial direction away from charge *Q*), we can write Eq. (8.4) as:

Note that when we integrate over *r*, the limits of integration are from $r_{a'}$ to r_b.

$$W' = 0 + \int_{a'}^{b} \vec{\mathbf{E}}.d\vec{\mathbf{l}} = \int_{r_{a'}}^{r_b} \vec{\mathbf{E}}.d\vec{\mathbf{r}} = \int_{r_{a'}}^{r_b} \frac{Q}{4\pi\varepsilon_0}\frac{\hat{\mathbf{r}}.\hat{\mathbf{r}}}{r^2}(dr) = \int_{r_a}^{r_b} \frac{Q}{4\pi\varepsilon_0}\frac{dr}{r^2} = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r_a} - \frac{1}{r_b}\right]$$

since $r_{a'} = r_a$.

Therefore, Eq. (8.4) becomes

$$W' = \int_{a}^{b} \vec{\mathbf{E}}.d\vec{\mathbf{l}} = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r_a} - \frac{1}{r_b}\right] \tag{8.5}$$

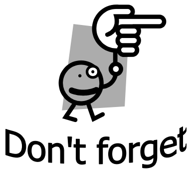
for the path shown in Fig. 8.3a between points *a* and *b*.

Now, you may recall from Sec. 3.4.1 of Unit 3, Block 1 of this course that a scalar potential can be associated with a conservative vector field. Since our aim here is to define an electric potential associated with electric field, we should establish that it is a conservative vector field. To do so, we examine if the electric field of a charge is conservative.

One of the characteristics of a conservative force is that the work done by this force in moving a particle from one point to another is independent of the path chosen to move the particle between the two points. The converse of this statement is also true: if the work done by a force in moving a particle from one point to another is independent of the path, the force is a conservative force.

This characteristic is exhibited by gravitational force (for a particle) as well as by electrostatic force (for a charged particle).

From Eq. (8.5), we note that the work done in moving a unit positive charge between any two points in the electric field of charge *Q* depends only on the distance of those points from charge *Q* and is independent of the path we choose to move the unit charge from one point to the other. So, **the line integral of the electric field is independent of the path**. Therefore, the electric field is a conservative vector field.



The electric field of a stationary charge is conservative.

8.2.2 Electrostatic Potential Energy

Recall from Sec. 10.3 of Unit 10, Block 2 of the Semester 1 course entitled Mechanics (BPHCT-131) that gravitational force is conservative. **You have learnt that we can define potential energy of an object moving under the influence of a conservative force.** For example, you have learnt that the change in gravitational potential energy, ΔU in moving an object from point a to point b is equal to the negative of the work done by the gravitational force in moving it from point a to b, that is,

$$(\Delta U)_{ba} = -W_{ab}$$

Now, you have learnt in Sec. 8.2.1 of Unit 8 that the electric field is a conservative vector field. Thus, we can say that the electrostatic force is a conservative force. So, we can also define electrostatic potential energy in the same way as we defined gravitational potential energy.

Thus, we can say that the change in electrostatic potential energy of a charge q in moving it from point *a to b* in an electric field of a charge Q is equal to the negative of the work done by the electrostatic force in moving the charge from point a to point b. If U_a and U_b are the initial and final electrostatic potential energy, respectively, of charge q, then we can write

$$\Delta U = (U_b - U_a) = -W_{ab} \tag{8.6}$$

where, W_{ab} is the work done by the electrostatic force in moving the positive charge q from point a to point b in the electric field $\vec{\mathbf{E}}$ due to charge Q. Now, from Eqs. (8.3) and (8.5), we can write:

$$W_{ab} = qW' = \frac{Qq}{4\pi\varepsilon_0}\left[\frac{1}{r_a} - \frac{1}{r_b}\right] \tag{8.7}$$

where W' is the work done in moving a unit positive charge from point a to point b. So, from Eqs. (8.6) and (8.7), we can write

$$\Delta U = (U_b - U_a) = -\frac{Qq}{4\pi\varepsilon_0}\left(\frac{1}{r_a} - \frac{1}{r_b}\right) = \frac{Qq}{4\pi\varepsilon_0}\left(\frac{1}{r_b} - \frac{1}{r_a}\right) \tag{8.8}$$

Eq. (8.8) gives the change in electrostatic potential energy of a positive charge q when it is moved from point a to b in the electric field due to charge Q. To fix these ideas, you may like to go through the following example.

EXAMPLE 8.1 : ELECTROSTATIC POTENTIAL ENERGY

The magnitude of a uniform electric field $\vec{\mathbf{E}}$ along the positive *x*-axis is 120 NC^{-1}. Calculate the change in electrostatic potential energy of a proton moving along a path parallel but opposite to the direction of $\vec{\mathbf{E}}$ through a distance 25 m.

SOLUTION ■ We know that the work done by a constant force $\vec{\mathbf{F}}$ in moving a particle through a displacement $\vec{\mathbf{d}}$ is given as

$$W = \vec{\mathbf{F}}.\vec{\mathbf{d}}$$

In the instant case, the electrostatic force on charge q due to an electric field $\vec{\mathbf{E}}$ is $\vec{\mathbf{F}} = q\vec{\mathbf{E}}$. Thus, the work done on the proton is

$$W = q\vec{\mathbf{E}}.\vec{\mathbf{d}} = qEd\cos\theta$$

where θ is the angle between $\vec{\mathbf{E}}$ and $\vec{\mathbf{d}}$. Now, the displacement of proton is parallel and opposite to the direction of $\vec{\mathbf{E}}$, i.e. $\theta = 180°$. Thus, we have

$$W = qEd\cos 180° = (1.6\times10^{-19}\,\text{C})\times(120\,\text{NC}^{-1})\times(25\,\text{m})\times\cos 180°$$

$$= -4.8\times10^{-16}\,\text{J}$$

If U_i and U_f are the initial and final electrostatic potential energy of the proton, we can write

$$\Delta U = U_f - U_i = -W = 4.8\times10^{-16}\,\text{J}$$

So, we discover that the electrostatic potential energy of proton increases (as $U_i < U_f$) when it moves opposite to the direction of electric field.

Before proceeding further, you should answer an SAQ.

SAQ 2 - Electrostatic potential energy

In a region, the uniform electric field is $200\,\hat{\mathbf{i}}\,\text{NC}^{-1}$. Calculate the work done in moving i) an electron, ii) a proton through a distance 30 m along the field direction. What will be the change in electrostatic potential energy of these charged particles?

With this understanding of work done by electrostatic force in moving a charge and the related concept of electrostatic potential energy, you can learn about electric potential.

8.3 ELECTRIC POTENTIAL DUE TO POINT CHARGES

In Unit 5 of this Block, you have learnt that the electric field, defined as electrostatic force per unit charge, is a very useful concept for determining the forces experienced by a charge or a group of charges of any sign and magnitude. **Now, let us ask ourselves: Can we define a simpler concept which enables us to determine the electrostatic force and electric field**

due to a charge or a system of charges? The answer is, yes, we can. **The electric potential** is such a concept. Let us elaborate it with the help of the relation between work done and electrostatic potential energy discussed in Sec. 8.2.2.

8.3.1 Electric Potential due to a Point Charge

Let us first define electric potential. **The electric potential is defined as electrostatic potential energy** per unit charge, that is,

$$V = \frac{U}{q} \tag{8.9}$$

where V is electric potential at a given point in the electric field and U is the electrostatic potential energy of charge q at that point. You know that the difference in electrostatic potential energy of charge q, when it is moved from point a to b in the electric field of charge Q is given by Eq. (8.8). Thus, on the basis of the definition of electric potential given above, we can write the difference in electric potential between points a and b as

$$\Delta V = \frac{\Delta U}{q}$$

or $$V_b - V_a = \frac{U_b - U_a}{q} = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r_b} - \frac{1}{r_a}\right] \tag{8.10}$$

You know that ΔU is related to the work done by the electrostatic force in moving charge q from point a to b through Eq. (8.6). Also, the work done per unit charge is related to the electric field by Eq. (8.3). Thus, on the basis of Eqs. (8.6) and (8.3), we can write Eq. (8.10) in terms of the line integral of electric field $\vec{\mathbf{E}}$ as

$$V_b - V_a = -\frac{W_{ab}}{q} = -\int_a^b \vec{\mathbf{E}}.d\vec{\mathbf{l}} \tag{8.11}$$

Further, from Eq. (8.10) we note that the difference in electric potential is a difference between two numbers (or scalars): $\left(\frac{Q}{4\pi\varepsilon_0 r_b}\right)$ and $\left(\frac{Q}{4\pi\varepsilon_0 r_a}\right)$.

Let us now see what happens if we assume that initial point a is at infinity (that is, $r_a = \infty$) and the electric potential at infinity is zero, that is, $V_a = 0$. Then, we can write Eq. (8.10) as

$$V_b = \frac{Q}{4\pi\varepsilon_0 r_b} \tag{8.12}$$

Eq. (8.12) gives the **electric potential** at point b at a distance r_b from a point charge Q. Further, for the condition that point a is located at infinity, i.e., $r_a = \infty$ and $V_a = 0$, Eq. (8.11), which defines electric potential at point b at a distance r_b in terms of line integral of $\vec{\mathbf{E}}$, reduces to

$$V_b = -\int_\infty^{r_b} \vec{\mathbf{E}}.d\vec{\mathbf{l}} \tag{8.13}$$

Note that Eqs. (8.12) and (8.13) are equivalent definitions of electric potential. Eq. (8.12) signifies that electric potential is a scalar quantity. Eq. (8.13) helps us understand what we mean when we say that the electric potential at a point in an electric field has some finite value. The RHS of Eq. (8.13) tells us that the electric potential at any point b at a distance r_b is the work done in bringing a unit positive charge from infinity up to that point (see Fig. 8.3). The SI unit for electric potential is the joule / coulomb (JC^{-1}). This combination occurs so often that a special unit, the volt (abbreviation V named after Alessandro Volta), is used to represent the unit of electric potential.

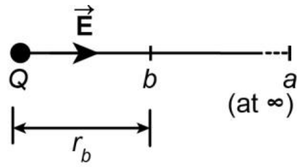


Fig. 8.3: **Work done in moving a unit positive charge from point *a* (at infinity) to point *b* in the electric field of charge *Q*.**

ELECTRIC POTENTIAL DUE TO A POINT CHARGE

Electric potential associated with electric field $\vec{\mathbf{E}}$ due to a charge Q at a point at distance r from it is defined as

$$V = \frac{Q}{4\pi\varepsilon_0 r} \tag{8.14}$$

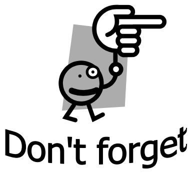
Electric potential V associated with the electric field $\vec{\mathbf{E}}$ due to a point charge Q at a distance r from it is given in terms of line integral as

$$V = -\int_{\infty}^{r} \vec{\mathbf{E}}.d\vec{\mathbf{l}} \tag{8.15}$$

From Eq. (8.14), we note that in the electric field of a positive charge Q, the potential at a point at distance r is positive; while for a negative charge, it is negative. Now, let us pause for a moment and ask ourselves: What is the physical meaning of this statement?

Note that in the electric field due to a positive charge, **work is done on** the unit positive test charge to move it from infinity to the given point **against the repulsive force** between the positive charge and the test charge. This **work done by an external agent** increases the electrostatic potential energy of the system and hence electric potential due to a positive charge at some finite distance is positive. On the other hand, in the electric field due to a negative charge, **the work is done by the electric field** in bringing the unit positive charge from infinity and the electrostatic potential energy of the system decreases. Therefore, the electric potential due to a negative charge at some finite distance is negative.

It is, therefore, clear that, **when work is done against the force field** (in this case **electric field**), **potential energy of the system increases**. This can be easily understood by considering an example in the case of gravitational field. When a body of finite mass is raised to a height against the force of gravity acting downwards, then the potential energy of the body increases. Here, work is done against gravity. And when work is done by the force of gravity as in case of free fall of a body, the potential energy decreases. The difference in potential energy gets converted into kinetic energy of the freely falling object.



A positive point charge produces positive electric potential and a negative point charge produces negative electric potential.

So far, you have learnt the concept of work done in moving a charge in an electric field, electrostatic potential energy and electric potential and how these concepts are related to each other.

Now, to concretise these ideas, you should go through the following example.

EXAMPLE 8.2 : ELECTRIC POTENTIAL DUE TO A POINT CHARGE

A particle of charge 5.0 μC is located on the *x*-axis at the point $x = 6.0\,\text{cm}$. Calculate the electric potential due to this charge at the origin, $x = 0$. Also calculate the work done in moving a charge $-6.0\,\mu\text{C}$ from infinity to the origin keeping the first charge fixed.

SOLUTION ■ From Eq. (8.14), we write the electric potential as

$$V = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}$$

Substituting the values of $Q = 5.0\times10^{-6}\,\text{C}$, $r = 6.0\times10^{-2}\,\text{m}$ and $(1/4\pi\varepsilon_0) = 9.0\times10^{9}\,\text{Nm}^2\text{C}^{-2}$ we have

$$V = (9.0\times10^{9}\,\text{Nm}^2\text{C}^{-2})\times\frac{5.0\times10^{-6}\,\text{C}}{6.0\times10^{-2}\,\text{m}} = 7.48\times10^{5}\,\text{V}$$

Further, to calculate the work done in moving the charge $-6.0\,\mu\text{C}$ from infinity to the origin, we use Eq. (8.11) with the understanding that potential at infinity is zero:

$$V = W/q \Rightarrow W = qV = (-6.0\times10^{-6}\,\text{C})\times(7.48\times10^{5}\,\text{V}) = -4.48\,\text{J}$$

In electrostatics, we associate three quantities with a static electric charge. The magnitude of the electrostatic force on a test charge *q* at a distance *r* from the point charge *Q* is given as

$$F = \frac{1}{4\pi\varepsilon_0}\frac{Qq}{r^2}$$

The magnitude of the electric field at a point distance *r* is given as

$$E = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}$$

The electric potential at distance *r* from the point charge *Q* is given as

$$V = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}$$

Note the nature of dependence of these quantities on the distance *r*, point charge *Q* and test charge *q*.

<u>**Potential Difference and Zero Potential**</u>

The way we have defined the electric potential at a point by Eq. (8.14) may give you an impression that it is an absolute quantity. It is, however, not true because we have **arbitrarily chosen a reference point at infinity and assumed that the electric potential at infinity is zero.** The more fundamental quantity is the **potential difference** as given by Eq. (8.10) and Eq. (8.11) which refers to the change in electrostatic potential energy or the work done when a unit positive charge is moved from one point to another in an electric field. To determine the potential difference between any two points in an electric field, we do not need any reference point.

Potential difference is a very important concept in the field of electrostatics and current electricity. Its knowledge helps us in determining the exact value of the current which flows between any two points in an electric circuit, provided the (electric) resistance between the two points is known.

Though potential difference is a more fundamental concept than absolute potential, it is of immense practical importance to define a

reference point where the value of potential can be taken to be zero. Such a reference point with zero potential enables us to assign an absolute value of electric potential to a point in electric field. We did that by choosing the reference point at infinity with zero potential and defined electric potential at a point by Eq. (8.14).

You should, however, remember that the choice of the reference point with zero potential is arbitrary and it is done in such a manner which makes the mathematical treatment of the problem simpler. For example, in most of the problems involving electric potential in electrical circuits, the potential of the Earth is taken as reference point with zero potential. This choice of reference potential is guided by the fact that the potential of the Earth remains constant even if it gains or loses electricity. This choice of reference with zero potential for electric situations is similar to our choice of sea level as reference point for describing the height of a place or a mountain on the Earth.

Before studying further, try to solve the following SAQ.

SAQ 3 - **Calculating electric potential, potential difference and work done**

a) Refer to Fig. 8.4 which shows two points *X* and *Y* located at distances 8 m and 12 m, respectively, from a point charge $+7\mu C$. (i) Calculate the electric potential at points *X* and *Y* and the potential difference between points *X* and *Y*. (ii) Suppose that the point charge $+7\mu C$ is replaced by a point charge $-7\mu C$. Calculate the electric potential at points *X* and *Y* and the potential difference between *X* and *Y*. (iii) If the point charge $+7\mu C$ is fixed at its position, calculate the work done in moving a charge $+3\mu C$ from infinity to the point *X*.

b) The radius of a gold nucleus is 6.6×10^{-15} m and the atomic number, *Z* of gold is 79. Assuming that the nucleus acts as a point charge, and electronic charge $e = 1.6 \times 10^{-19}$ C, calculate the electric potential at the surface of a gold nucleus.

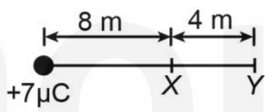


Fig. 8.4: **Diagram for SAQ 3a.**

From Eq. (8.14), you know how to determine electric potential due to an isolated charge at a point located at distance *r* from the charge. Now, suppose that we have many discrete changes located at different points in space. How do we determine electric potential at some given point due to this system of discrete charges? You will learn it now.

8.3.2 Electric Potential due to a System of Discrete Charges

From Unit 5 of this course, you know that electric field obeys superposition principle which enables us to calculate $\vec{\mathbf{E}}$ at a given point due to a system of discrete charges. The superposition principle for electric fields implies that (a) the electric field at a given point due to any one charge of the system is unaffected by the presence of the remaining charges, and (b) the net value of $\vec{\mathbf{E}}$ at a given point is the **vector sum** of the fields due to individual charges of the system, at that point.

You may, therefore, ask: **Can we use the superposition principle to determine the value of electric potential due to a system of charges?** The answer is: Yes, we can. Since electric potential is a scalar quantity, its value at a given point is the **algebraic sum** of the electric potential due to individual charges of the system. Thus, using the superposition principle for electric potential is much simpler than using it for $\vec{\mathbf{E}}$ because, in case of $\vec{\mathbf{E}}$, we have to deal with vector sum of the fields due to individual charges.

Suppose we have a system of charges $q_1, q_2, ..., q_N$ located at distances $r_1, r_2, ..., r_N$, respectively, from the point *P*. So, according to the superposition principle, the potential at point *P* can be written as the algebraic sum of the potential at *P* due to $q_1, q_2, ..., q_N$:

$$V_P = \frac{q_1}{4\pi\varepsilon_0 r_1} + \frac{q_2}{4\pi\varepsilon_0 r_2} + ... + \frac{q_N}{4\pi\varepsilon_0 r_N}$$

Note that here each individual charge is acting as if the other charges are not present. The above expression may be written in a summation form as:

$$V_P = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{N} \frac{q_i}{r_i} \qquad (8.16)$$

As a caution, you may keep in mind that the sum given in Eq. (8.16) is an algebraic sum and not a vector sum as the potential at a point is a scalar quantity. To get a feel for the value of potential due to a system of discrete charges, go through the following example.

EXAMPLE 8.3 : ELECTRIC POTENTIAL DUE TO MANY DISCRETE CHARGES

Three point charges are placed on the *x*-axis: $2\mu C$ at $x = 20$ cm, $-3\mu C$ at $x = 30$ cm, $-4\mu C$ at $x = 40$ cm. Calculate the electric potential at $x = 0$.

SOLUTION ■ To calculate the electric potential at a point due to many discrete charges, we use Eq. (8.16):

$$V = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{3} \frac{q_i}{r_i}$$

On substituting the numerical values of q_i and r_i, we get

$$V = 9 \times 10^9 \text{ Nm}^2 \text{ C}^{-2} \times \left[\frac{2 \times 10^{-6} \text{ C}}{0.20 \text{ m}} - \frac{3 \times 10^{-6} \text{ C}}{0.30 \text{ m}} - \frac{4 \times 10^{-6} \text{ C}}{0.40 \text{ m}} \right]$$

$$= 9\times10^{9}\ \text{Nm}^2\text{C}^{-2}\times[10^{-5}\ \text{m}^{-1} - 10^{-5}\ \text{m}^{-1} - 10^{-5}\ \text{m}^{-1}]$$

or $$V = -9\times10^{4}\ \text{Nm C}^{-1} = -9\times10^{4}\ \text{V}$$

Note that each of the three charges are placed at different points on the same line (x-axis). But, the electric potential at a given point $(x = 0)$ on the same line due to one charge is not affected by the presence of the other two charges.

Before proceeding further, answer an SAQ.

SAQ 4 - Electric potential due to many charges

Two point charges $+q$ and $-2q$ are placed along a straight line at a distance of 9 m from each other. Determine the distance of a point, from the charge $+q$, between the two charges where the electric potential is zero.

On the basis of the discussion so far, you have learnt that the electric field $\vec{\mathbf{E}}$ at a point in space gives us the magnitude and direction of electrostatic force and electric potential gives the work done by the electrostatic force in moving a unit positive charge from one point to another. So, if we have a relation which enables us to compute electric field at a point if the potential at that point is known, solving problems of electrostatics becomes far easier. It is far easier to use the concept of electric potential since it is a scalar. You will agree that working with vectors is more complicated than working with scalars. Let us now learn the relation between electric field and electric potential.

8.4 RELATION BETWEEN ELECTRIC FIELD AND ELECTRIC POTENTIAL

You know from Eq. (8.11) that the difference in electric potential, $V_{ba} (= V_b - V_a)$ between two points b and a in the electric field $\vec{\mathbf{E}}$ of charge Q is equal to the negative of the line integral of $\vec{\mathbf{E}}$ between the same two points:

$$V_{ba} = V_b - V_a = -\int_a^b \vec{\mathbf{E}}.d\vec{\mathbf{l}}$$

If the separation $d\vec{\mathbf{l}}$ between the two points a and b is small, we can write the potential difference dV between any two points as

$$dV = -\vec{\mathbf{E}}.d\vec{\mathbf{l}} \tag{8.17}$$

or $$dV = -E\cos\theta\left|d\vec{\mathbf{l}}\right|$$

or $$-E\cos\theta = \frac{dV}{\left|d\vec{\mathbf{l}}\right|} \tag{8.18}$$

The presence of cos θ term in Eq. (8.18) indicates that the electric field is not a simple derivative of the potential function V; rather, it is some special kind of derivative of the potential. We call it directional derivative about which you studied in Unit 1, Block 1 of this course.

As you have studied in Sec. 1.3, Unit 1, Block 1 of this course, the rates of change of scalar fields such as temperature and potential in different directions can be expressed by using the gradient operator. From Eq. (1.8), you know that the difference *df* in the value of a scalar function *f* between two points separated by $d\vec{\mathbf{r}}$ is given as

$$df = (\vec{\nabla}V).d\vec{\mathbf{r}}$$

Since electric potential is a scalar function, we can use the above general relation and write the electric potential difference between two points separated by $d\vec{\mathbf{l}}$ as

$$dV = (\vec{\nabla}V).d\vec{\mathbf{l}} \tag{8.19}$$

So, comparing Eqs. (8.17) and (8.19), we can write

$$\vec{\mathbf{E}} = -\vec{\nabla}V = -\left(\hat{\mathbf{i}}\frac{\partial V}{\partial x} + \hat{\mathbf{j}}\frac{\partial V}{\partial y} + \hat{\mathbf{k}}\frac{\partial V}{\partial z}\right) \tag{8.20}$$

The components of $\vec{\mathbf{E}}$ along *x, y* and *z* directions are

$$E_x = -\frac{\partial V}{\partial x}, \quad E_y = -\frac{\partial V}{\partial y}, \quad E_z = -\frac{\partial V}{\partial z} \tag{8.21}$$

Thus, we find that the electric field $\vec{\mathbf{E}}$ is the negative of the gradient of the electric potential *V at any point*.

Eq. (8.20) or Eq. (8.21) enables us to calculate the electric field at a point if we know the value of electric potential at that point. To understand this method, go through the following example.

EXAMPLE 8.4 : ELECTRIC FIELD FROM ELECTRIC POTENTIAL

The electric potential at a point is given by the relation $V = Ax + By - Cz$ where *A*, *B* and *C* are constants. Determine the electric field $\vec{\mathbf{E}}$ at that point.

SOLUTION ■ From Eq. (8.20), we have

$$\vec{\mathbf{E}} = -\vec{\nabla}V = -\left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right)V$$

Substituting the value of *V*, we get

$$\vec{\mathbf{E}} = -\left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right)(Ax + By - Cz)$$

$$\therefore \quad \vec{\mathbf{E}} = -[A\hat{\mathbf{i}} + B\hat{\mathbf{j}} - C\hat{\mathbf{k}}]$$

Now apply this method yourself to solve SAQ 5.

SAQ 5 - Electric field from electric potential

The electric potential at any point is given by $V = x(y^2 - 4x^2)$. Calculate the electric field $\vec{\mathbf{E}}$ at that point.

In Unit 5 of this block, you have learnt how to calculate electric field due to multiple discrete charges and, especially the electric dipole. In the following section, you will learn how to determine the electric potential due to an electric dipole.

8.5 POTENTIAL DUE TO AN ELECTRIC DIPOLE

In Unit 5, you have learnt about the electric dipole. You know that it is a pair of equal and opposite charges, $\pm q$, separated by some distance, $2a$. **Then $2\vec{\mathbf{a}}$ is a vector along the axis of the dipole, drawn *from* the negative *to* the positive charge (Fig. 8.5).**

Let us now determine the electric potential due to a dipole. We shall use polar coordinates for mathematical convenience. Refer to Fig. 8.5 which shows point *P* at a distance *r* from the midpoint *C* of the dipole *AB*. The line joining *P* and *C* makes an angle θ with the dipole axis. So, the polar coordinates of point *P* are *r* and θ with the origin at *C*, the midpoint of dipole. We now determine the electric potential at *P* due to the two charges $-q$ and $+q$ of the dipole.

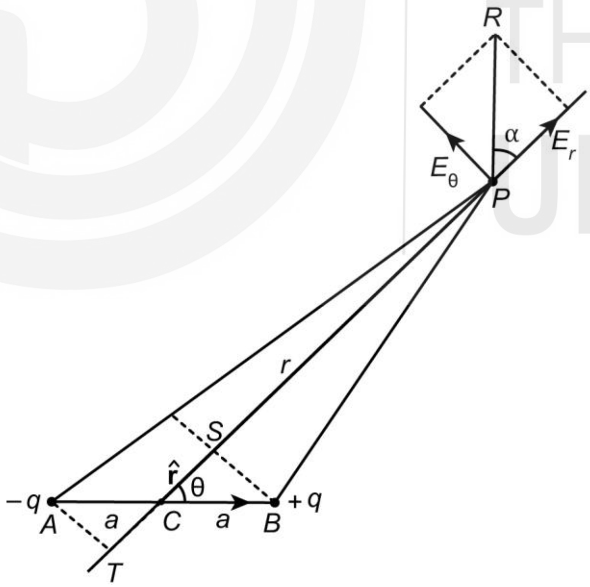


Fig. 8.5: **An electric dipole** *AB* **of length** $2a$ **and point** *P* **at a distance** *r* **from the mid-point** *C* **of the dipole.**

Study Fig. 8.5. Note that the distances of point *P* from $-q$ and $+q$ are *AP* and *BP*, respectively. Also note we have drawn perpendiculars from *B* to *S* and *A* to *T*. Thus, under the condition that point *P* is far away from the dipole so that $r >> 2a$, you can see from the figure that

$$BP = SP = PC - CS = r - a\cos\theta$$

and $$AP = TP = TC + CP = r + a\cos\theta$$

Thus, using the superposition principle [Eq. (8.16)], we can write the potential at P due to charges q and $-q$ of the dipole as:

$$V = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{(r - a\cos\theta)} - \frac{1}{(r + a\cos\theta)}\right] = \frac{2q\,a\cos\theta}{4\pi\varepsilon_0(r^2 - a^2\cos^2\theta)} \quad (8.22)$$

Now, let us suppose that $\vec{\mathbf{r}}$ is a vector from C to P and the unit vector along $\vec{\mathbf{r}}$ is $\hat{\mathbf{r}}$. Also, you know [Eq. (5.11)] that the dipole moment, $\vec{\mathbf{p}} = 2q\,\vec{\mathbf{a}}$. Since $\vec{\mathbf{p}}.\hat{\mathbf{r}} = 2q\,\vec{\mathbf{a}}.\hat{\mathbf{r}} = 2qa\cos\theta$, we can write Eq. (8.22) for V as

$$V = \frac{\vec{\mathbf{p}}.\hat{\mathbf{r}}}{4\pi\varepsilon_0(r^2 - a^2\cos^2\theta)} \quad (8.23)$$

When point P is far away from the dipole, r^2 is large compared to $a^2\cos^2\theta$. So, we can neglect $a^2\cos^2\theta$ in the denominator in comparison to r^2, and write Eq. (8.23) as

$$V = \frac{\vec{\mathbf{p}}.\hat{\mathbf{r}}}{4\pi\varepsilon_0 r^2} = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2} \quad (8.24)$$

Eq. (8.24) gives the general expression for the electric potential due to dipole at a distance r from its mid point.

On the basis of Eq. (8.24), you can conclude that:

- **The electric potential due to dipole varies with r as $1/r^2$ whereas the potential due to point charge varies as $1/r$. The comparison of these variations shows that the potential decreases more rapidly with r for a dipole than for a point charge.**
- **The electric potential due to dipole is zero for all points which lie on the perpendicular bisector of the dipole axis because, for any such point, $\theta = 90°$ and $\cos\theta = 0$. Hence, no work is done in moving a test charge along the perpendicular bisector.**

We will now determine the electric field of a dipole from its electric potential. But before studying further, you may like to solve an SAQ.

SAQ 6 – Electric potential due to an electric dipole

A straight line from the centre of an electric dipole and along the axis of the dipole first passes through point P_1 and then through point P_2. The distances of points P_1 and P_2 from the centre of the dipole are 40 cm and 60 cm, respectively. The dipole length is much smaller than 40 cm. If the potential at point P_1 is 60 V, calculate the potential at point P_2.

To determine the electric field from electric potential, we will use the relation given by Eq. (8.20). However, since we have used polar coordinates to specify the location of point P, we must use the expression for the del operator in

Eq. (8.20) in polar coordinates. In polar coordinates, the operator $\vec{\nabla}$ is given as

$$\vec{\nabla} = \hat{\mathbf{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial \theta}$$

Thus, Eq. (8.20) can be expressed in polar coordinates as

$$\vec{\mathbf{E}} = -\vec{\nabla}V = -\left[\hat{\mathbf{r}}\frac{\partial}{\partial r} + \frac{\hat{\boldsymbol{\theta}}}{r}\frac{\partial}{\partial \theta}\right]V$$

Now, substituting the value of V from Eq. (8.24), we can write

$$\vec{\mathbf{E}} = -\left[\hat{\mathbf{r}}\frac{\partial}{\partial r} + \frac{\hat{\boldsymbol{\theta}}}{r}\frac{\partial}{\partial \theta}\right]\left[\frac{p\cos\theta}{4\pi\varepsilon_0 r^2}\right] = -\left[\hat{\mathbf{r}}\frac{\partial}{\partial r}\left(\frac{p\cos\theta}{4\pi\varepsilon_0 r^2}\right) + \frac{\hat{\boldsymbol{\theta}}}{r}\frac{\partial}{\partial \theta}\left(\frac{p\cos\theta}{4\pi\varepsilon_0 r^2}\right)\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{p}{r^3}\left[\hat{\mathbf{r}}(2\cos\theta) + \hat{\boldsymbol{\theta}}\sin\theta\right] \qquad (8.25)$$

From Eq. (8.25), we can write the radial (E_r) and tangential (E_θ) components of electric field $\vec{\mathbf{E}}$ at point P (see Fig. 8.5) as

$$E_r = \frac{1}{4\pi\varepsilon_0}\frac{2p\cos\theta}{r^3} \qquad (8.26)$$

$$E_\theta = \frac{1}{4\pi\varepsilon_0}\frac{p\sin\theta}{r^3} \qquad (8.27)$$

The radial and tangential components of the electric field $\vec{\mathbf{E}}$ at point P are shown in Fig. 8.5. From the figure, note that the resultant electric field $\vec{\mathbf{E}}$ at point P is directed along PR and it makes an angle α with the (extended) line CP, i.e. the direction of the radial component E_r.

Thus, the magnitude of the electric field is given as

$$|\vec{\mathbf{E}}| = \sqrt{E_r^2 + E_\theta^2} = \frac{p}{4\pi\varepsilon_0 r^3}\sqrt{4\cos^2\theta + \sin^2\theta} = \frac{p}{4\pi\varepsilon_0 r^3}\sqrt{3\cos^2\theta + 1} \qquad (8.28)$$

To determine the direction of the resultant field $\vec{\mathbf{E}}$, we make use of Eqs. (8.26) and (8.27) and note from the geometry of Fig. 8.5:

$$\tan\alpha = \frac{E_\theta}{E_r} = \frac{\sin\theta}{2\cos\theta} = \frac{1}{2}\tan\theta \qquad (8.29)$$

The advantage of using polar coordinates for obtaining expressions for potential and hence electric field at a point due to dipole can be understood on the basis of Eqs. (8.26) and (8.27). Refer to Fig. 8.5. If we take $\theta = 0$, then the point P will shift to a point along the axis of the dipole. For any such point, Eqs. (8.26) and (8.27) show that only the radial component will be present; the tangential component, E_θ will be zero because of the $\sin\theta$ term. So, the magnitude of the electric field due to the dipole at a point along its axis can be written as

$$|\vec{\mathbf{E}}| = \frac{1}{4\pi\varepsilon_0}\frac{2p}{r^3}$$

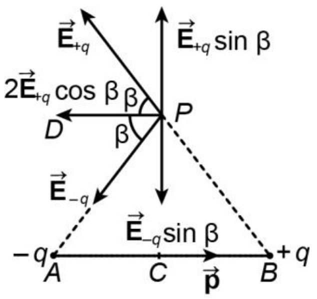


Fig. 8.6: **Direction of electric field at a point on the perpendicular bisector of dipole.**

And, Eq. (8.29) indicates that direction of the electric field will be along the axis of the dipole because, for $\theta = 0$, $\alpha = 0$ and α is the angle between the resultant electric field and the dipole axis. Thus, the electric field due to dipole at a point along its axis at a distance *r* from the mid point of dipole, such that *r* >> *a*, is given as

$$\vec{\mathbf{E}} = \frac{1}{4\pi\varepsilon_0}\frac{2\vec{\mathbf{p}}}{r^3} \tag{8.30}$$

Eq. (8.30) is the same as Eq. (i) of Example 5.4, Unit 5 obtained for electric field due to dipole at a point along its axis.

For $\theta = \pi/2$, point *P* will be a point on the perpendicular bisector of the dipole axis (Fig. 8.6). In this case, the radial component of electric field will be zero as $\cos\theta = 0$ in Eq. (8.26). Thus, the magnitude of the electric field at such a point will have contribution only from the tangential compound, E_θ. Thus, we can write Eq. (8.28):

$$|\vec{\mathbf{E}}| = \sqrt{E_\theta^{\,2}} = \frac{1}{4\pi\varepsilon_0}\frac{p}{r^3} \tag{8.31}$$

We cannot use Eq. (8.29) for determining the direction of $\vec{\mathbf{E}}$ at a point on the bisector of the dipole because, $\tan\theta = \tan(\pi/2)$ is not defined. We can, however, make use of the fact that the value of potential at every point on the bisector is zero [see Eq. (8.24)]. This means that no work is done in moving a charge along the bisector of a dipole. Further, the work done in moving a unit charge by distance $d\vec{\mathbf{l}}$ is given as $\vec{\mathbf{E}}.d\vec{\mathbf{l}}$. Thus, $\vec{\mathbf{E}}.d\vec{\mathbf{l}} = 0$ implies that field $\vec{\mathbf{E}}$ is perpendicular to $d\vec{\mathbf{l}}$, the direction of the perpendicular bisector. Now, to determine whether $\vec{\mathbf{E}}$ is along or opposite to $\vec{\mathbf{p}}$, refer to Fig. 8.6 which shows the electric field due to the dipole at point *P*. The components $E_{+q}\sin\beta$ and $E_{-q}\sin\beta$ of $\vec{\mathbf{E}}_{+q}$ and $\vec{\mathbf{E}}_{-q}$ respectively will cancel each other. However, the component $E_{+q}\cos\beta$ and $E_{-q}\cos\beta$ will add up along *PD*, a direction perpendicular to the bisector and opposite to the direction of dipole moment $\vec{\mathbf{p}}$. Thus, the electric field due to the dipole at any point on its perpendicular bisector is anti-parallel to $\vec{\mathbf{p}}$. Thus, we can write

$$\vec{\mathbf{E}} = -\frac{1}{4\pi\varepsilon_0}\frac{\vec{\mathbf{p}}}{r^3} \tag{8.32}$$

Eq. (8.32) is same as Eq. (i) of Example 5.5, Unit 5 obtained by computing electric fields due to dipole at a point on its bisector.

We mentioned in the beginning of this section that understanding the behaviour of an electric dipole under the influence of an external electric field is very useful in analysing the effect of electric field on dielectric materials.

So, let us now study the effect of electric field on a dipole.

8.6 DIPOLE IN AN ELECTRIC FIELD

Let us consider a dipole of length 2*a* in a uniform external electric field $\vec{\mathbf{E}}$ as shown in Fig. 8.7. A uniform electric field means that its magnitude and direction are the same everywhere. Let the dipole moment vector $\vec{\mathbf{p}}(=2q\vec{\mathbf{a}})$ makes an angle θ with the electric field, $\vec{\mathbf{E}}$.

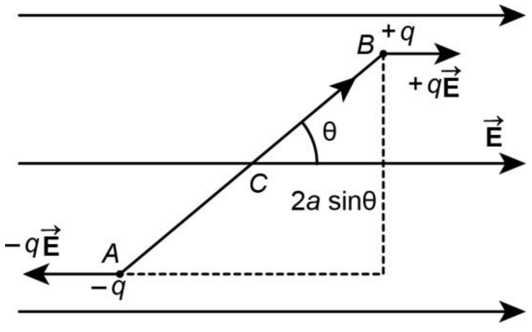


Fig. 8.7: **Torque experienced by a dipole placed in a uniform electric field $\vec{\mathbf{E}}$.**

Due to the external electric field $\vec{\mathbf{E}}$, the charge $+q$ of the dipole experiences a force $\vec{\mathbf{F}}_+ = q\vec{\mathbf{E}}$ while the charge $-q$ experiences an equal and opposite force $\vec{\mathbf{F}}_- = -q\vec{\mathbf{E}}$. Since the field is uniform, the net force $\vec{\mathbf{F}}$ on the dipole is zero, i.e.,

$$\vec{\mathbf{F}}_{net} = \vec{\mathbf{F}}_+ + \vec{\mathbf{F}}_- = q\vec{\mathbf{E}} - q\vec{\mathbf{E}} = \vec{\mathbf{0}} \tag{8.33}$$

As the net force on the dipole is zero, the centre of mass of the dipole is not accelerated, that is, there is no effect on its translational motion.

You may, therefore, ask: Does it mean that the external electric field has no effect on the dipole? No, it is not so. The dipole still experiences a **turning effect** due to the torque about its centre of mass *C*. This turning effect arises because the two equal and opposite forces, which cancel each other as free vectors, are acting at different points. That is, the forces experienced by charges $+q$ and $-q$ of the dipole do not have same line of action and hence they provide a turning effect.

From Fig. 8.7, note that the centre of mass C of the dipole is at a distance *a* from each charge of the dipole. Thus, we can write the magnitude of net torque $\vec{\boldsymbol{\tau}}$ as

$$\tau = qEa\sin\theta + qEa\sin\theta = 2qaE\sin\theta = pE\sin\theta$$

The above expression can be written in vector form as

$$\vec{\boldsymbol{\tau}} = \vec{\mathbf{p}} \times \vec{\mathbf{E}} \tag{8.34}$$

You know that the unit of torque is Newton metre (N m). The direction of the torque is obtained from right-hand rule (refer Sec. 12.3, Unit 12 of 1st semester course BPHCT-131) and is along $-\hat{\mathbf{k}}$ if the electric field $\vec{\mathbf{E}}$ and dipole are in the *xy*-plane.

Under the action of the torque, the dipole tend to align itself along the field direction with dipole moment vector $\vec{\mathbf{p}}$ parallel to $\vec{\mathbf{E}}$ vector. So, when $\vec{\mathbf{p}}$ is aligned along $\vec{\mathbf{E}}$, the torque on the dipole is zero because for $\theta = 0°, \sin\theta = 0$. **The system (that is, the dipole) is in stable equilibrium when $\vec{\mathbf{p}}$ is aligned with $\vec{\mathbf{E}}$.**

From Fig. 8.7, we note that the torque acting on the dipole tends to align it along $\vec{\mathbf{E}}$. So, the rotation of the dipole is in the clockwise direction.

Potential Energy of an Electric Dipole

Now, let us ask ourselves: What will happen to the potential energy of the dipole if it is rotated from its stable position? Whenever the dipole is rotated from its stable configuration ($\vec{\mathbf{p}}$ parallel to $\vec{\mathbf{E}}$) external work must be done. This external work is stored as potential energy of the dipole.

To obtain an expression for the potential energy of a dipole we need to calculate the work done by the electric field to rotate the dipole from some initial value of θ to final value of θ. The work done, in terms of torque and angular displacement $d\theta$ is

$$dW = -\tau\, d\theta$$

$$= -pE \sin\theta\, d\theta \qquad (8.35)$$

The negative sign in Eq. (8.35) indicates that the torque opposes any increase in θ. Thus, the work done by $\vec{\mathbf{E}}$ to rotate the dipole from an angle θ_0 to θ is

$$W = \int_{\theta_0}^{\theta} dW$$

$$= \int_{\theta_0}^{\theta} (-pE \sin\theta)\, d\theta$$

$$= pE(\cos\theta - \cos\theta_0) \qquad (8.36)$$

The change in potential energy ΔU of the dipole is the negative of the work done by the electric field. Thus, we have

$$\Delta U = U_f - U_i = -W = -pE(\cos\theta - \cos\theta_0) \qquad (8.37)$$

Note that $U_i = -pE\cos\theta_0$ is the potential energy at the initial or reference orientation of the dipole. As in the case of point charge for which we define potential energy to be zero at infinity, we need to define the orientation of dipole with respect to $\vec{\mathbf{E}}$ for which we can consider its potential energy to be zero. It turns out that when the dipole is aligned perpendicular to $\vec{\mathbf{E}}$, that is, when $\theta = \pi/2$ in Fig. 8.7, potential energy of the dipole can be taken to be zero.

Thus, the initial potential energy $U_i = 0$. So, we can write Eq. (8.37) as

$$U = -pE\cos\theta$$

or

$$U = -\vec{\mathbf{p}}.\vec{\mathbf{E}} \qquad (8.38)$$

Eq. (8.38) gives the potential energy of a dipole in a uniform electric field. It shows that the potential energy is minimum (most negative) when the dipole is aligned along the field direction (i.e., $\theta = 0°$), and is maximum (most positive) when it is aligned opposite to the field direction (i.e., $\theta = 180°$).

Let us now sum up what we have learnt in this unit.

8.7 SUMMARY

Concept	Description
Work done and line integral	■ The work W' done by the electric field $\vec{\mathbf{E}}$ in moving a unit positive charge from point *a* to *b*, is equal to the line integral of $\vec{\mathbf{E}}$: $$W' = \int_a^b \vec{\mathbf{E}}.d\vec{\mathbf{l}}$$
Path independence	■ The work done, that is, the line integral of $\vec{\mathbf{E}}$, in moving a unit positive charge from one point to another in an electric field is independent of the path between the two points.
Electrostatic potential energy	■ The difference in electrostatic potential energy of a charge between two points *a* and *b* in an electric field is equal to the negative of the work done by the field in moving the charge from *a* to *b*: $$\Delta U = U_b - U_a = -W'_{ab}$$
Electric potential as line integral	■ The negative of the work W' done by the electric field in carrying a unit positive charge from infinity to some point at distance *r* from the charge giving rise to the field is defined as the electric potential *V* at that point: $$V = -W' = -\int_{-\infty}^{r} \vec{\mathrm{E}}.d\vec{\mathbf{l}}$$
Electric potential	■ The electric potential *V* at a point at a distance *r* from a point charge *Q* is given as: $$V = \frac{Q}{4\pi\varepsilon_0 r}$$
Relation between V and $\vec{\mathbf{E}}$	■ The electric field $\vec{\mathbf{E}}$ at a point is the negative gradient of the electric potential *V* at that point: $$\vec{\mathbf{E}} = -\vec{\nabla}V$$
Electric potential due to dipole	■ The electric potential at any point *P*, at a distance *r* from the midpoint of the dipole, on a line which makes an angle θ with the axis of the dipole is given by: $$V = \frac{\vec{\mathbf{p}}.\hat{\mathbf{r}}}{4\pi\varepsilon_0 r^2} = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2}$$ where $\hat{\mathbf{r}}$ is a unit vector from the centre of dipole to the point *P* where potential is to be determined and $\vec{\mathbf{p}}(= 2q\vec{\mathbf{a}})$ is the dipole moment vector.
Torque on a dipole in electric field	■ Electric dipole in a uniform electric field experiences a turning effect. The torque $\vec{\boldsymbol{\tau}}$ experienced by the dipole is given by: $$\vec{\boldsymbol{\tau}} = \vec{\mathbf{p}} \times \vec{\mathbf{E}}$$
Electrostatic potential energy of the dipole	■ The electrostatic potential energy of an electric dipole in an electric field is given by $U = -\vec{\mathbf{p}}.\vec{\mathbf{E}}$. Its value is minimum when dipole moment vector $\vec{\mathbf{p}}$ is parallel to electric field $\vec{\mathbf{E}}$ and maximum when dipole moment vector $\vec{\mathbf{p}}$ is anti-parallel to $\vec{\mathbf{E}}$.

8.8 TERMINAL QUESTIONS

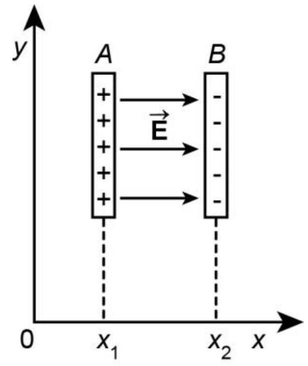


Fig. 8.8: **Diagram for TQ 2.**

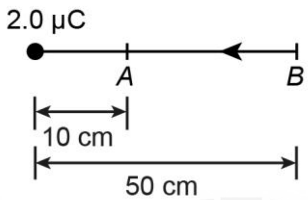


Fig. 8.9: **Diagram for TQ 3.**

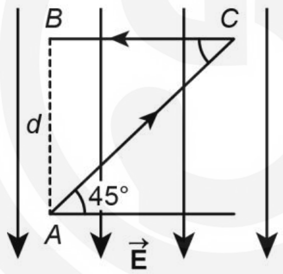


Fig. 8.10: **Diagram for TQ 4.**

1. Show that the line integral of the electric field $\vec{\mathbf{E}}$ over a closed path is equal to zero.

2. Show that, in a pair of oppositely charged plane parallel plates, the electric field $\vec{\mathbf{E}}$ is equal to the potential difference between the plates divided by their separation. You may assume that the electric field is confined between the plates as shown in Fig. 8.8.

3. Calculate the electric potential at two points *A* and *B* at distances of 10 cm and 50 cm from a charge $2.0\,\mu C$ as shown in Fig. 8.9. Also calculate the work done in bringing a charge of $0.05\,\mu C$ from point *B to A.*

4. Calculate the potential difference between points *A* and *B* assuming that a test charge q_0 is moved without acceleration from *A* to *B* along the path shown in Fig. 8.10.

5. Mark the following statements as True or False:

   a) If the electric field is zero in some region of space, the electric potential must also be zero in that region.

   b) If the electric potential is zero at a point, the electric field must also be zero at that point.

   c) The value of potential can be chosen to be zero at any convenient point.

   d) Electric field at a point is negative of the gradient of electric potential at that point.

   e) The electric field and potential due to an electric dipole decrease much faster with distance as compared to a point charge.

6. A uniform electric field of $3 \times 10^3\ NC^{-1}$ is in the positive *x*-direction. A positive point charge 2 μC is released from rest at the origin.

   a) Calculate the potential difference *V* (5 m) – *V* (0).

   b) What is the change in electrostatic potential energy of the charge when it is moved from $x = 0$ to $x = 5\,m$?

   c) Calculate the kinetic energy of the charge when it is at $x = 5\,m$.

   d) Calculate the value of the potential *V*(*x*) if electric potential is chosen to be zero at i) $x = 0$ and ii) $x = 1\,m$.

7. A uniform electric field is in the negative *x*-direction. Two points *a* and *b* are at $x = 3\,m$ and $x = 7\,m$, respectively.

   a) Is the potential difference $V_b - V_a$ positive or negative?

   b) If the value of the potential difference of $(V_b - V_a)$ is $10^4\,V$, calculate the magnitude of the electric field.

8. How much work needs to be done to transport an electron from the positive terminal of a 12 V battery to its negative terminal?

8.9 SOLUTIONS AND ANSWERS

Self-Assessment Questions

1. Let the electric field be $\vec{\mathbf{E}}$ and $d\vec{\mathbf{l}}$ be the element of path length. Since both $\vec{\mathbf{E}}$ and $d\vec{\mathbf{l}}$ are parallel, the angle θ between the two vectors is zero. Thus, using Eq. (8.3), we write the work done in moving a unit positive charge as

$$W' = \int \vec{\mathbf{E}}.d\vec{\mathbf{l}} = \int E(\cos\theta)dl = \int E(\cos 0^\circ)dl = \int Edl = El$$

2. The work done by a constant force $\vec{\mathbf{F}}$ in moving a particle through displacement $\vec{\mathbf{l}}$ is $W = \vec{\mathbf{F}}.\vec{\mathbf{l}} = Fl\cos\theta$ where θ is the angle between $\vec{\mathbf{F}}$ and $\vec{\mathbf{l}}$.

   i) As per the problem, the electron is moving along the direction of $\vec{\mathbf{E}}$. So, $\theta = 0°$. Thus, the work done is

$$W = (qE)l\cos 0°$$

$$= (-1.6\times10^{-19}\,\text{C})\times(200\,\text{NC}^{-1})\times(30\,\text{m}) = -9.6\times10^{-16}\,\text{J}$$

   The change in electrostatic potential energy of the electron is

$$\Delta U = U_f - U_i = -W = 9.6\times10^{-16}\,\text{J}$$

   Thus, the electrostatic potential energy of electron increases as it moves along the direction of the electric field.

   ii) As per the problem, the proton is moving along $\vec{\mathbf{E}}$. So, $\theta = 0°$. Thus, we have

$$W = qEl\cos 0°$$

$$= (1.6\times10^{-19}\,\text{C})\times(200\,\text{NC}^{-1})\times(30\,\text{m}) = 9.6\times10^{-16}\,\text{J}$$

   So, the change in electrostatic potential energy of proton is

$$\Delta U = U_f - U_i = -W = -9.6\times10^{-16}\,\text{J}$$

   Thus, we find that the electrostatic potential energy of proton decreases as it moves along the direction of the electric field.

3. a) From Eq. (8.14), we have the electric potential due to a point charge

$$V = \frac{1}{4\pi\varepsilon_0}.\frac{Q}{r}$$

   i) So, the electric potential at point X is

$$V_X = [(9\times10^9\,\text{Nm}^2\text{C}^{-2})\times(7\times10^{-6}\,\text{C})]/(8\,\text{m})$$

$$= 7.87\times10^3\,\text{V} = 8\times10^3\,\text{V}$$

   up to one significant digit. And the electric potential at point Y is

$$V_Y = [(9\times10^9\,\text{Nm}^2\text{C}^{-2})\times(7\times10^{-6}\,\text{C})]/(12\,\text{m})$$

$$= 5.25\times10^3\,\text{V} = 5\times10^3\,\text{V}$$

   Thus, the potential difference between the points X and Y is

$$V_X - V_Y = (7.87\times10^3\,\text{V} - 5.25\times10^3\,\text{V}) = 2.62\times10^3\,\text{V} = 3\times10^3\,\text{V}$$

ii) When the point charge $+7\,\mu\text{C}$ is replaced by $-7\,\mu\text{C}$, we have electric potential at point X

$$V_X = [(9\times10^9\,\text{Nm}^2\text{C}^{-2})\times(-7\times10^{-6}\,\text{C})]/(8\,\text{m}) = -7.87\times10^3\,\text{V}$$

$$V_Y = [(9\times10^9\,\text{Nm}^2\text{C}^{-2})\times(-7\times10^{-6}\,\text{C})]/(12\,\text{m}) = -5.25\times10^3\,\text{V}$$

Thus, the potential difference between the points X and Y is

$$V_X - V_Y = -7.87\times10^3\,\text{V} - (-5.25\times10^3\,\text{V}) = -2.62\times10^3\,\text{V} = -3\times10^3\,\text{V}$$

iii) The work done in moving the charge $+3\,\mu\text{C}$ from infinity to point X is

$$W = qV_X = (3\times10^{-6}\text{C})\times(7.87\times10^3\,\text{V}) = 2.36\times10^{-2}\,\text{J} = 2\times10^{-2}\,\text{J}$$

b) Charge on the nucleus $Q = Ze = 79\times1.6\times10^{-19}\,\text{C}$ and $r = 6.6\times10^{-15}\,\text{m}$

Thus, from Eq. (8.14), we have

$$V = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r} = \frac{(9\times10^9\,\text{Nm}^2\text{C}^{-2})\times(79\times1.6\times10^{-19}\,\text{C})}{6.6\times10^{-15}\,\text{m}}$$

$$= 1.7\times10^7\,\text{Nm C}^{-1} = 1.7\times10^7\,\text{V} \quad (\because\ \text{NmC}^{-1} = \text{JC}^{-1})$$

4. Let the point P be at a distance x from the point charge $+q$ and the electric potential at P due to the two charges be zero (Fig. 8.11). The electric potential at point P is

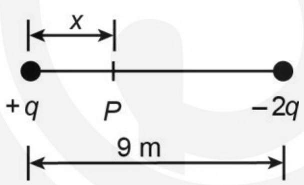


Fig. 8.11: **Diagram for answer to SAQ 4.**

$$V = \frac{1}{4\pi\varepsilon_0 r}\left[\frac{q}{x} + \frac{(-2q)}{(9-x)}\right]$$

Since $V = 0$ at P, we have

$$\frac{q}{x} = \frac{2q}{(9-x)} \Rightarrow q(9-x) = 2qx \Rightarrow 9q = 3qx \Rightarrow x = 3\,\text{m}$$

5. The relation between $\vec{\mathbf{E}}$ and V is:

$$\vec{\mathbf{E}} = -\left[\hat{\mathbf{i}}\frac{\partial V}{\partial x} + \hat{\mathbf{j}}\frac{\partial V}{\partial y} + \hat{\mathbf{k}}\frac{\partial V}{\partial z}\right]$$

As per the problem, $V = x(y^2 - 4x^2)$

Thus,

$$\frac{\partial V}{\partial x} = \frac{\partial V}{\partial x}[xy^2 - 4x^3] = y^2 - 12x^2; \quad \frac{\partial V}{\partial y} = \frac{\partial V}{\partial y}[xy^2 - 4x^3] = 2xy$$

$$\frac{\partial V}{\partial z} = \frac{\partial V}{\partial z}[xy^2 - 4x^3] = 0$$

So, $\vec{\text{E}} = -[\hat{\mathbf{i}}(y^2 - 12x^2) + \hat{\mathbf{j}}(2xy) + \hat{\mathbf{k}}(0)] = (12x^2 - y^2)\hat{\mathbf{i}} - 2xy\,\hat{\mathbf{j}}$

6. The electric potential due to an electric dipole is given by Eq. (8.24):

$$V = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2}$$

So, for point P_1 we have

$$(V)_{P_1} = \frac{p\cos\theta}{4\pi\varepsilon_0 (0.40\,\text{m})^2} \Rightarrow \frac{p\cos\theta}{4\pi\varepsilon_0} = (60\text{ V})\times(0.40\,\text{m})^2 \quad \text{(i)}$$

For point P_2, we can write using Eq. (i)

$$(V)_{P_2} = \frac{p\cos\theta}{4\pi\varepsilon_0 \times (0.60\,\text{m})^2} = \frac{(60\text{ V})\times(0.40\text{ m})^2}{(0.60\text{ m})^2} = 27\text{ V}$$

Terminal Questions

1. Let us consider a closed path starting from and ending at a as shown in Fig. 8.12. Let b be some point on this closed path. A unit positive charge can be moved between points a and b through two paths: L and L'. If V_a and V_b are potentials at a and b, respectively, we can write

$$-\int_{\substack{a\\ \text{along } L}}^{b} \vec{\mathbf{E}}.d\vec{\mathbf{l}} = V_b - V_a \qquad \text{(i)}$$

also

$$-\int_{\substack{a\\ \text{along } L'}}^{b} \vec{\mathbf{E}}.d\vec{\mathbf{l}} = V_b - V_a \qquad \text{(ii)}$$

Now, by changing the limits of integration, we can write Eq. (ii) as:

$$-\int_{\substack{a\\ \text{along } L'}}^{b} \vec{\mathbf{E}}.d\vec{\mathbf{l}} = \int_{\substack{b\\ \text{along } L'}}^{a} \vec{\mathbf{E}}.d\vec{\mathbf{l}} = V_a - V_b \qquad \text{(iii)}$$

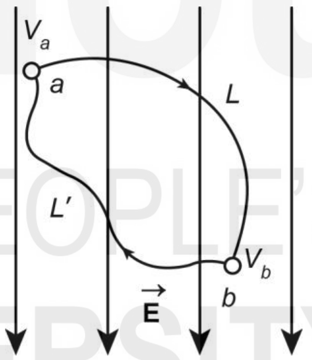


Fig. 8.12: **Diagram for answer to TQ 1.**

Adding Eqs. (i) and (ii) and making use of Eq. (iii), we can write

$$-\int_{\substack{a\\ \text{along } L}}^{b} \vec{\mathbf{E}}.d\vec{\mathbf{l}} - \int_{\substack{a\\ \text{along } L'}}^{b} \vec{\mathbf{E}}.d\vec{\mathbf{l}} = -\int_{\substack{a\\ \text{along } L}}^{b} \vec{\mathbf{E}}.d\vec{\mathbf{l}} + \int_{\substack{b\\ \text{along } L'}}^{a} \vec{\mathbf{E}}.d\vec{\mathbf{l}} = V_b - V_a + V_a - V_b = 0$$

That is, along a closed path, the line integral of the electric field is equal to zero.

Alternative method: We can also use the fact that the line integral of electric field is independent of the path. Thus, we can write

$$\int_{\substack{a\\ \text{along } L}}^{b} \vec{\mathbf{E}}.d\vec{\mathbf{l}} = \int_{\substack{a\\ \text{along } L'}}^{b} \vec{\mathbf{E}}.d\vec{\mathbf{l}}$$

or

$$\underset{\text{along } L}{\int_a^b \vec{\mathbf{E}}.d\vec{\mathbf{l}}} - \underset{\text{along } L'}{\int_a^b \vec{\mathbf{E}}.d\vec{\mathbf{l}}} = 0 \Rightarrow \underset{\text{along } L}{\int_a^b \vec{\mathbf{E}}.d\vec{\mathbf{l}}} + \underset{\text{along } L'}{\int_b^a \vec{\mathbf{E}}.d\vec{\mathbf{l}}} = 0$$

Note that $(L + L')$ implies a closed path between points a and b in Fig. 8.12.

2. Let A and B be two oppositely charged plates separated by a distance d (Fig. 8.8). Let $\vec{\mathbf{E}}$ be the uniform electric field between the two plates. Then, the potential difference between the two plates can be written as [Eq. (8.11)]:

$$-\int_A^B \vec{\mathbf{E}}.d\vec{\mathbf{l}} = (V_B - V_A)$$

where V_A and V_B are the potentials at the plates A and B respectively.

In the present case, writing $\int_A^B \vec{\mathbf{E}}.d\vec{\mathbf{l}}$ *as* $\int_{x_1}^{x_2} \vec{\mathbf{E}}.\hat{\mathbf{i}}\,dx$, and noting that both $\vec{\mathbf{E}}$ and $\hat{\mathbf{i}}\,dx$ are parallel, we can write

$$V_B - V_A = -\int_{x_1}^{x_2} \vec{\mathbf{E}}.\hat{\mathbf{i}}\,dx = -E\,[x]_{x_1}^{x_2} = -E\,(x_2 - x_1) = -E\,d$$

That is, the magnitude of the electric field between two oppositely charged parallel plates is equal to the difference of potential between them divided by their separation.

3. The electric potential V at a point distant r from a charge Q is given by Eq. (8.14):

$$V = \frac{Q}{4\pi\varepsilon_0 r}$$

As per the problem, $Q = 2.0\,\mu\text{C} = 2.0\times10^{-6}\,\text{C}$, $1/(4\pi\,\varepsilon_0) = 9\times10^9\,\text{Nm}^2\text{C}^{-2}$ and $r = 0.10\,\text{m}$ and $0.50\,\text{m}$. Substituting these values, we get

$$V_A = (9\times10^9\,\text{Nm}^2\,\text{C}^{-2})\times\frac{2.0\times10^{-6}\,\text{C}}{0.10\text{ m}} = 1.8\times10^5\,\text{V}$$

$$V_B = (9\times10^9\,\text{Nm}^2\,\text{C}^{-2})\frac{2.0\times10^{-6}\,\text{C}}{0.50\text{ m}} = 0.36\times10^5\,\text{V}$$

Work done in moving charge 0.5 μC from point B to A is $W = q\,(V_A - V_B)$, where $q = 0.05\times10^{-6}\,\text{C}$,

$$\therefore \qquad W = (0.05\times10^{-6}\,\text{C})\,(1.8\times10^5 - 0.36\times10^5)\,\text{V} = 7.2\times10^{-3}\,\text{J}$$

4. We can write

$$V_B - V_A = (V_B - V_C) + (V_C - V_A) = -\int_C^B \vec{\mathbf{E}}.d\vec{\mathbf{l}} - \int_A^C \vec{\mathbf{E}}.d\vec{\mathbf{l}}$$

For path C to B, $\vec{\mathbf{E}}$ and $d\vec{\mathbf{l}}$ are perpendicular to each other. Therefore,

$$\int_C^B \vec{\mathbf{E}}.d\vec{\mathbf{l}} = \left|\vec{\mathbf{E}}\right| \, d\vec{\mathbf{l}} \cos 90° = 0$$

For path A to C, the angle between $\vec{\mathbf{E}}$ and $d\vec{\mathbf{l}} = 135°$. Thus,

$$\int_A^C \vec{\mathbf{E}}.d\vec{\mathbf{l}} = \int_A^C Edl \cos 135°$$

$$= -\frac{E}{\sqrt{2}} \int_A^C dl = -\frac{E}{\sqrt{2}}(AC) = -\frac{E}{\sqrt{2}} \sqrt{2}\, d = -Ed$$

since $AC = d/\cos 45° = \sqrt{2}\, d.$ Thus, we have

$$V_B - V_A = -\int_A^C \vec{\mathbf{E}}.d\vec{\mathbf{l}} = Ed$$

You may note that this is also the value obtained via the direct path from A to B (shown in Fig. 8.10 by dotted lines.)

5. a) False
   b) False (See Eq. (8.24) and (8.32) for any point on the perpendicular bisector of an electric dipole.)
   c) True
   d) True
   e) True

6. a) $V(5\,\text{m}) - V(0) = -\int_A^B \vec{\mathbf{E}}.d\vec{\mathbf{l}} = -\int_0^{5\text{m}} Edl = -(3 \times 10^3\,\text{NC}^{-1}) \times (5\,\text{m}) = -15 \times 10^3\,\text{V}$

   b) The difference in electrostatic potential energy and potential difference is related by

   $$\Delta U = q\Delta V = (2 \times 10^{-6}\,\text{C}) \times (-15 \times 10^3\,\text{V}) = -3.0 \times 10^{-2}\,\text{J}$$

   c) From the conservation of energy, we know that

   $$\Delta U + \Delta K = 0$$

   where ΔU is the change in potential energy and ΔK is the change in kinetic energy. So,

   $$[K(5\,\text{m}) - K(0)] + \Delta U = 0 \Rightarrow K(5\,\text{m}) = -\Delta U = 3.0 \times 10^{-2}\,\text{J}$$

   d) We know that for uniform electric field, $V = Ed$. So

   $$V(x) - V(0) = -E_x(x - x_0) = -(3 \times 10^3\,\text{NC}^{-1})(x - x_0)$$

   i) for $V(0) = 0$

   $$V(x) = -(3 \times 10^3\,\text{NC}^{-1})\,x$$

ii) for $V(1\text{m}) = 0$

$$V(x) - 0 = -(3 \times 10^3\ \text{NC}^{-1}) \times (x - 1)$$

or $$V(x) = 3 \times 10^3\ \text{V} - (3 \times 10^3\ \text{Vm}^{-1})\,x$$

7. Note that the electric field is along negative x-direction. So, the relation between potential and electric field can be written as

$$E_x = \frac{dV}{dx}$$

So, the value of potential will be higher for larger value of x. So, $(V_B - V_A)$ is positive.

Further, to determine the magnitude of E_x for $V_B - V_A = 10^4$ V, we can write

$$E_x = \frac{\Delta V}{\Delta x} = \frac{V_B - V_A}{(7\,\text{m} - 4\,\text{m})} = \frac{10^4\ \text{V}}{4\ \text{m}} = 2.5 \times 10^3\ \text{Vm}^{-1}$$

8. In going from the positive terminal of a battery to the negative terminal, the electron (a negatively charged particle) moves from a point at a higher potential to a point at a lower potential. Thus, if A and B are, respectively, the positive and negative terminals of the battery, we have

$$V_B - V_A = -12\,\text{V}$$

Thus, the work done in moving an electron from the positive to the negative terminal is

$$W = q\,(V_B - V_A) = (-1.6 \times 10^{-19}\ \text{C}) \times (-12\ \text{V}) = 1.92 \times 10^{-18}\ \text{J}$$

Particle accelerators utilise very high potential differences to produce high energy charged particles used in *atom smashing* experiments for studying nuclear structure. This is a picture of the Large Hadron Collider located at CERN, near Geneva.

(Picture source: Wikimedia Commons)

UNIT 9

ELECTRIC POTENTIAL OF CONTINUOUS CHARGE DISTRIBUTIONS

Structure

STUDY GUIDE

In this unit, we will continue our discussion on electric potential begun in the previous unit. You will learn how to determine the electric potential of continuous charge distributions such as charged wire, spherical shell and non-conducting solid sphere. While studying this unit, you should focus on how to calculate the total charge for a given continuous charge distribution. The mathematical tools used for these calculations are similar to those you have learnt in Block 1 of this course. However, you will do better if you revise Units 3 and 4 of Block 1 on vector integral calculus and school integral calculus. Further, you should also focus on how the value of electric field can be calculated at a point using the expression for potential at that point due to a given continuous charge distribution. To help you understand and practice the method of determining electric potential better, we have given several examples, SAQs and TQs. Try to solve them yourself to check your understanding of the concepts and methods discussed in the unit.

“I have not failed. I've just found 10,000 ways that won't work.”

Thomas A. Edison

9.1 INTRODUCTION

In the previous units of this block, you have learnt how to determine the electric field $\vec{E}$ and electric potential V due to a point charge and a system of discrete charges. You have learnt how to calculate potential by evaluating the line integral of $\vec{E}$. You have also learnt how to calculate $\vec{E}$ from potential V by taking its gradient. In this unit, we shall extend these ideas to determine electric potential of continuous charge distributions.

You know that the electrical appliances we use in our homes work on a potential difference of 220 V. Apart from these appliances, the concept of potential difference plays an important role in the design and manufacturing of high voltage sources used by physicists to do interesting experiments. For example, if a charged particle is allowed to fall through a potential difference, it accelerates and its kinetic energy increases. The machines called *particle accelerators* have been designed on this basic principle to produce high energy charged particles used in *atom smashing* experiments for studying nuclear structure. In electrical appliances and machines, the desired potential difference is created by charging objects of appropriate geometry. Therefore, it is important to study electric potential of continuous charge distributions.

We begin the discussion by determining the electric potential of three types of continuous charge distributions, namely, line charge, spherical shell and non-conducting solid sphere (Sec. 9.2). In Sec. 9.3, you will learn about equipotential surface which is a useful concept because it is characterised by the fact that no net work is done in moving a charge from one point to other on this surface. In Sec. 9.4, you will learn how to calculate electrostatic potential energy of a system of discrete charges as well as continuous charge distributions if electric potential is known.

In the next unit (Block 3), you will study the macroscopic properties of the dielectrics kept in an electric field. The understanding of the concepts of electric field and potential studied in this block will help you appreciate the properties of dielectrics better.

Expected Learning Outcomes

After studying this unit, you should be able to:

- obtain the expression for electric potential of a line charge;
- determine the electric potential of a uniformly charged spherical shell;
- derive the expression of electric potential of uniformly charged non-conducting sphere;
- explain the concept of equipotential surface; and
- calculate the electrostatic potential energy for a given charge distribution.

9.2 ELECTRIC POTENTIAL OF CONTINUOUS CHARGE DISTRIBUTIONS

In the previous unit, you have learnt how to determine the electric potential of a point charge at a given point. You have also learnt how to use the superposition principle to obtain the expression for electric potential of multiple discrete charges.

Now, suppose that we need to determine the electric potential of a charged object such as a metal rod or a solid sphere. **In general, for determining the electric potential of a continuous charge distribution, we first calculate the potential due to a small element of the charge distribution and then integrate this expression over appropriate limits to include the effect of total charge in it.** We now determine the electric potential of three types of continuous charge distributions: line charge, uniformly charged spherical shell and uniformly charged non-conducting sphere.

NOTE

Can we consider a charged object as a collection or distribution of discrete charges and use the method described in the previous unit for determining its electric potential? No, we cannot. This is so because for such uniformly charged objects, we can only know the total charge on them. There is no way to ascertain the position of individual charges because the charge is uniformly distributed all over the object. A uniformly charged object is called continuous charge distribution because the separation between individual charges on such objects is very, very small.

9.2.1 Line Charge

In Unit 7 of this block, you have learnt how to determine the electric field at a point near an **infinitely long charged wire** (or **a line charge**). It is given by:

$$\vec{\mathbf{E}} = \frac{\lambda}{2\pi\varepsilon_0 r}\hat{\mathbf{r}} \tag{9.1}$$

where λ is the charge per unit length on the wire or the **linear charge density**, r is the perpendicular distance of the point from the wire, ε_0 is the permittivity of free space, and $\hat{\mathbf{r}}$ is the unit vector along the direction of increasing r from the line charge (Fig. 9.1).

The question now is: What is the potential of this wire at a point a situated at a perpendicular distance of r_a from the wire? From Eq. (8.15) of Unit 8, you can write

$$V = -\int_{\infty}^{r_a} \vec{\mathbf{E}}.d\vec{\mathbf{l}} \tag{9.2}$$

Let us evaluate the line integral in Eq. (9.2) by first moving a unit positive charge from a finite distance r_b instead of infinity, to point a at distance r_a and then let r_b go to infinity. Here r_b is the distance of point b from the wire (see Fig. 9.1). This integral then gives us the difference in potentials between points a and b, i.e.

$$V_a - V_b = -\int_{r_b}^{r_a} \vec{\mathbf{E}}.d\vec{\mathbf{l}} = \int_{r_a}^{r_b} \vec{\mathbf{E}}.d\vec{\mathbf{l}} = \int_{r_a}^{r_b} \vec{\mathbf{E}}.d\vec{\mathbf{r}} \tag{9.3}$$

because for the path a to b, $d\vec{\mathbf{l}}$ is parallel to $d\vec{\mathbf{r}}$. Inserting the expression for $\vec{\mathbf{E}}$ from Eq. (9.1) we get

$$V_a - V_b = \frac{\lambda}{2\pi\varepsilon_0}\int_{r_a}^{r_b} \frac{\hat{\mathbf{r}}.d\vec{\mathbf{r}}}{r}$$

Since $\hat{\mathbf{r}}$ and $d\vec{\mathbf{r}}$ are in the same direction, we have

$$V_a - V_b = \frac{\lambda}{2\pi\varepsilon_0}\int_{r_a}^{r_b} \frac{dr}{r}$$

$$= \frac{\lambda}{2\pi\varepsilon_0}[\ln r]_{r_a}^{r_b} = -\frac{\lambda \ln r_a}{2\pi\varepsilon_0} + \frac{\lambda \ln r_b}{2\pi\varepsilon_0} = -\frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{r_a}{r_b}\right) \tag{9.4}$$

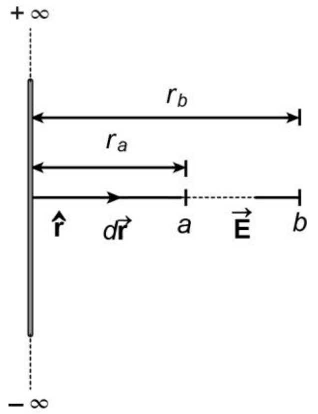


Fig. 9.1: **Electric potential at a point *a* due to an infinitely long charged wire.**

As you have learnt in the previous unit, we take a point at infinity with zero potential as our reference point and calculate the potential at a given point with respect to infinity. In the instant case, if we consider point *b* to be located at infinity, i.e. $r_b = \infty$ and take the potential V_b equal to zero, then the RHS of Eq. (9.4) tells us that the potential V_a at point *a* will be infinite. This is expected also because the infinitely long line charge having uniform charge distribution means an infinite amount of charge. Therefore, the sum of finite contributions from each part or element of an infinite line charge leads to an infinite potential.

Thus, to have a physically meaningful expression for potential at a point finite distance away from the line charge, we cannot take infinity with zero potential as our reference point. However, the inability to have a reference point with zero potential does not cause any problem because in practical situations, we are interested in difference in potential between two points rather than its absolute value at a given point. Thus, Eq. (9.4) which gives the potential difference between points *a* and *b* (Fig. 9.1) with both r_a and r_b having finite values meets our requirement.

Note that an infinite line charge contain infinite amount of charge. Thus, we cannot calculate *V* at a point for such a continuous charge distribution by taking total charge into consideration. That method will not work as it will give infinite potential everywhere. That is why we have used the relation between *V* and $\vec{\mathbf{E}}$ and the expression for $\vec{\mathbf{E}}$ for an infinite line charge to obtain a physically meaningful expression for *V*.

Further, to check whether we can obtain the value of electric field at a point, say *a*, using Eq. (9.4), let us assume that point *b* located at a finite distance r_b is the reference point with zero potential. This implies that r_b is fixed and $V_b = 0.$ Hence, the second term on the RHS of Eq. (9.4) is constant. Thus, we can write Eq. (9.4) as

$$V_a = -\frac{\lambda \ln r_a}{2\pi\varepsilon_0} + \text{const} \qquad (9.5)$$

You may recall that electric field $\vec{\mathbf{E}}$ and electric potential *V* are related by Eq. (8.20):

$$\vec{\mathbf{E}} = -\vec{\nabla} V = -\hat{\mathbf{r}}\,\frac{dV_a}{dr} = \frac{\lambda}{2\pi\varepsilon_0 r}\,\hat{\mathbf{r}} \qquad (9.6)$$

Note that Eq. (9.6) is the same as Eq. (9.1).

To concretise the ideas discussed above, go through the following example.

EXAMPLE 9.1 : ELECTRIC POTENTIAL OF A LINE CHARGE

An infinite line charge has linear charge density $\lambda = 2.0\,\mu\text{C m}^{-1}$. Calculate the electric potential at a point on a line perpendicular to the line charge, at a distance of 3.0 m from the line charge. Assume that the electric potential of the line charge is zero at the perpendicular distance of 4.0 m.

SOLUTION ■ From Eq. (9.4), note that the potential difference between two points *a* and *b* due to an infinite line charge is given as

$$V_a - V_b = -\frac{\lambda}{2\pi\varepsilon_0} \ln(r_a / r_b) \qquad \text{(i)}$$

From the problem, we have $r_a = 3.0\,\text{m}$ and $r_b = 4.0\,\text{m}$. It is also given that $V_b = 0$ at $r_b = 4.0\,\text{m}$. Substituting these values and $\lambda = 2.0 \times 10^{-6}\,\text{C m}^{-1}$ in Eq. (i), we get

$$V_a - 0 = -\frac{2.0 \times 10^{-6}\,\text{C m}^{-1}}{2 \times 3.14 \times (8.85 \times 10^{-12}\,\text{F m}^{-1})} \ln\left(\frac{3.0\,\text{m}}{4.0\,\text{m}}\right)$$

or $$V_a = 10.93 \times 10^3\,\text{V}$$

Before proceeding further, solve an SAQ.

***SAQ 1* - Electric potential of a line charge**

The linear charge density of an infinite line charge is $3.0 \times 10^{-6}\,\text{C m}^{-1}$. Assuming that the electric potential at a perpendicular distance of 5.0 m from the wire is zero, calculate the potential at the perpendicular distance of 6.0 m.

Now, let us discuss how to determine the electric potential of a uniformly charged spherical shell at a given point.

9.2.2 Uniformly Charged Spherical Shell

You know that a spherical shell is a hollow sphere. For determining the electric potential of a uniformly charged spherical shell, there are two regions of interest: one at a point inside the spherical shell and the other at a point outside.

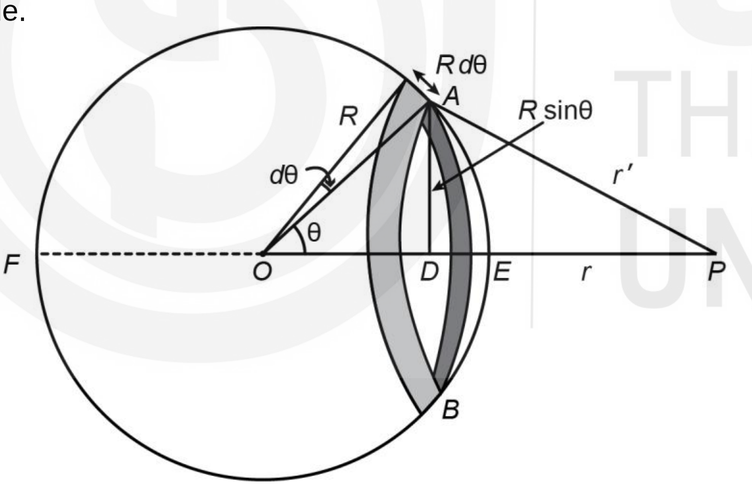


Fig. 9.2: **A uniformly charged spherical shell of radius *R* and point *P* is an external point.**

Study Fig. 9.2, which shows a uniformly charged spherical shell of radius *R*. To obtain an expression for the potential at an external point *P*, we first identify a suitable element of charged shell. The charged surface of the shell can be considered as a collection of a large number of thin rings such as the ring *AB*. The orientation of these rings are so selected that the axis of the rings is along *OP*, the line joining the centre *O* of the shell with the point *P*.

Now, let the ring *AB* be contained between the directions θ and $\theta + d\theta$ with respect to the axis *OP*. Let it be of infinitesimal width so that every point on it is at the same distance, say r', from *P*. The angular width of the ring is $d\theta$, its width is $R d\theta$ and its radius is $R\sin\theta$. The circumference of the ring is $2\pi R\sin\theta$ and hence, its area is given by

$$dA = (2\pi R \sin\theta) R\, d\theta = 2\pi R^2 \sin\theta d\theta \tag{9.7}$$

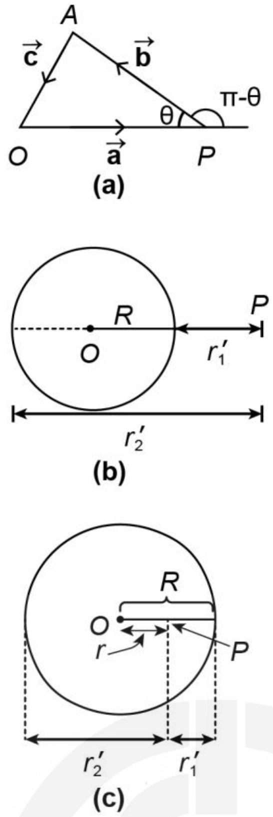


Fig. 9.3: **Diagrams for calculating electric potential of charged spherical shell.**

What is the charge on the ring? If the total charge on the shell is Q, then charge per unit area, $\sigma = Q/(4\pi R^2)$. Thus, using Eq. (9.7) we can write the charge on the ring

$$Q_{ring} = \frac{Q}{4\pi R^2} \times (2\pi R^2 \sin\theta) d\theta = \frac{Q}{2} \sin\theta d\theta \tag{9.8}$$

We shall now determine the electric potential at point P due to the ring AB. The ring is made up of a large number of point charges each having charge equal to, say δQ. So, the electric potential for one such point charge is $\frac{1}{4\pi\varepsilon_0} \frac{\delta Q}{r'}$. So, the electric potential due to the ring will be

$$dV_{ring} = \sum \left(\frac{1}{4\pi\varepsilon_0} \frac{\delta Q}{r'} \right) = \frac{1}{4\pi\varepsilon_0 r'} \sum \delta Q = \frac{1}{4\pi\varepsilon_0} \frac{Q_{ring}}{r'}$$

So, on using Eq. (9.8), we get

$$dV_{ring} = \left(\frac{1}{4\pi\varepsilon_0 r'} \right) \left(\frac{Q}{2} \sin\theta d\theta \right) \tag{9.9}$$

As we mentioned above, the shell can be imagined to be made of rings like AB having a common axis OP. Since electric potential is a scalar quantity, we shall integrate Eq. (9.9) to get the electric potential V of the shell.

Note that on the RHS of Eq. (9.9), we have two variables θ and r'. It will be convenient if we can express it in terms of a single variable. For this, we shall consider the relation between r', r and R. To do so, refer to Fig. 9.3a. From triangle OAP, we have (see Margin Remark):

For triangle OAP (Fig. 9.3a):

$$\vec{\mathbf{a}} + \vec{\mathbf{b}} + \vec{\mathbf{c}} = 0$$

$$\Rightarrow \vec{\mathbf{a}} + \vec{\mathbf{b}} = -\vec{\mathbf{c}}$$

$$\Rightarrow (\vec{\mathbf{a}} + \vec{\mathbf{b}}).(\vec{\mathbf{a}} + \vec{\mathbf{b}}) = c^2$$

$$\Rightarrow a^2 + b^2 + 2\vec{\mathbf{a}}.\vec{\mathbf{b}} = c^2$$

or

$$a^2 + b^2 + 2ab\cos(\pi - \theta) = c^2$$

or

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

$$r'^2 = r^2 + R^2 - 2rR\cos\theta$$

On differentiating with respect to θ, we get

$$2r' \frac{dr'}{d\theta} = 2rR\sin\theta$$

or
$$\frac{dr'}{rR} = \frac{\sin\theta d\theta}{r'} \tag{9.10}$$

Substituting Eq. (9.10) in Eq. (9.9), we get

$$dV_{ring} = \frac{1}{4\pi\varepsilon_0} \left(\frac{Q}{2} \right) \left(\frac{dr'}{rR} \right) \tag{9.11}$$

To obtain the electric potential due to the entire shell, we need to integrate Eq. (9.11) over appropriate limits of integration to include the contribution of every ring of the shell:

$$V = \int dV_{ring} = \frac{1}{4\pi\varepsilon_0} \frac{Q}{2rR} \int_{r_1'}^{r_2'} dr' \tag{9.12}$$

where r_1' and r_2' are, respectively, the minimum and maximum values of r'. To write the values of r_1' and r_2' in terms of r and R, we consider the two cases – point P outside the shell and point P inside the shell – separately :

a) *Point* P *outside the shell*

In this case, as shown in Fig. 9.3b, the values of r_1' and r_2' are

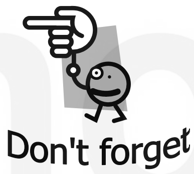
$r_1' = r - R$ and $r_2' = r + R$

So, Eq. (9.12) becomes

$$V = \frac{1}{4\pi\varepsilon_0}\frac{Q}{2rR}\int_{(r-R)}^{(r+R)} dr' = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{2rR}\right)[2R] = \frac{Q}{4\pi\varepsilon_0 r} \qquad (9.13)$$

Eq. (9.13) gives the electric potential due to a uniformly charged spherical shell at a point outside the shell. Note that Eq. (9.13) is same as Eq. (8.14) which is for the electric potential of a point charge at a point at distance r.

Thus, we may conclude that, for an external point, the uniformly charged spherical shell behaves as a point charge located at the centre of the shell.



b) *Point* P *inside the shell*

Refer to Fig. 9.3c which depicts the point P inside the shell. From the figure, you may note that for $r < R$

$r_1' = R - r$ and $r_2' = R + r$

Substituting these values of r_1' and r_2' as limits of integration in Eq. (9.12), we get

$$V = \frac{1}{4\pi\varepsilon_0}\frac{Q}{2rR}\int_{(R-r)}^{(R+r)} dr' = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{2rR}\right)[2r] = \frac{Q}{4\pi\varepsilon_0 R} \qquad (9.14)$$

From Eq. (9.14), which gives electric potential at an internal point P, we note that the electric potential is independent of r, the distance of point P from the centre O of the shell. This means that the electric potential at every point inside the shell is same and its value is equal to its value at the surface. If we plot the variation of potential for a spherical shell with distance from its centre, we obtain a curve as shown in Fig. 9.4.

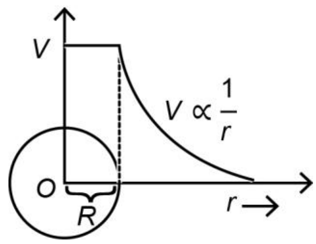
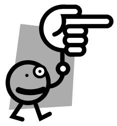


Fig. 9.4: **The variation of electric potential due to a spherical shell shell with distance from its centre.**

On the basis of Eq. (9.14) and Fig. 9.4, can you guess what will the value of electric field inside the uniformly charged spherical shell be? Note from Fig. 9.4 that the electric potential is constant everywhere inside the shell. So, if we move a test charge from one point to another inside the shell, no work is to be done because both the points are at the same potential. This is possible only if the value of the electric field inside the shell is zero. Thus, we conclude that:



The value of electric field inside a uniformly charged spherical shell is zero.

Now, before proceeding further, solve an SAQ.

***SAQ 2* - Electric potential of a uniformly charged spherical shell**

The radius and surface charge density of a uniformly charged spherical shell are 20 cm and $3.0\,\mu\text{C m}^{-2}$, respectively. Calculate the electric potential at a distance (a) 40 cm and (b) 15 cm from the centre of the shell.

9.2.3 Uniformly Charged Non-conducting Sphere

Let ρ be the volume charge density (charge per unit volume) of a uniformly charged non-conducting sphere. Let the radius of the sphere be R (see Fig. 9.5). As in the case of spherical shell, here also we have two regions of interest for determining electric potential: one at a point outside the sphere and the other at a point inside it.

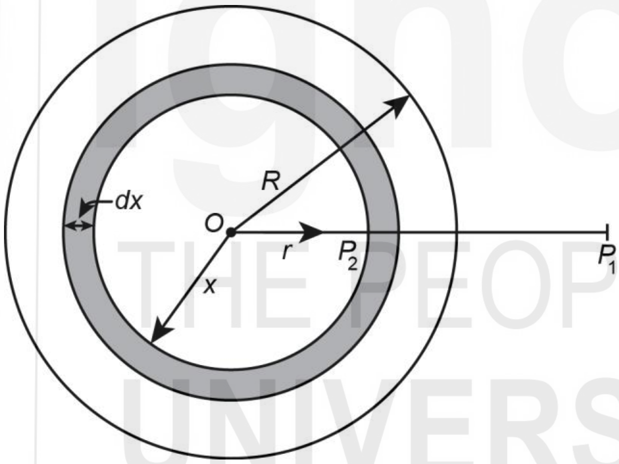


Fig. 9.5: **A uniformly charged non-conducting sphere of radius R with point P_1 outside the sphere and point P_2 inside the sphere.**

a) Electric potential at a point outside the sphere

For points outside the non-conducting sphere, such as P_1, located at distance r from the centre O of the sphere, the whole charge spread throughout the volume of the sphere behaves like a point charge located at its centre O. This fact can easily be deduced on the basis of the derivation of the potential at an external point due to a spherical shell discussed in the previous section. We can divide the non-conducting sphere into a large number of thin concentric shells as shown in Fig. 9.5. For each of these shells, the charge can be regarded as concentrated at the centre O for points outside the shell. Thus, for a point outside the sphere, such as P_1 in Fig. 9.5, the whole charge of the sphere can be regarded as a point charge located at its centre O. Hence, for points outside the sphere, the expression for electric potential due to a non-conducting charged sphere will be the same as for a uniformly charged spherical shell [(Eq. (9.13)]:

$$V = \frac{Q}{4\pi\varepsilon_0 r} \tag{9.15}$$

where $Q\left(= (4\pi/3)R^3\rho\right)$ is the total charge on the sphere and r is the distance of external point P_1 from the centre O of the sphere. **You must, however, note that the expression for the total charge, Q is different for a uniformly charged non-conducting solid sphere from that for the uniformly charged spherical shell** (it is spread in a volume, whereas for a shell, it is spread on its surface)**.**

b) Electric potential at a point inside the sphere

Let point P_2 be an internal point at a distance r from the centre O such that $r < R$ (see Fig. 9.5). If we divide the sphere into a large number of thin concentric shells with centre O, then for shells with radii $\leq r$, point P_2 is outside and for shells which have radii between r and R, point P_2 is inside. For shells with radii less than or equal to r, potential V_1 at P_2 can be written as if point P_2 is an external point and hence it is given by Eq. (9.15):

$$V_1 = \frac{Q_1}{4\pi\varepsilon_0 r} = \frac{4\pi}{3} \times \frac{r^3\rho}{4\pi\varepsilon_0 r} = \frac{\rho r^2}{3\varepsilon_0} \tag{9.16}$$

To evaluate the contribution to electric potential by the shells for which P_2 is inside the sphere, let us consider a shell of radius x and thickness dx as shown in Fig. 9.5. For this shell, the total charge Q_2 is equal to volume times charge density, i.e. $Q_2 = 4\pi x^2 dx\rho$. This charge contributes a constant electric potential dV_2 at any internal point and is given by (see Eq. (9.14)):

$$dV_2 = \frac{4\pi x^2\, dx\, \rho}{4\pi\varepsilon_0\, x} = \frac{\rho x\, dx}{\varepsilon_0} \tag{9.17}$$

For adding the contributions from all such shells for which P_2 is an internal point, we integrate Eq. (9.17) for x varying from r to R. This gives the electric potential V_2 at P_2 due to shells for which point P_2 is internal as

$$V_2 = \int_r^R dV_2 = \frac{\rho}{\varepsilon_0}\int_r^R x\, dx = \frac{\rho}{\varepsilon_0}\left(\frac{R^2 - r^2}{2}\right) \tag{9.18}$$

Thus, adding Eqs. (9.16) and (9.18), we can write the electric potential V of the non-conducting sphere at an internal point P_2 as:

$$V = V_1 + V_2 = \frac{\rho}{3\varepsilon_0} r^2 + \frac{\rho}{\varepsilon_0}\left(\frac{R^2 - r^2}{2}\right)$$

$$= \frac{\rho}{3\varepsilon_0}\left(\frac{3R^2 - r^2}{2}\right) = \frac{4\pi R^3 \rho}{3 \times 4\pi\varepsilon_0}\left(\frac{3R^2 - r^2}{2R^3}\right) = \frac{Q(3R^2 - r^2)}{8\pi\varepsilon_0 R^3} \tag{9.19}$$

where $Q[= (4/3)\pi R^3\rho]$ is the total charge on the uniformly charged non-conducting sphere.

To fix these ideas, you may like to go through the following example, which is for conducting sphere.

EXAMPLE 9.2 : ELECTRIC POTENTIAL DUE TO CHARGED CONDUCTING SPHERE

Two charged spherical conductors of radius $r_1 = 8.0\,\text{cm}$ and $r_2 = 2.0\,\text{cm}$ are separated by a distance much larger than 10 cm. These spheres are connected by a conducting wire and a total of 60 nC charge is placed on one of the spheres. (a) Calculate the charge on each sphere. b) Calculate the electric potential of each sphere at a point on their surfaces.

SOLUTION ■ Since the charged conducting sphere is connected through a conducting wire to the uncharged sphere, the 60 nC charge will redistribute between the two sphere in such a manner so that both sphere have same electric potential. Let the final charge be q_1 (on the larger sphere) and q_2 on the smaller sphere.

(a) From the conservation of charge, we have

$$q_1 + q_2 = 60\,\text{nC} \qquad \text{(i)}$$

Further, since the electric potential of both spheres are equal, we can write,

$$\frac{1}{4\pi\varepsilon_0}\frac{q_1}{r_1} = \frac{1}{4\pi\varepsilon_0}\frac{q_2}{r_2} \Rightarrow q_2 = \frac{r_2}{r_1}q_1 \qquad \text{(ii)}$$

From Eqs. (i) and (ii), we can write

$$q_1 + \frac{r_2}{r_1}q_1 = 60\,\text{nC}$$

$$\Rightarrow \quad q_1 = \left(\frac{r_1}{r_1 + r_2}\right) \times 60\,\text{nC} = \frac{80\,\text{cm}}{10\,\text{cm}} \times 60\,\text{nC} = 48\,\text{nC}$$

So, $\quad q_2 = (60\,\text{nC} - 48\,\text{nC}) = 12\,\text{nC}$

b) Using the values of q_1 and q_2, we can write the potential V_1 and V_2 of the two spheres at a point on their surfaces as

$$V_1 = \frac{1}{4\pi\varepsilon_0}\frac{48\,\text{nC}}{(8.0 \times 10^{-2}\,\text{m})} = \frac{(9 \times 10^9\,\text{Nm}^2\text{C}^{-2}) \times (48 \times 10^{-9}\,\text{C})}{(8.0 \times 10^{-2}\,\text{m})} = 5.4\,\text{kV}$$

$$V_2 = \frac{1}{4\pi\varepsilon_0}\frac{12\,\text{nC}}{(2.0 \times 10^{-2}\,\text{m})} = \frac{(9 \times 10^9\,\text{Nm}^2\text{C}^{-2}) \times (12 \times 10^{-9}\,\text{C})}{(2.0 \times 10^{-2}\,\text{m})} = 5.4\,\text{kV}$$

Note that the spheres in Example 9.2 are conducting spheres.

Now, you may like to solve an SAQ.

SAQ 3 - Electric potential of a charged conducting sphere

An isolated solid sphere of aluminium having radius 7.0 cm is at a potential of 500 V. Calculate the number of electrons which have been removed from the sphere to raise it to this potential.

Let us now learn about the concept of equipotential surface.

9.3 EQUIPOTENTIAL SURFACES

To understand the concept of equipotential surface, recall that electric potential of a point charge *Q*, at a point at distance *r* is given as:

$$V = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{r}\right)$$

From the above equation, we note that electric potential depends only on *r.* Now, you know that the locus of points having the same value of *r* is the surface of a sphere of radius *r* with the point charge as its centre. For a different value of *r,* we get a different surface of the sphere (see Fig. 9.6a). On any such surface, the value of electric potential will be the same everywhere because *r* is same for all points of this surface. Such a surface is called an **equipotential surface. Formally, we define equipotential surface as the locus of all points having the same electric potential.** Further, the geometry of Fig. 9.6a suggests that the electric field lines due to the point charge *Q* located at the centre of the concentric spheres are everywhere perpendicular to the equipotential surfaces. The consequences of this fact are very important which we shall discuss shortly.

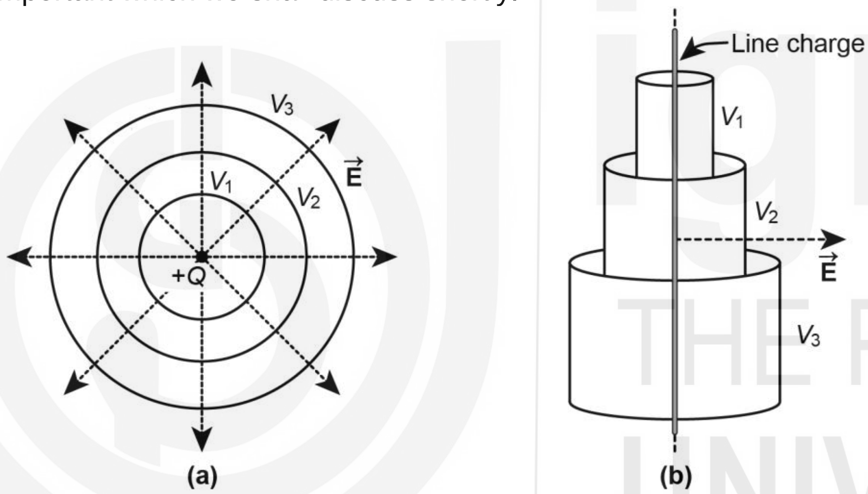


Fig. 9.6: a) Equipotential surfaces of a point charge $+Q$; the electric field lines are radial (dashed). Solid circles are intersections of equipotential surfaces on the plane of paper; b) The equipotential surfaces (cylindrical surfaces) of a uniform infinite line charge.

Can you guess the nature of equipotential surfaces for a uniform infinite line charge? From Eq. (9.5), you may note that for a uniform infinite line charge, the electric potential is same at all points equidistant from the line charge. Therefore, for such a charge distribution, equipotential surfaces are cylindrical with the line charge as the axis of the cylinder (Fig. 9.6b).

Yet another example of an equipotential surface is a conducting surface. **An ideal conducting surface must be an equipotential surface.** Can you guess why it is so? This is because if there were any potential difference between two points on the conducting surface, charges would move from higher to lower electric potential (or *vice-versa*) until the electric potential everywhere became equal. You will see later in Unit 11 that this property of conductors helps us determine the electric field and potential in the space between the plates of a capacitor easily.

Since an equipotential surface is a surface having **constant electric potential**, the potential difference between any two points on it is zero. This

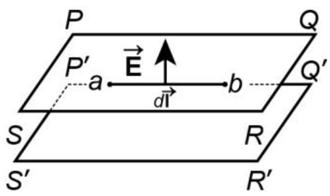


Fig. 9.7: **Direction of electric field vector $\vec{\mathbf{E}}$ relative to equipotential surfaces.** *PQRS* **and** *P′Q′R′S′* **are part of equipotential surfaces.**

implies that the work done in moving a unit charge from one point to another on such a surface is also zero. Thus, if *a* and *b* are two points on a equipotential surface, we can write

$$V_b - V_a = -\int_a^b \vec{\mathbf{E}}.d\vec{\mathbf{l}} = 0 \tag{9.20}$$

where V_a and V_b are potential at points *a* and *b*, respectively. You will agree that Eq. (9.20) will hold only when the electric field $\vec{\mathbf{E}}$ and the small displacement vector $d\vec{\mathbf{l}}$ are perpendicular to each other. Since $d\vec{\mathbf{l}}$ is an infinitesimal displacement on the equipotential surface, $\vec{\mathbf{E}}$ has to be perpendicular at all points on such a surface (see Fig. 9.7). It is for this reason that we have drawn the electric field lines as perpendicular to the equipotential surfaces in Fig. 9.6.

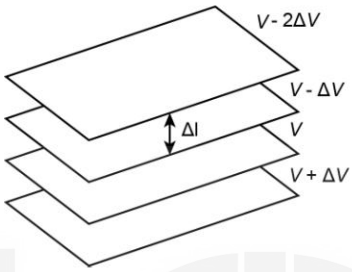


Fig. 9.8: **Separation between equipotential surfaces for arbitrary distribution of charges. Portions of four equipotential surfaces are shown.**

For an arbitrary charge distribution, the equipotential surfaces may look like the ones drawn in Fig. 9.8. **By convention, the equipotential surfaces are drawn such that there is a constant difference of potential, say** ΔV, **between the adjacent surfaces as shown in Fig. 9.8.**

Further, you may note in Fig. 9.9 which depicts the equipotential surfaces for an arbitrary charge distribution, that the equipotential surfaces may or may not be parallel to each other. They are relatively closer where the magnitude of $\vec{\mathbf{E}}$ is large, and are relatively far apart where the magnitude of $\vec{\mathbf{E}}$ is small. It is so because the difference in potential, ΔV between any two given equipotential surfaces is constant and we know that

$$\Delta V = Ed = E\Delta l \tag{9.21}$$

Thus, for constant ΔV, if Δl decreases, *E* must increase.

> **The magnitude of electric field is greater in the region where equipotentials are closer to each other.**

So, we have seen that the sketch of equipotential surfaces gives us a fairly good idea about the magnitude of electric field in that region. You have also learnt that the electric field is directed perpendicular to an equipotential surface. Can we also draw some inference about the sense of the direction of electric field on the basis of equipotentials? Yes, we can. To find out, refer again to Fig. 9.9. Note that, on the left hand side of the figure, equipotential surfaces are closer to each other as compared to the right hand side. Now, you may recall from Unit 8 that the relation between $\vec{\mathbf{E}}$ and electric potential is given by:

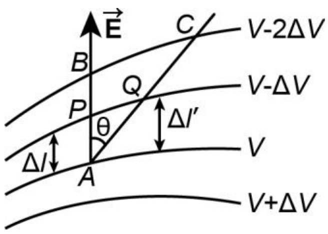


Fig. 9.9: **Direction of electric field $\vec{\mathbf{E}}$ from equipotential surfaces.**

$$\vec{\mathbf{E}} = -\vec{\nabla} V \tag{9.22}$$

The negative sign in Eq. (9.22), along with the fact that the electric field $\vec{\mathbf{E}}$ is always perpendicular to equipotential surfaces, implies that $\vec{\mathbf{E}}$ always points in the direction of decreasing *V*. To understand this better, let us consider two probable directions *APB* and *AQC* for the electric field $\vec{\mathbf{E}}$ (Fig. 9.9). Let the separation between two adjacent equipotentials along *APB* and *AQC* be Δl

and $\Delta l'$ respectively. Since ΔV is constant and the geometry of the figure suggests that $\Delta l' > \Delta l$, we have

$$\frac{\Delta V}{\Delta l} > \frac{\Delta V}{\Delta l'}$$

This implies that $\vec{\mathbf{E}}$ is directed along $\Delta\vec{\mathbf{l}}$, that is, along *APB* because the decrease in *V* is fastest along this line. **Thus, we conclude that the electric field $\vec{\mathbf{E}}$ is always along the direction of maximum (or the steepest) decrease of potential, *V*.**

Thus, a sketch of the equipotential surfaces gives us a visual picture of both the direction and the magnitude of $\vec{\mathbf{E}}$ in a region of space containing a single charge, a group of charges, or a charge distribution of some particular form (or shape).

On the basis of the above discussion, we can summarise the properties of equipotential surfaces as follow:

Recap

- **The electric field is perpendicular to equipotential surface.**
- **The electric field is directed along the maximum (steepest) decrease of potential. That is, it points from surface at higher electric potential to lower electric potential.**
- **No work is done in moving a charge between any two points on an equipotential surface.**
- **The tangential component of electric field along an equipotential surface is zero. If it were not so, a finite work would be required to be done in moving a charge along the surface.**

So far, we have described the electrostatic field in terms of electric field vector, potential and equipotential surfaces. In the next section we shall discuss the electrostatic energy associated with discrete and continuous charge distributions. But, before studying the next section, you may like to try an SAQ.

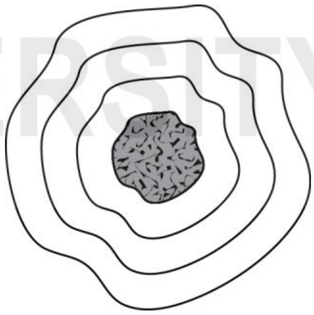


Fig. 9.10: **Diagram for SAQ 4b.**

SAQ 4 – Equipotential surfaces

a) Suppose you are given a sketch of electric field lines due to a group of charges and asked to draw the equipotential surfaces. List the various points you will keep in mind while drawing equipotential surfaces.

b) The equipotential surfaces for a charged solid metal object are shown in Fig. 9.10. Draw the electric field lines.

9.4 ELECTROSTATIC POTENTIAL ENERGY

In the previous unit, you have learnt about the electrostatic potential energy of charge *q* in the field of another charge *Q*. We now extend this discussion to discrete and continuous charge distributions.

Let us first consider two charged particles q_1 and q_2 very far apart from one another as shown in Fig. 9.11a. Now, if we bring these two particles slowly towards each other to a distance between them be r_{21}, then how much work is done in this process? Recall that the work done will be the same whether we move q_2 and keep q_1 fixed or *vice-versa*. The work done is the integral of the product of the force between the charges and displacement in the direction of force.

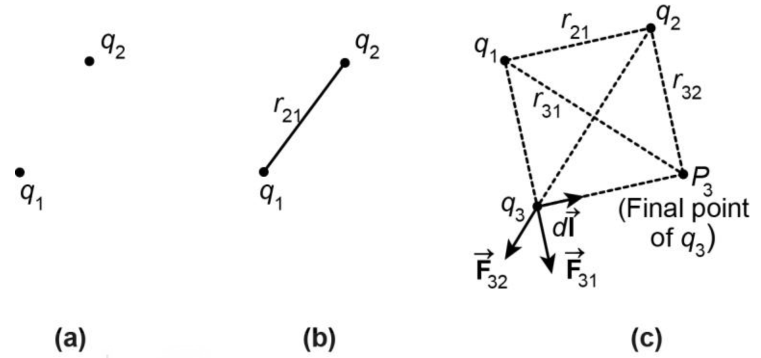


Fig. 9.11: **a) Two charged particles q_1 and q_2 at a very large distance from each other; b) the two charges at a separation r_{21} from each other; c) three charges q_1, q_2 and q_3 are brought near one another.**

The work done in bringing the charges q_1 and q_2 separated by a large distance to a separation r_{21} from each other is:

$$W_1 = \int \vec{\mathbf{F}}_{21}.d\vec{\mathbf{l}} = \int_{r=\infty}^{r_{21}} \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2(-dr)}{r^2} = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{r_{21}} \qquad (9.23a)$$

Note that we have taken distance as (– dr) because r is changing from $\propto$ to r_{21}. We have taken q_1 and q_2 to be positive; so the charges must be pushed together and the displacement is opposite to the direction of Coulomb force.

You know from Unit 8 that the work done in moving a charge from infinity to a finite distance r in a field due to another charge is independent of the path we take. With this understanding, let us now bring a third charge q_3 from infinity (that is, from very large distance from charges q_1 and q_2) and bring it to a position such that its distance from q_1 is r_{31} and from q_2, r_{32} (Fig. 9.11c). So, the work done in moving charge q_3 to this position is

$$\int \vec{\mathbf{F}}.d\vec{\mathbf{l}} = \int (\vec{\mathbf{F}}_{31} + \vec{\mathbf{F}}_{32}).d\vec{\mathbf{l}} = -\int \vec{\mathbf{F}}_{31}.d\vec{\mathbf{r}} - \int \vec{\mathbf{F}}_{32}.d\vec{\mathbf{r}} \qquad (9.23b)$$

Eq. (9.23b) is written due to the fact that the work done to bring q_3 to point P_3 is the sum of the work needed when q_1 alone is present and the work needed when q_2 alone is present. So,

$$W_2 = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1 q_3}{r_{31}} + \frac{q_2 q_3}{r_{32}}\right)$$

So, the total work done in assembling this arrangement of three charges q_1, q_2 and q_3 is

$$W = W_1 + W_2 = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1q_2}{r_{21}} + \frac{q_1q_3}{r_{31}} + \frac{q_2q_3}{r_{32}}\right) \qquad (9.24)$$

The work done given by Eq. (9.24) is defined as electrostatic potential energy of the system.

We can now generalise the result contained in Eq. (9.24) to any number of charges. If we have N different charges in any configuration in space, the electrostatic potential energy of the system can be written as sum over all pairs. So, for a system of N charges $q_1, q_2, q_3, ..., q_N$, the electrostatic potential energy can be written as

$$P.E. = \frac{1}{2}\sum_{j=1}^{N}\sum_{\substack{k=1\\k\neq j}}^{N}\frac{q_jq_k}{4\pi\varepsilon_0\, r_{jk}} \qquad (9.25)$$

Note that the double summation notation. $\sum_{j=1}^{N}\sum_{\substack{k=1\\k\neq j}}^{N}$ implies that when we take $j = 1$, we need to sum over all values of k except 1; that is, we sum over $k = 2, 3, ..., N$; then we take $j = 1$ and sum over $k = 1, 3, 4, ..., N$ (leaving $k = 2$), and so on. So, we find that the double summation includes every pair twice and the factor of (1/2) has been included in Eq. (9.25) to correct this double counting.

In terms of electric potential V_j at the position of charge q_j, Eq. (9.25) may be written as

$$\boxed{P.E. = \frac{1}{2}\sum_{j=1}^{N} q_j V_j} \quad \text{where} \quad V_j = \sum_{\substack{k=1\\j\neq k}}^{N}\frac{q_k}{4\pi\varepsilon_0 r_{jk}} \qquad (9.26)$$

Eq. (9.26) implies that, for calculating the electrostatic potential energy for a group of point charges, one may consider each charge by turn, and the corresponding potential at its position due to all other charges except the one under consideration.

<u>Continuous Charge Distribution</u>

Since most of the charged real physical systems such as the plates of parallel plate capacitor are described as continuous charge distributions, you may like to know how to determine their electrostatic or electrical potential energy. To learn the method, take a simple example of adding point charges gradually, in steps, on an isolated conductor. In such a situation, the work done can be calculated as follows.

Let the charge on a conductor at a given time be q. Then, the potential V of this charged conductor is proportional to q. Thus, the work done δW in adding an additional charge δq on q (isolated conductor) is

$$\delta W = V\delta q$$

Further, we can write V as $V = kq$ where k is the constant of proportionality. Hence

$$\delta W = kq\,\delta q$$

As we go on adding more and more charges to this conductor, the total work done is the electrical potential energy of the charged body. The total work done can be calculated by integration (equivalent to summation). Thus, if Q is the final charge on the isolated conductor, then its electrical potential energy can be expressed as:

$$P.E. = \int_0^Q \delta W = \int_0^Q k\,q\,\delta q = k\left[\frac{q^2}{2}\right]_0^Q = k\frac{Q^2}{2} = \frac{Q}{2}V_f \qquad (9.27)$$

where $(V_f = kQ)$ is the final electric potential of the charged isolated conductor.

Eq. (9.27) gives the electrical potential energy of a charged conductor. We can write this expression in terms of charge density. For example, if in an infinitesimal volume $d\tau$, we assemble point charges such that the volume charge density is ρ and the electric potential is V then Eq. (9.27) for electrical potential energy can be written as

$$P.E. = \frac{1}{2}\int_{volume} \rho V d\tau \qquad (9.28)$$

Note that $\rho d\tau$ in Eq. (9.28) gives charge in the volume element $d\tau$ and when we integrate it over volume, we get Q, total charge on the conductor.

Similarly, for a charge distribution on a surface, if σ is the charge per unit area, then Eq. (9.28) takes the form

$$P.E. = \frac{1}{2}\int_{surface} \sigma V\ dS \qquad (9.29)$$

where dS is the element of surface area. And for a line charge distribution, if λ is the charge per unit length, then Eq. (9.27) for potential energy becomes

$$P.E. = \frac{1}{2}\int_{line} \lambda V\ dl \qquad (9.30)$$

where dl is line element.

Now, let us work out an example on electrical potential energy.

EXAMPLE 9.3 : ELECTRICAL POTENTIAL ENERGY

Three charges are arranged as shown in Fig. 9.12. Calculate the electrical potential energy of the system. Assume $q = 1.0\times10^{-5}$ C, and $d = 0.10$ m.

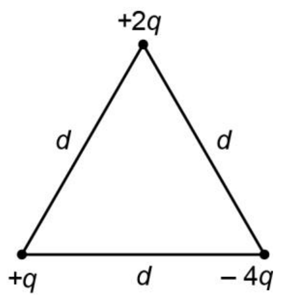


Fig. 9.12: **Diagram for Example 9.3.**

SOLUTION ■ The total electrical potential energy ($P.E.$) of the system is the algebraic sum of the electrical potential energies of all pair of charges, viz.,

$$P.E. = \frac{1}{4\pi\varepsilon_0}\left[\frac{(+q)\times(-4q)}{d} + \frac{(+q)\times(+2q)}{d} + \frac{(-4q)\times(+2q)}{d}\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{-10q^2}{d}\right] = \frac{-(9.0\times10^9\ \text{N m}^2\ \text{C}^{-2})\times(10)\times(1.0\times10^{-5}\ \text{C})^2}{0.10\ \text{m}}$$

$$= -90\ \text{J}$$

***SAQ 5* - Electrical potential energy**

With the help of a suitable diagram, estimate the number of terms that will contribute to the electrical potential energy for a system of five point charges.

We now sum up what you have learnt in this unit.

9.5 SUMMARY

Concept	Description
Potential due to infinite line charge	■ The potential difference between two points *a* and *b* on the line perpendicular to infinite line charge at distance r_a and r_b, respectively is: $$V_a - V_b = -\frac{\lambda \ln r_a}{2\pi\varepsilon_0} + \frac{\lambda \ln r_b}{2\pi\varepsilon_0}$$ If we assume that point *b* is at finite distance r_b and it is the reference point having zero electric potential (that is, $V_b = 0$), then $$V_a = -\frac{\lambda \ln r_a}{2\pi\varepsilon_0} + \text{const}$$
Potential due to uniformly charged spherical shell	■ At a point distant *r* from the centre and outside the spherical shell of radius *R*: $$V = \frac{Q}{4\pi\varepsilon_0 r}$$ where $Q = 4\pi R^2 \sigma$. At a point inside the shell: $$V = \frac{Q}{4\pi\varepsilon_0 R}$$
Potential due to uniformly charged non-conducting sphere	■ At an external point at distance *r* from the centre of the sphere: $$V = \frac{Q}{4\pi\varepsilon_0 r}$$ where $Q = (4\pi/3)R^3\rho$. At an internal point: $$V = \frac{Q}{8\pi\varepsilon_0 R^3}(3R^2 - r^2)$$ where *R* is the radius of the sphere and *r* (< *R*) is the distance of the internal point from the centre.
Equipotential surface	■ Equipotential surfaces are surfaces on which the potential at each point is same. The electric field $\vec{\mathbf{E}}$ is always directed perpendicular to an equipotential surface. It is always along the direction of the fastest decrease of the electric potential. No work is done in moving a charge between any two points on an equipotential surface. Equipotential surfaces are closer to each other in regions of strong electric field and are relatively far apart in regions of weak electric field.

Electrical potential energy

- The electrical potential energy is the energy stored in a system of charges. It is equal to the amount of work done in assembling the system together by bringing the charges from infinity.

  The electrical potential energy for a group of N discrete point charges is given as:

  $$P.E. = \frac{1}{2}\sum_{j=1}^{N} q_j V_j$$

  where V_j is the potential at the position of charge q_j due to all the charges except the charge q_j.

  The electrical potential energy of a charged conductor is

  $$P.E. = \frac{1}{2}\int_{\text{volume}} \rho V d\tau$$

  where ρ is volume charge density.

  The electrical potential energy of a charge distribution on a surface is

  $$P.E. = (1/2)\int_{\text{surface}} \sigma V dS$$

- The electrical potential energy of a line charge is

  $$P.E. = (1/2)\int_{\text{line}} \lambda V dl$$

9.6 TERMINAL QUESTIONS

1. If electric field $\vec{\mathbf{E}}$ equals zero at a given point, must V (electric potential) equal zero at that point? Give one example to justify your answer.

2. An infinite charged sheet has a surface charge density σ of 1.0×10^{-7} Cm^{-2}. How far apart are the equipotential surfaces whose potentials differ by 5.0 V?

3. A uniformly charged sphere has electric potential of 375 V on its surface. At a radial distance of 25 cm from the surface of the sphere, the electric potential is 125 V. Calculate the radius and charge on the sphere.

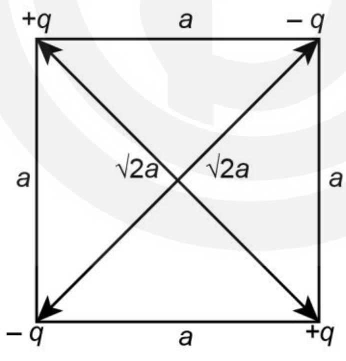


Fig. 9.13: **Diagram for TQ 4.**

4. Derive an expression for the work required to put the four charges together as indicated in Fig. 9.13.

5. Calculate the gain or loss of electrical potential energy when a droplet of radius R carrying a charge Q splits into two equal sized droplets of charge $Q/2$ and radius r. Assume that the droplets are repelled to a large distance compared to r because of electrostatic repulsion.

6. There are two charged conducting spheres of radii a and b. Suppose that they are connected by a conducting wire. What will happen? Using the result from this arrangement, explain why charge density on sharp and pointed ends of a conductor is higher than on its flatter portions.

7. Devise an arrangement of three point charges, separated by finite distances, that has zero potential energy.

9.7 SOLUTIONS AND ANSWERS

Self-Assessment Questions

1. We know that the potential difference between two points P and Q due to an infinite line charge is given as [Eq. (9.4)]:

$$V_P - V_Q = -\frac{\lambda}{2\pi\varepsilon_0}\ln(r_P / r_Q)$$

As per the problem, let point Q be at a distance of 5.0 m from the line charge where potential $V_Q = 0.$ So, the potential at point P located at 6.0 m can be written as

$$V_P = -\frac{2\lambda}{4\pi\varepsilon_0}\ln(r_P / r_Q) = -2\times(3.0\times10^{-16}\,\text{C m}^{-1})\times(9\times10^{9}\,\text{N m}^2\text{C}^{-2})\times\ln\left(\frac{6.0\,\text{m}}{5.0\,\text{m}}\right)$$

$$= -9.8\times10^{3}\,\text{V}$$

2. a) The point located at 40 cm from the centre of the shell where potential is to be calculated is an external point because radius is 20 cm. So, we can write

$$V = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r} = \frac{(9\times10^{9}\text{Nm}^2 C^{-2})\times4\pi\times(0.20\,\text{m})^2\times(3.0\times10^{-6}\,\text{C m}^{-2})}{(0.40\,\text{m})}$$

$$= 8.4\times10^{4}\,\text{V}$$

b) The point located at 15 cm from the centre is an internal point. For any such point, potential has a constant value given by

$$V = \frac{1}{4\pi\varepsilon_0}\frac{Q}{R} = \frac{(9\times10^{9}\,\text{N m}^2\text{C}^{-2})\times4\pi\times(0.20\,\text{m})^2\times(3.0\times10^{-6}\,\text{C m}^{-2})}{(0.20\,\text{m})}$$

$$= 6.7\times10^{4}\,\text{V}$$

3. Let Q be the charge on the aluminium sphere and n number of electrons have been removed to raise it to potential of 500 V. So, $Q = ne,$ where e is electronic charge. So, $n = (Q/e).$ Further, the potential of the sphere is given as

$$V = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r} = \frac{1}{4\pi\varepsilon_0}\frac{ne}{r}$$

So, $$n = \frac{(4\pi\varepsilon_0)Vr}{e} = \frac{(500\,\text{V})\times(.07\,\text{m})}{(9\times10^{9}\text{N m}^2 C^{-2})\times(1.6\times10^{-19}\text{C})} = 2.4\times10^{8}$$

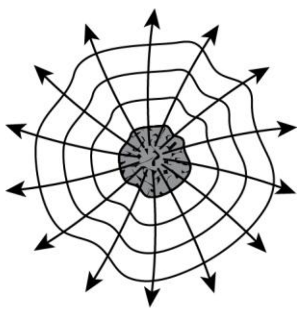


Fig. 9.14: **Diagram for answer to SAQ 4b.**

4. a) i) Equipotentials are always perpendicular to the electric field lines.

ii) Separation between the equipotentials depends on the strength of the electric field.

b) Electric field lines for the charged metal object are shown in Fig. 9.14.

5. The diagram for a system of five charges is shown in Fig. 9.15. Since each pair of charge has a potential energy and there are 10 pairs between 5 point charges, 10 terms would be contributing to the potential energy of 5 charges.

   (Rule: If there are n charges, the number of terms (pairs) contributing to the potential energy is $\frac{n(n-1)}{2}$).

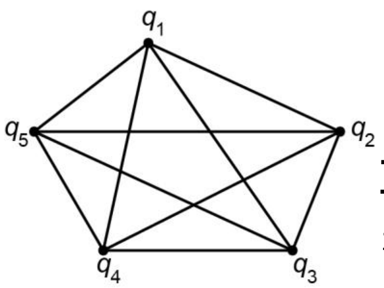


Fig. 9.15: **Diagram for answer to SAQ 5**.

Terminal Questions

1. We know that the electric field is related to potential as $\vec{\mathbf{E}} = -(dV/dx)$. Thus, if $|\vec{\mathbf{E}}| = 0$, electric potential has to be a constant. It is not necessary that V be equal to zero when $|\vec{\mathbf{E}}| = 0$. Consider, for example two identical charges separated by a distance $2a$. At the mid-point between the charges,

$$|\vec{\mathbf{E}}| = 0, \text{ but } V = \frac{1}{2\pi\varepsilon_0}\frac{q}{a}$$

2. The magnitude of electric field near an infinite charged sheet is given by (see Unit 6):

$$|\vec{\mathbf{E}}| = \frac{\sigma}{2\varepsilon_0}$$

   where σ is the surface charge density. Therefore, for the problem under consideration,

$$|\vec{\mathbf{E}}| = \frac{1.0 \times 10^{-7}\ \text{Cm}^{-2}}{2 \times 8.9 \times 10^{-12}\ \text{C}^2\,\text{N}^{-1}\text{m}^{-2}} = 5.6 \times 10^3\ \text{NC}^{-1}$$

   The spacing Δl between the equipotential surface is given by $\Delta l = (\Delta V / E)$ where ΔV is the potential difference between the adjacent surfaces. With $\Delta V = 5.0\,\text{V}$, we have

$$\Delta l = \frac{5.0\,\text{V}}{5.6 \times 10^3\ \text{NC}^{-1}} = 0.89 \times 10^{-3}\,\text{m} = 0.89\ \text{mm}$$

3. The potential of a charged sphere is

$$V = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r} \quad \Rightarrow \quad 375\ \text{V} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r} \qquad \text{(i)}$$

   Also, as per the problem

$$125\ \text{V} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{(4 + 0.25\,\text{m})} \qquad \text{(ii)}$$

   From Eqs. (i) and (ii)

$$\frac{375\ \text{V}}{125\ \text{V}} = \frac{(r + 0.25\,\text{m})}{r}$$

$$3r - r = 0.25\,\text{m} \quad \Rightarrow \quad r = 0.13\,\text{m}$$

And, total charge on the sphere is

$$Q = V(4\pi\varepsilon_0)r = \frac{375\text{ V} \times 0.13\text{ m}}{9 \times 10^9\text{ Nm}^2\text{ C}^{-2}} = 5.2 \times 10^{-9}\text{C}$$

4. The work required to assemble four charges together as shown in Fig. 9.13 is equal to the electric potential energy of the system. The electrical potential energy of the system may be obtained by considering the charges in pairs:

$$= \frac{1}{4\pi\,\varepsilon_0}\left[\frac{(q)(-q)}{a} + \frac{(-q)\,(q)}{a} + \frac{(-q)(q)}{a} + \frac{(-q)(q)}{a}\right] + \left[\frac{(-q)(-q)}{\sqrt{2}\,a} + \frac{(q)(q)}{\sqrt{2}\,a}\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{-4q^2}{a} + \frac{2q^2}{\sqrt{2}\,a}\right] = \frac{1}{4\pi\varepsilon_0}(-4 + \sqrt{2})\frac{q^2}{a}$$

5. Total volume of 2 droplets after splitting $= 2 \times (4\pi/3)\,r^3$. Volume of the original droplet $= (4\pi/3)\,R^3$. Since volumes have to be equal, we have

$$2 \times (4\pi/3)\,r^3 = (4\pi/3)R^3 \quad \Rightarrow r = (1/2)^{1/3}R \qquad \text{(i)}$$

Electrical potential energy (P.E.) of the original droplet with charge Q is

$$P.E. = \frac{1}{2}QV = \frac{1}{2}Q\frac{Q}{4\pi\varepsilon_0 R} = \frac{Q^2}{8\pi\varepsilon_0 R} \qquad \text{(ii)}$$

Total electrical potential energy of 2 droplets after splitting is

$$(P.E.)_{\text{split}} = 2 \times \frac{1}{2}\,\frac{Q}{2}\,\frac{Q/2}{4\pi\varepsilon_0 r} = \frac{Q^2/2}{8\pi\varepsilon_0 r}$$

Using Eq. (i), we have

$$(P.E.)_{\text{split}} = \frac{Q^2}{8\pi\varepsilon_0 R}\left(\frac{1}{2}\right)^{2/3} \qquad \text{(iii)}$$

Thus, the loss in electrical potential energy after splitting is

$$\frac{Q^2}{8\pi\varepsilon_0 R}\left[1 - \frac{1}{(2)^{2/3}}\right]$$

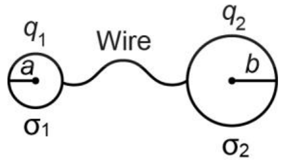


Fig. 9.16: **Diagram for answer to TQ 6.**

6. When two charged conducting spheres are connected by a wire as shown in Fig. 9.16, the charges redistribute themselves till both spheres are at the same potential, i.e.,

$$V = \frac{1}{4\pi\varepsilon_0}\,\frac{q_1}{a} = \frac{1}{4\pi\varepsilon_0}\,\frac{q_2}{b}$$

where q_1 and q_2 are charges on spheres of radii a and b respectively. This gives

$$\frac{q_1}{q_2} = \frac{a}{b} \qquad \text{(i)}$$

The surface charge densities σ_1 and σ_2 on these spheres are:

$$\sigma_1 = \frac{q_1}{4\pi a^2} \quad \text{and} \quad \sigma_2 = \frac{q_2}{4\pi b^2}$$

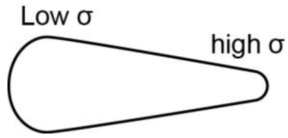


Fig. 9.17: **Diagram for the answer of TQ 6.**

Thus, we can write

$$\frac{\sigma_1}{\sigma_2} = \frac{q_1}{q_2} \times \frac{b^2}{a^2} \qquad \text{(ii)}$$

Combining Eqs. (i) and (ii), we get

$$\frac{\sigma_1}{\sigma_2} = \frac{a}{b} \times \frac{b^2}{a^2} = \frac{b}{a}$$

Thus, we see that the surface charge densities of conducting spheres are inversely proportional to their radii. For sharp and pointed ends, the radii are small, resulting in high surface charge densities. For flatter ends, the radii are larger which result in smaller surface charge densities. See Fig. 9.17.

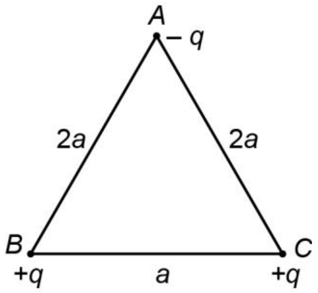


Fig. 9.18: **Diagram for the answer to TQ 7.**

7. If we devise an arrangement as shown in Fig. 9.18, the electrical potential energy (P.E.) turns out to be zero because the P.E. of the arrangement is:

$$P.E. = \frac{1}{4\pi\varepsilon_0}\left[\frac{(-q)\,q}{2a} + \frac{(-q)\,(q)}{2a} + \frac{(q)\,(q)}{a}\right] = 0$$