Indira Gandhi National
Open University
School of Sciences

BPHCT-133
ELECTRICITY AND MAGNETISM

Block

3

ELECTROSTATICS IN MEDIUM AND MAGNETISM

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Dielectric material is used in a transformer (shown above) to withstand high voltages present within it as it is an insulator, and dissipate heat generated in the transformer windings. You will learn about dielectrics in this unit. (Picture source: Wikimedia Commons)

UNIT 10

MACROSCOPIC PROPERTIES OF DIELECTRICS

Structure

STUDY GUIDE

In this unit, you will study the effect of electric field on a dielectric (or insulator). Before studying this unit, you should refresh your knowledge about electrical conductors and insulators from your school physics. Dielectric materials are characterised by the fact that there are no free charges in them to move around under the influence of electric field. To understand the mathematical treatment given in this unit, you should revise vector analysis given in Block 1 of this course. Also, we advise you that you should work out the mathematical derivations while you are studying this unit. Passive reading of the mathematical derivations will not be of much help. We also advise you to try to solve SAQs and TQs yourself. This will help you understand and be more familiar with mathematical derivations involving vector calculus.

"Science is a way of thinking much more than it is a body of knowledge."

Carl Sagan

10.1 INTRODUCTION

In Block 2, you have studied electrostatics in free space. You have learnt the concepts of electrostatic force, electric field, electric potential and electrostatic energy when charges are placed in vacuum. However, in real life, we mostly have situations in which electric phenomena take place in matter. Matter, as you know, can be in any form: solid, liquid or gas. Different kinds of matter behave differently in the electric field. From school physics, you know that on the basis of their electrical properties, we can broadly classify most materials around us into two categories: **conductors** and **insulators**. Insulators are also called **dielectrics**. In this unit, you will study how dielectric materials behave in the presence of electric fields and learn how Gauss's law is modified in a dielectric medium.

You may like to know: **Why do we need to study about macroscopic properties of dielectrics?** This is what we shall explain in the beginning of this unit when we introduce dielectrics (Sec. 10.2). In Sec. 10.3, we shall use a simple model of dielectric materials to explain **what happens when a dielectric is placed in an external electric field**. You will learn that this results in the phenomenon of **polarisation of dielectrics**. We shall define electric polarisation $\vec{\mathbf{P}}$ and introduce the displacement vector $\vec{\mathbf{D}}$ to determine the electric field in a dielectric material. In Sec. 10.4, we shall obtain the electric field due to a polarised object and explain its physical meaning. Finally, in Sec. 10.5, we shall deduce the electrostatic equations or Gauss's law in a dielectric medium in terms of $\vec{\mathbf{D}}$.

In Unit 11, you will study about capacitors in detail, wherein dielectric materials find major applications.

Expected Learning Outcomes

After studying this unit, you should be able to:

- ❖ explain the behaviour of dielectrics in an electric field;
- ❖ define electric polarisation and explain the mechanism of polarisation in polar and non-polar dielectrics;
- ❖ define displacement vector $\vec{\mathbf{D}}$ and deduce Gauss's law in a dielectric medium;
- ❖ relate $\vec{\mathbf{D}}$ to the electric field $\vec{\mathbf{E}}$; and
- ❖ define dielectric constant.

10.2 DIELECTRICS

Dielectrics (or insulators) are an important class of materials used in a variety of applications such as in electrical insulation, capacitors, radio frequency transmission lines, printed circuit boards, etc. The study of dielectrics helps us understand how a proper dielectric is chosen for a capacitor, as well as many optical phenomena such as reflection, refraction and double refraction in quartz or calcite crystals. Natural rubber, cotton, wood are some examples of good electrical insulators. Paper, mica, glass and a large number of plastics are good dielectrics which are used in capacitors. You may have studied in your school physics that **dielectrics are used in capacitors to increase their capacitance manifold**. **Why does this happen**? Since the physics of capacitors is well known to you from School, we would like to explain why you need to learn about dielectrics by answering this question.

Fig. 10.1: **Spherical capacitors used by Faraday. Faraday showed that when dielectric material was placed between the central brass ball and a concentric brass shell, the capacitance of the spherical capacitor increased manifold. The factor by which it increased was different for different dielectric materials.** (Source: Collectionsonline.nmsi.ac.uk)

In fact, this effect was demonstrated in 1837 by Faraday. Faraday repeated independently the experiments performed by Cavendish in about 1770. He showed that when a slab of dielectric material (such as glass or mica) was introduced between a central ball and a concentric brass shell of a spherical capacitor (see Fig. 10.1), its capacitance **increased** manifold (by a factor called the **dielectric constant**). The value of this factor is 1 for vacuum and greater than 1 for various dielectrics. The dielectric constants for a few materials are given in Table 10.1.

Table 10.1: Dielectric constants of some common materials.

Material	Dielectric constant
Air	1.0006
Mica	5 – 9
Glass	4.5 – 7.00
Paper	2 – 2.3
Water	80.4

The dielectric constant is one of the important macroscopic electrical properties of a dielectric material and its value varies widely for different dielectrics. For example, for water, it is 80.4 and for different types of glass, it is around 6. So the capacitance increases according to the dielectric being used in it. The choice of a dielectric in a capacitor depends on the application for which it is to be designed. You will learn more about this aspect of capacitors in Unit 11.

For now, we are interested in knowing: **How do we explain the increase in the capacitance of a capacitor when a dielectric is placed between two conductors**?

To understand this phenomenon, let us consider a parallel plate capacitor with some free charge Q on its conducting plates (Fig. 10.2). Let us assume a negative charge on the upper plate and a positive charge on the lower plate. If A is the area of the plates and d the distance between them, you know from school physics that the capacitance of the capacitor is given by:

$$C = \varepsilon_0 \frac{A}{d} \qquad (10.1)$$

and a charge Q on the plates results in a potential difference given by:

$$V = \frac{Q}{C} \qquad (10.2)$$

Now it is an **experimental fact that if we put a dielectric slab between the plates, we find that the capacitance increases**. **How is that possible**?

From Eq. (10.2), you can see that an increase in capacitance means that the potential difference between the plates decreases. How does this happen? We can try and understand this using the concepts you have studied in Block 2.

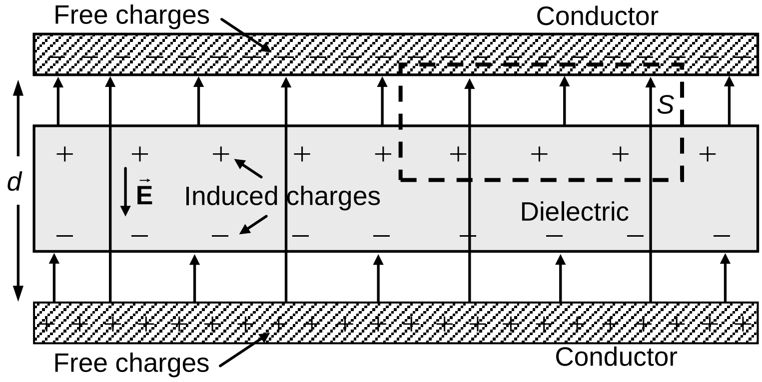


Fig. 10.2: **A parallel plate capacitor with a dielectric material inserted between its plates**.

Study Fig. 10.2. Consider a Gaussian surface *S* (a rectangular box lying partially inside the dielectric material and partially inside the conducting plate) as shown in the figure. Recall from Unit 6 that Gauss's law tells us that the electric flux out of the surface is related to the enclosed charge q_{en} :

$$\oint_S \vec{\mathbf{E}}.d\vec{\mathbf{S}} = \frac{q_{en}}{\varepsilon_0}$$

Since the capacitance of the capacitor is found to increase, from Eq. (10.2) this means that the potential difference or the voltage between the plates is decreased. You know that **potential difference or voltage is proportional to the electric field**. Therefore, when capacitance increases, the electric field must decrease. From Gauss's law, this, in turn, implies that the charge should decrease. This can happen only if (somehow) **a positive charge has appeared on the upper surface of the dielectric**. This positive charge, of course, has to be smaller than the negative charge placed on the plate of the capacitor.

Thus, we can explain the increase in capacitance due to a dielectric material only if we can understand: How is a positive charge induced on one surface of a dielectric material and a negative charge on the other surface when it is placed in an electric field? This requires an understanding of the behaviour of dielectric materials in an electric field, which we now discuss. In doing so, we arrive at an understanding of polarisation of dielectrics in electric fields.

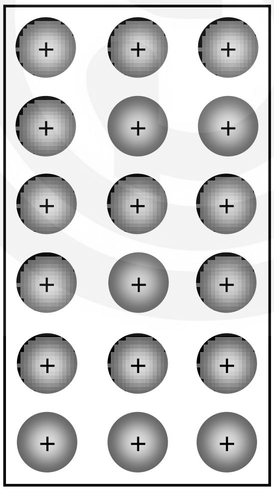


Fig. 10.3: **Model of a dielectric made up of atoms**. **The charges in the dielectric are not free to move around**; **they are bound to the atoms. The + sign depicts the positive nucleus and the grey sphere represents the negatively charged electron cloud**.

10.3 DIELECTRIC MATERIALS IN ELECTRIC FIELDS: POLARISATION

From school physics, you know very well the behaviour of conductors which have free electrons to conduct electricity. In Block 2, you have learnt that the electric field inside a conductor is zero and the total charge of a conductor resides on its surface. The electric field due to this charge is normal to the surface of a conductor. Also the surface of a conductor is an equipotential surface.

What can you say about dielectric materials? We use the following simple model for describing dielectric materials that you may have learnt in school (see Fig. 10.3):

- Like all matter, dielectrics are made up of a large number of atoms and molecules.
- Every atom consists of positively charged nucleus and negatively charged electron cloud distributed around it.
- The total positive charge is equal to the total negative charge so that the atom/molecule is electrically neutral.
- In contrast to conductors, the charges attached to atoms and molecules are not free to move around in dielectric materials: at most they can move *within* the atom/molecule. The charges are bound in the atoms and molecules of a dielectric.
- A molecule may be made of atoms of similar kind or of a different kind.

We will now understand the behaviour of dielectrics in electric fields with the help of this model. Let us ask: **What happens when we put a dielectric material in an external electric field?**

Since the dielectric is made up of atoms and molecules as described above, to answer this question, we need to first understand: **What happens to a neutral atom or molecule of the dielectric when it is placed in an external electric field?** Your first reaction could be that nothing would happen since the atoms and molecules are charge neutral. But this is not correct. The behaviour of a dielectric in an electric field depends on whether it is made up of **neutral atoms/non-polar molecules** or **polar molecules**.

Broadly there are two types of dielectrics: **non-polar dielectrics** made up of **neutral atoms** or **non-polar molecules** and **polar dielectrics** made up of **polar molecules**. You will learn about non-polar and polar molecules in the next two sections.

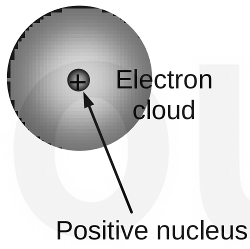
There are two mechanisms by which an external electric field affects the charge distribution in a dielectric:

1. By **inducing dipoles** in neutral atoms or non-polar molecules in a non-polar dielectric.
2. By **aligning the permanent dipoles** of the polar molecules in a polar dielectric.

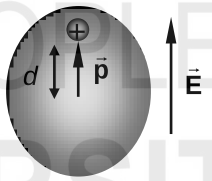
We now explain both these mechanisms.

10.3.1 Induced Dipoles in Neutral Atoms and Non-polar Molecules

Let us first ask: When a dielectric is placed in an electric field $\vec{E}$, **what happens to a neutral atom in it**? You may think that nothing will happen because the atom is electrically neutral. But this is not true. To find out what happens, let us consider the following crude model of an atom (Fig. 10.4a):



(a)



(b)

Fig. 10.4: **a) An atom with a positively charged nucleus and a cloud of negatively charged electrons such that the centres of the positive and negative charge coincide**; **b) In the presence of an electric field, the centres of positive and negative charge in the atom no longer coincide. These are separated and an induced dipole appears. The figure, of course, is not to scale and has been magnified by many orders of magnitude.**

- the positive nucleus is present at the centre of the atom,
- the negatively charged electrons are distributed in a spherical cloud about the centre,
- the centre of positive charges and the centre of negative charges coincide, and
- the atom is electrically neutral and also has no electric dipole moment.

Now when the neutral atom is placed in an external electric field, the two regions of positive and negative charges within the atom are influenced by the electric field: the positive nucleus is displaced in the direction of the field and the negatively charged electrons in the atom are displaced in the opposite direction. But the positive and negative charges also attract each other, and the atom is held together. If the electric field is very large, it ionizes the atom.

If the external electric field is not very large, equilibrium is soon established. The two opposing forces, one due to the external electric field pushing the positive nucleus and the electron cloud apart, and the other due to their mutual electrostatic attraction pulling them closer, reach a balance. When this happens, the centre of positive charge is shifted slightly in one direction and the centre of negative charge is shifted in the opposite direction. This results in a **small separation of the centres of positive and negative charges and an induced dipole appears** (Fig. 10.4b). Note that the electron cloud is distorted by the external electric field.

Thus, a dipole moment is induced in the neutral atom in the presence of an external electric field $\vec{\mathbf{E}}$. The dipole moment is in the same direction as the electric field. Let us determine the expression of the induced dipole moment. Suppose *d* is the distance between the centres of positive and negative charges in the atom. Then the dipole moment of the atom is given by:

$$\vec{\mathbf{p}} = q\,d\,\hat{\mathbf{n}} = q\,\vec{\mathbf{d}} \qquad (10.3)$$

where $\hat{\mathbf{n}}$ is a unit vector in the direction of $\vec{\mathbf{d}}$ and points in the same direction as $\vec{\mathbf{E}}$. Typically, the **displacement $\vec{\mathbf{d}}$ is proportional to the external electric field** unless the fields are very large. (In that case, it could even result in the ionization of the atom.) Since $\vec{\mathbf{d}} \propto \vec{\mathbf{E}}$ (as long as $\vec{\mathbf{E}}$ is not too large) and $\vec{\mathbf{p}} = q\,\vec{\mathbf{d}}$, the **induced dipole moment is proportional to the electric field**:

$$\vec{\mathbf{p}} \propto \vec{\mathbf{E}} \quad \text{and} \quad \vec{\mathbf{p}} = \alpha\vec{\mathbf{E}} \qquad (10.4)$$

The constant of proportionality α is called the **atomic polarisability** and its unit is $\text{C}^2\,\text{mN}^{-1}$. Its value depends on the structure of the atom. Let us take up an example to estimate the atomic polarisability of an atom using Eq. (10.4).

EXAMPLE 10.1: ATOMIC POLARISABILITY

Consider an atom of radius *a* in the presence of an external electric field *E* (Fig. 10.5). Calculate the separation between the positive nucleus and the centre of the negatively charged electron cloud and its dipole moment. Calculate its atomic polarisability and estimate its value for the hydrogen atom of radius $a_0 \approx 10^{-10}$ m. Take the value of *E* to be 10^6 V m^{-1}.

SOLUTION ■ You have learnt that in the presence of an external electric field, the positive nucleus is pulled opposite to the centre of negative charge. For keeping the calculation simple, we assume that at equilibrium, the negative charge cloud keeps its spherical shape and is merely displaced by an amount *b* with respect to the positive nucleus (see Fig. 10.5b). At equilibrium, the force on the nucleus due to the external electric field $\vec{\mathbf{E}}$ is balanced by the attractive force due to the negative charge cloud. You know from Unit 5 that the force on the nucleus due to the external electric field is $+q\vec{\mathbf{E}}$, where *q* is the charge on the nucleus.

Now, let us calculate the value of the electric field E_e due to electron cloud at the new location of the nucleus. To do so, recall from Unit 6 that the electric field due to a uniformly charged non-conducting sphere at an internal point is given as:

$$E_e = \frac{1}{4\pi\varepsilon_0}\frac{qb}{a^3}$$

where *b* is the distance between the centre of the electron cloud and the nucleus, *q* is the magnitude of the total charge of the electron cloud and *a* is the radius of the uniformly charged spherical electron cloud.

Thus, at equilibrium, we can write the magnitude of the external electric field as

$$E = E_e$$

or $$E = \frac{1}{4\pi\varepsilon_0}\frac{qb}{a^3}$$

or $$b = \frac{4\pi\varepsilon_0 a^3 E}{q}$$

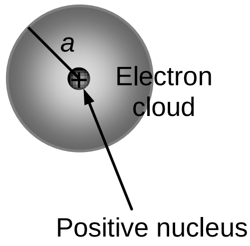
From Eq. (10.3), the dipole moment $p = qb = 4\pi\varepsilon_0 a^3 E$.

Therefore, from Eq. (10.4), the atomic polarisability is given by:

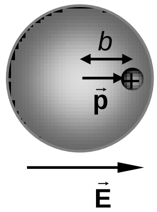
$$\alpha = \frac{p}{E} = 4\pi\varepsilon_0 a^3$$

For the hydrogen atom, $a = a_0 \approx 10^{-10}$m. Let us estimate the atomic polarisability of the hydrogen atom based on this crude model. It is:

$$\alpha = 4\pi \times 8.85 \times 10^{-12} \times 10^{-30}\, \text{C}^2\text{mN}^{-1} = 1.11 \times 10^{-40}\, \text{C}^2\text{mN}^{-1}$$



(a)



(b)

Fig. 10.5: **a) A neutral atom of radius *a*; b) A dipole is induced in the atom in the presence of an electric field. We assume that at equilibrium, the electron cloud is spherical in shape.**

The separation b in the presence of a modest electric field $E = 10^6\ \text{Vm}^{-1}$ is

$$b \approx \frac{4\pi \times 8.85 \times 10^{-12} \times 10^{-30} \times 10^6}{1.6 \times 10^{-19}}\ \text{m} \approx 6.9 \times 10^{-15}\ \text{m}$$

Although this atomic model is very crude, the estimated value of the atomic polarisability given by Eq. (10.4) based on this model is not too bad. It is accurate to within a factor of four for simple atoms. Compare the value of α obtained from this crude model with its experimental value for the hydrogen atom. In units of $10^{-30}\ \text{m}^3$, the experimental value of $\frac{\alpha}{4\pi\varepsilon_0}$ for the hydrogen atom is 0.667.

So far we have seen how an electric dipole moment is induced in a neutral atom in the presence of an external electric field. What happens in the case of a molecule? Is the situation the same or different? Let us find out.

In one type of molecules called **non-polar** molecules, the centres of positive and negative charges always coincide. Such molecules have zero dipole moment in the absence of external electric fields. The dielectrics made up of such molecules are called **non-polar dielectrics**. Some examples of non-polar molecules are air, hydrogen, oxygen (Fig. 10.6), benzene, carbon tetrachloride, etc.

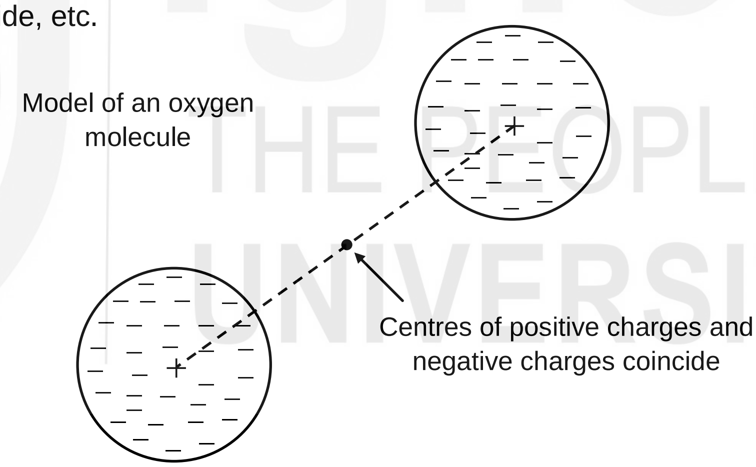


Fig. 10.6: **In an oxygen molecule, the centres of positive and negative charges coincide and it has zero dipole moment in the absence of an external electric field**.

So the non-polar molecules do not possess any permanent dipole moment (their dipole moment is zero in the absence of an external electric field). Hence, you can immediately say that their behaviour in the presence of an external electric field should be the same as that of a neutral atom.

Thus, when a neutral atom or non-polar molecule is placed in an external electric field, it acquires (by induction) a tiny dipole moment in the direction of the electric field.

10.3.2 Alignment of Polar Molecules in Electric Fields

Some molecules are so made that the centres of positive and negative charges do not coincide. Such molecules, e.g., water and glass, are electrically neutral but have electric dipole moments even in the absence of external electric fields and are called **polar molecules**. We say that polar molecules have permanent dipole moments. Dielectrics made up of such molecules are called polar dielectrics.

For example, study the simple structure of the hydrogen chloride (HCl) molecule shown in Fig. 10.7. It is a diatomic molecule made up of dissimilar atoms. Originally, the H and Cl atoms are spherical. When the HCl molecule is formed from these atoms, the electron of the H atom shifts partially over to the Cl structure, leaving the positive hydrogen nucleus behind. Thus, there is an excess of negative charge at the chlorine end and an excess of positive charge at the hydrogen end of the molecule. This separation of the centres of positive and negative charge gives rise to a permanent dipole moment in the HCl molecule.

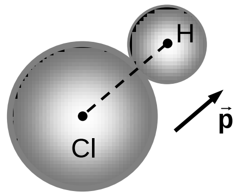


Fig. 10.7: **The hydrogen chloride molecule in which the positive and negative charge centres do not coincide and the molecule has a permanent electric dipole moment.**

You have just learnt that **polar molecules** have permanent electric dipoles. If these molecules are placed in an **external electric field**, **the force $\vec{F}_+$ on the positive charge will exactly cancel the force $\vec{F}_-$ on the negative charge**. But notice from Fig. 10.8 that these forces form a **couple**. Therefore, each electric dipole would experience a **torque** that will tend **to align it along the electric field**.

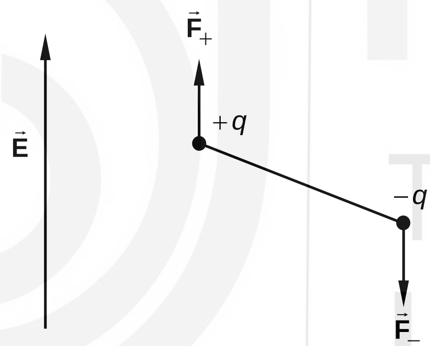


Fig. 10.8: **A polar molecule experiences a torque arising due to forces on the separated positive and negative charges in the presence of external electric field**. **Due to this torque**, **the molecule tends to orient itself in the direction of the field**.

In Secs. 10.3.1 and 10.3.2, you have learnt how neutral atoms/non-polar molecules and polar molecules behave when they are placed in an external electric field. We can now answer the question: **What happens when a dielectric material is placed in an external electric field**?

10.3.3 Polarisation of Dielectrics and Polarisation Vector $\vec{P}$

You have learnt that there are two types of dielectrics, non-polar dielectrics made up of neutral atoms/non-polar molecules and polar dielectrics made up of polar molecules. So, we need to actually understand: **How do the polar and non-polar dielectrics behave in the presence of an external electric field**? On the basis of what you have learnt in Secs. 10.3.1 and 10.3.2, we can summarise the answer as follows:

1. Non-polar dielectrics: When a non-polar dielectric, which is made up of neutral atoms or non-polar molecules is placed in an external electric field, the atoms/non-polar molecules of the dielectric acquire (by induction) a tiny dipole moment in the direction of the electric field.

2. Polar dielectrics: When a polar dielectric, which is made up of polar molecules (having permanent dipole moments) is placed in an external electric field, the permanent dipole moments of the polar molecules experience a torque tending to align them in the direction of the electric field.

Thus, both mechanisms (**induction of dipoles** in neutral atoms/non-polar molecules and **alignment of permanent dipoles** in polar molecules) produce **atomic/molecular dipoles in the dielectric pointing along the direction of the field**. We say that the dielectric material is **polarised**. The dielectric as a whole remains electrically neutral and inside the dielectric slab there is no excess charge in any volume element.

However, the atoms/molecules are constantly in random thermal motion and collide with each other. Therefore, alignment of the electric dipoles is not complete and increases as the electric field is increased or the temperature decreases. The random motion of the molecules tends to destroy the alignment of the dipoles at higher temperature, and particularly when the electric field is removed. Thus, to sum up, qualitatively we can say that

Recap

The polarisation of a dielectric due to an applied external field results from

- The induction of dipole moments due to relative displacement of the centres of negative and positive charges in neutral atoms/non-polar molecules in a non polar dielectric

or

- The alignment of permanent dipoles (in polar molecules) in polar dielectric material.

In a homogeneous dielectric, its properties e.g., permittivity and susceptibility, are the same at all points in the dielectric, i.e., they do not vary with position. In an isotropic dielectric, its properties are the same in all directions.

We now give a quantitative definition of the polarisation of a dielectric. Let us consider a **homogeneous** and **isotropic dielectric slab**. This means that **the properties of the dielectric are the same at all points and in all directions**. Let the dielectric slab be placed in an external electric field $\vec{\mathbf{E}}_0$. The external electric field could be applied by any means, e.g., due to charges on the plates of a parallel plate capacitor as shown in Fig. 10.9a. If the dielectric material is made up of neutral atoms/non-polar molecules, **a dipole moment will be induced in each atom or molecule** of the dielectric. If the dielectric material is made up of polar molecules, **each permanent dipole would experience a torque tending to align it along the electric field**. You have learnt in Secs. 10.3.1 and 10.3.2 that the direction of the dipole moments in either case will be the same as that of the electric field.

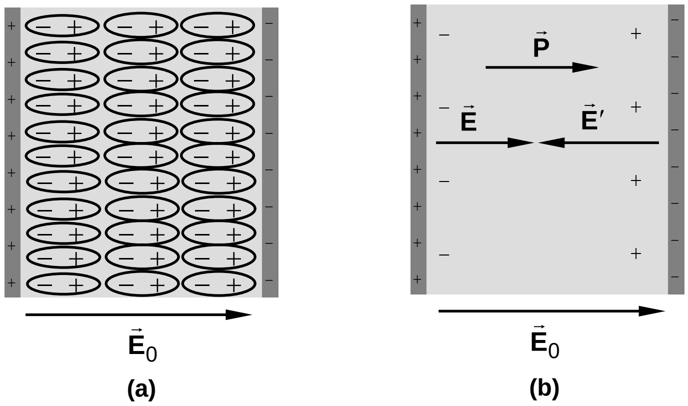


Fig. 10.9: **a) In a dielectric slab placed in an external electric field $\vec{E}_0$, the centres of positive and negative charges are separated and it gets polarised**; **b) the separation of charges produces surface charges on the slab faces**, **which set up a field $\vec{E}'$ opposite to $\vec{E}_0$. This is an idealised picture**. **In reality**, **the atoms/molecules in the dielectric are in random motion**.

The separation of the centres of positive and negative charges produces surface charges on the faces of the dielectric slab as shown in Fig. 10.9b. The surface charges on the dielectric faces produce an electric field, say $\vec{E}'$ in the direction opposite to the external electric field. The resultant electric field $\vec{E}$ inside the dielectric is given by the vector sum of the electric fields $\vec{E}_0$ and $\vec{E}'$. It is in the same direction as $\vec{E}_0$ but smaller in magnitude.

To describe this phenomenon mathematically, we define the **polarisation** $\vec{P}$ as the **total dipole moment per unit volume**:

$$\vec{P} = \text{Dipole moment per unit volume}$$

Defined in this manner, polarisation is simply the mean dipole moment **averaged** over a large volume that contains a very large number of atoms/molecules. It is thus an **average macroscopic property** of the dielectric, which is a large scale manifestation of the electric dipole moments of the atoms and molecules the dielectric is made up of. If there are N polarised molecules per unit volume in the dielectric, we have

$$\vec{P} = N\vec{p} \tag{10.5}$$

For an ideal, homogeneous and isotropic dielectric, polarisation $\vec{P}$ is proportional to the electric field $\vec{E}$ in the dielectric and we can write

$$\vec{P} \propto \vec{E} \qquad \text{or} \qquad \vec{P} = \chi\, \varepsilon_0 \vec{E} \tag{10.6}$$

The constant of proportionality χ in Eq. (10.6) is called the **electric susceptibility**. The constant ε_0 appears in Eq. (10.6) so that χ is dimensionless. Dielectric materials that satisfy Eq. (10.6) are called **linear dielectrics**.

Eq. (10.6) is found to be experimentally true for many substances, provided that the electric field is not too strong. Eq. (10.6) tells us that the susceptibility of a dielectric provides a measure of the extent to which it can be polarised when it is kept in an external electric field. The susceptibility χ of a dielectric depends on the microscopic structure of the material and also on external factors such as temperature. Note that in Eq. (10.6), $\vec{\mathbf{E}}$ is the net electric field in the dielectric. It is due to both free charges and the polarisation of the dielectric. **So if we put a dielectric material in an external electric field $\vec{\mathbf{E}}_0$, we cannot calculate $\vec{\mathbf{P}}$ directly from Eq. (10.6).**

This is because the external electric field will polarise the material; the resulting polarisation will produce its own electric field. This contributes to the net electric field, which gets modified. The modified electric field again modifies polarisation and this process continues. Thus, in reality, the phenomenon of polarisation of a dielectric is far more complex and we shall not go into the details here. For the time being, we are interested in knowing: **What field does a polarised dielectric itself produce**? This is what you will learn in the next section. But you may like to work out a simple SAQ before studying further.

SAQ 1 - **Unit of polarisation vector**

Determine the unit of $\vec{\mathbf{P}}$.

10.4 ELECTRIC FIELD OF A POLARISED OBJECT

Consider a polarised dielectric object, which contains a large number of atomic/molecular dipoles aligned in the direction of the applied electric field. Let the dipole moment per unit volume of this material be given by $\vec{\mathbf{P}}$. We now ask: **What is the electric field produced by this object at a given point**? Remember that from Unit 5 we know the electric field of an individual dipole at a given point.

So to find the answer, we divide the material into a large number of such infinitesimal dipoles and integrate their electric fields to get the net electric field of the object. This is a standard method in physics about which you have studied in Block 1 of this course. And you may know that the problem is easier to solve for the electric potential since it is a scalar. Then we can obtain the electric field from its expression.

So we consider a small volume element $d\tau$ of this material which has a dipole moment $\vec{\mathbf{P}}d\tau$. We first calculate the electric potential due to this dipole element at the given point. The total electric potential is then obtained by integrating over the entire material. You know from Unit 8 that for a single dipole $\vec{\mathbf{p}}$, the **electric potential** produced at a point $\vec{\mathbf{r}}$ from the dipole is given by:

$$V = \frac{1}{4\pi\varepsilon_0}\frac{\hat{\mathbf{r}}\,.\,\vec{\mathbf{p}}}{r^2} \qquad (10.7)$$

Study Fig. 10.10. It shows a volume element $d\tau'$ of the dielectric material. It is situated at $\vec{\mathbf{r}}'(x', y', z')$ and has dipole moment $\vec{\mathbf{P}}\, d\tau'$. The electric potential dV due to this volume element at the point P (x, y, z) is given by:

$$dV = \frac{1}{4\pi\varepsilon_0}\frac{\vec{\mathbf{p}}.\hat{\mathbf{r}}_1}{r_1^2} = \frac{1}{4\pi\varepsilon_0}\frac{\vec{\mathbf{P}}.\hat{\mathbf{r}}_1}{r_1^2}d\tau' \quad (\because\ \vec{\mathbf{p}} = \vec{\mathbf{P}}\, d\tau') \tag{10.8}$$

where $$\vec{\mathbf{r}}_1 = \vec{\mathbf{r}} - \vec{\mathbf{r}}' \tag{10.9}$$

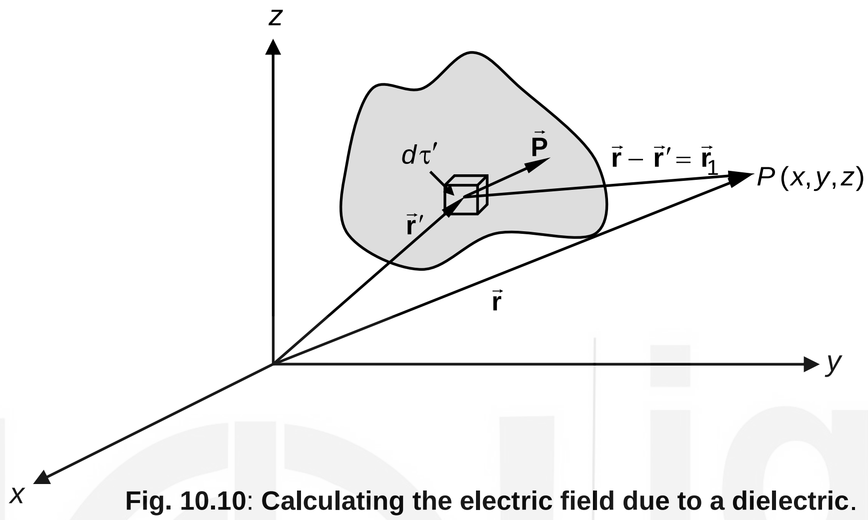


Fig. 10.10: **Calculating the electric field due to a dielectric**.

Integrating over the entire volume τ of the material, we get the **total electric potential** as

$$V(\vec{\mathrm{r}}) = \frac{1}{4\pi\varepsilon_0}\int_\tau \frac{\vec{\mathbf{P}}\,.\,\hat{\mathbf{r}}_1}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|^2}d\tau' \tag{10.10}$$

Now, you can show that

$$\vec{\nabla}'\left(\frac{1}{r_1}\right) = \frac{\hat{\mathbf{r}}_1}{r_1^2} \tag{10.11}$$

where $\vec{\nabla}'$ is evaluated at (x', y', z'). In fact, you can do this calculation and arrive at Eq. (10.11) yourself. Solve SAQ 2 before studying further.

SAQ 2 - Electric potential of a polarized dielectric

Derive Eq. (10.11).

Using Eq. (10.11) in Eq. (10.10), we can write

$$V(\vec{\mathrm{r}}) = \frac{1}{4\pi\varepsilon_0}\int_\tau \vec{\mathbf{P}}.\vec{\nabla}'\left(\frac{1}{r_1}\right)d\tau' \tag{10.12}$$

We now make use of the following vector identity in Eq. (10.12):

$$\vec{\nabla}'.(f\vec{\mathbf{A}}) = f\,\vec{\nabla}'.\vec{\mathbf{A}} + \vec{\mathbf{A}}.\vec{\nabla}' f$$

where f and $\vec{\mathbf{A}}$ are scalar and vector fields, respectively. We substitute

$f = \dfrac{1}{r_1}$ and $\vec{\mathbf{A}} = \vec{\mathbf{P}}$ in the vector identity. Then we get

From Units 8 and 9, you know that the electric potential due to a point charge q at a distance r from it is

$$V(\vec{\mathbf{r}}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$$

and the electric potential of a distribution of charges is

$$V(\vec{\mathbf{r}}) = \frac{1}{4\pi\varepsilon_0}\sum\frac{q_i}{r_i}$$

For a continuous distribution of charges:

$$V(\vec{\mathbf{r}}) = \frac{1}{4\pi\varepsilon_0}\int\frac{dq}{r}$$

For a charge distribution having volume charge density $\rho(\vec{\mathbf{r}})$:

$$V(\vec{\mathbf{r}}) = \frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\vec{\mathbf{r}}')}{r}\,d\tau'$$

For a surface charge distribution having surface charge density $\sigma(\vec{\mathbf{r}})$:

$$V(\vec{\mathbf{r}}) = \frac{1}{4\pi\varepsilon_0}\int\frac{\sigma(\vec{\mathbf{r}}')}{r}\,dS'$$

$$\vec{\mathbf{P}}\cdot\vec{\nabla}'\left(\frac{1}{r_1}\right) = \left(\vec{\nabla}'\cdot\frac{\vec{\mathbf{P}}}{r_1}\right) - \frac{1}{r_1}(\vec{\nabla}'\cdot\vec{\mathbf{P}})$$

Substituting this result in Eq. (10.12), we can write the expression for the electric potential as

$$V = \frac{1}{4\pi\varepsilon_0}\left[\int_\tau\left(\vec{\nabla}'\cdot\frac{\vec{\mathbf{P}}}{r_1}\right)d\tau' - \int_\tau\frac{1}{r_1}(\vec{\nabla}'\cdot\vec{\mathbf{P}})\,d\tau'\right] \qquad (10.13)$$

We now apply the divergence theorem to the first term in Eq. (10.13) and rewrite the expression of the potential. Thus we get

$$V = \frac{1}{4\pi\varepsilon_0}\oint_S\frac{1}{r_1}\vec{\mathbf{P}}\cdot d\vec{\mathbf{S}}' - \frac{1}{4\pi\varepsilon_0}\int_\tau\frac{1}{r_1}(\vec{\nabla}'\cdot\vec{\mathbf{P}})\,d\tau' \qquad (10.14)$$

The first term in Eq. (10.14) is equivalent to the electric potential produced by a **surface charge density** σ_b (see the last equation in the margin remark) if we define σ_b as

$$\boxed{\sigma_b = \vec{\mathbf{P}}\cdot\hat{\mathbf{n}}} \qquad (10.15)$$

where $\hat{\mathbf{n}}$ is the unit vector normal to the surface. The second term in Eq. (10.14) is equivalent to the electric potential produced by a **volume charge density** ρ_b (see the equation in the margin remark) if we define ρ_b as

$$\boxed{\rho_b = -\vec{\nabla}\cdot\vec{\mathbf{P}}} \qquad (10.16)$$

With these definitions, we can write Eq. (10.14) as

$$V = \frac{1}{4\pi\varepsilon_0}\oint_S\frac{\sigma_b}{r_1}\,dS' + \frac{1}{4\pi\varepsilon_0}\int_\tau\frac{\rho_b}{r_1}\,d\tau' \qquad (10.17)$$

Thus, **the electric potential and hence the electric field of a polarised object is the same as that produced by a volume charge density $\rho_b = -\vec{\nabla}\cdot\vec{\mathbf{P}}$ and a surface charge density $\sigma_b = \vec{\mathbf{P}}\cdot\hat{\mathbf{n}}$ of bound charges** in the dielectric. So we do not need to calculate the contributions of all infinitesimal dipoles in a polarised object to solve Eq. (10.10). Instead, we can determine the bound charges and then calculate the electric fields they produce. This is the same as calculating the electric field of any volume or surface charge density using Gauss's law. You may now like to understand the physical meaning of these bound charge densities in a polarised dielectric.

10.4.1 Physical Interpretation of Bound Charge Density

So far you have learnt that the electric field of a polarised object is the same as that produced by a certain distribution of "bound charges" having densities σ_b and ρ_b. But we had just defined these quantities to recast the integral of Eq. (10.12) in a certain form without explaining the physical basis for these bound charge densities. This is what we do now. We will now demonstrate that the bound electric charge densities σ_b and ρ_b represent **actual accumulation of charge**.

Let us first consider surface charge density σ_b. Let *N* be the number of molecules per unit volume in the dielectric. In the presence of an external electric field $\vec{\mathbf{E}}$, the centres of positive and negative charges are separated by a distance *d*. Let us assume for simplicity that the centres of negative charges remain fixed and the centres of positive charges move to produce a dipole moment $\vec{\mathbf{p}}$ per molecule. Now consider an element of surface area $d\vec{\mathbf{A}}$ in the dielectric (Fig. 10.11). In the presence of an external electric field $\vec{\mathbf{E}}$, the centres of positive charges would cross the element of surface area $d\vec{\mathbf{A}}$ by moving in the direction of $\vec{\mathbf{E}}$.

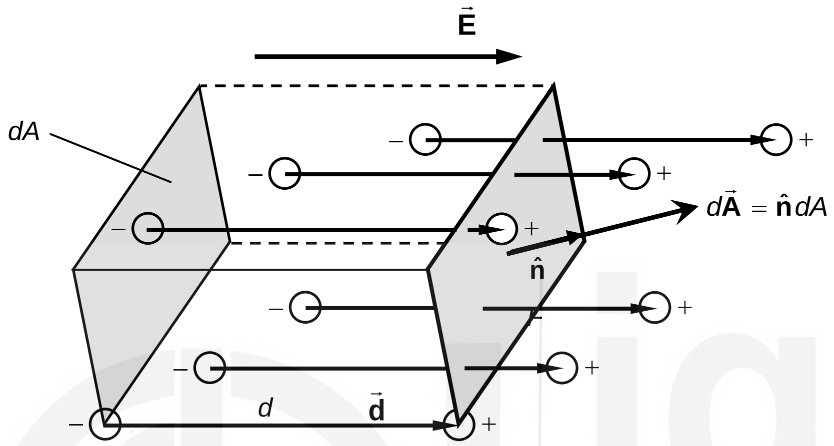


Fig. 10.11: $d\vec{\mathbf{A}}$ **is along the normal to the shaded surface. The circles represent positive and negative charges in the molecule, which are separated by a distance** *d* **in the direction of the electric field.**

The number of centres of positive charges that will cross the element of surface area $d\vec{\mathbf{A}}$ will be the number of molecules contained in a parallelepiped of volume

$$dV = d\vec{\mathbf{A}}.\vec{\mathbf{d}} \qquad (10.18)$$

Therefore, the charge in the volume *dV* is

$$dQ = Nq\,dV = Nq\vec{\mathbf{d}}.d\vec{\mathbf{A}} = N\vec{\mathbf{p}}.d\vec{\mathbf{A}} = \vec{\mathbf{P}}.d\vec{\mathbf{A}} = \vec{\mathbf{P}}.\hat{\mathbf{n}}\,dA \qquad (10.19)$$

where $\vec{\mathbf{p}} = q\vec{\mathbf{d}}$ is the dipole moment, $\vec{\mathbf{P}} = N\vec{\mathbf{p}}$ is the polarisation and $\hat{\mathbf{n}}$ is the unit vector normal to the surface. From Fig. 10.11, you can see that if $d\vec{\mathbf{A}}$ is an element of area on the surface of the dielectric, the charge *dQ* will accumulate there in a layer of thickness $\vec{\mathbf{d}}.\hat{\mathbf{n}}$. Since *d* is of the order of molecular size, we can consider the charge to be present on the surface of the dielectric. Therefore, the surface charge density, i.e., surface charge per unit area, is given by

$$\sigma_b = \frac{dQ}{dA} = \vec{\mathbf{P}}.\hat{\mathbf{n}} \qquad (10.20)$$

The effect of polarisation is, therefore, to give rise to a bound charge over the surface of the material.

Next, consider the case when the **polarisation is non-uniform**, that is, it is different at different points in the dielectric. This means that the dielectric is not a linear dielectric. It is anisotropic (polarisation is not the same in all directions) and non-homogeneous (polarisation varies with position).

In this case we get an accumulation of bound charge *within* the material along with the bound charge on its surface. Let us calculate the bound charge density in this case. Since polarisation is non-uniform, a net charge Q flows out of the volume dV of the parallelepiped through the element of surface area $d\vec{\mathbf{A}}$. We can obtain its value by integrating dQ given by Eq. (10.19) over the entire surface:

$$Q = \int \vec{\mathbf{P}}.d\vec{\mathbf{A}} \tag{10.21}$$

The net bound charge that remains inside a given volume is equal and opposite to the charge that flows out of it. Therefore,

$$-Q = -\oint_S \vec{\mathbf{P}}.d\vec{\mathbf{A}} \tag{10.22a}$$

As you know, we can express this net bound charge in terms of the bound volume charge density as follows:

$$-Q = \int_V \rho_b dV \quad \Rightarrow \quad -\oint_S \vec{\mathbf{P}}.d\vec{\mathbf{A}} = \int_V \rho_b dV \tag{10.22b}$$

You know from Gauss's divergence theorem (Unit 4) that

$$\oint_S \vec{\mathbf{P}}.d\vec{\mathbf{A}} = \int_V \vec{\nabla}.\vec{\mathbf{P}} dV \tag{10.23}$$

Substituting Eq. (10.22b) in Eq. (10.23), we get

$$\int_V \vec{\nabla}.\vec{\mathbf{P}} dV = -\int_V \rho_b dV \tag{10.24}$$

Since this result is true for all volume elements, we have

$$\rho_b = -\vec{\nabla}.\vec{\mathbf{P}} \tag{10.25}$$

Eq. (10.25) tells us that **if the polarisation of the dielectric is non-uniform, its divergence results in the net pile-up of bound charges in the material**. The volume charge density is associated with this bound charge. (The volume charge density is zero for isotropic dielectrics since $\vec{\nabla}.\vec{\mathbf{P}} = 0$ for them.) These are perfectly **real charge densities** which we have called here (surface and volume) bound or polarisation charge density. You may now like to calculate these charge densities for a concrete situation.

SAQ 3 – Calculating bound charge densities

a) A dielectric block is polarised such that $\vec{\mathbf{P}} = 2.5 \times 10^{-7} (2x\,\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}})\ \text{Cm}^{-2}$. Calculate the bound volume charge density for the block.

b) Consider a polarised rectangular block of a dielectric (Fig. 10.12) whose polarisation $\vec{\mathbf{P}} = 2.0 \times 10^{-6}\,\hat{\mathbf{k}}\ \text{Cm}^{-2}$. Calculate the bound surface charge density on the six faces of the block.

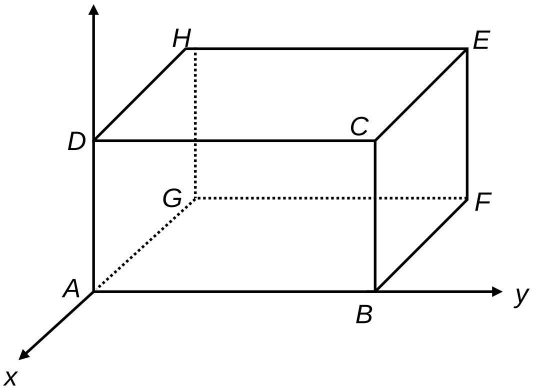


Fig. 10.12: **Diagram for SAQ 3b.**

We now discuss how Gauss's law is modified inside a dielectric.

10.5 ELECTROSTATIC EQUATIONS IN DIELECTRICS: DISPLACEMENT VECTOR $\vec{\mathbf{D}}$ AND GAUSS'S LAW

You have studied the fundamental equation of electrostatics, namely, Gauss's law in Unit 6. Recall from Unit 6 that the differential form of Gauss's law is given by:

$$\vec{\nabla}\cdot\vec{\mathbf{E}} = \frac{\rho}{\varepsilon_0} \tag{10.26}$$

Here ρ is the volume charge density of **all** electric charges and $\vec{\mathbf{E}}$ is the total electric field of all these charges. For modifying Gauss's law for dielectric materials, we find that it is convenient to separate the electric field of Eq. (10.26) into two parts:

1. one that results from the bound polarisation charge density (ρ_b), and
2. the other that is due to everything *else* (which, for want of a better term, we call **free charge**).

The free charge is any other charge in the material that is not the result of polarisation; it could be due to electrons on a conductor or ions embedded in a dielectric material or due to any other factor. Let us not at the moment worry about the source of the free charge. Then we can express the total volume charge density ρ within the dielectric as the sum of bound polarisation charge density ρ_b and the free charge density ρ_f :

$$\rho = \rho_f + \rho_b \tag{10.27a}$$

Eq. (10.26) or Gauss's law then becomes

$$\vec{\nabla}\cdot\vec{\mathbf{E}} = \frac{\rho_f + \rho_b}{\varepsilon_0} = \frac{\rho_f - \vec{\nabla}\cdot\vec{\mathbf{P}}}{\varepsilon_0} \quad (\because\ \rho_b = -\vec{\nabla}\cdot\vec{\mathbf{P}}) \tag{10.27b}$$

or $$\vec{\nabla}\cdot\left(\vec{\mathbf{E}} + \frac{\vec{\mathbf{P}}}{\varepsilon_0}\right) = \frac{\rho_f}{\varepsilon_0} \tag{10.28a}$$

or $$\vec{\nabla}\cdot\left(\varepsilon_0\vec{\mathbf{E}} + \vec{\mathbf{P}}\right) = \rho_f \tag{10.28b}$$

We define a new vector $\vec{\mathbf{D}}$ called the **electric displacement** as follows:

$$\boxed{\vec{\mathbf{D}} = \varepsilon_0 \vec{\mathbf{E}} + \vec{\mathbf{P}}} \tag{10.29}$$

Substituting Eq. (10.29) in Eq. (10.28b), we get Gauss's law for a dielectric medium in terms of $\vec{\mathbf{D}}$:

$$\boxed{\vec{\nabla}.\vec{\mathbf{D}} = \rho_f} \tag{10.30a}$$

In the integral form, Gauss's law for a dielectric medium is given by:

$$\boxed{\int \vec{\mathbf{D}}.d\vec{\mathbf{S}} = (Q_f)_{enclosed}} \tag{10.30b}$$

where $(Q_f)_{enclosed}$ is the total free charge in the volume. This is a very useful way of expressing Gauss's law for dielectric materials as it refers to only free charges enclosed in the volume. So we can calculate $\vec{\mathbf{D}}$ by the standard methods using Gauss's law for charge distributions having some kind of symmetry (linear, planar, spherical or cylindrical). The equation for curl of $\vec{\mathbf{E}}$ remains unchanged:

$$\vec{\nabla} \times \vec{\mathbf{E}} = \vec{\mathbf{0}} \tag{10.31}$$

From Eq. (10.6) of Sec. 10.3, you know that for **linear dielectrics**, the polarisation $\vec{\mathbf{P}}$ is proportional to the electric field and is given by:

$$\vec{\mathbf{P}} = \varepsilon_0 \chi \, \vec{\mathbf{E}} \tag{10.32}$$

provided $\vec{\mathbf{E}}$ is not too strong. You have learnt that χ is called the **electric susceptibility of the dielectric material/medium**. **It is a macroscopic property of the material and depends on the microscopic structure of the medium. It is a measure of the extent to which a dielectric is polarised by an external electric field**. The greater the susceptibility of the dielectric, the greater is the polarisation of the material in response to the electric field, thereby reducing the electric field inside the material.

Using Eq. (10.32) in Eq. (10.29), we can write $\vec{\mathbf{D}}$ for **linear dielectrics** as

$$\vec{\mathbf{D}} = \varepsilon_0 \vec{\mathbf{E}} + \varepsilon_0 \chi \vec{\mathbf{E}} = \varepsilon_0 (1+\chi) \vec{\mathbf{E}} \tag{10.33}$$

If we define a new parameter ε given by

$$\boxed{\varepsilon \;=\; \varepsilon_0 \, (1+\chi)} \tag{10.34}$$

Then we can write the $\vec{\mathbf{D}}$ field inside dielectrics given by Eq. (10.33) as

$$\boxed{\vec{\mathbf{D}} = \varepsilon \vec{\mathbf{E}}} \tag{10.35}$$

The constant of proportionality ε defined by Eq. (10.34) is called the **permittivity** of the dielectric material. Eq. (10.35) tells us that the displacement $\vec{\mathbf{D}}$ is proportional to the total electric field $\vec{\mathbf{E}}$. If we divide Eq. (10.34) by the factor ε_0, we get a dimensionless quantity ε_r or K:

You may like to memorise Eqs. (10.29) to (10.37), since these will be used quite often in this block.

$$\boxed{\varepsilon_r = \frac{\varepsilon}{\varepsilon_0} = 1 + \chi = K} \tag{10.36}$$

The constant ε_r or K is called the **relative permittivity** or the **dielectric constant** of the material/medium. Henceforth, we shall use the symbol K for the dielectric constant in this block. Thus, we can write Eq. (10.35) as

$$\boxed{\vec{\mathbf{D}} = \varepsilon_0 K \vec{\mathbf{E}}} \qquad (10.37)$$

The susceptibility and dielectric constant are important macroscopic properties of dielectric materials. The dielectric constant is also a measure of the extent to which a dielectric is polarised by an external electric field. If a material with a large dielectric constant is placed in an electric field, the magnitude of the electric field will be significantly reduced inside the dielectric. This property is used for increasing the capacitance and is important in the design of capacitors for various applications. You may like to pause and reflect over what you have studied so far. You may also like to try an SAQ to calculate the value of dielectric constant.

SAQ 4 - Dielectric constant of a dielectric material

Two parallel plates, which have cross-sectional area of $100\ \text{cm}^2$, carry equal and opposite charge of 1.0×10^{-7} C. The space between the plates is filled with a dielectric material and the electric field within the dielectric is $3.3 \times 10^5\ \text{Vm}^{-1}$. What is the dielectric constant of the dielectric if the electric field across the plates without the dielectric is given by $E_0 = \frac{\sigma}{\varepsilon_0}$, where σ is the surface charge density of the plates?

Thus, the laws of electrostatics in vacuum given by

$$\boxed{\vec{\nabla}.\vec{\mathbf{E}}_0 = \frac{\rho_f}{\varepsilon_0} \quad \text{and} \quad \vec{\nabla} \times \vec{\mathbf{E}} = \vec{0}} \qquad (10.38a)$$

are modified as follows for **linear dielectrics**, (i.e., when polarisation is proportional to the electric field):

$$\boxed{\vec{\nabla}.\vec{\mathbf{D}} = \rho_f \quad \text{or} \quad \vec{\nabla}.(K\vec{\mathbf{E}}) = \frac{\rho_f}{\varepsilon_0} \quad \text{and} \quad \vec{\nabla} \times \vec{\mathbf{E}} = \vec{\mathbf{0}}} \qquad (10.38b)$$

If K is same everywhere, i.e., it is a constant, then we can write

$$\vec{\nabla} \times \vec{\mathbf{E}} = \vec{\nabla} \times (K\vec{\mathbf{E}}) = \vec{\nabla} \times \vec{\mathbf{D}} = \vec{\mathbf{0}} \qquad (10.38c)$$

Note that Eqs. (10.38b and c) for KE are of the same form as Eq. (10.38a) for E_0, the electric field in vacuum. We, therefore, have the solution

$$KE = E_0 \qquad (10.39a)$$

Eq. (10.39a) implies that in a dielectric medium with dielectric constant K the electric field is **everywhere reduced** by a factor K.

Recall from Eq. (8.15) of Unit 8 that the potential difference between any two points a and b is just the negative of the line integral of the electric field:

$$V_b - V_a = -\int_a^b \vec{\mathbf{E}}.d\vec{\mathbf{r}} \qquad (10.39b)$$

Therefore, the potential difference or voltage is reduced by the same factor K. For a parallel plate capacitor, the charge placed on the capacitor plates is the same. Hence, its capacitance $C = \frac{Q}{V}$ is increased by a factor K. From Eq. (10.1), it is given by:

$$C = \frac{\varepsilon_0 K A}{d} \quad (10.39c)$$

You may like to apply these equations to a parallel plate capacitor.

SAQ 5 - Electric displacement and polarisation

Consider a parallel plate capacitor made up of two rectangular plates of area of cross-section $6.45 \times 10^{-4} \text{m}^2$ and separated by a distance of 2.0×10^{-3} m. A voltage of 100 V is applied across the plates. If a dielectric material of dielectric constant 6.0 is introduced between the plates of the capacitor, calculate the

a) capacitance of the capacitor;
b) charge stored on each plate of the capacitor;
c) displacement D; and
d) polarisation P.

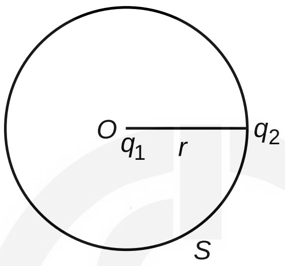


Fig. 10.13: **Two charges q_1 and q_2 situated in a homogeneous dielectric medium.**

You may now like to know: **What is the force between two point charges placed in a dielectric**?

To answer this question, consider two charges q_1 and q_2 situated in a homogeneous dielectric like a liquid or gas. We take a Gaussian spherical surface in this material centred around the charge q_1 and of radius r, the distance between the two charges q_1 and q_2 (see Fig. 10.13).

Let us apply Gauss's law to this surface. For a spherical surface, $\vec{\mathbf{D}}$ is along the radius vector. Thus, it is parallel to $\hat{\mathbf{n}}$, the unit vector normal to the surface S and we have

$$\oint_S \vec{\mathbf{D}} . d\vec{\mathbf{S}} = \oint_S \vec{\mathbf{D}} . \hat{\mathbf{n}} \, dS = \oint_S D \, dS = Q_{enclosed}$$

For the Gaussian sphere of radius r enclosing the charge q_1, applying Gauss's law for constant D, we get

$$4\pi r^2 D = q_1 \quad \text{or} \quad D = \frac{q_1}{4\pi r^2}$$

From Eq. (10.37), we can write this result as

$$D = \frac{q_1}{4\pi r^2} = \varepsilon_0 K E$$

or

$$E = \frac{q_1}{4\pi \varepsilon_0 r^2 K} \quad \text{and} \quad \vec{\mathbf{E}} = \frac{q_1}{4\pi \varepsilon_0 r^2 K} \hat{\mathbf{r}} \quad (10.40)$$

Here $\hat{\mathbf{r}}$ is the unit vector along the radius pointing from q_1 to q_2.

The force experienced by the charge q_2 is, therefore,

$$\vec{\mathbf{F}} = q_2 \vec{\mathbf{E}} = \frac{q_1 q_2}{4\pi \varepsilon_0 K r^2} \hat{\mathbf{r}} \quad (10.41)$$

From Eq. (10.41), you can see that the force between any two charges in a dielectric medium is reduced by the factor K.

We now take up an example to calculate the electric field in a dielectric.

EXAMPLE 10.2: CALCULATION OF ELECTRIC FIELD IN A DIELECTRIC

A metal sphere of radius a carries charge Q. It is surrounded by a linear dielectric material of dielectric constant K up to distance b. Calculate the electric field in the three regions (i) $r < a$, (ii) $a < r < b$ and (iii) $r > b$ and the electric potential at the centre of the sphere.

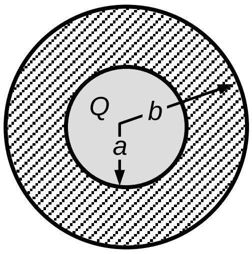


Fig. 10.14: **Diagram for Example 10.2**

SOLUTION ■ Refer to Fig. 10.14. Since the charge in a conductor resides on its surface, we have from Gauss's law:

$$E = 0 \qquad \text{for} \qquad r < a$$

For calculating the electric field in the region $a < r < b$ where the dielectric medium is present we use Eq. (10.30b) given as $\oint_S \vec{\mathbf{D}}.d\vec{\mathbf{S}} = Q_{enclosed}$

$$\therefore \qquad D(4\pi r^2) = Q \quad \Rightarrow \vec{\mathbf{D}} = \frac{Q}{4\pi r^2}\hat{\mathbf{r}} \quad \text{for} \quad a < r < b \qquad (10.42)$$

From Eq. (10.37), $\vec{\mathbf{D}} = \varepsilon_0 K \vec{\mathbf{E}}$ or $\vec{\mathbf{E}} = \dfrac{\vec{\mathbf{D}}}{\varepsilon_0 K}$

$$\text{Thus,} \qquad \vec{\mathbf{E}} = \frac{Q}{4\pi\varepsilon_0 K r^2}\hat{\mathbf{r}} = \frac{Q}{4\pi\varepsilon r^2}\hat{\mathbf{r}} \quad \text{for} \quad a < r < b \qquad (10.43a)$$

In the region $r > b$ where the dielectric material is not present, the electric field is given by:

$$\vec{\mathbf{E}} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}} \qquad \text{for} \qquad r > b \qquad (10.43b)$$

The electric potential at the centre of the sphere is, therefore,

$$V = -\int_{\infty}^{0} \vec{\mathbf{E}}.d\vec{\mathbf{l}} = -\int_{\infty}^{b} \frac{Q}{4\pi\varepsilon_0 r^2} dr - \int_{b}^{a} \frac{Q}{4\pi\varepsilon r^2} dr - \int_{a}^{0} 0\, dr \qquad (10.44)$$

$$\text{Hence,} \qquad V = \frac{Q}{4\pi}\left(\frac{1}{\varepsilon_0 b} + \frac{1}{\varepsilon a} - \frac{1}{\varepsilon b}\right) \qquad (10.45)$$

In solving the integrals in Eq. (10.44), we have used the result

$$\int \frac{1}{r^2} dr = -\frac{1}{r}$$

Hence,

$$\int_b^a \frac{Q}{4\pi\varepsilon r^2} dr = -\frac{Q}{4\pi\varepsilon r}\Bigg|_a^b$$

$$= \frac{Q}{4\pi\varepsilon}\left(\frac{1}{a} - \frac{1}{b}\right)$$

Other integrals are special cases.

You may now like to solve a problem on your own.

SAQ 6 - Electric field in a dielectric

A large metal plate of area $1.0\,\text{m}^2$ carries a charge $4.4 \times 10^{-10}\,\text{C}$. Calculate the electric field at a point near the plate.

We now summarise what you have studied in this unit.

10.6 SUMMARY

Concept	Description
Dielectric in electric field	■ When an insulating material called a **dielectric** is placed in an external electric field it gets **polarised**.
Dielectric polarisation	■ Electric dipole moment per unit volume $\vec{\mathbf{P}}$ is called **polarisation**. At the atomic level, polarisation of the dielectric material or medium takes place in two ways: i) If the dielectric material is made up of neutral atoms/molecules in which the centres of positive and negative charges coincide, the neutral atom/molecule (called **non-polar** molecule) does not have any electric dipole moment. The effect of external electric field on neutral atoms/molecules is that the centres of positive and negative charges in them are separated and the material develops a net electric dipole moment. ii) If the dielectric material is made up of polar molecules, then in the absence of the electric field the permanent dipole moments move randomly due to the thermal motion of the molecules. However, in the presence of external electric field, the permanent dipole moments tend to align along the direction of the electric field and the dielectric material develops a net electric dipole moment.
Atomic polarisability	■ The **electric dipole moment acquired by an atom/molecule is proportional to the electric field** and can be written as $\vec{\mathbf{p}} = \alpha \vec{\mathbf{E}}$ where α is called the **atomic/molecular polarisability**.
Bound charge	■ The **electric field produced by a polarised dielectric** is equivalent to the electric field produced by a **bound surface charge density** $\sigma_b = \vec{\mathbf{P}}.\hat{\mathbf{n}}$ and a **bound volume charge density** $\rho_b = -\vec{\nabla}.\vec{\mathbf{P}}$.
Gauss's law for dielectric	■ **Gauss's law of electrostatics gets modified in a dielectric medium** and it is convenient to introduce a **displacement vector** $\vec{\mathbf{D}}$ for the medium given by: $\vec{\mathbf{D}} = \varepsilon_0 \vec{\mathbf{E}} + \vec{\mathbf{P}}$ In terms of $\vec{\mathbf{D}}$, Gauss's law states that the flux of $\vec{\mathbf{D}}$ through a closed surface is equal to the total free charge enclosed in the volume bounded by the closed surface: $\oint_S \vec{\mathbf{D}}.d\vec{\mathbf{S}} = (Q_f)_{enclosed}$ or $\vec{\nabla}.\vec{\mathbf{D}} = \rho_f$
Electric susceptibility, permittivity and dielectric constant	■ For ideal, homogeneous and isotropic dielectrics, called **linear dielectrics** $\vec{\mathbf{P}} = \varepsilon_0 \chi \vec{\mathbf{E}}$

where χ is called the **electric susceptibility** of the medium. The **displacement vector $\vec{\mathbf{D}}$ for linear dielectrics** is

$$\vec{\mathbf{D}} = \varepsilon_0 (1+\chi)\,\vec{\mathbf{E}} = \varepsilon \vec{\mathbf{E}}$$

where ε is called the **permittivity** of the medium. We define a dimensionless quantity K called the **dielectric constant** as

$$K = \frac{\varepsilon}{\varepsilon_0} \text{ and hence } \vec{\mathbf{D}} = \varepsilon_0 K \vec{\mathbf{E}}$$

$\vec{\mathbf{D}}$ depends only on the free charges and can be obtained without any reference whatsoever to the bound charges in a dielectric.

Effect of dielectrics on capacitance

■ **In a dielectric**, **the electric field due to a distribution of free charges is reduced by a factor** K. This has the effect of **increasing the capacitance of a capacitor** filled with a dielectric by a factor equal to the dielectric constant of the material.

10.7 TERMINAL QUESTIONS

1. Two parallel conducting plates of area of cross-section $2.0\,\text{m}^2$ are separated by a distance of $1.0 \times 10^{-2}\,\text{m}$. The potential difference (V_0) between them in vacuum is 3000 V. When a dielectric sheet of thickness 1.0 cm is introduced between them, the voltage is found to decrease to 1000 V. Calculate

   a) the dielectric constant K, the permittivity ε of the dielectric and its susceptibility χ,
   b) the electric field between the plates in vacuum,
   c) the electric field in the dielectric, and
   d) the electric field produced by the bound charges.

2. Consider two isotropic dielectric mediums A and B of permittivity ε_1 and ε_2, respectively, separated by a charge free boundary as shown in Fig. 10.15. The electric field $\vec{\mathbf{E}}_1$ is incident at the boundary of the mediums at an angle of incidence θ_i and the electric field $\vec{\mathbf{E}}_2$ in medium B makes an angle of refraction θ_r. Assuming that at the interface of the two dielectrics, the normal component of $\vec{\mathbf{D}}$ and tangential component of $\vec{\mathbf{E}}$ are continuous, show that $\dfrac{\tan\theta_i}{\tan\theta_r} = \dfrac{\varepsilon_1}{\varepsilon_2}$

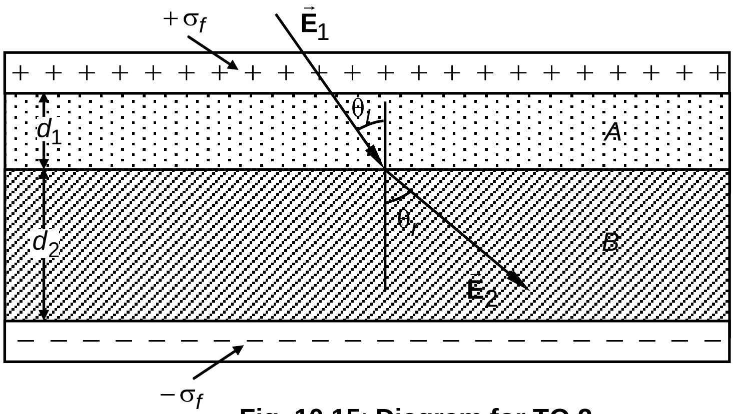


Fig. 10.15: **Diagram for TQ 2.**

3. Show that the polarisation (bound) charge density at the interface of two charge free dielectrics of permittivity ε_1 and ε_2 is given by

$$\sigma_b = \hat{\mathbf{n}}.(\vec{\mathbf{P}}_1 - \vec{\mathbf{P}}_2) = \varepsilon_0 \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_2} \vec{\mathbf{E}}_1.\hat{\mathbf{n}}$$

where $\hat{\mathbf{n}}$ is a unit vector normal to the surface. Assume that the normal component of $\vec{\mathbf{D}}$ and tangential component of $\vec{\mathbf{E}}$ are continuous at the interface of the two dielectrics.

4. A thin dielectric rod of cross-section A extends along the x-axis from $x = 0$ to $x = L$. The polarisation of the rod is along its length and is given by $\vec{\mathbf{P}} = (ax^2 + b)\,\hat{\mathbf{i}}$. Obtain the bound volume charge densities and the surface charge densities at each end of the rod. Show explicitly that the total bound charges vanish.

10.8 SOLUTIONS AND ANSWERS

Self-Assessment Questions

1. $\vec{\mathbf{P}}$ is dipole moment per unit volume. Therefore, unit of $\vec{\mathbf{P}}$ is:

$$\frac{\text{Coulomb} \times \text{metre}}{(\text{metre})^3} \quad \text{or} \quad \text{Cm}^{-2}$$

2. From the definition of the gradient, $\vec{\nabla}'\left(\frac{1}{r_1}\right) = \left(\hat{\mathbf{i}}\frac{\partial}{\partial x'} + \hat{\mathbf{j}}\frac{\partial}{\partial y'} + \hat{\mathbf{k}}\frac{\partial}{\partial z'}\right)\left(\frac{1}{r_1}\right)$

where from Eq. (10.9), $r_1 = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$. Now

$$\frac{\partial}{\partial x'}\left(\frac{1}{r_1}\right) = \frac{\partial}{\partial x'}\left(\frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}}\right) = \frac{(x-x')}{r_1^3},$$

$$\frac{\partial}{\partial y'}\left(\frac{1}{r_1}\right) = \frac{\partial}{\partial y'}\left(\frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}}\right) = \frac{(y-y')}{r_1^3} \quad \text{and}$$

$$\frac{\partial}{\partial z'}\left(\frac{1}{r_1}\right) = \frac{\partial}{\partial z'}\left(\frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}}\right) = \frac{(z-z')}{r_1^3}$$

Therefore,

$$\left(\hat{\mathbf{i}}\frac{\partial}{\partial x'} + \hat{\mathbf{j}}\frac{\partial}{\partial y'} + \hat{\mathbf{k}}\frac{\partial}{\partial z'}\right)\left(\frac{1}{r_1}\right) = \frac{(x-x')\hat{\mathbf{i}} + (y-y')\hat{\mathbf{j}} + (z-z')\hat{\mathbf{k}}}{r_1^3}$$

$$= \frac{\vec{\mathbf{r}}_1}{r_1^3} = \frac{\hat{\mathbf{r}}_1}{r_1^2} \qquad (\because \vec{\mathbf{r}}_1 = r_1\hat{\mathbf{r}}_1)$$

where we have used Eq. (10.9) for the expression of $\vec{\mathbf{r}}_1$. Hence, we get

Eq. (10.11): $\vec{\nabla}'\left(\frac{1}{r_1}\right) = \frac{\hat{\mathbf{r}}_1}{r_1^2}$

3. a) From Eq. (10.25), the volume charge density ρ_b is given by

$$\rho_b = -\vec{\nabla}\,.\,\vec{\mathbf{P}} = -\left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right).\,(2x\,\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}) \times 2.5 \times 10^{-7}\,\text{Cm}^{-3}$$

$$= -2 \times 2.5 \times 10^{-7}\,\text{Cm}^{-3} = -5.0 \times 10^{-7}\,\text{Cm}^{-3}$$

b) Surface charge density is the component of $\vec{\mathbf{P}}$ normal to the surface. Now refer to Fig. 10.12. The faces *BFEC* and *AGHD* have normals along $\hat{\mathbf{j}}$ and $-\hat{\mathbf{j}}$, respectively. The charge densities on these surfaces are zero since $\hat{\mathbf{j}}.\hat{\mathbf{k}} = 0$: $\vec{\mathbf{P}}.\hat{\mathbf{j}} = 0$ and $-\vec{\mathbf{P}}.\hat{\mathbf{j}} = 0$. The faces *ABCD* and *GFEH* have normals along $\hat{\mathbf{i}}$ and $-\hat{\mathbf{i}}$, respectively. The charge densities on these surfaces too are zero because $\hat{\mathbf{i}}.\hat{\mathbf{k}} = 0$.

Charge density on the face *DCEH* $= \sigma = \vec{\mathbf{P}}.\hat{\mathbf{k}} = 2.0 \times 10^{-6}\ \text{Cm}^{-2}$

Charge density on the face *ABFG* $= \sigma = -\vec{\mathbf{P}}.\hat{\mathbf{k}} = -2.0 \times 10^{-6}\ \text{Cm}^{-2}$

4. The surface charge density on the plates is

$$\sigma = \frac{Q}{A} = \frac{1.0\times10^{-7}\,\text{C}}{100\times10^{-4}\,\text{m}^2} = 1.0\times10^{-5}\,\text{C}\,\text{m}^{-2}$$

The electric field between the plates in the absence of any dielectric is

$$E_0 = \frac{\sigma}{\varepsilon_0} = \frac{1.0\times10^{-5}\,\text{C}\,\text{m}^{-2}}{8.85\times10^{-12}\,\text{C}^2\text{N}^{-1}\text{m}^{-2}} = \frac{1.0\times10^{7}}{8.85}\,\text{Vm}^{-1}$$

In the presence of the dielectric, the field is reduced by a factor equal to the dielectric constant. Therefore, from Eq. (10.39a),

$$E = \frac{E_0}{K}$$

$$\therefore \quad K = \frac{E_0}{E} = \frac{1.0\times10^{7}}{8.85\times3.3\times10^{5}} \approx 3.4$$

5. a) From Eq. (10.39c), the capacitance *C* of a parallel plate capacitor filled with a dielectric material of dielectric constant *K* is given by

$$C = K\frac{\varepsilon_0 A}{d} = \frac{6.0\times8.85\times10^{-12}\times6.45\times10^{-4}}{2.0\times10^{-3}}\,\text{F} = 1.7\times10^{-11}\,\text{F}$$

b) The voltage applied is 100 V. Therefore,

Charge stored on each plate $= CV$

$$= 1.7\times10^{-11}\times100\,\text{C} = 1.7\times10^{-9}\,\text{C}$$

c) Applying Gauss's law for the dielectric $\left[\oint_S \vec{\mathbf{D}}.d\vec{\mathbf{S}} = (Q_f)_{enclosed}\right]$ to this case, we get $DA = Q$ or $D = \frac{Q}{A}$

$$\therefore\ D = \frac{Q}{A} = \frac{1.7\times10^{-9}}{6.45\times10^{-4}}\,\text{C}\,\text{m}^{-2} = 2.6\times10^{-6}\,\text{C}\,\text{m}^{-2}$$

d) $\because\ D = \varepsilon_0 E + P$ and $E = \frac{V}{d}$, we get

$$P = D - \varepsilon_0 E = \left[2.6\times10^{-6} - \frac{8.85\times10^{-12}\times100}{2.0\times10^{-3}}\right]\text{C}\,\text{m}^{-2}$$

or $P = (2.6\times10^{-6} - 4.4\times10^{-7})\,\text{C}\,\text{m}^{-2} = 2.2\times10^{-6}\,\text{C}\,\text{m}^{-2}$

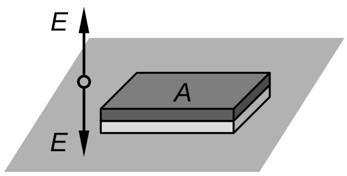


Fig. 10.16: **Diagram for SAQ 6.**

6. Let σ be the charge density on the surface of the plates. Now consider a "Gaussian pill box" which extends to equal distances above and below the plane of the positively charged plate (Fig. 10.16). Let us apply Gauss's law to this surface:

$$\oint_S \vec{\mathbf{E}}.d\vec{\mathbf{S}} = \frac{(Q_f)_{enclosed}}{\varepsilon_0} \qquad \text{(i)}$$

If A is the area of the lid of the pill box, then for this case

$$(Q_f)_{enclosed} = \sigma A \qquad \text{(ii)}$$

Only the top and bottom surfaces of the pill box contribute to the integral since for other surfaces, $\vec{\mathbf{E}}$ and $d\vec{\mathbf{S}}$ are perpendicular to each other and their scalar product is zero.

For both the top and bottom surfaces of the pill box, the electric field points away from the plane (since the vector $d\vec{\mathbf{S}}$ is normal to the surfaces). It is upwards for the points above the plane and downwards for the points below the plane.

Thus, we take the contributions of only the top and bottom surfaces of the pill box to the electric field into account. Then using Eq. (ii), the value of the integral of Eq. (i) is given by

$$\oint_S \vec{\mathbf{E}}.d\vec{\mathbf{S}} = 2AE = \frac{\sigma A}{\varepsilon_0} \quad \text{or} \quad E = \frac{\sigma}{2\varepsilon_0}$$

Since $A = 1.0\text{ m}^2$, $\sigma = \dfrac{Q}{A} = \dfrac{4.4\times 10^{-10}\text{ C}}{1.0\text{ m}^2} = 4.4\times 10^{-10}\text{ C m}^{-2}$

$$\therefore \quad E = \frac{\sigma}{2\varepsilon_0} = \frac{4.4\times 10^{-10}\text{ C m}^{-2}}{2\times 8.85\times 10^{-12}\text{ C}^2\text{ N}^{-1}\text{m}^{-2}} = 24.9\text{ Vm}^{-1} \approx 25\text{ Vm}^{-1}$$

Terminal Questions

1. a) From Eq. (10.39a), the dielectric constant

$$K = \frac{E_0}{E} = \frac{V_0}{V} = \frac{3000}{1000} = 3$$

From Eq. (10.36): $\varepsilon = K\varepsilon_0 = 3\varepsilon_0$ and $1+\chi = K \quad \Rightarrow \quad \chi = 2$

b) $$E_0 = \frac{V_0}{d} = \frac{3000}{1.0\times 10^{-2}}\text{ V m}^{-1} = 3.0\times 10^5\text{ V m}^{-1}$$

c) $$E = \frac{V}{d} = \frac{1000}{1.0\times 10^{-2}}\text{ V m}^{-1} = 1.0\times 10^5\text{ V m}^{-1}$$

d) The electric field E is the resultant of the electric field E_0 and the field E_b set up by bound charges.

$$\therefore \quad E_b = E_0 - E = 2.0\times 10^5\text{ V m}^{-1}$$

2. As given in the problem, at the interface of the two dielectrics, the normal component of $\vec{\mathbf{D}}$ and the tangential component of $\vec{\mathbf{E}}$ are continuous. Therefore,

$$E_1 \sin\theta_i = E_2 \sin\theta_r \qquad \text{(i)}$$

and $$D_1 \cos\theta_i = D_2 \cos\theta_r \qquad \text{(ii)}$$

But from Eq. (10.35), $D = \varepsilon E$ and therefore,

$$D_1 = \varepsilon_1 E_1 \quad \text{and} \quad D_2 = \varepsilon_2 E_2$$

Hence, Eq. (ii) becomes $\varepsilon_1 E_1 \cos\theta_i = \varepsilon_2 E_2 \cos\theta_r$ (iii)

Dividing Eq. (i) by Eq. (iii), we get

$$\frac{1}{\varepsilon_1}\tan\theta_i = \frac{1}{\varepsilon_2}\tan\theta_r \qquad \text{or} \qquad \frac{\tan\theta_i}{\tan\theta_r} = \frac{\varepsilon_1}{\varepsilon_2}$$

3. The surface charge density of a polarised medium is given by $\sigma_b = \vec{\mathbf{P}}\cdot\hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is the unit vector normal to the face on which polarisation (bound) charges appear. Let $\vec{\mathbf{P}}_1$ and $\vec{\mathbf{P}}_2$ be the polarisation vectors in the two media.

At the interface, the net surface charge density σ_b is given by:

$$\sigma_b = \hat{\mathbf{n}}.(\vec{\mathbf{P}}_1 - \vec{\mathbf{P}}_2)$$

Now $$\vec{\mathbf{D}} = \varepsilon_0 \vec{\mathbf{E}} + \vec{\mathbf{P}} \quad \Rightarrow \vec{\mathbf{P}} = \vec{\mathbf{D}} - \varepsilon_0 \vec{\mathbf{E}}$$

and $$\vec{\mathbf{P}}_1 - \vec{\mathbf{P}}_2 = (\vec{\mathbf{D}}_1 - \vec{\mathbf{D}}_2) - \varepsilon_0(\vec{\mathbf{E}}_1 - \vec{\mathbf{E}}_2)$$

$$\therefore \quad \sigma_b = \hat{\mathbf{n}}.(\vec{\mathbf{D}}_1 - \vec{\mathbf{D}}_2) - \varepsilon_0 \hat{\mathbf{n}}.(\vec{\mathbf{E}}_1 - \vec{\mathbf{E}}_2) \qquad \text{(i)}$$

As per the problem, the normal components of $\vec{\mathbf{D}}$ are continuous at the interface. Thus, we have

$$\hat{\mathbf{n}}.(\vec{\mathbf{D}}_1 - \vec{\mathbf{D}}_2) = 0$$

Therefore, from Eq. (i) $\sigma_b = -\varepsilon_0 \hat{\mathbf{n}}.(\vec{\mathbf{E}}_1 - \vec{\mathbf{E}}_2) = \varepsilon_0 \hat{\mathbf{n}}.(\vec{\mathbf{E}}_2 - \vec{\mathbf{E}}_1)$. But $\vec{\mathbf{D}} = \varepsilon\vec{\mathbf{E}}$ and therefore,

$$\hat{\mathbf{n}}.(\vec{\mathbf{D}}_1 - \vec{\mathbf{D}}_2) = 0 \quad \Rightarrow \hat{\mathbf{n}}.(\varepsilon_1 \vec{\mathbf{E}}_1 - \varepsilon_2 \vec{\mathbf{E}}_2) = 0$$

or $$\hat{\mathbf{n}}\cdot\vec{\mathbf{E}}_2 = \frac{\varepsilon_1}{\varepsilon_2}\hat{\mathbf{n}}\cdot\vec{\mathbf{E}}_1$$

$$\therefore \quad \sigma_b = \varepsilon_0\left(\frac{\varepsilon_1}{\varepsilon_2} - 1\right)\hat{\mathbf{n}}\cdot\vec{\mathbf{E}}_1 = \varepsilon_0\left(\frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_2}\right)\vec{\mathbf{E}}_1\cdot\hat{\mathbf{n}}$$

4. It is given that $\vec{\mathbf{P}} = (ax^2 + b)\hat{\mathbf{i}}$. See Fig. 10.17.

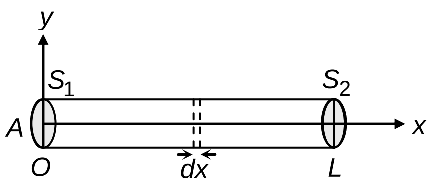


Fig. 10.17: **Diagram for TQ 4.**

The volume charge density is given by

$$\rho_b = -\vec{\nabla}\cdot\vec{\mathbf{P}} = -\left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right)\cdot(ax^2 + b)\hat{\mathbf{i}} = -2ax \qquad \text{(i)}$$

Since $\hat{\mathbf{n}} = -\hat{\mathbf{i}}$ at the face at $x = 0,$ the surface charge density at $x = 0$ is given by

$$\sigma_b\big|_{x=0} = \vec{\mathbf{P}}.\hat{\mathbf{n}}\big|_{x=0} = -\vec{\mathbf{P}}.\hat{\mathbf{i}} = -(ax^2 + b)\big|_{x=0} = -b \qquad \text{(ii)}$$

Since $\hat{\mathbf{n}} = \hat{\mathbf{i}}$ at the face $x = L,$ the surface charge density at $x = L$ is

$$\sigma_b\big|_{x=L} = \vec{\mathbf{P}}.\hat{\mathbf{n}}\big|_{x=L} = \vec{\mathbf{P}}.\hat{\mathbf{i}} = (ax^2 + b)\Big|_{x=L} = (aL^2 + b) \quad \text{(iii)}$$

Since $dV = A\,dx,$ using Eq. (i), we get the total bound volume charge as

$$Q_b^V = \int_V \rho_b\, dV = \int_0^L (-2ax)\, A\,dx = -aAL^2$$

Using Eq. (ii), we get the bound surface charge on the surface S_1 at $x = 0$ as

$$Q_b^{S_1} = \sigma_b\big|_{x=0} A = -bA$$

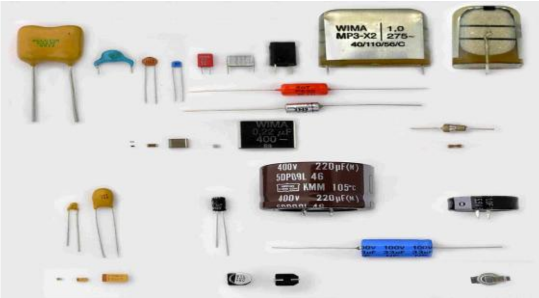
Using Eq. (iii), we get the bound surface charge on the surface S_2 at $x = L$ as

$$Q_b^{S_2} = \sigma_b\big|_{x=L} A = (aL^2 + b)A$$

Thus the total bound charge on the rod is

$$Q_b^{total} = Q_b^V + Q_b^{S_1} + Q_b^{S_2} = -aAL^2 - bA + (aL^2 + b)A = 0$$

as expected.



Capacitors of different shapes and sizes shown here are integral components of electrical and electronic circuits. The use of dielectric materials in capacitors enhances their capacitance and helps in miniaturisation of electrical and electronic appliances. (Picture source: Wikimedia Commons)

UNIT 11

CAPACITORS

Structure

STUDY GUIDE

In this unit, you will study about capacitor which is an electrical component used in a variety of electrical and electronic applications. The utility of capacitors is enhanced manifold when dielectric materials are filled between its plates. Thus, we shall refer to dielectrics and its behaviour in electric field (Unit 10) very frequently in this unit. You should, therefore, read Unit 10 before studying this unit.

For calculating capacitance of capacitors of different geometrical shapes, you will need to determine electric field and electric potential. We have used Gauss's law for calculating electric field due to charge on the capacitors. You should refresh the applications of Gauss's law form Units 6 and 7. Also, you should revise the concept of electric potential discussed in Unit 9.

You know the mathematics used in this unit as you have studied it in the previous units of this course. However, you should work through the mathematical derivations yourself as you study the unit. We also advise you to try to solve the SAQs and TQs yourself before looking up their answers given at the end.

> ***"No amount of experimentation can ever prove me right; a single experiment can prove me wrong."***
>
> ***Albert Einstein***

11.1 INTRODUCTION

In Unit 10, you have studied the behaviour of dielectric materials in electric fields and electric fields of polarised dielectrics. You have deduced Gauss's law for dielectric materials and learnt about their susceptibility and dielectric constant. As you know, dielectric materials are used very widely in the fabrication of capacitors for a variety of applications. Therefore, we focus on capacitors in this unit. In school physics and in Unit 10, you have learnt that the potential of a conductor increases as the charge placed on it is increased. This means that the charge on a conductor is directly proportional to the voltage across it and the constant of proportionality is called the **capacitance** of the conductor. Mathematically, we write this condition as $Q \propto V$ or $Q = CV$, where the constant C is the capacitance. You know that any device that has capacitance is called a capacitor. You are already familiar with this device from your school physics.

Capacitors have many applications in our daily lives. When we turn the 'tuning' knob on a radio receiver to get the radio station of our choice, we actually change the capacitance. Capacitors are used in many electrical or electronic circuits. They are used to provide coupling between amplifier stages and to smoothen the output of power supplies. Capacitors are commonly used in motors and fans. In combination with inductances, they are used to produce oscillations which when transmitted become radio signals/TV signals, etc. Besides these, capacitors have a variety of applications in electrical power transmission.

In Sec. 11.2 of this unit, you will learn about capacitance and the charging of a capacitor. We also determine the energy stored in a capacitor. In Sec. 11.3, we discuss the parallel plate capacitor and calculate its capacitance when a dielectric material is inserted between its plates. We also determine the energy stored in a dielectric medium. In Sec. 11.4, we determine the capacitance of spherical capacitor and cylindrical capacitor. In electrical circuits, capacitors are connected in parallel and/or in series. Therefore, in Sec. 11.5, you will learn how to calculate the resultant capacitance of these two types of combinations of capacitors in a circuit. Finally, in the last section (Sec. 11.6) of this Unit, you will learn about the applications of dielectrics in capacitors and some capacitors used in practical applications. We also briefly talk about the voltage rating of a capacitor.

In the next unit we discuss the magnetic field and its relation with electric current.

Expected Learning Outcomes

After studying this unit, you should be able to:

- ❖ define the capacitance of a capacitor;
- ❖ calculate the energy stored in a capacitor and in a dielectric medium;
- ❖ calculate the capacitance of a capacitor when a dielectric material is inserted in a capacitor;
- ❖ determine the capacitance of parallel plate, spherical and cylindrical capacitors;

- obtain the effective capacitance of a given combination of capacitors in series and in parallel; and
- describe practical capacitors and list their applications.

11.2 CAPACITANCE

Consider two conductors carrying charges $+Q$ and $-Q$, respectively (see Fig. 11.1). You have studied in Unit 9 that the voltage V on a conductor is constant because a conductor is an equipotential surface.

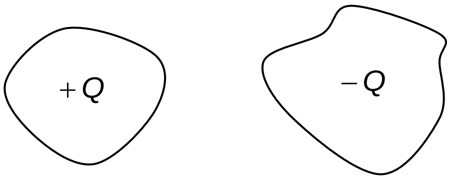


Fig. 11.1: **Two conductors carrying charges $+Q$ and $-Q$, respectively**.

By definition, the potential difference between the two conductors is

$$V = V_+ - V_- = -\int_-^+ \vec{\mathbf{E}}.d\vec{\mathbf{l}} \qquad (11.1)$$

where V_+ is the potential of the conductor carrying positive charge and V_-, the potential of the conductor carrying the negative charge. You know from school physics that the potential on the conductor increases if the charge on it increases. You also know that for a system of two conductors for which the potential difference is given by V as defined by Eq. (11.1), charge Q is proportional to V:

$$Q \propto V \qquad \text{or} \qquad Q = CV \qquad (11.2)$$

The constant of proportionality is called the **capacitance** of the system.

From Eq. (11.2), we get

$$C = \frac{Q}{V} \qquad (11.3)$$

You have learnt in school physics that **capacitance is determined by the shape and size of the conductors as well as the separation between them**. Notice that, by definition, V is the difference between the potential of the **positive conductor** and the potential of the **negative conductor** and Q is the charge of the **positive conductor**. Thus C is a positive quantity.

We can also talk of the capacitance of a **single conductor**. In this case, the **second conductor is an imaginary spherical shell of infinite radius surrounding the conductor and it contributes nothing to the field**.

For example, consider an insulated conducting spherical shell of radius R. Let us place a charge Q on the surface of this shell which is an equipotential surface. The potential on the outer surface of the shell (see Sec. 9.2 of Unit 9) is given by:

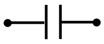
$$V = \frac{Q}{4\pi\varepsilon_0 R} \qquad (11.4)$$

under the assumption that a shell having (very large) infinite radius is at zero potential. Instead of a shell of infinity radius we can regard the ground (Earth) at zero potential. Then the capacitance of this shell (with respect to the ground) is

$$C = \frac{Q}{V} = 4\pi\varepsilon_0 R \,\frac{\text{coulomb}}{\text{volt}} \qquad (11.5)$$

where R is in m. The unit of capacitance C in SI system is farad (denoted by F) and is defined as:

$$1\,\text{farad} = \frac{1\ \text{coulomb}}{1\ \text{volt}} \qquad (11.6)$$

If $R = 1.0\,\text{m}$, the capacitance of the above shell is
$4\pi\varepsilon_0\,(1.0\,\text{m}) = 1.1\times 10^{-10}\,\text{F}.$ We notice that farad is an inconveniently large unit. The more practical units of capacitance are microfarad $(10^{-6}\,\text{F})$ and picofarad $(10^{-12}\,\text{F}).$ The symbol of a capacitor used in electric circuits is: 

Before studying further, you may like to answer an SAQ based on the concepts discussed so far.

SAQ 1 **– Capacitance of the Earth**

Consider a spherical shell of radius equal to that of the Earth (6000 km). What is its capacitance?

11.2.1 Charging of a Capacitor and Energy Stored in It

In order to "charge up" a capacitor, we have to remove electrons from the positive conductor and carry them to the negative conductor. In doing so, we have to do **work against the electric field** which is pulling them back to the positive conductor and pushing them away from the negative conductor. Suppose we start putting charge on the conductor. At an intermediate stage when the charge on the conductor is q, the potential difference is $V = \frac{q}{C}$ and to bring an additional charge dq to the conductor, we have to do an amount of work dW given by:

$$dW = V\,dq = \frac{q}{C}\,dq \qquad (11.7)$$

In a nutshell, a capacitor is an electronic device for storing electrical energy by allowing charges to accumulate on metal conductors. The electrical energy stored in the capacitor is recovered when these charges are allowed to move from these conductors to the electrical circuit they are a part of.

The total work done in charging a capacitor from zero charge $(q = 0)$ to some final charge $q = Q$ is

$$W = \int_0^Q \left(\frac{q}{C}\right) dq = \frac{1}{2}\frac{Q^2}{C} \qquad (11.8)$$

This work is stored as the **electric potential energy** U in a capacitor.
Since $Q = CV$, we have

$$U = \frac{1}{2}CV^2 \qquad (11.9)$$

where V is the final potential of the capacitor. Eqs. (11.8) and (11.9) hold no matter what the geometry (that is, the shape or size) of the capacitor is.

The energy stored in a $1\mu\text{F}$ capacitor when charged to a potential of 10 V is thus

$$U = \frac{1}{2}(10^{-6}\,\text{F})\times(10\,\text{V})^2 = 50\times 10^{-6}\,\text{J}$$

We now discuss the parallel plate capacitor, which is a capacitor of the simplest geometry and the most familiar to you.

11.3 PARALLEL PLATE CAPACITOR

A parallel plate capacitor consists of two metal plates (rectangular or circular) arranged parallel to each other and separated by a distance *d* (see Fig. 11.2). **The distance *d* is usually very small compared to the size of the plates**.

If we put a charge $+Q$ on the top plate and the charge $-Q$ on the bottom plate, the charges will spread uniformly over the surfaces. The surface charge density is then $\sigma = \frac{Q}{A}$ on the top plate, where *A* is the area of the plate.

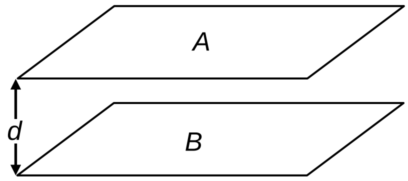


Fig. 11.2: **Geometry of a parallel plate capacitor**.

We can calculate the electric field between the plates by using Gauss's law. It is given by:

$$E = \frac{\sigma}{\varepsilon_0} \tag{11.10}$$

The electric field is normal to the surface and is uniform between the plates provided the distance between the plates is very small compared to the size of the plates. You may like to prove this result before studying further. Attempt the following SAQ.

SAQ 2 **- Electric field in a parallel plate capacitor**

Using Gauss's law calculate the electric field between the plates of a parallel plate capacitor carrying a surface charge density σ.

From Unit 8 (solution of TQ 2), you know that the potential difference between the plates *A* and *B* of the capacitor is given by

$$V = -\int_B^A \vec{\mathbf{E}}.d\vec{\mathbf{l}} = Ed = \frac{\sigma}{\varepsilon_0} d = \frac{Q}{\varepsilon_0 A} d \tag{11.11}$$

and, therefore,

$$C = \frac{\varepsilon_0 A}{d} \tag{11.12}$$

If, for example, the plates of the capacitor are square in shape with sides of 10 cm and are held 1.0×10^{-4} m (= 0.1 mm) apart, then its capacitance is

$$C = \frac{(1.0 \times 10^{-1}\text{m})^2}{1.0 \times 10^{-4}\text{m}} \times 8.85 \times 10^{-12}\text{C}^2\text{N}^{-1}\text{m}^{-2} = 885\,\text{pF} \approx 8.9 \times 10^{-10}\text{F}$$

SAQ 3 **- Capacitance of a parallel plate capacitor**

Calculate the capacitance of a parallel plate capacitor having plates of size 1.0 cm^2 with a separation of 1.0×10^{-4} m between the plates. What is the energy stored in it when it is connected across a cell of voltage 1.5 V?

What happens when we place a dielectric material between the plates of a parallel plate capacitor? Let us find out.

11.3.1 Parallel Plate Capacitor with Dielectric Material Inserted between its Plates

You have already learnt in Sec. 10.5 of Unit 10 [Eq. 10.39a)] that in a dielectric medium, the electric field is reduced by an amount *K* called the dielectric constant. We can extend this result to a parallel plate capacitor: Whenever a dielectric material is inserted between the plates of a parallel plate capacitor, the electric field between plates is reduced by the dielectric constant *K*.

To verify this result, let us consider a parallel plate capacitor of area of cross-section *A* with a dielectric material inserted between the plates (Fig. 11.3).

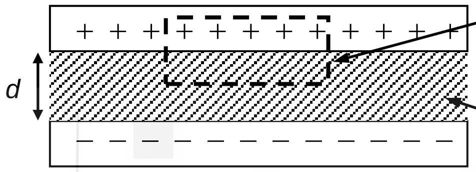


Fig. 11.3: **A parallel plate capacitor with dielectric material inserted between the plates.**

Let *d* be the distance between the plates and σ the surface charge density on the plates. We now apply Gauss's law for dielectric materials [Eq. (10.30b) of Unit 10] to a Gaussian surface of area *S* as shown in Fig. 11.3:

$$\oint \vec{\mathbf{D}}.d\vec{\mathbf{S}} = (Q_f)_{\text{enclosed}}$$

Since electric displacement $\vec{\mathbf{D}}$ is perpendicular to the plates of the capacitor and only the free surface charge density contributes to it, we have

$$D = (Q / A) = \sigma \qquad (11.13)$$

Note that as you have learnt in Unit 10, the bound surface charges do not contribute to the flux of *D*. Further, from Eq. (10.37), we also have

$$\vec{\mathbf{D}} = \varepsilon_0 K \vec{\mathbf{E}} \quad \text{or} \quad \vec{\mathbf{E}} = \frac{\vec{\mathbf{D}}}{\varepsilon_0 K} \qquad (11.14a)$$

Therefore, from Eqs. (11.13) and (11.14a), we get

$$E = \frac{D}{\varepsilon_0 K} = \frac{\sigma}{\varepsilon_0 K} \qquad (11.14b)$$

The potential difference *V* between the plates is given by $V = Ed$ and using Eq. (11.14b), we get the capacitance as

$$C = \frac{Q}{V} = \frac{\sigma A}{E d} = \frac{\sigma A \varepsilon_0 K}{\sigma d}$$

or $$C = \frac{\varepsilon_0 K A}{d} \qquad (11.15)$$

Comparing Eq. (11.15) with Eq. (11.12), we find that the value of the capacitance of the parallel plate capacitor increases by a factor *K*, the dielectric constant of the material.

Thus, the effect of introducing a dielectric between plates is to increase the capacitance of a capacitor.

The dielectric constant of a material can thus be determined by measuring the ratio of the capacitance of a parallel plate capacitor with the dielectric and without it, viz.

$$K = \frac{\text{Capacitance with dielectric between plates}}{\text{Capacitance with free space between plates}}$$

Table 11.1: Relative permittivity/dielectric constants of some common materials.

Material	Dielectric constant
Air	1.0006
Castor oil	4.7
Mica	5 – 9
Glass	4.5 – 7.0
Bakelite	4.5 – 7.5
Paper	2.0 – 2.3
Porcelain	5.5
Quartz	1.5
Water	80.4

Table 11.1 gives the relative permittivity ε_r or the dielectric constant K of some common materials. Since we can write the capacitance given by Eq. (11.15) as

$$C = \frac{\varepsilon_0 A}{(d/K)} \qquad (11.16)$$

we can say that a dielectric of thickness d has an **equivalent free space thickness** $\frac{d}{K}$.

The concept of the equivalent free space thickness allows us to answer the question: **What is the capacitance of a parallel plate capacitor when the space between the plates is only partially filled by the dielectric**? Let us solve this problem in an Example.

EXAMPLE 11.1: CAPACITOR PARTIALLY FILLED WITH A DIELECTRIC

Consider a parallel plate capacitor of area A. Let a dielectric slab of thickness t and dielectric constant K be kept between its plates as shown in Fig. 11.4. Notice that the upper surface of the dielectric slab is at a distance d_1 from the upper plate of the capacitor and the distance between the lower surface of the dielectric and the lower plate of the capacitor is d_2. Calculate the capacitance of this capacitor.

SOLUTION ■

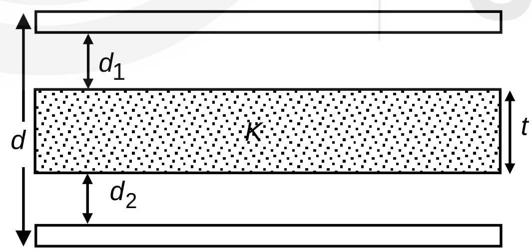


Fig. 11.4: **Calculation of capacitance in terms of the equivalent free space thickness**.

To calculate the capacitance of the partially filled capacitor [Fig. 11.4], we will use the concept of the equivalent free space thickness. From Fig. 11.4, you can see that the free space between the plates is $(d - t)$. From Eq. (11.16), the slab of dielectric constant K of thickness t has equivalent free space of thickness (t/K). Thus, the total free space between the plates is $(d - t + t/K)$ and the capacitance C of this capacitor is

$$C = \frac{\varepsilon_0 A}{d - t + t/K} = \frac{\varepsilon_0 A K}{Kd - Kt + t} \qquad (11.17)$$

A note regarding symbols. In some books, dielectric constant, K is called relative permittivity and is denoted by ε_r. We shall use the term dielectric constant and denote it by K. However, for the sake of completeness, we have made a mention of equivalent term relative permittivity and its symbol ε_r.

We can also calculate the capacitance by calculating the potentials. From Fig. 11.4, you can also see that

$$d = d_1 + d_2 + t \qquad (11.18)$$

Now suppose

V_1 is the potential difference between the upper plate of the capacitor and the upper surface of the dielectric,

V_2 is the potential difference between the upper and lower surfaces of the dielectric, and

V_3 is the potential difference between the lower surface of the dielectric and the lower plate of the capacitor.

The three potential differences are shown in Fig. 11.5.

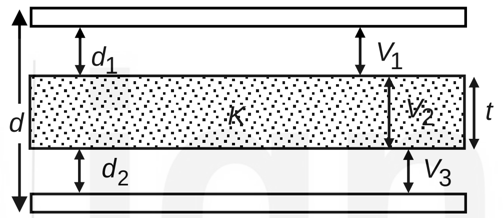


Fig. 11.5: **Calculation of capacitance from the potential differences**.

The total voltage V across the capacitor is the sum of these three potential differences.

Thus, $$V = V_1 + V_2 + V_3 \qquad (11.19)$$

If $\vec{\mathbf{E}}$ is the electric field inside the capacitor, we have

$$V_1 = d_1 E, \qquad V_2 = E\frac{t}{K} \qquad \text{and} \qquad V_3 = d_2 E \qquad (11.20)$$

$$\therefore \qquad V = d_1 E + \frac{Et}{K} + d_2 E = (d_1 + d_2)E + \frac{Et}{K}$$

Using the result $d_1 + d_2 = d - t$ from Eq. (11.18), we can write

$$V = (d - t)E + \frac{Et}{K} = (d - t + \frac{t}{K})E \qquad (11.21)$$

Comparing Eq. (11.21) with the general expression of V in terms of E and d $(V = Ed)$, we note that the equivalent free space thickness of the dielectric is

$$(d - t + \frac{t}{K}) \qquad (11.22)$$

Therefore, the capacitance C is,

$$C = \frac{\varepsilon_0 A}{d - t + (t/K)} \qquad (11.23)$$

as before [Eq. (11.17)].

You may now like to solve a problem based on what you have learnt so far.

SAQ 4 – Capacitor partially filled with dielectric

A dielectric of dielectric constant 3 is filled in the gap between the plates of a capacitor. Calculate the factor by which the capacitance is increased, if the dielectric is only sufficient to fill up 3/4 of the gap.

You have learnt in Sec. 11.2.1 that capacitors can be used to store charge and energy. You also know that their capacitance (and hence the capacity to store charge and energy) increases if a dielectric material is inserted in the space between the two conductors forming the capacitor. A logical question that follows is: **What is the energy stored in a dielectric medium?** Let us answer this question for a parallel plate capacitor.

11.3.2 Energy Stored in a Dielectric Medium

From Eq. (11.9), you know that the energy stored in a parallel plate capacitor is given by:

$$U = \frac{1}{2}CV^2 = \frac{Q^2}{2C}$$

From Eq. (11.12), you know that the capacitance of a parallel plate capacitor with free space between its plates is given by:

$$C = \frac{\varepsilon_0 A}{d}$$

Also $\quad V = Ed$

Putting these values in the expression of energy U given by Eq. (11.9), we get

$$U = \frac{1}{2}\frac{\varepsilon_0 A}{d}E^2 d^2$$

$$= \frac{1}{2}\varepsilon_0 (Ad)E^2$$

or $\quad \frac{U}{\tau} = \frac{1}{2}\varepsilon_0 E^2$ where the volume $\tau = Ad$ (11.24)

Eq. (11.24) gives the energy per unit volume in the capacitor.

When a dielectric of dielectric constant K fills the space between the plates of the capacitor, then the effective capacitance is given by Eq. (11.15) as

$$C_{dielectric} = \frac{\varepsilon_0 KA}{d}$$

The energy stored in a capacitor with a dielectric material of dielectric constant K inserted between its plates is given by:

$$U = \frac{1}{2}C_{dielectric}V^2$$

or

$$U = \frac{1}{2}\frac{\varepsilon_0 KA}{d}E^2 d^2$$

$$= \frac{1}{2}\varepsilon_0 (Ad) KE^2$$

$$\therefore \quad \frac{U}{\tau} = \frac{1}{2}\varepsilon_0 KE^2 = \frac{1}{2}\vec{\mathbf{E}}.\vec{\mathbf{D}} \qquad (11.25)$$

Here $\vec{\mathbf{D}}$ is the electric displacement in the dielectric. Thus, in the case of a parallel plate capacitor with free space between its plates, the energy stored per unit volume is $\frac{1}{2}\varepsilon_0 E^2$ which becomes $\frac{1}{2}\varepsilon_0 K E^2 = \frac{1}{2}\vec{\mathbf{E}}.\vec{\mathbf{D}}$ when the dielectric material is inserted between the plates of the capacitor.

Thus, the energy stored per unit volume in a dielectric medium is given by:

$$\frac{U}{\tau} = \frac{1}{2}\vec{\mathbf{E}}.\vec{\mathbf{D}}\,\text{Jm}^{-3} \tag{11.26}$$

We have considered here the case of a linear dielectric for which $\vec{\mathbf{E}}$ and $\vec{\mathbf{D}}$ are in the same direction.

So far, we have calculated the capacitance of a parallel plate capacitor. We now determine the capacitance of a spherical capacitor and a cylindrical capacitor.

11.4 CAPACITANCE OF SPHERICAL AND CYLINDRICAL CAPACITORS

Let us first consider a spherical capacitor.

a) Spherical capacitor

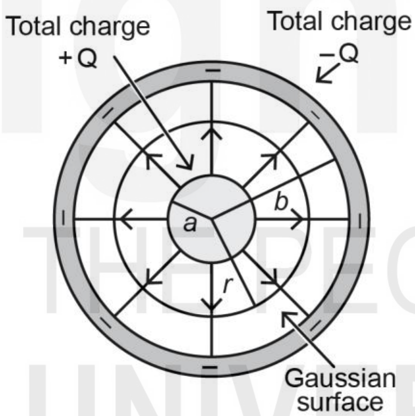


Fig. 11.6: **Spherical capacitor comprising two concentric conducting spheres of radius** *a* **and** *b.*

Fig. 11.6 shows a spherical capacitor comprising two concentric spherical shells of radii *a* and *b*, respectively, such that $b > a$. Suppose that the inner shell carries positive charge *Q*. If σ is the surface charge density, then $Q = 4\pi a^2 \sigma$. Now consider a spherical Gaussian surface *S* of radius *r* lying between the concentric spherical shells. Let us apply Gauss's law to this surface. Note that $\vec{\mathbf{E}}$ is in the radial direction and hence it is parallel to the normal $d\vec{\mathbf{S}}$ to the surface and $\oint_S \vec{\mathbf{E}}.d\vec{\mathbf{S}} = EA,$ where *A* is the area of the spherical shell's surface. Thus, we get

$$\oint_S \vec{\mathbf{E}}.d\vec{\mathbf{S}} = EA = E\,4\pi r^2 = \frac{Q_{en}}{\varepsilon_0} = \frac{4\pi a^2\sigma}{\varepsilon_0}$$

or $$E = \frac{a^2\sigma}{\varepsilon_0}\frac{1}{r^2}$$

In solving the integral for *V* we have used the result

$$\int \frac{dr}{r^2} = -\frac{1}{r}$$

The potential difference $(V_a - V_b)$ is given by

$$V = V_a - V_b = -\int_b^a \vec{\mathbf{E}}.d\vec{\mathbf{r}} = \int_a^b \vec{\mathbf{E}}.d\vec{\mathbf{r}} = \int_a^b E dr \text{ (since } \vec{\mathbf{E}} \text{ is along } d\vec{\mathbf{r}}\text{)}$$

or $$V = \int_a^b \frac{a^2\sigma}{\varepsilon_0}\frac{1}{r^2}dr = -\frac{a^2\sigma}{\varepsilon_0}\frac{1}{r}\bigg|_a^b = \frac{a^2\sigma}{\varepsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right) \quad (11.28)$$

Thus, $$C = \frac{Q}{V} = \frac{4\pi a^2\sigma}{V} = \frac{4\pi\varepsilon_0}{\left(\frac{1}{a}-\frac{1}{b}\right)} = \frac{4\pi\varepsilon_0 ab}{(b-a)} \quad (11.29)$$

When the outer surface is at ∞, $b \to \infty$ and we have $C = 4\pi\varepsilon_0 a$.

b) Cylindrical capacitor

Fig. 11.7a shows the schematic diagram of a cylindrical capacitor. It is made up of two hollow coaxial cylindrical conductors of radii *a* and *b*, respectively. The space between the cylinders is filled with a dielectric of dielectric constant *K*. A slightly enlarged cross-section of this capacitor is shown in Fig. 11.7b.

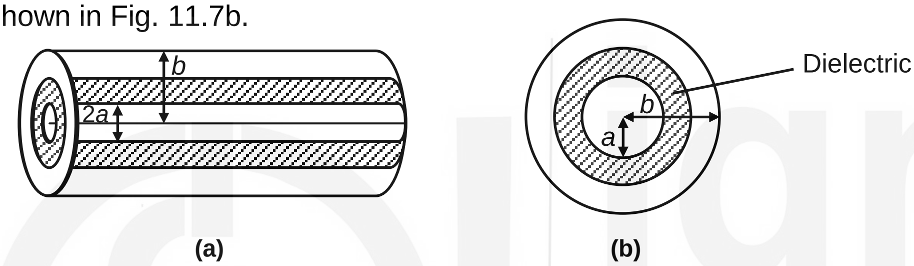


Fig. 11.7: **a) Cylindrical capacitor**; **b) cross-section of the cylindrical capacitor**.

Some examples of such capacitors are:

i) A coaxial cable, in which the inner conductor is a wire and the outer conductor is normally a mesh of conducting wire separated from the inner conductor by an insulator (usually plastic).

ii) The submarine cable, in which a copper conductor is covered by polystyrene (the outer conductor is sea water).

These capacitors are used quite commonly around us and, therefore, it is important to determine their capacitance. We now determine the capacitance per unit length for a cylindrical capacitor.

Since both the inner and outer cylinders of a coaxial capacitor are conductors, they are equipotential surfaces (see Sec. 9.3 of Unit 9). The electric field is radial (normal to the surface of the cylinder). Let λ be the charge per unit length on the inner cylinder of the capacitor shown in Fig. 11.8. The outer cylinder is grounded. An equal and opposite amount of charge will appear on the inner side of the outer cylinder (not shown in the figure). This is because the electric field inside the conductor is zero. To evaluate the electric field for the cylindrical capacitor, we consider a coaxial closed cylindrical Gaussian surface *ABCD* of length ΔL and radius *r*.

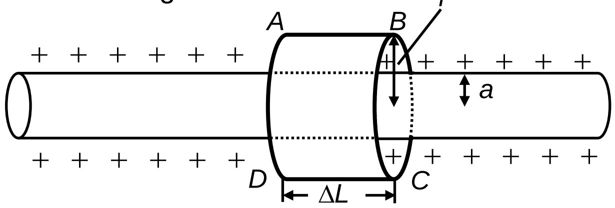


Fig. 11.8: **Calculating the capacitance of a cylindrical capacitor**. *ABCD* **is the Gaussian surface**.

The electric field is normal to the surface of the inner cylinder. It is also confined in the space between the inner and outer cylinders. The flux of electric displacement vector $\vec{\mathbf{D}}$ through the bottom and top circular surfaces of the Gaussian cylinder *ABCD* is zero as $\vec{\mathbf{D}}$ is parallel to these faces and therefore perpendicular to $d\vec{\mathbf{S}}$ (since by definition, $d\vec{\mathbf{S}}$ is perpendicular to the surface) so that $\vec{\mathbf{D}}.d\vec{\mathbf{S}}$ is zero.

The flux of $\vec{\mathbf{D}}$ is only through the curved part of the surface of *ABCD*. Since $\vec{\mathbf{D}}$ is in the radial direction, it is normal to this part of the surface at all points and parallel to $d\vec{\mathbf{S}}$ so that $\vec{\mathbf{D}}.d\vec{\mathbf{S}}$ is non-zero. Therefore, the flux through the curved part of the closed Gaussian surface is given by:

$$\vec{\mathbf{D}}.d\vec{\mathbf{S}} = \vec{\mathbf{D}}.\hat{\mathbf{n}}d\vec{\mathbf{S}} = D(2\pi r)\Delta L = \text{charge enclosed } = \lambda\Delta L \qquad (11.30)$$

or
$$D = \frac{\lambda}{2\pi r} = \varepsilon E = \varepsilon_0 KE \qquad (11.31)$$

where $\lambda\ \Delta L$ is the free charge enclosed by the Gaussian surface and K is the dielectric constant. Hence,

$$E = \frac{\lambda}{2\pi r\varepsilon_0 K} \qquad (11.32)$$

To find the capacitance, we need to calculate the potential difference between the inner and outer cylinders. In Unit 9, you have studied that the general expression for the potential difference for a continuous charge distribution is given by:

$$V_a - V_b = -\int_b^a \vec{\mathbf{E}}.d\vec{\mathbf{r}} = \int_a^b \vec{\mathbf{E}}.d\vec{\mathbf{r}} \qquad (11.33)$$

Since for cylindrical symmetry, $\vec{\mathbf{E}}$ and $d\vec{\mathbf{r}}$ are in the same direction, $\vec{\mathbf{E}}.d\vec{\mathbf{r}} = E\,dr$ and, therefore,

$$V = \int_a^b E.dr$$

or
$$V = \int_a^b \left(\frac{\lambda}{2\pi\varepsilon_0 K}\right)\frac{dr}{r} = \left(\frac{\lambda}{2\pi\varepsilon_0 K}\right)[\ln r]_a^b$$

or
$$V = \frac{\lambda}{2\pi\varepsilon_0 K}\ln\left(\frac{b}{a}\right) \qquad (11.34)$$

Thus, the capacitance per unit length of the cylindrical capacitor is given by:

$$C = \frac{\lambda}{V} = \frac{2\pi\varepsilon_0 K}{\ln(b/a)} \qquad (11.35)$$

In the expression for the capacitance per unit length of a cylindrical capacitor, Eq. (11.35), we find that the capacitance depends **only on the ratio of the radii of the inner and outer cylinders of the capacitor and not on their absolute values**. You may like to solve a couple of problems to calculate the capacitance of different capacitors. Try the following SAQ.

SAQ 5 - Capacitance of cylindrical and spherical capacitors

a) Two cylindrical capacitors are of equal length and have the same dielectric. In one of them, the radii of the inner and outer cylinders are 8 cm and 10 cm, respectively, and in the other they are 4 cm and 5 cm. Determine the ratio of their capacitances.

b) The thickness of air layer in a spherical capacitor is 4.0 cm. The capacitor has the same capacitance as the capacitance of an insulated conducting sphere of diameter 30 cm. Calculate the radii of the surfaces of the spherical capacitor.

11.5 CAPACITORS IN SERIES AND IN PARALLEL

Just like resistors, capacitors are connected in many different ways in electrical circuits, for example, in series, in parallel or in their combinations. In this section, we shall determine the equivalent capacitor of capacitors connected in series and in parallel. The underlying principle is that the **equivalent capacitor holds the same charge when kept at the same potential difference as the combination of the capacitors**. The capacitance of that capacitor is known as the **effective capacitance** of the combination.

We first determine the effective capacitance of **capacitors connected in parallel**.

11.5.1 Combination of Capacitors in Parallel

Fig. 11.9 shows two capacitors connected in parallel. In this combination, we find that

- the potential difference between the plates remains the same; and
- the total charge is the sum of the charge on each capacitor (since more area is available for storing charges).

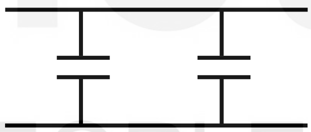


Fig. 11.9: **Two capacitors connected in parallel**.

We now determine the effective capacitance of the combination of three capacitors in parallel shown in Fig. 11.10.

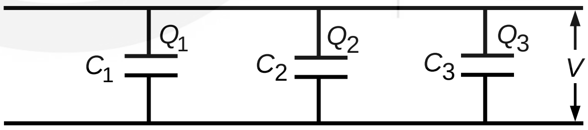


Fig. 11.10: **Three capacitors connected in parallel**.

Here C_1, C_2 and C_3 are the capacitances of the individual capacitors. The charges on them are Q_1, Q_2 and Q_3 respectively and V is the potential difference between the plates of each capacitor. Let C be the **effective capacitance** of the combination.

The total charge Q of the parallel combination is

$$Q = Q_1 + Q_2 + Q_3 \tag{11.36}$$

Since the potential difference V for the parallel combination of the capacitors is the same as for individual capacitors, we have

$$C = \frac{Q}{V} = \frac{Q_1 + Q_2 + Q_3}{V} = \frac{Q_1}{V} + \frac{Q_2}{V} + \frac{Q_3}{V}$$

or

$$\boxed{C = C_1 + C_2 + C_3} \tag{11.37}$$

Thus, the effective capacitance of a parallel combination of capacitors is equal to the sum of the individual capacitances.

Let us now consider the case of capacitors connected in series in a circuit.

11.5.2 Combination of Capacitors in Series

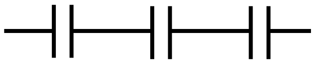


Fig. 11.11: **Capacitors in series**.

Fig. 11.11 shows the combination of capacitors connected in series. In this combination, we find that

- If a voltage source is connected across the two end plates of the first and last capacitor of the series, equal charges are induced in each capacitor; and
- the potential difference across each capacitor depends upon its capacitance.

Let us determine the effective capacitance of a combination of three capacitors in series shown in Fig. 11.12.

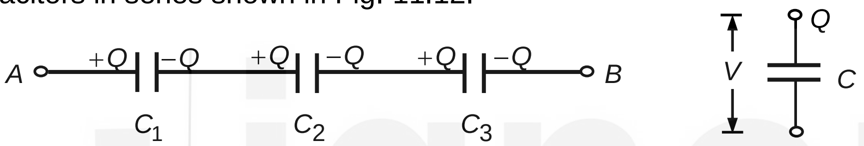


Fig. 11.12: **Capacitors in series**.

Here C_1, C_2 and C_3 are the capacitances of the individual capacitors. When a voltage V is applied across this combination at terminals A and B, a charge $+Q$ is induced on one plate, which induces a charge $-Q$ on the other plate. The other plates acquire equal and opposite charges, because of electrostatic induction. The potential drop across each capacitor is inversely proportional to its capacitance (since $C = Q/V, V = Q/C$). Since Q is fixed, $V \propto 1/C$.

Thus, for the potential drops across the capacitors, we have

$$V_1 \propto 1/C_1, \quad V_2 \propto 1/C_2 \quad \text{and} \quad V_3 \propto 1/C_3$$

Now, we replace the three capacitors by a single capacitor of capacitance C that holds the charge Q when subjected to the potential difference $V = V_1 + V_2 + V_3$. The capacitance C is known as the **effective capacitance** of the combination. Thus, we have

$$C = \frac{Q}{V} \quad \text{or} \quad \frac{1}{C} = \frac{V}{Q}$$

But $V = V_1 + V_2 + V_3$. Therefore,

$$\frac{1}{C} = \frac{V_1 + V_2 + V_3}{Q} = \frac{V_1}{Q} + \frac{V_2}{Q} + \frac{V_3}{Q}$$

or

$$\boxed{\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}} \tag{11.38}$$

Thus, **for capacitors connected in series, the reciprocals of the capacitances add to give the reciprocal of the effective capacitance**.

Before studying the last section of this unit, you may like to work out an SAQ.

SAQ 6 – Capacitors in combinations

a) Determine the equivalent capacitance of the combination of capacitors shown in Fig. 11.11 and the voltage drop across each capacitor given that $C_1 = 0.05\ \mu F$, $C_2 = 0.02\ \mu F$, $C_3 = 0.01\ \mu F$ and $V = 220$ V.

b) Calculate the effective capacitance of three capacitors arranged in such a way that two of them (C_1 and C_2) are in series and the third (C_3) is in parallel with this series combination.

11.6 APPLICATIONS OF DIELECTRICS IN PRACTICAL CAPACITORS

You have learnt in Unit 10 and this unit that dielectrics are used very widely in capacitors. Although the actual requirements vary depending on the application, there are certain characteristics which are desirable for their use in capacitors. In general, a practical capacitor should be small, have high resistance, be capable of being used at high temperatures and have long life. From a commercial point of view it should also be cheap.

A variety of dielectric materials such as kraft paper, thin films, ceramics, etc. are used in different capacitors having varied functions. For example, specially prepared **thin kraft paper**, free from holes and conducting particles, is used in **power capacitors** where withstanding high voltages is more important than incurring dielectric losses. In addition, the kraft paper is impregnated with a suitable liquid such as chlorinated di phenyl. This increases the dielectric constant. This reduces the size of the capacitor and in addition, the breakdown voltage is increased.

Thin films of Teflon, mylar or polythene used in capacitors not only reduce their sizes but also have high resistivity. Teflon is used at high frequencies as it has low loss. In such capacitors, an electrolyte is deposited on the impregnating paper. The size of such a capacitor is small as the film is very thin. Polarity and the maximum operating voltage are important specifications for these capacitors.

Some ceramics can be used as temperature compensators in electronic circuits. High dielectric constant materials, where small variations in dielectric constant with temperature can be tolerated, help in miniaturising capacitors. Barium titanate and its modifications are the best examples of such materials.

Let us now study some of the common capacitors that use such dielectrics.

Capacitors may be broadly classified into two groups: **fixed capacitors** and **variable capacitors**. They may be further classified according to their construction and use as follows:

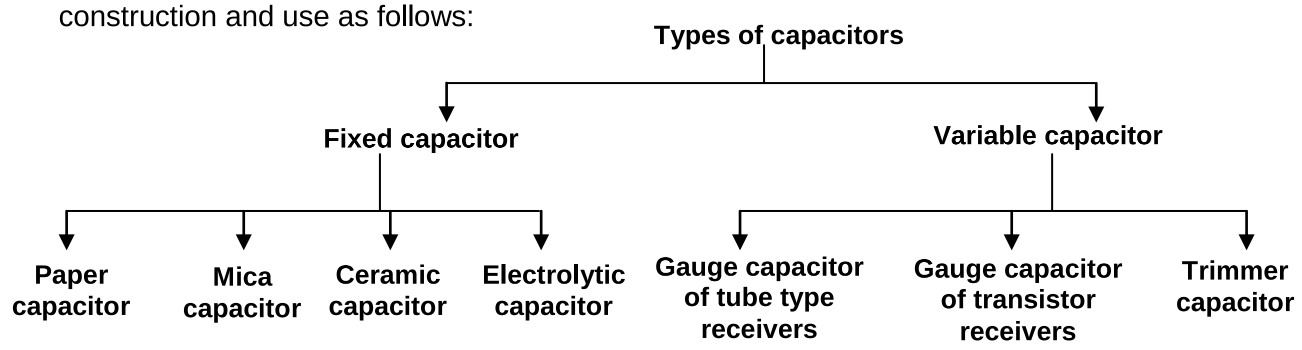


Fig. 11.13: **Classification of Practical Capacitors**.

We now briefly discuss some practical capacitors listed in Fig. 11.13.

Fixed Capacitors

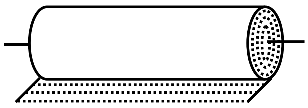


Fig. 11.14: **Fixed capacitors**.

As their name suggests, these capacitors have fixed capacitance. These are essentially parallel plate capacitors, but compact enough to occupy much less space. These capacitors consist of two very thin layers of metal coated on the two surfaces of mica or paper having a uniform coating of paraffin. The mica or paper forms the dielectric between the conductors. This arrangement is rolled up so that the capacitor is in a compact form (Fig. 11.14).

Though paraffin-wax paper capacitors are cheaper, they absorb a good amount of power. For this reason these capacitors are used in alternating current circuits, radio-sets, etc.

Ceramic Capacitors

These are low loss capacitors at all frequencies. Ceramic materials can be made to have very high relative permittivity. For example, for Teflon $\varepsilon = 8$ but when titanium is added to it, the value of ε becomes 100 and when barium titanate is added to it, the value of ε may be increased to 5000. Each piece of such dielectric is coated with silver on the two sides to form a capacitor of large capacitance.

Yet another advantage with these ceramic dielectrics is that they have negative temperature coefficient. Ceramic capacitors are widely used in transistor circuits.

Electrolytic Capacitors

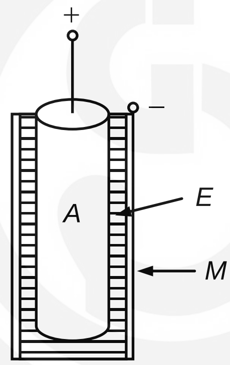


Fig. 11.15: **Wet type electrolytic capacitor.**

An electrolytic capacitor consists of two electrodes of aluminium, called the positive and negative plates. The positive plate is electrolytically coated with a thin layer of aluminium oxide. This coating serves as the dielectric. The two electrodes are in contact through the electrolyte which is a solution of glycerine and sodium (or a paste of borates, for example, ammonium borate). There are two types of electrolytic capacitors – the wet type and the dry type.

In the wet type electrolytic capacitor (Fig. 11.15), the positive plate (*A*) is in the form of a cylinder and presents a large surface area. This is immersed in the electrolyte (*E*) contained in a metal can (*M*). This can act as a negative plate.

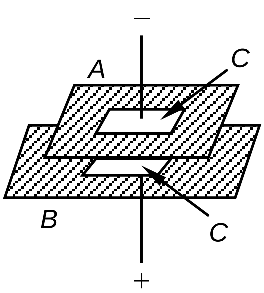


Fig. 11.16: **Dry type electrolytic capacitor**.

In the dry type electrolytic capacitor (Fig. 11.16), both plates are in the form of long strips of aluminium foils. Aluminium oxide is deposited electrically on one (*A*) of the foils. This is kept separated from the other (*B*) by cotton gauze (*C*) soaked in the electrolyte. It is then rolled up into a cylindrical form. The oxide films on aluminium offer a low resistance to current in one direction and a very high resistance in the other direction. Hence **an electrolytic capacitor must be placed in a DC circuit such that the potential of the oxide plate is always positive relative to the other plate**.

Variable Air Capacitor/Gang Capacitor

A very common capacitor whose capacitance can be varied continuously is used for tuning in a radio station. The capacitance of this capacitor can be uniformly varied by rotating a knob (Fig. 11.17).

Fig. 11.17: **Variable air capacitor**. (Source: freewebs.com)

The capacitor consists of two sets of semicircular aluminium plates. One set of the plates is fixed and the other set of plates can be rotated by the knob. As it is rotated, the moving set of plates gradually gets into (or comes out of) the space between the fixed set of plates. The area of overlap between the two sets of plates can thus be uniformly varied. This changes the capacitance of the capacitor. The air between the plates acts as the dielectric. Usually it consists of two capacitors attached to the same knob (ganged). When the knob is rotated, the variation of capacitance in both the plates takes place simultaneously. This type of capacitor is widely used in wireless sets and electronic circuits.

Voltage Rating of a Capacitor

Capacitors are designed and manufactured to operate at a certain maximum voltage which depends on the distance between the plates of the capacitor. If the voltage is exceeded, the electrons jump across the space between the plates and this can result in permanent damage to the capacitor. The maximum safe voltage is called the working voltage. The capacitance and the working voltage (WV) are marked on the capacitor in the case of bigger capacitors. These are indicated by the colour code (similar to that of resistance) in the case of capacitors having low values of capacitance.

In Table 11.2, we give the capacitance range, maximum rating voltage and use of different types of capacitors.

Table 11.2: **Range, ratings and uses of different types of capacitors**

Type of Dielectric	Capacitance Range	Maximum Rating Voltage	Remarks
Paper	250 pF – 10 μF	150 kV	Cheap, used in circuits where losses are not important.
Mica	25 pF – 0.25 μF	2 kV	High quality, used in low loss circuit.
Ceramic	0.5 pF – 0.01 μF	500 kV	High quality used in low loss precision circuit where miniaturisation is important.
Electrolytic (Aluminium oxide)	1μF – 1000 μF	600 V at small capacitance	Used where large capacitance is needed.

We now summarise what you have studied in this unit.

11.7 SUMMARY

Concept	Description
Capacitance	■ Any device which can store charges is a **capacitor**. The **capacitance** of a parallel plate capacitor with free space between its plates is given by: $$C = \frac{Q}{V} = \frac{\varepsilon_0 A}{d}$$ The **capacitance** of a parallel plate capacitor with a dielectric material of dielectric constant K inserted between its plates is given by: $$C = \frac{\varepsilon_0 AK}{d}$$
Energy stored in capacitor	■ The energy stored in a capacitor is given by: $$U = \frac{1}{2}CV^2 = \frac{Q^2}{2C}$$
Effect of dielectric on capacitance	■ If an insulator of thickness 't' is introduced between the two plates of a parallel plate capacitor, its resultant capacitance is given by: $$C = \frac{\varepsilon_0 A}{[d - t + (t/K)]}$$
Energy stored in dielectric medium	■ The energy stored per unit volume in a dielectric medium is given by: $$\frac{1}{2}\vec{\mathbf{E}}.\vec{\mathbf{D}}$$
Spherical capacitor	■ The capacitance of a spherical capacitor is $$C = \frac{4\pi\varepsilon_0 ab}{(b-a)}$$ where a and b are the radii of inner and outer spherical shells.
Cylindrical capacitor	■ The capacitance per unit length of a cylindrical capacitor with dielectric is given by: $$C = \frac{2\pi\varepsilon_0 K}{\ln(b/a)}$$
Capacitors in series	■ The resultant capacitance of two capacitors C_1 and C_2 connected in series is given by: $$C = \left[\frac{1}{C_1} + \frac{1}{C_2}\right]^{-1} = \frac{C_1C_2}{C_1 + C_2}$$
Capacitors in parallel	■ The resultant capacitance of two capacitors C_1 and C_2 connected in parallel is given by: $$C = C_1 + C_2$$

11.8 TERMINAL QUESTIONS

1. A parallel plate capacitor has n similar plates of cross-sectional area A at equal spacing d, with the alternate plates connected together. If dielectric of dielectric constant K is filled between these plates, show that its capacitance is equal to $(n-1)\varepsilon_0 KA/d$.

2. The plates of a parallel plate capacitor are separated by a distance of 0.5 cm. What should the potential difference between the plates be so that

the force of gravity on a proton equals the force on it due to the electric field? Take the mass of proton as 1.67×10^{-27} kg.

3. A capacitor is made up of two hollow concentric metal spheres of radii a and b such that $b > a$ (Fig. 11.18). The space between the concentric spheres is filled with a dielectric material of dielectric constant K and the outer sphere is earthed. Determine the capacitance of the capacitor.

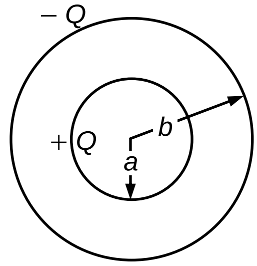


Fig. 11.18: Diagram for TQ 3.

4. In the arrangement shown in Fig. 11.19, obtain the condition on the capacitances such that when a voltage is applied between the terminals A and B, the voltage between terminals C and D is zero.

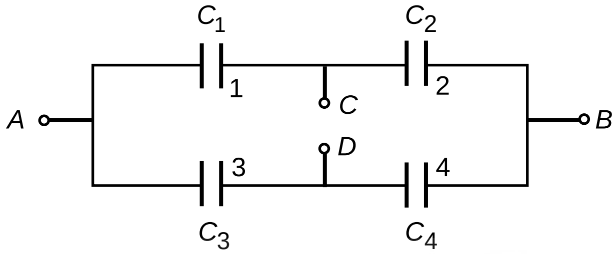


Fig. 11.19: **Diagram for TQ 4.**

5. Two capacitors, one charged and the other uncharged, are connected in parallel. Show that the final energy of the combination is less than the sum of the initial energy of the individual capacitors. Derive the formula for the loss of energy in terms of the initial charges and the capacitances of the two capacitors.

6. Consider a parallel plate capacitor of area A with distance d between the plates. The capacitor is filled equally with two dielectric materials of dielectric constants K_1 and K_2 as shown in Fig. 11.20. Calculate the capacitance of the arrangement. What is its capacitance when $K_1 = K_2$?

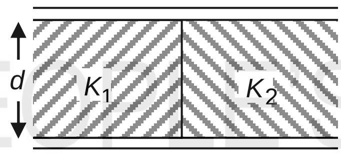


Fig. 11.20: **Diagram for TQ 6.**

7. The space of thickness d between the plates of a parallel plate capacitor is filled with two charge-free slabs of dielectric material each of thickness $d/2$ and dielectric constants K_1 and K_2, respectively (Fig. 11.21). Here d is the distance between the capacitor's plates. The free charge densities on the upper and lower plates are $+\sigma$ and $-\sigma$, respectively.

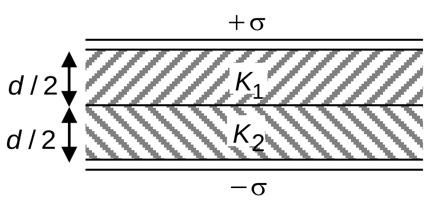


Fig. 11.21: **Diagram for TQ 7**

Obtain the

a) electric displacement in each slab;
b) electric field in each slab;
c) potential difference between the plates; and
d) capacitance of the capacitor assuming the area of the plates to be A.

8. Consider two concentric metallic spherical shells of radii a and c, respectively. The region between the shells is partially filled with a dielectric as shown in Fig. 11.22. Calculate the surface charge densities at $r = b$ and $r = c$. Calculate the potential difference between the outer

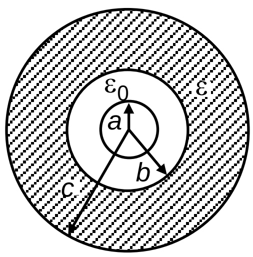


Fig. 11.22: **Diagram for TQ 7.**

and inner shells when a charge Q is placed on the inner shell. What is the capacitance of this arrangement?

11.9 SOLUTIONS AND ANSWERS

Self-Assessment Questions

1. From Eq. (11.5), the capacitance of a spherical shell of radius R (in m) is

$$C = 4\pi\varepsilon_0 R = 4\pi \times 8.85 \times 10^{-12} \times 6000 \times 10^3 \text{ F} = 6.67 \times 10^{-4} \text{ F}$$

2. Let $+\sigma$ and $-\sigma$ be the surface charge densities on the upper and lower plates of a parallel plate capacitor (Fig. 11.23). Consider a Gaussian surface lying inside the upper plate and in the space between the plates. To apply Gauss's law to this surface, we first calculate the value of the integral $\oint_S \vec{\mathbf{E}}.d\vec{\mathbf{S}}$ for it.

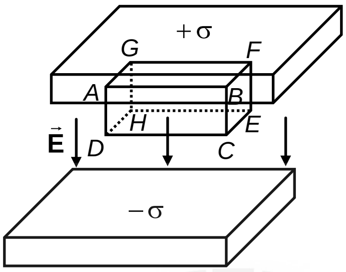


Fig. 11.23: **Diagram for answer of SAQ 2.**

Its value through the surface $ABFG$ is zero as the electric field $\vec{\mathbf{E}}$ inside the conducting surface is zero.

Similarly its values through the surfaces $ABCD$, $EFGH$, $BCEF$ and $ADHG$ are zero since these surfaces are parallel to the electric field.

Since $\vec{\mathbf{E}}$ is normal to the surface $DCEH$, the value of the integral through the surface $DCEH$ is equal to EA where A is area of the surface. From Gauss's law this integral is equal to $\dfrac{Q_{en}}{\varepsilon_0}$, where Q_{en} is the charge enclosed by the Gaussian surface. Thus, we have

$$EA = \frac{Q_{en}}{\varepsilon_0} = \frac{\sigma A}{\varepsilon_0} \quad \Rightarrow \quad E = \frac{\sigma}{\varepsilon_0}$$

3. The capacitance of the capacitor is given by

$$C = \frac{\varepsilon_0 A}{d} = \frac{8.85 \times 10^{-12} \times (1.0 \times 10^{-2})^2}{1.0 \times 10^{-4}} \text{ F} = 8.85 \text{ pF}$$

Energy stored is given by $U = \dfrac{1}{2}CV^2$. Thus,

$$U = [\frac{1}{2} \times 8.85 \times 10^{-12} \times (1.5)^2] \text{ J} = 9.95 \times 10^{-12} \text{ J} \approx 1.0 \times 10^{-11} \text{ J}$$

4. We know that $K = \dfrac{\text{Capacitance with the dielectric}}{\text{Capacitance with free space}}$

Here $K = 3$. Thus the capacitance increases by a factor of 3 when the dielectric is introduced. From Eq. (11.15), the capacitance of the capacitor without the dielectric is given by $C_{free\,space} = \dfrac{\varepsilon_0 A}{d}$. If a dielectric material of thickness t is introduced in the parallel plate capacitor with gap d, its capacitance is given by Eq. (11.17)/(11.23) as

$$C_{dielectric} = \frac{\varepsilon_0 A}{(d - t + t/K)}$$

$$\therefore \frac{C_{dielectric}}{C_{free\,space}} = \frac{d}{(d - t + t/K)}$$

Here $t = \frac{3}{4}d$ and $K = 3$.

$$\therefore \quad d - t + t/K = d - \frac{3}{4}d + \frac{3d}{4\times 3} = \frac{d}{2}$$

and $$\frac{C_{dielectric}}{C_{free\,space}} = \frac{d}{\frac{d}{2}} = 2$$

That is, the capacitance will get doubled.

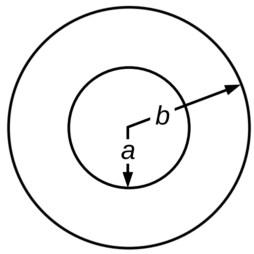


Fig. 11.24: Diagram for answer of SAQ 5b.

5. a) From Eq. (11.35), $C_1 = \frac{2\pi\varepsilon_0 K}{\ln(10/8)}$ and $C_2 = \frac{2\pi\varepsilon_0 K}{\ln(5/4)}$

$$\therefore \quad \frac{C_1}{C_2} = \frac{\ln(5/4)}{\ln(10/8)} = 1 \qquad \text{or} \qquad C_1 = C_2$$

b) Refer to Fig. 11.24 which shows two concentric spherical shells of radii a and b. As per the problem,

$$(b-a) = 4.0\,\text{cm} = 0.04\,\text{m}. \qquad \text{(i)}$$

From Eq. (11.29), we know that the capacitance of spherical capacitor is given as

$$C_1 = \frac{4\pi\varepsilon_0 ab}{(b-a)} = \frac{4\pi\varepsilon_0 ab}{(0.04\,\text{m})} \qquad \text{(ii)}$$

Also, from Eq. (11.5), we know that the capacitance of an insulated conducting sphere having radius a can be written as

$$C_2 = 4\pi\varepsilon_0 a$$

As per the problem, the radius a of the isolated sphere is $[(30\,\text{cm})/2] = 15\,\text{cm} = .15\,\text{m}.$

So,

$$C_2 = 4\pi\varepsilon_0\,(0.15\,\text{m}) \qquad \text{(iii)}$$

As per the problem,

$$C_1 = C_2$$

Thus, from Eqs. (ii) and (iii), we have

$$\Rightarrow \quad \frac{ab}{(0.04\,\text{m})} = (0.15\,\text{m})$$

$$\Rightarrow \quad ab = 0.006\,\text{m}$$

And, we can write

$$(b+a)^2 = (b-a)^2 + 4ab$$

$$= (0.04\,\text{m})^2 + 4\times(0.006\,\text{m})$$

$$b + a = 0.16\,\text{m} \qquad \text{(iv)}$$

So, we have from Eqs. (i) and (iv),

$$a = 6.0\,\text{cm} \quad \text{and} \quad b = 10\,\text{cm}$$

6. a) When the capacitors are connected in series, the equivalent capacitance C is given by:

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}$$

$$\therefore \quad \frac{1}{C} = \left(\frac{1}{0.05} + \frac{1}{0.02} + \frac{1}{0.01}\right) \quad \Rightarrow \quad C = \frac{1}{170}\,\mu\text{F}$$

and $$Q = CV = \frac{1}{170} \times 10^{-6} \times 220\ \text{C} = 1.3 \times 10^{-6}\ \text{C}$$

$$V_1 = \frac{Q}{C_1} = \frac{1.3 \times 10^{-6}}{0.05 \times 10^{-6}}\ \text{V} = 26\ \text{V}$$

$$V_2 = \frac{Q}{C_2} = \frac{1.3 \times 10^{-6}}{0.02 \times 10^{-6}}\ \text{V} = 65\ \text{V}$$

$$V_3 = \frac{Q}{C_3} = \frac{1.3 \times 10^{-6}}{0.01 \times 10^{-6}}\ \text{V} = 1.3 \times 10^{2}\ \text{V}$$

b) The arrangement is shown in Fig. 11.25. Let C_4 be the effective capacitance of C_1 and C_2. Using the result for capacitors in series, we have

$$\frac{1}{C_4} = \frac{1}{C_1} + \frac{1}{C_2} \quad \text{or} \quad C_4 = \frac{C_1 C_2}{C_1 + C_2}$$

The capacitance C_4 then adds to C_3 to give the total capacitance C of the combination:

$$C = C_4 + C_3 \quad \text{or} \quad C = C_3 + \frac{C_1 C_2}{C_1 + C_2}$$

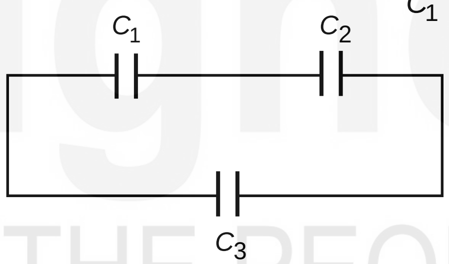


Fig. 11.25: **Diagram for answer to SAQ 6b.**

Terminal Questions

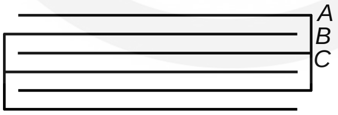


Fig. 11.26: **Diagram for answer to TQ 1.**

1. From Fig. 11.26, you can see that *n* plates provide (*n* –1) capacitors connected in parallel. Dielectric of dielectric constant *K* is filled between these plates. For example in Fig. 11.26, the first 3 plates *A*, *B*, *C* give two capacitors *AB* and *BC*, and so on.

   The effective capacitance *C* of (*n*–1) capacitors, of equal capacitance connected in parallel is equal to the sum of the individual capacitance of all capacitors:

$$C = (n-1) \times \text{capacitance of a single capacitor} = (n-1)K\frac{\varepsilon_0 A}{d}$$

2. Let *V* be the required potential. Then $E = V/d = V/(5\times10^{-3})\ \text{Vm}^{-1}$

   and the force on the proton due to the electric field is

$$qE = (200 \times 1.6 \times 10^{-19}\ V)\text{N}$$

   The gravitational force on the proton is

$$mg = (1.67 \times 10^{-27} \times 9.8)\text{N}$$

   Equating the two we get

$$V = \frac{(1.67 \times 10^{-27} \times 9.8)}{(200 \times 1.6 \times 10^{-19})}\ \text{V} = 5 \times 10^{-10}\ \text{V}$$

3. If a charge $+Q$ is placed on the inner sphere of radius 'a', an equal and opposite amount of charge appears on the inner side of the outer sphere. The electric field gets confined to the space between the concentric spheres. To determine the capacitance, we have to calculate the displacement $\vec{\mathbf{D}}$. We consider a spherical Gaussian surface S of radius r lying in the dielectric (Fig. 11.27). The displacement $\vec{\mathbf{D}}$ is normal to this surface and so applying Gauss's law, we get

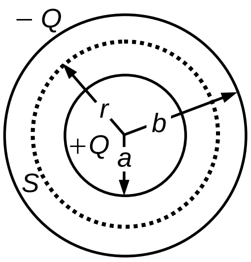


Fig. 11.27: **Diagram for answer to TQ 3.**

$$4\pi r^2 D = Q$$

Since $\vec{\mathbf{D}} = \varepsilon_0 K \vec{\mathbf{E}}$, we have

$$E = \frac{D}{\varepsilon_0 K} = \left(\frac{Q}{4\pi\varepsilon_0 K}\right)\left(\frac{1}{r^2}\right)$$

The potential of the inner sphere with reference to the outer sphere is

$$V = V_a - V_b = -\int_b^a \vec{\mathbf{E}}.d\vec{\mathbf{r}} = \int_a^b \vec{\mathbf{E}}.d\vec{\mathbf{r}} = \int_a^b \frac{Q}{4\pi\varepsilon_0 K}\frac{dr}{r^2}$$

because $\vec{\mathbf{E}}.d\vec{\mathbf{r}} = E\,dr$. The outer sphere is earthed and is, therefore, at zero potential and $V_b = 0$.

$$\therefore \qquad V = V_a = \frac{Q}{4\pi\varepsilon_0 K}\left(-\frac{1}{r}\Big|_a^b\right)$$

or $$V = \frac{Q}{4\pi\varepsilon_0 K}\left(\frac{1}{a} - \frac{1}{b}\right)$$

Therefore, the capacitance is given by $C = \dfrac{Q}{V} = \dfrac{4\pi\varepsilon_0 abK}{(b-a)}$

4. The potentials of the two plates of the capacitors 1 and 2 connected to the point C are the same. Hence if a charge q_1 is placed on one of these plates, the other plate will have an equal and opposite charge. Let a voltage be applied between A and B. Suppose a charge q_1 accumulates on capacitor 1 of capacitance C_1 and a charge q_2 accumulates on capacitor 3 of capacitance C_3. Then the potential differences between the plates of the capacitors 1, 2, 3 and 4 are, respectively, given by:

$$\frac{q_1}{C_1}, \frac{q_1}{C_2}, \frac{q_2}{C_3} \text{ and } \frac{q_2}{C_4}$$

If the potential difference between C and D is equal to zero, then the potential difference across C_2 = potential difference across C_4 and the potential difference across C_1 = potential difference across C_3

$$\therefore \qquad \frac{q_1}{C_2} = \frac{q_2}{C_4} \quad \text{and} \quad \frac{q_1}{C_1} = \frac{q_2}{C_3}$$

or $$\frac{q_1}{q_2} = \frac{C_2}{C_4} = \frac{C_1}{C_3}$$

This is the required condition for zero potential difference between *C* and *D*.

5. Let the initial charge on the charged capacitor be *q* and its capacitance be C_1. When this capacitor is connected to the uncharged capacitor of capacitance C_2, then the charge *q* is distributed on both the capacitors.

   Since the capacitors are connected in parallel, the charge is distributed until the potentials of the capacitors are equal. Suppose in this process, a charge q_2 flows from the charged capacitor to the uncharged one.

   Suppose a charge q_1 remains on the initially charged capacitor after the potentials on the capacitors are equal. Then $q_1 = q - q_2$. Since the potentials are equal, we have

$$\frac{q_1}{C_1} = \frac{q - q_2}{C_1} = \frac{q_2}{C_2}$$

   Solving these two equations for q_1 and q_2, we get

$$q_2 = \frac{C_2 q}{(C_1 + C_2)} \qquad \text{(i)}$$

   and

$$q_1 = \frac{C_1 q}{(C_1 + C_2)} \qquad \text{(ii)}$$

   The initial energy E_i of the charged capacitor (before the distribution of the charges) is

$$E_i = \frac{q^2}{2C_1}$$

   The final energy E_f of the two capacitors is given by:

$$E_f = \frac{q_1^2}{2C_1} + \frac{q_2^2}{2C_2}$$

   Substituting for q_1 and q_2 from Eqs. (ii) and (i), we get

$$E_f = \frac{C_1 q^2}{2(C_1 + C_2)^2} + \frac{C_2 q^2}{2(C_1 + C_2)^2} = \frac{q^2}{2(C_1 + C_2)}$$

   Hence the loss in energy is $\frac{q^2}{2}\left[\frac{1}{C_1} - \frac{1}{C_1 + C_2}\right] = \frac{q^2 C_2}{2C_1(C_1 + C_2)}$

6. The arrangement shown in Fig. 11.20 is equivalent to two capacitors of area $\frac{A}{2}$, thickness *d*, which are filled with dielectric materials of dielectric constants K_1 and K_2, respectively. These capacitors are arranged in parallel (because the upper and lower plates of one capacitor are joined with the respective upper and lower plates of the other capacitor). Now from Eq. (11.15), we get

$$C_1 = \frac{\varepsilon_0 K_1 A/2}{d} \quad \text{and} \quad C_2 = \frac{\varepsilon_0 K_2 A/2}{d}$$

$$\therefore \quad C = C_1 + C_2 = \frac{\varepsilon_0 A/2}{d}(K_1 + K_2) = \frac{\varepsilon_0 A}{2d}(K_1 + K_2)$$

   When $K_1 = K_2$, we get the well known result $C = \frac{\varepsilon_0 K A}{d}$.

7. a) Consider the Gaussian surfaces S_1 and S_2 each of area ΔA in the two slabs (Fig. 11.28).

Let the displacements in the two slabs be $\vec{\mathbf{D}}_1$ and $\vec{\mathbf{D}}_2$, respectively.

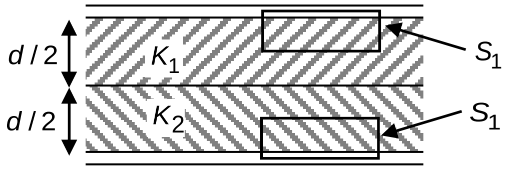


Fig. 11.28: **Diagram for answer to TQ 7.**

Applying Gauss's law to surface S_1, we get

$$\int_{S_1} \vec{\mathbf{D}}_1 . d\vec{\mathbf{S}} = Q_{en}\Big|_{S_1}$$

or $\qquad D_1 \Delta A = \sigma \Delta A \quad \Rightarrow \quad D_1 = \sigma \qquad$ (i)

Similarly for surface S_2, $\qquad D_2 = \sigma \qquad$ (ii)

b) Since $\vec{\mathbf{D}} = \varepsilon_0 K \vec{\mathbf{E}}$, we get $\quad E_1 = \dfrac{D_1}{\varepsilon_0 K_1} = \dfrac{\sigma}{\varepsilon_0 K_1} \qquad$ (iii)

and $\quad E_2 = \dfrac{D_2}{\varepsilon_0 K_2} = \dfrac{\sigma}{\varepsilon_0 K_2} \qquad$ (iv)

c) The potential difference between the plates is given by:

$$V = \int_0^d \vec{\mathbf{E}} . d\vec{\mathbf{r}} = \int_0^{d/2} \vec{\mathbf{E}}_1 . d\vec{\mathbf{r}} + \int_{d/2}^{d} \vec{\mathbf{E}}_2 . d\vec{\mathbf{r}} = \int_0^{d/2} E_1 dr + \int_{d/2}^{d} E_2 dr$$

or $\quad V = E_1 r\Big|_0^{d/2} + E_2 r\Big|_{d/2}^{d} = E_1 \dfrac{d}{2} + E_2 \dfrac{d}{2}$

Using Eqs. (iii) and (iv) in this expression, we get

$$V = \frac{\sigma}{\varepsilon_0} \frac{d}{2} \left(\frac{1}{K_1} + \frac{1}{K_2} \right) \qquad \text{(v)}$$

d) From Eq. (v), $\quad C = \dfrac{Q}{V} = \dfrac{\sigma A \varepsilon_0}{\dfrac{\sigma d}{2}\left(\dfrac{1}{K_1} + \dfrac{1}{K_2}\right)} = \dfrac{2A\varepsilon_0}{d} \dfrac{K_1 K_2}{(K_1 + K_2)}$

8. In order to calculate the surface charge densities at $r = b$ and $r = c$, we need to calculate the polarisation for both cases. We do it as follows: Due to spherical symmetry, the electric fields and the displacements are radial for both cases. Now consider a spherical Gaussian surface of radius r such that $a < r < b$ (Fig. 11.29). Since Q is the charge enclosed by this sphere, from Gauss's law, we have

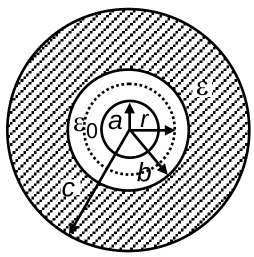


Fig. 11.29: **Diagram for answer to TQ 8.**

$$D_1 4\pi r^2 = Q \quad \Rightarrow \quad D_1 = \frac{Q}{4\pi r^2}$$

Now recall from Unit 6 that the electric field

$$\vec{\mathbf{E}}_1 = \frac{Q}{4\pi \varepsilon_0 r^2} \hat{\mathbf{r}} \qquad a < r < b \qquad \text{(i)}$$

In the same way, we can show that for the region $b < r < c$,

$$\vec{\mathbf{E}}_2 = \frac{Q}{4\pi \varepsilon r^2} \hat{\mathbf{r}} \qquad b < r < c \qquad \text{(ii)}$$

Now, you know that for linear dielectrics,

$$\vec{\mathbf{P}} = \vec{\mathbf{D}} - \varepsilon_0 \vec{\mathbf{E}} = (\varepsilon - \varepsilon_0)\vec{\mathbf{E}} = \varepsilon_0 (K-1)\vec{\mathbf{E}}$$

Therefore, at $r = b$, $\vec{\mathbf{P}}_1 = \varepsilon_0 (K-1)\vec{\mathbf{E}}_1$

Since the normal to the dielectric surface at $r = b$ is along $-\hat{\mathbf{r}}$, we have

$$\sigma_b\big|_{r=b} = -\vec{\mathbf{P}}_1 . \hat{\mathbf{r}}\Big|_{r=b} = -\varepsilon_0 (K-1)\vec{\mathbf{E}}_1 . \hat{\mathbf{r}}\Big|_{r=b}$$

$$= -\varepsilon_0 (K-1) \frac{Q}{4\pi\varepsilon_0 r^2} \hat{\mathbf{r}} . \hat{\mathbf{r}}\Bigg|_{r=b}$$

$$= -\varepsilon_0 (K-1) \frac{Q}{4\pi\varepsilon_0 b^2}$$

Notice that the unit vector normal to the dielectric's surface points *outward with respect to the dielectric sphere*, which is $+\hat{\mathbf{r}}$ at $r = c$ but $-\hat{\mathbf{r}}$ at $r = b$.

or $$\sigma_b\big|_{r=b} = -\frac{Q(K-1)}{4\pi b^2}$$

To determine σ_b at $r = c$, we follow the same method as above.

Since the normal to the dielectric surface at $r = c$ (see the margin remark) is along $\hat{\mathbf{r}}$, we have

$$\sigma_b\big|_{r=c} = \vec{\mathbf{P}}_2 . \hat{\mathbf{r}}\big|_{r=c} = \varepsilon_0 (K-1)\vec{\mathbf{E}}_2 . \hat{\mathbf{r}}\big|_{r=c}$$

$$= \varepsilon_0 (K-1) \frac{Q}{4\pi\varepsilon c^2}$$

$$= \varepsilon_0 (K-1) \frac{Q}{4\pi\varepsilon_0 K c^2} \qquad \left(\because K = \frac{\varepsilon}{\varepsilon_0}\right)$$

or $$\sigma_b\big|_{r=c} = \frac{Q(K-1)}{4\pi K c^2}$$

To determine the potential difference between the outer and inner shells, we begin from its definition

$$V = -\int_c^a \vec{\mathbf{E}} . d\vec{\mathbf{r}} = \int_a^c \vec{\mathbf{E}} . d\vec{\mathbf{r}} = \int_a^b \vec{\mathbf{E}}_1 . d\vec{\mathbf{r}} + \int_b^c \vec{\mathbf{E}}_2 . d\vec{\mathbf{r}}$$

Substituting $\vec{\mathbf{E}}_1$ and $\vec{\mathbf{E}}_2$ from Eqs. (i) and (ii), we get

$$V = \frac{Q}{4\pi\varepsilon_0} \int_a^b \frac{dr}{r^2} + \frac{Q}{4\pi\varepsilon_0 K} \int_b^c \frac{dr}{r^2} \qquad (\because \varepsilon = \varepsilon_0 K)$$

$$= -\frac{Q}{4\pi\varepsilon_0} \left(\frac{1}{r}\right)\Bigg|_a^b - \frac{Q}{4\pi\varepsilon_0 K} \left(\frac{1}{r}\right)\Bigg|_b^c$$

$$= -\frac{Q}{4\pi\varepsilon_0} \left(\frac{1}{b} - \frac{1}{a}\right) - \frac{Q}{4\pi\varepsilon_0 K} \left(\frac{1}{c} - \frac{1}{b}\right)$$

Simplifying the expression for V, we get

$$V = \frac{Q}{4\pi\varepsilon_0} \left[\frac{c-b}{K cb} + \frac{b-a}{ab}\right]$$

and $$C = \frac{Q}{V} = \frac{4\pi\varepsilon_0}{\dfrac{c-b}{K cb} + \dfrac{b-a}{ab}}$$

Compass was used for navigation even in ancient times, when there was not enough understanding about magnetism. You will learn about modern understanding of magnetism in this unit. (Picture source: Wikimedia Commons)

UNIT 12

MAGNETIC FIELD

Structure

STUDY GUIDE

So far, in this course, you have learnt the concept of electric field and electric potential. You have also learnt how dielectric materials respond to electric field. The focus of these discussions has been on learning laws, concepts and techniques which enable us to determine electrostatic force and electric field due to static electric charges and charge distributions.

We now shift our focus to magnetic field. In the present and the next two units, you will study about magnetic field and related concepts. To understand the contents of this unit better, you should look for analogies between magnetic field and electric field. For example, as electric field is produced by static electric charge(s), what produces magnetic field? Are there any laws to determine the value of magnetic field similar to Coulomb's law and Gauss's law for electric field?

You are, therefore, advised to refresh the basic laws of electrostatics such as Coulomb's law and Gauss's law given in Units 5 and 6 of this course. Since you will be using the concepts of vector calculus extensively in this unit, you must revise Units 1 to 4 of this course. Further, while studying this unit, you should notice the differences and similarities between the electric field and the magnetic field. You should try to solve all the SAQs and TQs yourself.

"In questions of science, the authority of a thousand is not worth the humble reasoning of a single individual."

Galileo Galilei

12.1 INTRODUCTION

In Unit 5 (Block 2) of this course, you have learnt the concept of electrostatic force between stationary charges and its description in terms of static electric field and electric potential. You have learnt how to calculate electrostatic force and field using Coulomb's law and Gauss's law. You may now like to know: **What happens when charges are moving?** A moving charge experiences two types of forces: (i) electrostatic force due to the electric field of other charges at rest; and (ii) a magnetic force when it is in the presence of steady flow of charge (i.e., a steady current) or a permanent magnet. Like electrostatic force, the magnetic force is described in terms of a vector field, called magnetic field, which is the topic of this unit. However, there are some major differences between the electric and magnetic fields, which you will discover as you study this unit.

In the science laboratory, during your school days, you must have been fascinated with magnets. Recall that when you tried to push two magnets together in a way they didn't want to go, you felt a mysterious force!

In the 19^{th} century, it was discovered that electric current produce magnetic field. In view of close relation between electric current and magnetic field, we begin this unit by first discussing the concept of electric current and current density in Sec. 12.2. In this section, you will also learn the continuity equation which expresses one of the basic laws of physics – **conservation of charge** – in differential form. In Sec. 12.3, we discuss the sources of magnetic field and define magnetic field in terms of the force experienced by current and charge. You have learnt Gauss's law for electrostatics in Unit 6 of this course. The form that Gauss's law takes for magnetism is discussed in Sec. 12.4. In Sec. 12.5, you will learn Biot-Savart law which gives us a method to determine the magnetic field produced by steady currents. You will also learn how to determine the magnetic field due to steady currents using this law. We end this unit by calculating the force between two parallel current carrying conductors which enables us to arrive at the definition of Ampere – the unit of electric current.

In the next unit, we will continue our discussion of magnetic field and you will learn Ampere's law and its applications for determining magnetic field due to steady currents flowing in different geometries.

Expected Learning Outcomes

After studying this unit, you should be able to:

- ❖ explain the concept of current density and derive continuity equation;
- ❖ describe the conduction mechanism and explain the concept of drift velocity;
- ❖ deduce the relation between electric current and the magnetic field;
- ❖ define the magnetic field at a point in terms of the force on a steady current element and also on a moving charge particle;

- use the expression for the force on a steady current element or on a charged particle due to a magnetic field to calculate the force on certain simple current carrying circuits;
- state and explain Gauss's law for magnetic field;
- use Biot-Savart law to determine the magnetic field generated by a simple current flow; and
- determine the force between two parallel current-carrying conductors.

12.2 ELECTRIC CURRENT AND CURRENT DENSITY

In Block 2 of this course, you have learnt that when a positive charge is placed in an electric field, it experiences the electrostatic force and moves in the direction of the field. If the ends of a conductor, say, a copper wire, are connected to a battery, an electric field $\vec{\mathbf{E}}$ is set up at every point within the conductor. Due to the presence of the field, the electrons present in the wire move in the wire. You know that electric current flows whenever charges move. In the case of a copper wire, the flow of **electrons** constitutes the **electric current**. **It is defined as the amount of charge moving across a given cross-section of the wire per unit time.**

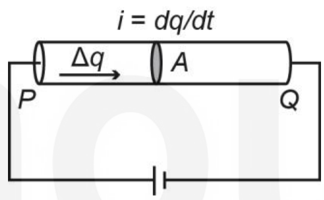


Fig. 12.1: **The instantaneous current along a wire is defined as the net rate at which the charge passes through an area perpendicular to the axis of the wire.**

When the current is not constant, i.e., the current varies with time, we define an instantaneous value of the current *i*(*t*). Refer to Fig. 12.1 which shows a conductor wire *PQ* connected to a battery. If a net charge Δq crosses the shaded area *A* which is perpendicular to the axis of the wire *PQ* (Fig. 12.1) in time Δt, the instantaneous current is given by

$$i(t) = \lim_{t \to 0} \frac{\Delta q}{\Delta t} = \frac{dq}{dt} \qquad (12.1)$$

Eq. (12.1) shows that the unit of current is coulomb per second (C s^{-1}). In the SI system of units, it has been given the name ampere (abbreviated as A).

NOTE

Note that current is a scalar quantity, because both *q* and *t* are scalars. It is not a vector quantity as it does not obey the vector laws. Often, a current in a wire is represented by an arrow. Such arrows are not vectors; they only show a direction (or sense) of flow of charges along a conductor, not a direction in space.

An electric current may consist of either positive or negative charge in motion, or it may involve both positive and negative charges. **By convention, the direction of current is defined as the direction in which the positive charge flows.** If the moving charge is negative, as with electrons in a metal, then the direction of current flow is opposite to the flow of the actual charges. When the current is due to both positive and negative charges, it is determined by the net charge motion; that is, by the algebraic sum of the currents associated with both kinds of charges. For example, when salt (NaCl) is dissolved in water, it splits up into Na^+ ions and Cl^- ion. The sodium ion is positively charged and the chlorine ion is negatively charged. Under the influence of the electric field established between the two electrodes, these ions move through the liquid in opposite direction. Thus the motion of both positive and negative ions contributes to the current in the same direction.

Current Density

As defined earlier, current is the total charge passing through the entire corss-section of a wire per unit time. Therefore, the current is determined by the total charge that flows through the wire. It does not matter whether or not the charge passing through every element of the cross-section of the wire is the same. It is for this reason that current is a **macroscopic quantity**. If the charge passing through various elements of the cross-section of the wire is not the same, it is necessary to define a quantity at every point of the conductor. This is called the **current density** which is a microscopic quantity and it is denoted by $\vec{\mathbf{J}}$. **It is defined as the charge flowing per unit time per unit area-normal-to-flow and has a direction in which the positive charge moves.**

Let us consider a simple system in which particles, each of charge q, are moving to the right as shown in Fig. 12.2. Imagine a small cylindrical surface of cross-sectional area dS around point P. So all the particles crossing this small cylinder may be assumed to have the same speed v. Then, the length of the cylinder through which charges flow in time dt is vdt as shown in Fig. 12.2. So, the volume of the cylinder is $dSvdt$. If n is the number of charged particles per unit volume, then the number of charged particles found in the cylinder is $ndSvdt$. Therefore, the average rate at which charge flows through dS, i.e. the current through dS, is given by

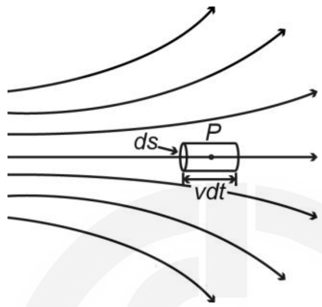


Fig. 12.2: **Calculation of current in terms of velocity of charge.**

$$i = \frac{q(ndSvdt)}{dt} = ndSvq \qquad (12.2)$$

Since current density J, is defined as the current per unit area held normal to the velocity of the current carriers, we have,

$$J = \frac{i}{dS} = nqv \qquad (12.3)$$

Since the direction of J is the direction of the actual flow of charges at that point, the above equation can be written in vector form as

$$\vec{\mathbf{J}} = nq\,\vec{\mathbf{v}} \qquad (12.4)$$

Thus, $\vec{\mathbf{J}}$ is a vector quantity. In SI system of units, $\vec{\mathbf{J}}$ is expressed in amperes per square meter. When the current carriers are electrons, $q = -e$ and Eq. (12.4) takes the form

$$\vec{\mathbf{J}} = -ne\,\vec{\mathbf{v}} \qquad (12.5)$$

The product nq in Eq. (12.4) represents the **volume charge density** ρ of the current carriers. Hence, in terms of ρ, the current density [Eq. (12.4)] is expressed as follows:

$$\vec{\mathbf{J}} = \rho\vec{\mathbf{v}} \qquad (12.6)$$

If current density is uniform over the cross section S of the wire, we can calculate the total current by multiplying the current density by the cross-section of the wire. If the current density is not at right angles to the cross-sectional area, we consider only that component of $\vec{\mathbf{J}}$ which is perpendicular to it. If we define a vector $\vec{\mathbf{S}}$ whose magnitude is the cross-sectional area S

and the direction is along the perpendicular to the area, then a uniform current density $\vec{\mathbf{J}}$ gives rise to a total current $i = \vec{\mathbf{J}}.\vec{\mathbf{S}}$ (Fig. 12.3). When the current density and/or surface orientation vary with position, we can follow the same process for many small areas $d\vec{\mathbf{S}}$, and then sum the result to get the total current (Fig. 12.4). The current through a small area $d\vec{\mathbf{S}}$ is $\vec{\mathbf{J}}.d\vec{\mathbf{S}}$, so that the total current, *i*, through the entire surface is

$$i = \iint_S \vec{\mathbf{J}}.d\vec{\mathbf{S}} \qquad (12.7)$$

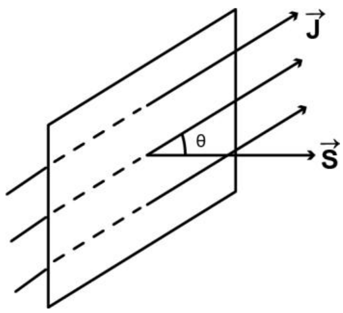


Fig. 12.3: **The current through a surface of area $\vec{\mathbf{S}}$ is given by JS cos θ, or $\vec{\mathbf{J}}.\vec{\mathbf{S}}$ where θ is the angle between the vectors $\vec{\mathbf{S}}$ and $\vec{\mathbf{J}}$.**

where the limit of the integral is chosen to cover the entire surface. Eq. (12.7) should remind you of the definition of the electric flux you have learnt in Unit 6 of Block 2. Indeed, the electric current though a surface is the flux of the current density through that surface. Eq. (12.7) shows that electric current is a scalar quantity because the integral $\vec{\mathbf{J}}.d\vec{\mathbf{S}}$ is a scalar.

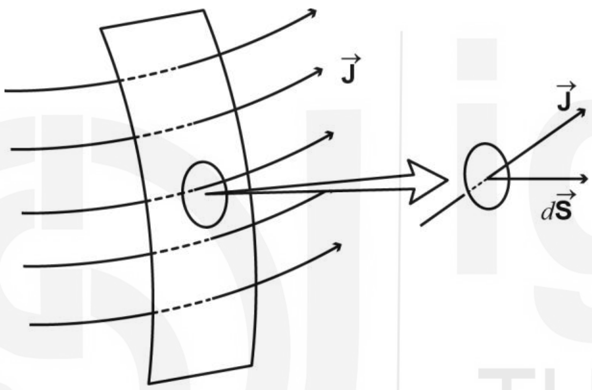


Fig. 12.4: **When the current density and/or surface orientation vary with position, the total current is written as** $i = \iint \vec{\mathbf{J}}.d\vec{\mathbf{S}}$.

In Fig. 12.4, we have taken the surface *S* to be an open surface. In such a situation, the vector $d\vec{\mathbf{S}}$ is taken to be positive in that direction along which the current through *S* is required.

But, when *S* is a closed surface, as shown in Fig. 12.5, the direction of every vector $d\vec{\mathbf{S}}$ is taken along the **outward normal to the surface**. For such closed surfaces, the integral of $\vec{\mathbf{J}}$ over *S* gives the rate at which the charge is going out of the volume enclosed by *S*.

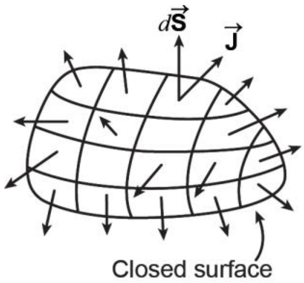


Fig. 12.5: **The integral $\oiint_S \vec{\mathbf{J}}.d\vec{\mathbf{S}}$ over a closed surface is the rate of change of total charge inside.**

Now one of the basic laws of physics is that an electric charge is indestructible; it is never destroyed or created. Electric charges can move from place to place but never appear from nowhere. **We say that the charge is conserved**. Hence, if there is a net current out of a closed surface, it must be equal to the rate at which the total charge within the volume is depleting.

Using Eq. (12.7), we can write the law of conservation of charge as follows:

$$\oiint_S \vec{\mathbf{J}}.d\vec{\mathbf{S}} = -\frac{d}{dt}(q_{\text{inside}}) \qquad (12.8)$$

You know that the charge within the volume can be written as a volume integral of the charge density ρ:

$$q_{\text{inside}} = \iiint_V \rho \, dV \tag{12.9}$$

where *V* is the volume enclosed by surface *S*. Using Eq. (12.9) in Eq. (12.8) we get

$$\oiint_S \vec{\mathbf{J}}.d\vec{\mathbf{S}} = -\frac{d}{dt}\iiint_V \rho \, dV \tag{12.10}$$

Since we are dealing with a fixed volume *V*, the time derivative operates only on the function ρ. Since ρ is a function of spatial coordinates as well as time, the time derivative of ρ is written as the partial derivative with respect to time when it is moved inside the integral. Hence, Eq. (12.10) can be written as

$$\oiint_S \vec{\mathbf{J}}.d\vec{\mathbf{S}} = -\iiint_V \frac{\partial \rho}{\partial t} dV \tag{12.11}$$

The surface integral on the left hand side of the Eq. (12.11) can be converted into a volume integral using the **divergence theorem** (see Sec. 4.7, Unit 4 of Block 1), leading to

According to the Divergence theorem,

$$\oiint_S \vec{\mathbf{J}}.d\vec{\mathbf{S}} = \iiint_V (\vec{\nabla}.\vec{\mathbf{J}})\, dV$$

$$\iiint_V (\vec{\nabla}.\vec{\mathbf{J}})\, dV = -\iiint_V \frac{\partial \rho}{\partial t}\, dV$$

or

$$\iiint_V \left(\vec{\nabla}.\vec{\mathbf{J}} + \frac{\partial \rho}{\partial t}\right) dV = 0 \tag{12.12}$$

Now, since the volume *V* is completely arbitrary, Eq. (12.12) will hold for an arbitrary volume element only when the integrand is zero. Thus, we have

$$\boxed{\vec{\nabla}.\vec{\mathbf{J}} + \frac{\partial \rho}{\partial t} = 0} \tag{12.13}$$

The differential equation [Eq. (12.13)] is known as the **continuity equation**. It expresses the conservation of charge in a differential form. Its meaning is clearer in Eq. (12.12), according to which the change in the quantity of charge in any arbitrary volume must be accompanied by a net flow of charge inwards or outwards across its surface. For steady currents, we have

$$\frac{\partial \rho}{\partial t} = 0 \tag{12.14}$$

This is because a steady current is one for which $\vec{\mathbf{J}}$ is constant in time at every point. In other words, equal charges flow in and flow out of a section and, hence, there cannot be any accumulation of charge at any point of the system. Thus, for steady currents, the continuity equation [Eq. (12.13)] becomes

$$\boxed{\vec{\nabla}.\vec{\mathbf{J}} = 0} \tag{12.15}$$

Before proceeding further, solve an SAQ.

SAQ 1 – Calculating electric current

The amount of charge passing through a cross section of a wire is given by

$$q(t) = (4.5\,\mathrm{C\,s^{-2}})t^2 + 2.5\,\mathrm{C}$$

for t varying between 0 and 5.0 s. a) Write the expression for instantaneous current $i(t)$ in this time interval. b) Calculate the value of current at t = 3.0 s.

Now let us discuss why metals conduct electricity. This will lead us to a relation between current density and electric field causing current flow.

Current Density and Electric Field

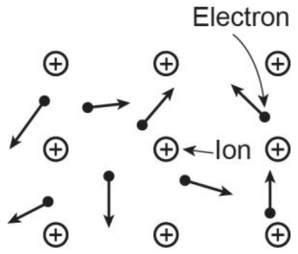


Fig. 12.6: A schematic view of the crystal structure of a metal. The positive metal ions exist on a rigid lattice. Each atom, on forming an ion, gives up one or more electrons, which are then free to wander through the crystal.

In an electrical conductor like metal, the metal ions are fixed in a regular array, known as lattice, making them relatively immobile. The metal ions are positively charged because the atoms forming the metal lose one or more electrons which become free in the sense that these electrons wander through the ion lattice as shown in Fig. 12.6. It is the motion of these negatively charged electrons that gives metals their conducting properties.

When a battery is connected between the ends of a metallic wire MN as shown in Fig. 12.7, we find that current flows through it from M to N (current flowing in the wire can be detected by putting an ammeter in series). Let us find out why and how the current starts flowing in a particular direction by taking a microscopic view of the situation.

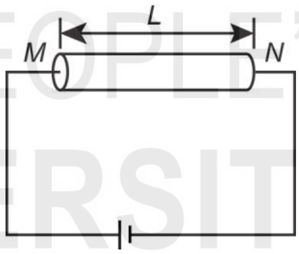


Fig. 12.7: A battery (source of emf) can maintain an electric field within a conducting wire.

When the metallic conductor is not connected to the battery, the free electrons present in the metal are in constant motion because of their thermal energy. Their motion is random and their velocities are oriented randomly as shown in Fig. 12.8a.

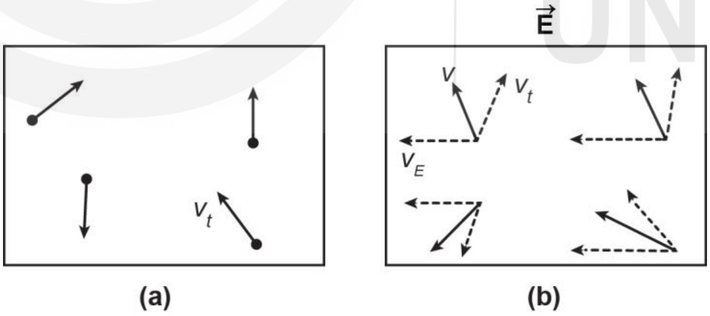


Fig. 12.8: Motion of some free electrons a) in the absence of an external field; b) in the presence of an external field. Here $\vec{v}_t$ represents thermal velocity, $\vec{v}_E$ is the velocity only in the presence of electric field and $\vec{v}$ is the net velocity (as shown by the solid lines).

In this state, the free electrons undergo frequent collisions with positive ions and impurity atoms (if any). In each collision, the velocity changes both in magnitude and direction. Since the motion is completely random, at any instant, the average thermal velocity $\langle \vec{v}_t \rangle$ along any direction in the bulk of the conductor is zero. Hence no current flows. But remember that **average speed** of these free electrons at any instant is not zero. Its value is of the order of $10^5\,\mathrm{m\,s^{-1}}$.

When a battery is connected between the ends of the metallic wire, it maintains a uniform electric field $\vec{\mathbf{E}}$ at each point in the wire. The electrons experience a force in a direction opposite to that of the applied electric field. Due to this force, besides having a thermal velocity $\vec{\mathbf{v}}_t$, an electron also experiences a constant acceleration $a = eE/m_e$, where m_e is the mass of the electron. You may now ask: In the presence of the electric field, does the electron velocity, $\vec{\mathbf{v}}_E$, increase continuously as it moves in the wire? Experiments show that this does not happen. As an electron picks up speed under the action of the field, it collides with the ions or the impurity atoms within the metal. The result of these collisions is that the electron loses its velocity acquired due to acceleration in the field. In other words, in each collision the velocity of the electron is randomised, and it begins afresh and accelerates in the direction of the field. If $\vec{\mathbf{u}}$ is the velocity of an electron just after a collision, its velocity $\vec{\mathbf{v}}_E$ just before the next collision will be

$$\vec{\mathbf{v}}_E = \vec{\mathbf{u}} + \frac{e\vec{\mathbf{E}}}{m_e}t \qquad (12.16)$$

where t is the time of travel between the two collisions. The average of the velocities of all electrons before collision can be written as

$$\langle \vec{\mathbf{v}}_E \rangle = \langle \vec{\mathbf{u}} \rangle + \frac{e\vec{\mathbf{E}}}{m_e}\langle t \rangle \qquad (12.17)$$

where the sign $<>$ denotes the average value of the parameter. Since the effect of each collision is to reduce the velocity to zero and to restore the random thermal motion, we can write $\langle \vec{\mathbf{u}} \rangle$ as $\langle \vec{\mathbf{v}}_t \rangle$ which is zero, as explained earlier. If $\langle t \rangle$ is represented by τ, then we can write Eq. (12.17) as

$$\langle \vec{\mathbf{v}}_E \rangle = \frac{e\vec{\mathbf{E}}}{m_e}\tau \qquad (12.18)$$

> You may know that a freely falling body in vacuum has a velocity $v = gt$ which increases continuously with time. But, if the body falls through a viscous fluid, its terminal motion becomes uniform with a constant limiting velocity. By analogy, the effect of the crystal lattice can be represented by a viscous force, acting on the conduction electrons when their natural motion is disturbed by the applied electric field.

So, $\vec{\mathbf{v}}_E$ does not increase continuously with time, but will rather have an average value $\langle \vec{\mathbf{v}}_E \rangle$ as given by Eq. (12.18). Here τ denotes the average time between successive collisions, i.e., the time over which the electron accelerated freely under the action of the electric field. This is called **mean free time**. The thermal motion of the free electrons is, therefore, modified as shown in Fig. 12.8b. It is clear from the figure that at any instant, the resultant velocity is $\vec{\mathbf{v}}_t + \vec{\mathbf{v}}_E$ and for each electron it is different. The average resultant velocity of all the electrons can be expressed as

$$\langle \vec{\mathbf{v}} \rangle = \langle \vec{\mathbf{v}}_t + \vec{\mathbf{v}}_E \rangle = \langle \vec{\mathbf{v}}_t \rangle + \langle \vec{\mathbf{v}}_E \rangle \qquad (12.19)$$

As explained above, $\langle \vec{\mathbf{v}}_t \rangle$ is zero, but $\langle \vec{\mathbf{v}}_E \rangle$ is not zero because of the fact that the $\vec{\mathbf{v}}_E$ for all the free electrons is in the same direction. Therefore, $\langle \vec{\mathbf{v}} \rangle = \langle \vec{\mathbf{v}}_E \rangle$. Hence, the free electrons in a metallic wire have an average velocity which is caused only by the applied electric field. This velocity is called the **drift velocity** of the electrons and it is denoted by $\vec{\mathbf{v}}_d$. Thus, we write Eq. (12.18) as

$$\vec{\mathbf{v}}_d = \frac{e\vec{\mathbf{E}}}{m_e}\tau \qquad (12.20)$$

The velocity $\vec{\mathbf{v}}$ of electrons which appeared in Eq. (12.4) is actually the drift velocity, $\vec{\mathbf{v}}_d$ given by Eq. (12.20). Thus, the current density in a conductor can be written as

$$\vec{\mathbf{J}} = nq\,\vec{\mathbf{v}}_d \qquad (12.21)$$

In most substances and over a wide range of electric field strengths, it has been experimentally found that the current density is proportional to the electric field that causes it. Thus, the relation may be written as

$$\boxed{\vec{\mathbf{J}} = \sigma\vec{\mathbf{E}}} \qquad (12.22)$$

where σ is the proportionality constant and is known as the **conductivity of the material**. Eq. (12.22) is a statement of **Ohm's law**. It is an empirical law, a generalisation derived from experiments for some materials under certain conditions. It is not a theorem that must be universally obeyed. The value of σ is very large for metallic conductors and extremely small for good insulators. It may also depend on the physical state of the material, for instance, on its temperature. But for many common conductors, for given conditions, it does not depend on the magnitude of $\vec{\mathbf{E}}$. Such materials are called **ohmic** or **linear** and for such materials Eq. (12.22) implies that the direction of $\vec{\mathbf{J}}$ is always the same as the direction of $\vec{\mathbf{E}}$. Instead of the conductivity, we can use its reciprocal, called **resistivity** ρ, in stating the relation between current density and electric field as follows:

Eq. (12.22) holds only for isotropic materials: materials in which the electric properties are the same in all directions.

$$\boxed{\vec{\mathbf{E}} = \rho\vec{\mathbf{J}}} \qquad (12.23)$$

It is customary to use ρ as the symbol for resistivity and σ as the symbol for conductivity inspite of their use in some of our other units for volume charge density and surface charge density, respectively. Thus, you should be careful about these symbols and take into consideration the context of their use in an expression.

The units of resistivity are Ωm. Since both $\vec{\mathbf{E}}$ and $\vec{\mathbf{J}}$ are microscopic parameters, ρ also defines a microscopic property of the conductor.

If we use Eq. (12.20) in Eq. (12.21) and replace charge q by electronic charge e, we can write

$$\vec{\mathbf{J}} = ne\,\vec{\mathbf{v}}_d = \frac{ne^2\tau}{m_e}\vec{\mathbf{E}}$$

By comparing the above expression with Eq. (12.22), we can write the expression for conductivity as follows:

$$\sigma = \frac{ne^2\tau}{m_e} \qquad (12.24)$$

Then the resistivity is given by

$$\rho = \frac{m_e}{ne^2\tau} \qquad (12.25)$$

Eqs. (12.24) and (12.25) show that the conductivity or resistivity of a metal depends on the density of the free electrons, their mass and charge, and on mean free time. With the above background knowledge about electric current and current density, you are now ready to study magnetic field. You will see later in this unit that the magnetic field is produced by electric current.

12.3 MAGNETIC FIELD

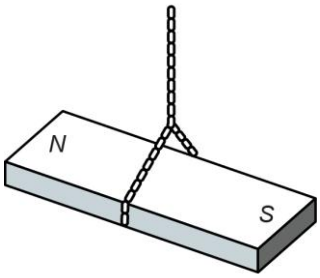


Fig. 12.9: **When a magnet is freely suspended, a particular end of it points towards north. This end of the magnet is defined as the north pole.**

Whenever we speak of magnetic field, we normally think of bar magnets since this is the way magnetic fields were first studied. You are already aware of the basic features of a magnetic field from your school days. For example, you know that the poles of a bar magnet experience force when placed in a magnetic field. If a bar magnet is suspended by a delicate fibre as shown in Fig. 12.9, a particular end of the magnet will always point towards north. This end of the magnet is called its north pole. The other end is called the south pole. Do you recall that this arrangement is a simple compass?

The north poles of two magnets repel each other. The south pole of a magnet is always attracted by the north pole of another magnet. If one tries to break off the north or south pole from a simple bar magnet, then this exercise proves to be futile. The broken magnet becomes two new bar magnets each having a north and a south pole. **This shows that an isolated magnetic pole does not exist.**

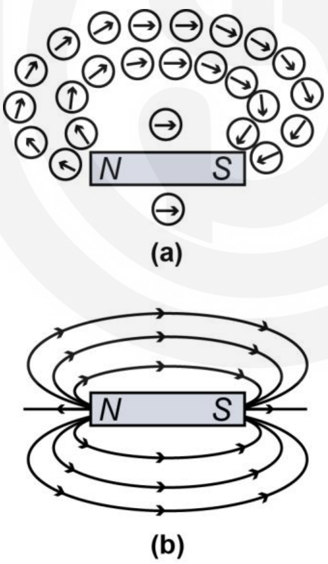


Fig. 12.10: **a) A compass needle points in the direction of the magnetic field;**
b) magnetic field lines of a magnet drawn using the fact that a compass needle should line up along the field lines.

In order to plot the direction of the magnetic field due to a bar magnet, we need only a compass needle. The direction in which the compass needle points is taken to be direction of the magnetic field. In your school physics classes, you must have used this fact to plot the magnetic field in the vicinity of the bar magnet as shown in Fig. 12.10a. The magnetic field lines are drawn in such a way that a compass needle placed on the line aligns itself tangentially to the line. Fig. 12.10b shows the typical magnetic field for the bar magnet. **Notice that the field lines emerge from the north pole and enter the south pole.**

These are some of qualitative features of magnetic field with which we are all familiar. Let us now discuss what causes magnetic field.

12.3.1 Source of Magnetic Field

As you know, the space near a rubbed glass rod (rubbed either by rubber or rabbit's fur) is characterised by an electric field which is denoted by $\vec{\mathbf{E}}$. Similarly, a magnetic field around a magnet may be represented by the symbol $\vec{\mathbf{B}}$. In electrostatics, the electric charges set up an electric field and the electric field, in turn, exerts an electrostatic force on another electric charge that may be placed in that field. Now, by analogy, can we think of a similar relation for magnetism? The answer is that we cannot. This is because a single isolated magnetic pole or a magnetic charge is not known to exist.

Thus, the question is: **If magnetic charges do not exist to give rise to magnetic field as electric charge gives rise to electric field, then how does the magnetic field arise?** Let us try to find out the answer to this question by considering a simple experiment.

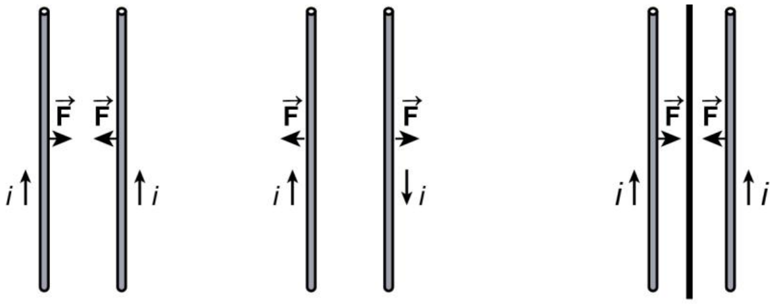


Fig. 12.11: **a) Parallel wires carrying currents in the same direction are pulled together; b) parallel wires carrying currents in opposite directions are pushed apart; c) a sheet of metal between the two wires does not affect these forces.**

Let us consider two conducting wires, running parallel to one another, as shown in Fig. 12.11a. If the current in the two wires flows in the same direction, then the wires are found to attract each other. If the direction of one of the currents is reversed, as shown in Fig. 12.11b, then the wires repel each other. If a sheet of metal is put between the two wires, the force with which wires attract or repel each other is not affected at all (Fig. 12.11c).

Now, the question is: **How do we explain the above observations?** Does electrostatic force account for the attraction or repulsion of the parallel wires? No, the force acting between the wires is not an electrostatic or Coulomb force. This is because (i) there is no net charge on the conductor (the charge density of conduction electrons just compensates for the positive charge on the lattice ions); (ii) the force is reversed in sign by reversing the direction of either current; (iii) the force ceases as soon as the circuit is broken; (iv) the force is not affected by the presence of a simple medium; (v) the attraction and repulsion of the electric currents is contrary to the attraction or repulsion of the electric charges.

The observations of the experiments depicted in Fig. 12.11 can be explained if we assume that there is an additional force associated with a moving charge, which is different from the electrostatic force. **This new force that comes into play when charges are moving is called the magnetic force.** A charge sets up an electric field whether the charge is at rest or is moving. However, a charge sets up a magnetic field **only if it is moving**.

You may now ask a simple question: **A bar magnet sets up a magnetic field in its vicinity, but where are the moving electric charges in a bar magnet?** Actually, the circulating electrons in the atoms of the bar magnet (magnetic material) are responsible for its magnetism. You will learn more about it in Unit 14 of this bock.

Thus, you have learnt that (i) a moving charge or a current sets up a magnetic field and also (ii) if we place a moving charge or a wire carrying a current in a magnetic field, a force will act on it. Now, with this qualitative understanding about the origin of magnetic field, let us define it. But before learning this, try to answer the following SAQ.

SAQ 2 **- Magnetic field and electric motor**

You have probably studied about an electric motor in your school, and you may know the principle on which it works. Briefly explain how an electric motor illustrates the relation between electric current and magnetic field.

12.3.2 Definition of Magnetic Field

In Unit 5 (Block 2), you have learnt that the electric field $\vec{\mathbf{E}}$ at a point, in terms of the electrostatic force $\vec{\mathbf{F}}_E$ that acted on a test charge q at rest at that point, is given by:

$$\vec{\mathbf{F}}_E = q\vec{\mathbf{E}} \qquad (12.26)$$

We can define the magnetic field in terms of the magnetic force exerted on a moving electric charge. It can also be defined in terms of the magnetic force on a current. Since current is a flow of electric charge, the two definitions are related. First, let us state the definition in terms of force on a current-carrying wire.

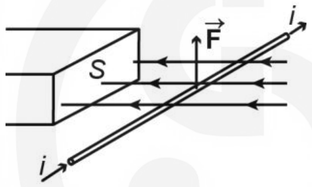


Fig. 12.12: **A straight current-carrying wire experiences force when it is placed in a magnetic field.**

a) Force on currents

Experiments show that a current-carrying wire placed in a magnetic field, experiences a force. Fig. 12.12 shows a wire carrying a current *i* in a magnetic field produced by a magnet. Since the field lines come out of the north pole and enter the south pole, the field in Fig. 12.12 is directed from right to left. It is found that the wire experiences a force, which is proportional to both the current and the strength of the magnetic field. When the wire is placed parallel (or anti parallel) to the field lines, it experiences no force. But when the wire is placed perpendicular to the field lines, the force on the wire is maximum. These two cases are shown in Fig. 12.13.

Fig. 12.13: **The force on a current carrying wire is a) zero if the wire is parallel or anti-parallel to the field lines; b) maximum when the wire is perpendicular to field lines.**

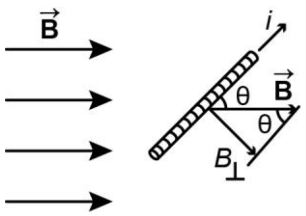


Fig. 12.14: **The force on the wire is $B \sin \theta$; that is, it is proportional to $B_{\perp}$.**

This shows that the force on a wire is due entirely to the component of the field that is perpendicular to the wire. In other words, the force also depends on the relative orientation of the wire and the field lines. In

Fig. 12.14, suppose, the angle between the field lines (represented by $\vec{\mathbf{B}}$) and the current-carrying wire is θ. As explained above, the force $\vec{\mathbf{F}}$ on the wire of length L is due entirely to the component of $\vec{\mathbf{B}}$ that is perpendicular to the wire. This component, represented as $B_{\perp}$ is given by (see Fig. 12.14):

$$B_{\perp} = B \sin \theta \qquad (12.27)$$

Further, the force on the wire of length L depends on L itself and the current i in the wire. We, therefore, conclude that the force F on a length L of the wire is given by

$$F = iLB \sin\theta \qquad (12.28)$$

Recall that the vector product $\vec{\mathbf{C}} \times \vec{\mathbf{D}}$ gives rise to a vector of magnitude $CD \sin\theta$ which is perpendicular to the plane containing $\vec{\mathbf{C}}$ and $\vec{\mathbf{D}}$. Using this in Eq. (12.28), we can write

$$\vec{\mathbf{F}} = i(\vec{\mathbf{L}} \times \vec{\mathbf{B}}) \qquad (12.29)$$

If a surface of area A is placed perpendicular to a uniform magnetic field B, then the product BA is an important physical quantity, which is called magnetic flux through the surface area A. It is denoted by Φ. Thus,

$$\Phi = BA$$

or $$B = \frac{\Phi}{A}$$

The unit of magnetic flux is weber. Hence the unit of B is also weber per square metre. It is also called tesla (T).

Here, $\vec{\mathbf{L}}$ is a vector of magnitude L which is the length of the wire and its direction is along the direction of current flow. Eq. (12.28) or (12.29) shows that the SI units of B are $NA^{-1}m^{-1}$. This unit is also given the name **weber per square meter or tesla** (abbreviated as T). One tesla is a strong magnetic field; thus, a smaller unit called the **gauss** (G) is often used.

$$\boxed{1\,\text{tesla} = 10^4\,\text{gauss}}$$

Since gauss is not an SI unit, we should always convert it to tesla before using it in equations. The quantity B has several names. **Its correct name is magnetic induction.** It is also designated as the magnetic field intensity. However, for historical reasons, we will call the quantity B as magnetic field. Also, we shall define another quantity in Unit 14 which we shall call magnetic field intensity or simply **magnetic intensity** and denote it by H.

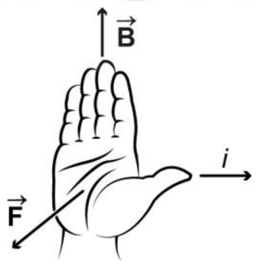
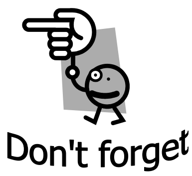


Fig. 12.15: **Right - hand rule.**

The direction of the force on the wire is always perpendicular to the plane defined by $\vec{\mathbf{B}}$ and i. To determine the direction of the force, we use the **right-hand rule** as shown in Fig. 12.15.

According to the right-hand rule: if one's right hand is held flat with the fingers pointing in the direction of the magnetic field and the thumb pointing in the direction of the current, then the palm of the hand will push in the direction of the force.



We now apply Eq. (12.28) or (12.29) to a simple situation so that you can understand its meaning better.

EXAMPLE 12.1: FORCE ON A CURRENT-CARRYING WIRE

A horizontal wire is carrying current from east to west. What is the direction of the force on this current-carrying wire if we assume that at this location the magnetic field of the Earth points due north? If the wire carries a current of 20 A, calculate the force per unit length on it due to the Earth's magnetic field, which is about 1.0 G.

SOLUTION ■ We use the right-hand rule to determine the force experienced by the wire for the orientation of the wire and the Earth's magnetic field. We find that when the thumb of the right hand points west and the fingers point north, the palm faces down. Hence, the force on this wire will be down (into the page).

Earth's magnetic field lines are in a direction perpendicular to the wire. Thus, we have $\theta = 90°$ and we can write Eq. (12.28) as

$$F = iLB\sin 90° = iLB$$

So, force per unit length on the wire is

$$\frac{F}{L} = iB = 20\,\text{A} \times 10^{-4}\,\text{T} = 2.0 \times 10^{-3}\,\text{Nm}^{-1}$$

Now, you should solve an SAQ to concretise the ideas discussed above.

SAQ 3 – Force on current-carrying wire in magnetic field

A current of 9.5 A is flowing in a wire which is oriented perpendicular to a uniform magnetic field. If the magnitude of the magnetic force on a 0.70 m length of the wire is 15 mN, what is the magnitude of the magnetic field? If the direction of the current flow in the wire is from east to west and the magnetic force acting on it is directed towards south, determine the direction of the magnetic field.

The conditions given in Example 12.1 and SAQ 3 were rather simple and straight forward. Suppose the wire carrying current is not straight so that, at each point, its orientation relative to the field changes. Another possible scenario is that the field changes in magnitude/direction over the length of the current-carrying wire. How do we calculate the magnetic force in such situations? We can still use Eq. (12.29) to calculate the force. For this purpose we imagine the wire to be broken up into small segments so that each of these segments can be considered straight, and the field is essentially constant over its length (see Fig. 12.16). Under these assumptions, Eq. (12.29) can be applied to each segment of the current-carrying wire.

If the length of a small segment of the current-carrying wire is $d\vec{\mathbf{L}}$ then we can write for a small magnetic force $d\vec{\mathbf{F}}$ on the segment as

$$d\vec{\mathbf{F}} = i(d\vec{\mathbf{L}} \times \vec{\mathbf{B}}) \qquad (12.30)$$

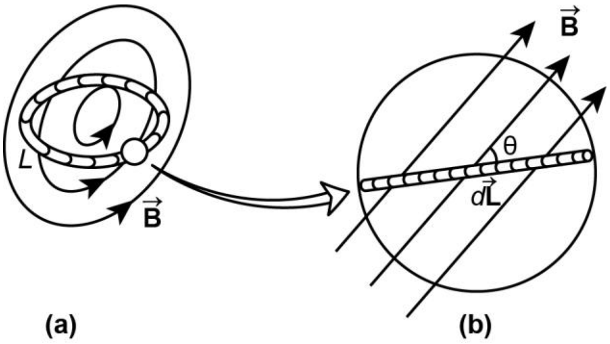


Fig. 12.16: **a) A curved wire** L **in a non-uniform magnetic field** $\vec{\mathbf{B}}$**; b) a small enough segment of the wire can be considered as straight wire in a uniform field.**

We can obtain the total magnetic force on this arbitrarily shaped long current-carrying wire placed in a non-uniform magnetic field $\vec{\mathbf{B}}$, by summing the expression for $d\vec{\mathbf{F}}$ in Eq. (12.30) over the whole wire:

$$\vec{\mathbf{F}} = \sum d\vec{\mathbf{F}} = \sum i\, d\vec{\mathbf{L}} \times \vec{\mathbf{B}}$$

If we let the length $d\vec{\mathbf{L}}$ approach zero, this sum becomes an integral, and we write the above expression as

$$\vec{\mathbf{F}} = i\int d\vec{\mathbf{L}} \times \vec{\mathbf{B}} \qquad (12.31)$$

Do you recognise the right hand side of the Eq. (12.31)? From Unit 3 of this course, you know that it is a **line integral** taken over the length of the wire. The current *i,* being a constant, is taken out of the integral. In particular, if the magnetic field is uniform, which means that $\vec{\mathbf{B}}$ is constant both in magnitude and direction at all points of the wire, then we can write Eq. (12.31) as

$$\vec{\mathbf{F}} = i\left(\int d\vec{\mathbf{L}}\right) \times \vec{\mathbf{B}}$$

In this expression, $d\vec{\mathbf{L}}$ is the vector joining the initial point of the segment of wire to its final point and the integral is over length of the wire. Further, if the current-carrying wire is straight and its length is L, then we have

$$\vec{\mathbf{F}} = i(\vec{\mathbf{L}} \times \vec{\mathbf{B}})$$

This expression for magnetic force is the same as Eq. (12.29).

So far, we considered the force on the current in a wire. An electric current is simply a group of charged particle sharing a common motion, so we should expect a moving charge to experience force in the magnetic field. This gives another way of defining the magnetic field.

b) Force on a moving charge

The force which a magnetic field exerts on a moving positive charge can be obtained from Eq. (12.29). Recall from Sec. 12.2 that the velocity v

of charge q in a wire of cross-section A is related to current i by Eq. (12.2) as follows:

$$i = qnAv$$

where n is the number of charges per unit volume. Substituting this expression for i into Eq. (12.30) gives

$$d\vec{\mathbf{F}} = (dL)Anq\,\vec{\mathbf{v}} \times \vec{\mathbf{B}} \qquad (12.32)$$

Here $(dL)\,A$ represents the volume of the wire segment of length (dL). So $(dL)\,An$ is the number of moving charges in that portion of the wire for which we are writing the force. Hence, the force $\vec{\mathbf{F}}$ on a single moving charge is given by $d\vec{\mathbf{F}}/(dL)An.$ Thus, we can write

$$\vec{\mathbf{F}} = q\,\vec{\mathbf{v}} \times \vec{\mathbf{B}} \qquad (12.33)$$

The magnitude of the force is given by $qvB\sin\theta.$ The direction of the force on the moving charge can be obtained by **right-hand rule** (Fig. 12.15 with i replaced by v). **Note that, if the particle is negatively charged, the direction of $\vec{\mathbf{F}}$ will be reversed.**

Now, go through the following example so that you can understand how the force on a charged particle is calculated. It also illustrates the use of the right-hand rule for determining the direction of force.

EXAMPLE 12.2: FORCE ON A CHARGED PARTICLE MOVING IN MAGNETIC FIELD

In a certain region, a magnetic field of 0.10 T points vertically upward. Three protons enter the region, two horizontally and one vertically upward as shown in Fig. 12.17. All the three protons are moving with the same speed $2.0 \times 10^3\,\text{m s}^{-1}$. Determine the force on each proton.

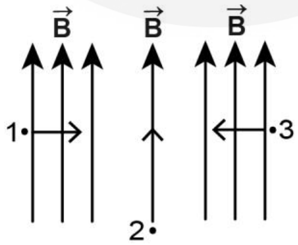


Fig. 12.17: **Diagram for Example 12.2.**

SOLUTION ■ Proton 2 is moving vertically upward, i.e., parallel to the field and sin θ = 0. So, from Eq. (12.33), we have $|\vec{\mathbf{F}}| = qvB\sin\theta = 0.$ Therefore, it experiences no force. Protons 1 and 3 are moving at right angles to the field, so sin θ = 1 in Eq. (12.33). Thus, the forces on these two protons have the same magnitude given by

$$F = qvB = (1.6 \times 10^{-19}\,\text{C}) \times (2.0 \times 10^3\,\text{m s}^{-1}) \times (0.10\,\text{T}) = 3.2 \times 10^{-17}\,\text{N}$$

Since the protons carry a positive charge, the direction of the force is the direction of the vector $\vec{\mathbf{v}} \times \vec{\mathbf{B}}.$ For proton 1, moving to the right, $\vec{\mathbf{v}} \times \vec{\mathbf{B}}$ points out of the page. For proton 3, moving to the left, the force points into the page. This example clearly shows that the magnetic field alone does not determine the force. Identical charged particles moving in the same field may experience different forces, if their velocities are not identical. If the particles were electrons, the negative sign of the electron charge would have indicated a force opposite to the direction of $\vec{\mathbf{v}} \times \vec{\mathbf{B}}.$

Now, answer an SAQ.

SAQ 4 – Force on a charged particle moving in magnetic field

Of the three vectors in the equation $\vec{\mathbf{F}} = q\,\vec{\mathbf{v}} \times \vec{\mathbf{B}}$, which pairs are always at right angles to one another? Which of these may have any angle between them?

Eq. (12.33) is equivalent to Eq. (12.29) so that either of them can be taken as the defining equation for $\vec{\mathbf{B}}$. In practice, we define $\vec{\mathbf{B}}$ from Eq. (12.29), because it is much easier to measure the force acting on a wire than that on a single moving charge.

In this section, you have learnt that a moving charge gives rise to a magnetic field. You have also learnt how to define magnetic field in terms of the force exerted by it on a current-carrying wire and a moving charge. Now, suppose there is a current-carrying wire, and you are asked to calculate the magnetic field produced due to it at any point of space. This is similar to the problem of calculating electric field at a point in space due to a charge or system of charges (in electrostatics). In electrostatics, you used Coulomb's law and Gauss's law to find the solution. So, you would like to have laws for magnetic field which are analogous to Coulomb's law and Gauss's law. Let us first find out Gauss's law for magnetism.

12.4 GAUSS'S LAW FOR MAGNETISM

Suppose magnetic charges – monopoles – exist. Then, they would give rise to magnetic fields like the electric fields due to electric point charges. In such a situation, we can describe the magnetic fields due to monopoles and due to those of magnetic charge distributions by laws analogous to Gauss's law for electrostatics. That is, Gauss's law for magnetic field would require that the flux of the magnetic field through any closed surface depend only on the enclosed magnetic charge. Thus, under the assumption that magnetic charges exist, we may write Gauss's law for magnetism as

$$\oiint \vec{\mathbf{B}}.d\vec{\mathbf{S}} = \mu_0 g \tag{12.34}$$

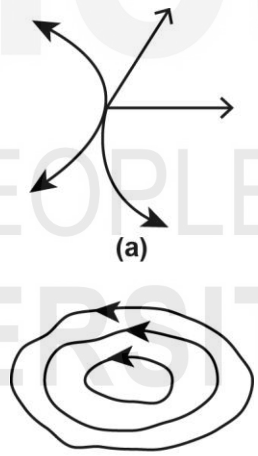
where the integral on the left is the flux of $\vec{\mathbf{B}}$ over a closed surface enclosing the magnetic charge or monopoles denoted by g and μ_0 is some constant.

But, the very existence of the magnetic monopoles is uncertain. And even if they do exist, they seem to play no significant role in our world. In the absence of the magnetic monopoles, we must put $g = 0$ and then the magnetic flux through any closed surface must be zero. We state this mathematically as Gauss's law for magnetism and write it as follows:

$$\boxed{\oiint \vec{\mathbf{B}}.d\vec{\mathbf{S}} = 0} \tag{12.35}$$

A consequence of Gauss's law for magnetism is that magnetic field lines can never begin or end (Fig. 12.18). Unlike the electric field lines, the magnetic field lines have to form closed loops. If we convert the surface integral of Eq. (12.35) into a volume integral using the divergence theorem, we obtain

$$\iiint \vec{\nabla}.\vec{\mathbf{B}}dV = 0 \tag{12.36}$$



(a)

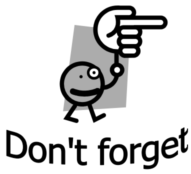
(b)

Fig. 12.18: **In the absence of magnetic monopoles, the magnetic flux through a closed surface must be zero. a) There can be no point where magnetic field lines begin or end because a closed surface surrounding such a point would have non-zero net flux; b) Instead, magnetic field lines form closed loops.**

The integration in Eq. (12.36) is over the volume enclosed by the closed surface of Eq. (12.35). Since Eq. (12.36) holds for any arbitrary volume of integration, we must have

$$\boxed{\vec{\nabla}.\vec{\mathbf{B}} = 0} \tag{12.37}$$

Eq. (12.37) is true even if $\vec{\mathbf{B}}$ varies with time. Eqs. (12.35 and 12.37) are the integral and differential forms of Gauss's law for magnetism.



Gauss's law for magnetism implies that the magnetic field does not have any source similar to electric charge(s) for electric field. In other words, magnetic monopoles do not exist. Mathematically, it means that divergence of $\vec{\mathbf{B}}$ is zero.

Now, let us discuss Biot-Savart law which is analogous to Coulomb's law.

12.5 BIOT-SAVART LAW

In the previous sections, you have learnt the effect of magnetic field on a current-carrying wire and moving charges and have calculated the magnetic force experienced by them. Now, the question is: **How do we calculate the magnetic fields produced by a current?** Can we show that a current loop has the magnetic field of a dipole? Interest in questions like these led the French scientists Jean-Baptiste Biot and Felix Savart to experimentally determine the form of the magnetic field arising from a **steady current**. Known as Biot-Savart law, its result gives the magnetic field at a point due to a small element of current.

In Unit 5, you have learnt how to calculate the electric field due to a given distribution of charges in the surrounding space. Our approach was to divide the charge distribution into charge elements dq as in Fig. 12.19a. We then calculated the field $d\vec{\mathbf{E}}$ due to a given charge element at an arbitrary point P. Finally, we calculated $\vec{\mathbf{E}}$ at point P by integrating $d\vec{\mathbf{E}}$ over the entire charge distribution. Recall that the magnitude of $d\vec{\mathbf{E}}$ is given as:

$$dE = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^2}$$

where r is the distance from the charge element to the point P.

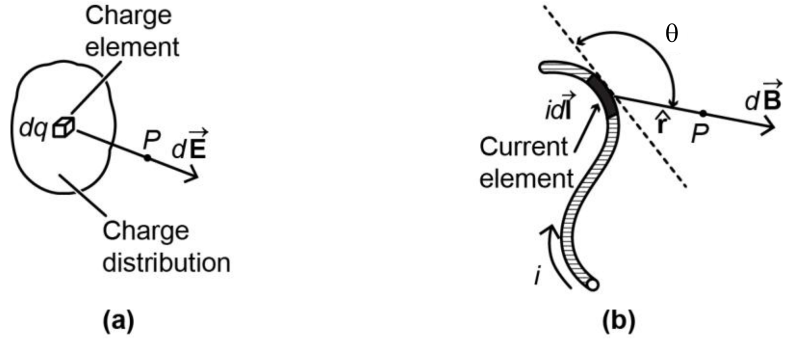


Fig. 12.19: **a) The electric field** $d\vec{\mathbf{E}}$ **at point** P **due to a charge element** dq**; b) the magnetic field** $d\vec{\mathbf{B}}$ **at point** P **due to a current element** $id\vec{\mathbf{l}}$.

In the case of magnetic field, our approach will be the same. Fig. 12.19b shows a wire of arbitrary shape carrying a steady current i. We wish to know:

What is the magnetic field $\vec{\mathbf{B}}$ at an arbitrary point P near this wire? We first break up the wire into differential current elements $id\vec{\mathbf{l}}$, corresponding to the charge elements dq of Fig. 12.19a. Here the vector $d\vec{\mathbf{l}}$ is a differential element of length, pointing along the tangent to the wire in the direction of the current. Note that the differential charge element dq is a scalar, but the differential current element $id\vec{\mathbf{l}}$ is a vector.

Under these conditions, the Biot-Savart law says that the magnitude of the magnetic field due to a given current element idl at point P at distance r is given as follows:

$$\left|d\vec{\mathbf{B}}\right| = \frac{\mu_0}{4\pi}\frac{idl\sin\theta}{r^2} \tag{12.38}$$

Here μ_0 is a constant, called the **permeability of free space**. Its value is $4\pi\times10^{-7}\,\mathrm{T\,m\,A^{-1}}$. This constant plays a role in magnetic problems, much like the role that the **permittivity** ε_0 plays in electrostatic problems.

The expression for $d\vec{\mathbf{B}}$ in vector form is

$$d\vec{\mathbf{B}} = \frac{\mu_0}{4\pi}\frac{id\vec{\mathbf{l}}\times\hat{\mathbf{r}}}{r^2} \tag{12.39}$$

where $\hat{\mathbf{r}}$ is a unit vector pointing from $d\vec{\mathbf{l}}$ towards P. Eq. (12.39) is the analog of Coulomb's law for electrostatics and is called **Biot-Savart law**. The direction of $d\vec{\mathbf{B}}$ in Fig. 12.19b is that of the vector $d\vec{\mathbf{l}}\times\hat{\mathbf{r}}$, where $\hat{\mathbf{r}}$ is a unit vector that points from the current element to the point P at which we wish to know the field.

Recall from Unit 5 that **Coulomb's law** gives the **electric field** of a point charge in terms of the charge and the distance from the charge to the field point. The electric field varies as the inverse square of the distance, and its direction lies along the line joining the charge with the field point. **Analogously, Biot-Savart law gives the magnetic field at a given point** in terms of the **current element** (source of the magnetic field) and the distance to the field point from the current element. Like the electric field of a point charge, the magnetic field of an isolated current element varies as the inverse square of the distance. But here the analogy ends.

Unlike the electric charge in Coulomb's law, the current element $id\vec{\mathbf{l}}$ in Biot-Savart law has associated with it a direction as well as a magnitude. Hence, the magnetic field of the current element is not symmetric about the element; it depends on the position of the field point relative to the direction of the current element. This directional character is expressed by the cross product in Eq. (12.39). So, in Fig. 12.19b, the magnetic field is at right angles to both the current element and the vector from the current element to the field point. Another significant difference between the magnetic and electric fields is that the magnetic field lines have no sources like the electric field lines which end or originate on electric charges; magnetic field lines are continuous and join back on themselves.

Let us see how Eq. (12.39) and Fig. 12.19b show that the magnetic field lines are continuous and join back on themselves. Let the point P move around the current axis at a constant distance from the axis. From Eq. (12.39), the magnitude of $d\vec{\mathbf{B}}$ is constant along this path, and at each point it has a direction tangent to the path. These are just the requirements for the lines to be concentric circles around the current. Hence, the magnetic field lines encircle the current as shown in Fig. 12.20. The direction in which the circular field lines point depends on the direction in which the current flows. If the direction of the current flow is reversed, the direction of the field line is also reversed as shown in Fig. 12.21.

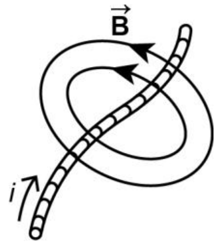


Fig. 12.20: **Magnetic field lines generally encircle a current.**

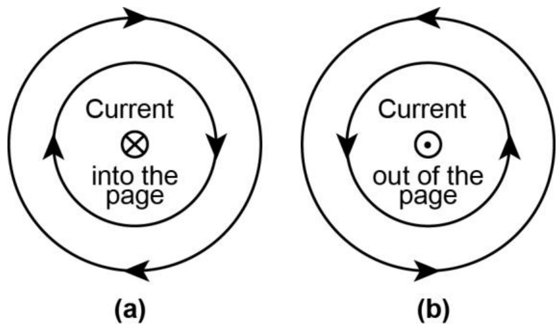


Fig. 12.21: **The direction in which the magnetic field lines point is determined by the direction in which the current flows. a) When the current flows into the page, denoted by symbol ⊗, the field lines form clockwise circles; b) when the current flows out of the page, indicated by symbol ʘ, the field lines form anticlockwise circles.**

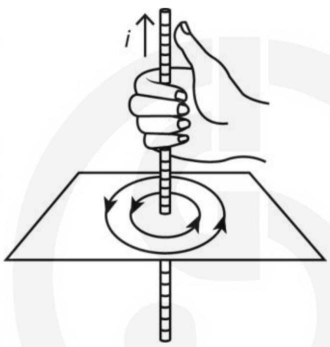


Fig. 12.22: **By using your right hand to 'grip' a current-carrying conductor, you can find out the direction of the magnetic field. When your thumb points in the direction of current flow, your fingers curl along the direction of the magnetic field.**

However, there is an easy way to remember these directions. Just close the palm of your right hand and point your thumb in the direction of the current as shown in Fig. 12.22. In either case, you will find that your fingers will naturally curl around in the direction of the magnetic field as illustrated in Fig. 12.22, and is referred to as the **right-hand rule.**

SAQ 5 – Direction of the magnetic field

a) Write one analogy and one difference between Coulomb's law and Biot-Savart law.

b) A horizontal wire carries a current from east to west. What is the direction of the magnetic field due to this current directly above and below the wire?

Refer again to Fig. 12.19b. Like the electric field, the magnetic field obeys the superposition principle. Therefore, the net magnetic field at *P* due to entire circuit, of which the wire is a part, will be the vector sum or line integral of the magnetic fields of individual current elements:

$$\vec{\mathbf{B}} = \int d\vec{\mathbf{B}} = \frac{\mu_0}{4\pi}\int_C \frac{i\,d\vec{\mathbf{l}} \times \hat{\mathbf{r}}}{r^2} \qquad (12.40)$$

where *C* represents the path of integration, i.e., the path through which current *i* flows. Let us now apply Biot-Savart law to calculate magnetic field for some simple situations.

a) $\vec{\mathbf{B}}$ due to a long current-carrying straight wire

Refer to Fig. 12.23 which shows a long straight wire carrying current *i.* Suppose we want to calculate the magnetic field at point *P*. Let *r* be the distance between the point *P* and the current element *idl* of the wire, and let *R* be the perpendicular distance between the wire and *P*.

From Biot-Savart law [Eq. (12.39)], we can write the magnitude of the differential magnetic field at point *P* due to the current element *idl* as:

$$dB = \frac{\mu_0}{4\pi} \frac{idl \sin\theta}{r^2} \qquad (12.41)$$

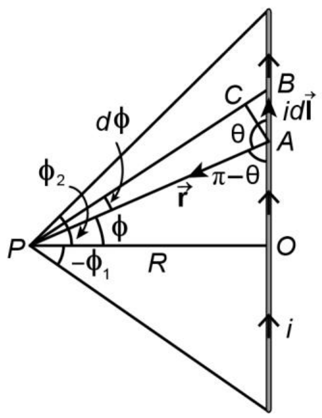


Fig. 12.23: **A long straight wire carrying current *i*.**

where θ is the angle between $\hat{\mathbf{r}}$ and $id\vec{\mathbf{l}}$. The direction of $d\vec{\mathbf{B}}$ is given by the right-hand rule. In the instant case, it points out of page at point *P*. You should convince yourself that this is true irrespective of the position of $d\vec{\mathbf{l}}$ along the wire. Thus, at point *P* all the differential magnetic fields due to all the current elements $id\vec{\mathbf{l}}$ point in the same direction. So, to find the magnitude of total magnetic field *B* at point *P*, we integrate Eq. (12.41):

$$B = \int dB = \frac{\mu_0 i}{4\pi} \int \frac{dl \sin\theta}{r^2} \qquad (12.42)$$

In order to sum up the contributions from all current elements of the long straight wire, we change the variables from θ and *r* to ϕ (see Fig. 12.23). From Fig. 12.23, note that

$$\sin\theta = \sin(\pi - \theta) = \cos\phi \qquad (12.43a)$$

Now, let us draw a line *AC* which is perpendicular to *PB*. Then we can write

$$\frac{AC}{AB} = \frac{rd\phi}{dl} = \cos\phi$$

or $$r\,d\phi = dl \cos\phi \qquad (12.43b)$$

Using Eq. (12.43a), we can write

$$\frac{dl \sin\theta}{r^2} = \frac{dl \cos\phi}{r^2}$$

$$= \frac{rd\phi}{r^2} \qquad \text{[from Eq. (12.43b)]} \qquad (12.44)$$

Substituting Eq. (12.44) in the expression for *B* [Eq. (12.42)], and since $\cos\phi = (R/r)$ from Fig. 12.23, we can write

$$B = \int dB = \frac{\mu_0 i}{4\pi} \int \frac{d\phi}{r} = \frac{\mu_0 i}{4\pi R} \int_{-\phi_1}^{\phi_2} \cos\phi \, d\phi$$

Note that for the given wire, the limits of integration are from $-\phi_1$ (since *PO* is the reference line) and ϕ_2. Thus,

$$B = \frac{\mu_0 i}{4\pi R} [\sin\phi_2 + \sin\phi_1] \qquad (12.45)$$

From Fig. 12.23, we note that if the straight wire is infinitely long, we can write $\phi_1 = \phi_2 = (\pi/2)$. Thus, we get

You know that
$$\sin\theta = \sin(\pi - \theta)$$
In Fig. 12.23,
$$\sin(\pi - \theta) = \frac{OP}{AP}$$
Note that
$$\cos\phi = \frac{OP}{AP}$$
Hence, we get Eq. (12.43a):
$$\sin\theta = \cos\phi$$

In Fig. 12.23, we can write
$$\angle CBA = \angle PAO = \pi - \theta$$
because $d\vec{\mathbf{l}}$ is very small. So,
$$\sin(\pi - \theta) = \frac{AC}{AB}$$
Thus, we have
$$\frac{AC}{AB} = \sin\theta = \cos\phi$$

$$B = \frac{\mu_0 i}{2\pi R} \tag{12.46}$$

for infinite wire. From Eq. (12.46), you may note that the magnitude of B falls off inversely as the first power of the distance from an infinitely long wire. Note that this expression for B is analogous to the expression for E due to a long charged wire given as $\frac{1}{4\pi\varepsilon_0}\left(\frac{2\lambda}{r}\right)$.

Now, you should go through an example on calculating magnetic field.

EXAMPLE 12.3: MAGNETIC FIELD DUE TO A LONG STRAIGHT WIRE

A long straight conducting wire carries a current of 15 A. Determine the magnitude of the magnetic field at a perpendicular distance of 0.20 m from the wire.

SOLUTION ■ The magnitude of the magnetic field due to a current-carrying long straight wire is given by [Eq. (12.46)]:

$$B = \frac{\mu_0 i}{2\pi R}$$

As per the problem, i = 15 A and R = 0.20 m. Since $(\mu_0 / 4\pi) = 10^{-7}\,\text{Tm A}^{-1}$, we get

$$B = \frac{(2\times 10^{-7}\,\text{Tm A}^{-1})\times(15\,\text{A})}{(0.20\,\text{m})} = 1.5\times 10^{-5}\,\text{T}$$

Before proceeding further, you may like to solve an SAQ.

SAQ 6 – Magnetic field due to two long straight wires

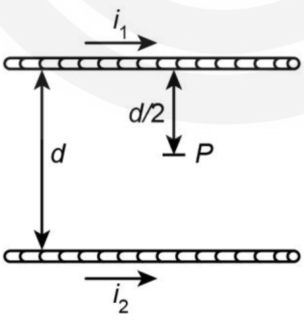


Fig. 12.24: **Diagram for SAQ 6.**

Two conducting long straight wires separated by a distance of 0.50 m are kept parallel to each other as shown in Fig. 12.24. If the wires carry currents $i_1 = 15$ A and $i_2 = 10$ A, respectively, determine the magnetic field in the plane of the two wires at a point P located half way between the wires.

b) $\vec{B}$ along the axis of a current loop

Let us consider a circular loop of radius a and carrying a current i as shown in Fig. 12.25. The x-axis has been chosen along the axis of the loop and we choose a point P on its axis at a distance R from its centre. The magnetic field $d\vec{\mathbf{B}}$ at P due to a current element of length $d\vec{\mathbf{l}}$ is given by Biot-Savart law:

$$d\vec{\mathbf{B}} = \frac{\mu_0}{4\pi}\frac{i\,d\vec{\mathbf{l}}\times\hat{\mathbf{r}}}{r^2} \tag{12.47}$$

For all current elements around the loop, $\vec{\mathbf{r}}$ is perpendicular to $i\,d\vec{\mathbf{l}}$. Hence, the value of $\sin\theta$ in the cross-product in Eq. (12.47) is 1 and we can write:

$$d\vec{\mathbf{B}} = \frac{\mu_0 i}{4\pi}\frac{dl}{r^2}$$

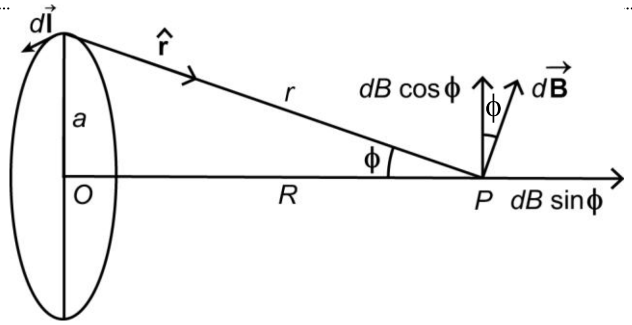


Fig. 12.25: **Magnetic field at point *P* along the axis of a current loop.**

From Eq. (12.47), you can see that since $d\vec{\mathbf{B}}$ is a cross product of $d\vec{\mathbf{l}}$ and $\vec{\mathbf{r}}$, it is always perpendicular to the plane consisting $\vec{\mathbf{r}}$ and $d\vec{\mathbf{l}}$. Thus $d\vec{\mathbf{B}}$ is perpendicular to $\vec{\mathbf{r}}$ at point *P* as shown in Fig. 12.25. It can be resolved into two components, one $dB\sin\phi$ along the axis of the loop and the other $dB\cos\phi$ at right angles to the axis. Here ϕ is the angle between $\vec{\mathbf{r}}$ and the axis of the loop. You will notice that the components of $d\vec{\mathbf{B}}$ perpendicular to the axis will cancel, due to opposite length elements in the entire current loop. Therefore, the resultant $\vec{\mathbf{B}}$ is in the direction of the axis and will be given by summing only the components $dB\sin\phi$. Thus, *B* due to entire loop is given by

$$B = \int dB\sin\phi = \int \frac{\mu_0 i}{4\pi}\frac{dl}{r^2}\sin\phi = \frac{\mu_0 i}{4\pi}\frac{\sin\phi}{r^2}\int dl$$

Since all the length elements *dl* constituting the current loop lie in a circle, both ϕ and *r* are constants. Therefore, they are taken outside the integral. Further, the length element *dl* integrated around the loop is equal to $2\pi a$. Thus, we can write

$$B = \frac{\mu_0 i \sin\phi}{4\pi r^2} 2\pi a$$

If we write $\sin\phi (= a/r)$ and $r\left[= (a^2 + R^2)^{1/2}\right]$ in terms of the constants *a* and *R*, we get

$$B = \frac{\mu_0 i}{2}\frac{a^2}{(a^2 + R^2)^{3/2}} \qquad (12.48)$$

When we choose the point *P* to lie far from the loop so that *R* >> *a*, Eq. (12.48) can be written as

$$B = \frac{\mu_0}{4\pi}\frac{2iA}{R^3} \qquad (12.49)$$

Here we have written $A = \pi a^2$, the area of the loop. Notice that the magnetic field due to the current loop at large distances on its axis is like the electric field due to an electric dipole $\left[E = \frac{1}{4\pi\varepsilon_0}\left(\frac{2p}{r^3}\right)\right]$ (recall Sec. 5.6 of Unit 5.) This shows that the term (*iA*) is analogous to electric dipole moment $\vec{\mathbf{p}}$ of the electric dipole. Therefore, the term *iA* is called the **magnetic dipole moment** of the loop and is represented by μ.

On the basis of the above discussion, you may note the similarity between a bar magnet which is magnetic dipole, and a current loop. The similarity can also be seen by plotting the magnetic field around the current loop. When a compass is used to plot the magnetic field due to a current loop, we obtain magnetic field lines as shown in Figs. 12.26a and b. You should convince yourself that this is reasonable by applying the right-hand rule to a portion of the loop. Now, refer to Fig. 12.26c which depicts the magnetic field lines due to a bar magnet. You may note that the magnetic field lines due to a current loop are quite like those of a bar magnet. The current loop can be considered to have north and south poles. We shall see in a later section that this is one aspect of a very important similarity between bar magnets and current loops.

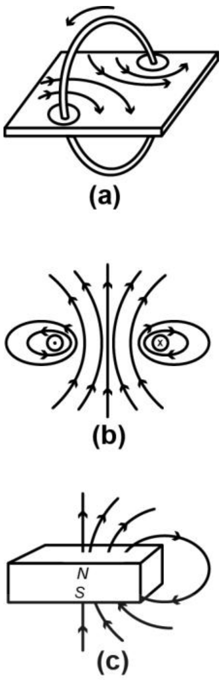


Fig. 12.26: **The magnetic field due to current- carrying loops in (a) and (b) are very similar to that due to small bar magnet shown in (c).**

After studying this section we hope that you can tell why the two parallel current carrying wires, shown in Fig. 12.11, are attracted in one case while they are repelled in another case. If not, study Sec. 12.6. It will also help you in defining the unit of current – ampere – which we have been using so far without defining it precisely. But before that, you should work out the following example.

EXAMPLE 12.4: MAGNETIC FIELD DUE TO A CURRENT LOOP

A current of 0.75 A is flowing in a circular coil of radius 0.02 m. Calculate the magnitude of the magnetic field due to this coil at a point 1.5 m away from the centre of the coil along its axis.

SOLUTION ■ For the case when the distance of the axial point is very large compared to the radius of the current loop ($R >> a$) the magnitude of the magnetic field due to a current loop (or coil) is given by Eq. (12.49):

$$B = \frac{\mu_0}{4\pi} \cdot \frac{2iA}{R^3}$$

We have $i = 0.75$ A, $A = \pi a^2 = 3.14 \times (0.02\text{ m})^2$, $R = 1.5$ m. So,

$$B = \frac{(1\times10^{-7}\,\text{Tm A}^{-1})\times(2\times0.75\,\text{A})\times(3.14)(0.02\,\text{m})^2}{(1.5\,\text{m})^3} = 5.6\times10^{-11}\text{T}$$

Now, you should solve an SAQ.

SAQ 7 - Magnetic field due to electron circulating around the nucleus in a hydrogen atom

As per the Bohr model of hydrogen atom, an electron circulates around the nucleus along a circular path of radius 3.1×10^{-11}m with a frequency, $f = 6.8\times10^{15}$Hz. Calculate the value of the magnetic field set up at the nucleus of the hydrogen atom due to the electron's motion.

12.6 FORCE BETWEEN TWO PARALLEL CONDUCTORS (DEFINITION OF AMPERE)

In this section, we will determine how much force one of the wires in Fig. 12.11 exerts on the other. We assume that the wires are linear, parallel and very long. Here, each wire experiences a force, because it is in the magnetic field due to current in the other wire.

Fig. 12.27 shows two long, parallel wires separated by a distance *d* and carrying currents i_1 and i_2 **in the same direction**. The current in wire 2 produces a magnetic field B_2 at all points around the wire. From Eq. (12.46) the magnitude of B_2 at the site of wire 1 is given by

$$B_2 = \frac{\mu_0 i_2}{2\pi d} \qquad (12.50)$$

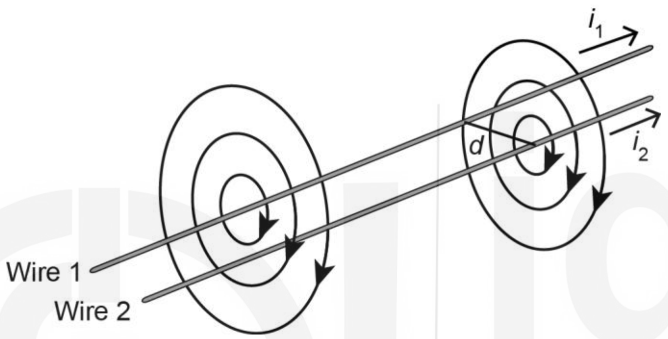


Fig. 12.27: **Two parallel wires carrying currents in the same direction attract each other.**

The **right-hand rule** tells us that the direction of B_2 at any point on wire 1, is out of the page, as shown in Fig. 12.27. Now, wire 1 which is carrying current i_1 is immersed in an external magnetic field B_2. If *L* is the length of this wire, it will experience a force given by Eq. (12.29), whose magnitude is

$$F_1 = i_1 L B_2 = \frac{\mu_0 \, i_1 \, i_2 L}{2\pi d} \qquad (12.51)$$

What is the direction of this force? The **right-hand rule** says that $\vec{\mathbf{F}}_1$ points towards the wire 2. This means that wire 1 is attracted towards wire 2.

Similarly, for currents in the two wires flowing in the opposite direction, you should be able to show that the wires repel each other. **The rule is that parallel currents attract and anti-parallel currents repel.**

The force between current-carrying conductors forms the basis for the definition of the ampere. **One ampere is that constant current which, if maintained in two straight parallel conductors of infinite length, of negligible circular cross-section, and placed one metre apart in vacuum, would produce on each of these conductors a force equal to 2×10^{-7} N per meter.**

In other words, suppose we have two straight parallel conductors of infinite length, of negligible circular cross-section, and placed one meter apart in vacuum. When constant current is made to flow in both the conductors, it is observed that each of these conductors experiences a force. The constant current which produces force equal to 2×10^{-7} N per metre of length of the conductors is known as one ampere (A).

Now, let us sum up what you have studied in this unit.

12.7 SUMMARY

Concept	Description
Electric current	■ **Electric current** is the flow of charge. The unit of electric current is the **ampere**. Current is defined as the amount of charge per unit time passing a given point. $$i = \frac{dq}{dt}$$
Current density	■ **Current density** $\vec{\mathbf{J}}$ is a vector specifying the current per unit area. The direction of $\vec{\mathbf{J}}$ at any point is the direction in which a positive charge-carrier would move if placed at that point. $$\vec{\mathbf{J}} = nq\,\vec{\mathbf{v}}_d$$ The total current through a surface is the flux of the current density over that surface. $$i = \iint_{\mathbf{S}} \vec{\mathbf{J}}.d\vec{\mathbf{S}}$$ where $d\vec{\mathbf{S}}$ is an element of area and the integral is taken over the surface.
Continuity equation	■ The total charge crossing a surface S in unit time is $\iint \vec{\mathbf{J}}.d\vec{\mathbf{S}}$. If S is a closed surface enclosing a volume V, the rate of loss of charge through S must be the same as the rate of depletion of charge contained in V, i.e. $$\oiint \vec{\mathbf{J}}.d\vec{\mathbf{S}} = -\frac{\partial}{\partial t}\iiint_V \rho dV$$ This result expresses conservation of charge and is known as the continuity equation. The differential form of the **continuity equation** is: $$\vec{\nabla}.\vec{\mathbf{J}} + \frac{\partial \rho}{\partial t} = 0$$ Conductivity σ is a property of a material which is equal to the ratio of current density to electric field in the material: $$\vec{\mathbf{J}} = \sigma\vec{\mathbf{E}}$$ Resistivity ρ is the inverse of conductivity.

Force on current in magnetic field

- A long straight wire carrying current i and placed in a uniform magnetic field $\vec{\mathbf{B}}$ experiences a force. The force on a section of the wire of length L is given by

$$\vec{\mathbf{F}} = i\vec{\mathbf{L}} \times \vec{\mathbf{B}}$$

where $\vec{\mathbf{L}}$ is a vector of magnitude L, pointing in the direction in which the current flows along the wire.

Force on moving charge in magnetic field

- A magnetic field $\vec{\mathbf{B}}$ is said to exist in any region in which a moving charge experiences a force that depends on its charge, its velocity $\vec{\mathbf{v}}$ and the magnetic field. If $\vec{\mathbf{B}}$ and $\vec{\mathbf{v}}$ make an angle θ with each other, the force on the moving charge is given by:

$$\vec{\mathbf{F}} = q\,\vec{\mathbf{v}} \times \vec{\mathbf{B}} \qquad \text{or} \qquad F = qvB\sin\theta$$

Gauss's law for magnetism

- Gauss's law for magnetism states that the magnetic flux through any closed surface is zero:

$$\oiint \vec{\mathbf{B}}.d\vec{\mathbf{S}} = 0$$

The differential form of **Gauss's law for magnetism** is

$$\vec{\nabla}.\vec{\mathbf{B}} = 0$$

This shows that magnetic lines have no beginning or end; they form closed loops.

Biot-Savart law

- Current gives rise to a magnetic field. The magnetic field due to a current-carrying conductor can be determined using Biot-Savart law:

$$d\vec{\mathbf{B}} = \frac{\mu_0}{4\pi}\frac{id\vec{\mathbf{l}} \times \hat{\mathbf{r}}}{r^2}$$

where $d\vec{\mathbf{B}}$ is the contribution to the magnetic field from a current i flowing along an infinitesimal current element $d\vec{\mathbf{l}}$. The constant μ_0 is called **magnetic permeability in free space**. Its value is $4\pi \times 10^{-7}\,\text{NA}^{-2}$. The unit vector $\hat{\mathbf{r}}$ points from the current element $id\vec{\mathbf{l}}$ towards the point where the field is being calculated.

Right-hand rule

- Right-hand rule is used for (i) determining the direction of magnetic field, and (ii) determining the direction of the magnetic force on a current-carrying conductor kept in a magnetic field.

For determining the direction of the magnetic field due to a current-carrying wire, if we point the thumb of right hand in the direction of current then our fingers will curl along the direction of $\vec{\mathbf{B}}$.

For determining the direction of magnetic force, if the right hand is held flat with the fingers pointing in the direction of the magnetic field and the thumb pointing in the direction of the current, then the palm of the hand will push in the direction of the magnetic force.

Magnetic field due to an infinite straight wire

- The magnetic field at a point at a perpendicular distance *R* from an infinite straight wire carrying a current *i* is given by

$$B = \frac{\mu_0 i}{2\pi R}$$

Magnetic field due to current loop

- The magnetic field at a point along the axis of a circular loop carrying current is given by

$$B = \frac{\mu_0 i}{2} \frac{a^2}{(a^2 + R^2)^{3/2}}$$

where *a* is the radius of the circular loop carrying current *i* and *R* is the distance of the point (along the axis of the loop) from the centre of the loop.

When the point is far away from the loop such that *R* >> *a* then

$$B = \frac{\mu_0}{4\pi} \frac{2iA}{R^3}$$

where $A = \pi a^2$ is the area of the current loop. The current loop behaves like a magnetic dipole.

Definition of Ampere

- Two parallel wires carrying currents in the same (or opposite) direction attract (or repel) each other. If these two wires are separated by a distance *d* in a vacuum, then the force (*F*) of attraction (or repulsion) on a segment of length *L* of either wire is given by

$$F = \frac{\mu_0 i_1 i_2 L}{2\pi d}$$

where i_1 and i_2 are the currents flowing in the two wires. The force between two current-carrying wires is used to define the ampere – the unit of electric current.

12.8 TERMINAL QUESTIONS

1. TV set shoots out a beam of electrons. The beam current is 10 μA. How many electrons strike the TV screen each second? How much charge strikes the screen in a minute?

2. In the Bohr model of the hydrogen atom, the electron follows a circular orbit centred on the nucleus. Its speed is *v* and the radius of the orbit is *r*. Show that the effective current in the orbit is $ev/2\pi r$. If the radius of the orbit is 5.3×10^{-11} m and the electron's speed is 2.2×10^6 ms^{-1} calculate its frequency *f* and the current *i* in the orbit.

3. What is the electric field in a copper conductor of resistivity $\rho = 1.72 \times 10^{-8}\, \Omega$m having a current density $J = 2.54 \times 10^6$ Am^{-2}?

4. Calculate the magnitude of the magnetic force exerted by the Earth's magnetic field, $B = 10^{-5}$ T, on an electron moving with speed 1.0×10^5 ms^{-1} near the Earth's surface. Compare this force with the weight of the electron on the Earth's surface. Assume that the Earth's magnetic field is perpendicular to the direction of motion of electron.

5. A 0.3 m length of current-carrying wire kept perpendicular to a magnetic field of magnitude 300 mT experiences a force of 2.5 mN. What is the current flowing in the wire?

6. In Chennai, the horizontal component of the Earth's magnetic field is $3.6\times10^{-5}\ \text{Wbm}^{-2}$. If a vertical wire carries a current of 30 A upward there, what is the magnitude and direction of the force on 1 m of the wire?

7. Calculate the force on each segment of the wire shown in Fig. 12.28, if B = 0.15 T. Assume that the current in the wire is 15 A. (It is given that $\sin 65° = 0.9063$.)

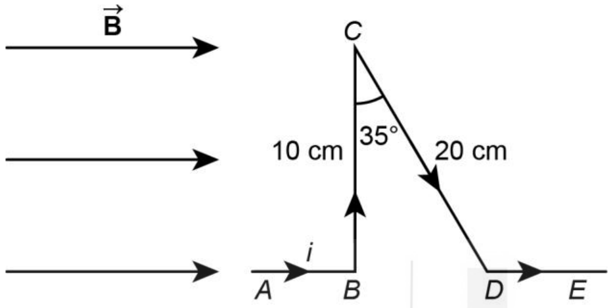


Fig. 12.28: **Diagram for TQ 7.**

8. Two long, straight parallel wires separated by a distance d carry currents i_1 and $i_2 (= 2i_1)$ along the same direction. Determine the distance from the wire carrying current i_1 where the value of the magnetic field is zero between the two wires.

9. For the Bohr model of the hydrogen atom, show that $\vec{\mu} = -(e/2m)\vec{\mathbf{L}}$, where $\vec{\mathbf{L}} = mr\,\vec{\mathbf{v}}$ is the angular momentum of the electron in its orbit.

10. Two long, straight, parallel wires carry equal current of 10 A in opposite directions – one out of the plane and the other into the plane of the paper as shown in Fig. 12.29. Determine the magnitude and direction of the magnetic field at a) point P and b) point Q. Take x = 0.25 m.

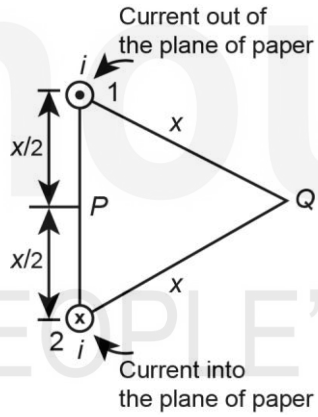


Fig. 12.29: **Diagram for TQ 10.**

12.9 SOLUTIONS AND ANSWERS

Self-Assessment Questions

1. a) The instantaneous current is given as [Eq. (12.1)]:

$$i(t) = \frac{dq}{dt}$$

We have $q(t) = (4.5\,\text{C s}^{-2})t^2 + 2.5\,\text{C} \Rightarrow \dfrac{dq}{dt} = (9.0\,\text{C s}^{-2})t$

Thus, instantaneous current, $i(t) = (9.0\,\text{C s}^{-2})t$

b) The value of current at $t = 3.0\,\text{s}$ is $i(t = 3.0\,\text{s}) = 9\times3\,\text{C s}^{-1} = 27\ \text{A}$

2. In most electric motors, current in a wire sets up a magnetic field. The magnetic field, in turn, exerts a force on a second current carrying wire causing the shaft to rotate.

3. The magnitude of the force on a current-carrying wire due to magnetic field is given by Eq. (12.28): $F = iLB\sin\theta$

Since the wire is oriented perpendicular to the magnetic field, $\theta = 90° \Rightarrow \sin\theta = 1$. So,

$$F = iLB \Rightarrow B = \frac{F}{iL} = \frac{15 \times 10^{-3}\,\text{N}}{(9.5\,\text{A}) \times (0.70\,\text{m})} = 2.3\,\text{mT}$$

The direction of the magnetic field is given by the right-hand rule. For current along west and magnetic force along south, the magnetic field will be vertically downward (that is, into the page).

4. The pair $\vec{\mathbf{F}}$ and $\vec{\mathbf{v}}$, $\vec{\mathbf{F}}$ and $\vec{\mathbf{B}}$ are always at right angles. Vectors $\vec{\mathbf{v}}$ and $\vec{\mathbf{B}}$ may have any angle between them.

5. a) Both are inverse square laws. In Coulomb's law, electric field is along $\hat{\mathbf{r}}$ or $-\hat{\mathbf{r}}$ depending on the sign of the charge. In Biot-Savart law, magnetic field acts perpendicular to the plane containing the current element and $\hat{\mathbf{r}}$.

   b) If we apply the right-hand rule to determine the direction of $\vec{\mathbf{B}}$ for the given direction of current, we find that directly above the wire, $\vec{\mathbf{B}}$ points into the page of the paper and directly below the wire, it points out of the page.

6. To solve the problem, we will use the superposition principle followed by magnetic field: the magnetic field at a point due to two or more current elements is the vector sum of the magnetic field at that point due to each individual current element.
   The magnitude of magnetic field due to i_1 and i_2 at point P is [Eq. (12.46)]:

$$B_1 = \frac{\mu_0 i_1}{2\pi R} = \frac{(2 \times 10^{-7}\,\text{Tm A}^{-1}) \times (15\,\text{A})}{(0.25\,\text{m})} = 1.20 \times 10^{-5}\,\text{T}$$

and $$B_2 = \frac{\mu_0 i_2}{2\pi R} = \frac{(2 \times 10^{-7}\,\text{Tm A}^{-1}) \times (10\,\text{A})}{(0.25\,\text{m})} = 8.0 \times 10^{-6}\,\text{T}$$

   The direction of $\vec{\mathbf{B}}_1$ and $\vec{\mathbf{B}}_2$ is determined by the right-hand rule. So, $\vec{\mathbf{B}}_1$ is directed into the page at point P and $\vec{\mathbf{B}}_2$ is directed out of the page. So, the resultant field, $\vec{\mathbf{B}} = \vec{\mathbf{B}}_1 + \vec{\mathbf{B}}_2$ will have the direction of $\vec{\mathbf{B}}_1$, the larger of the two fields. So, the magnetic field at point P is into the page and the magnitude of the resultant field is

$$B = B_1 - B_2 = \frac{2 \times 10^{-7}\,\text{Tm A}^{-1}}{(0.25\,\text{m})}(15\,\text{A} - 10\,\text{A}) = 4.0 \times 10^{-6}\,\text{T}$$

7. The motion of electron in a circular orbit/path around the nucleus of an hydrogen atom constitutes a electric current. The value of current is given by

$$i = \frac{\text{Charge}}{\text{Time}} = ef = (1.6 \times 10^{-19}\,\text{C}) \times (6.8 \times 10^{15}\,\text{Hz}) = 1.1 \times 10^{-3}\,\text{A}$$

   The magnitude of the magnetic field due to current loop is given by Eq. (12.48):

$$B = \frac{\mu_0}{2} \cdot \frac{ia^2}{(a^2 + R^2)^{3/2}}$$

   Notice that for the given problem, we cannot use Eq. (12.49) because we need to calculate B at the nucleus (that is, at the centre of the current loop) and the condition $R >> a$ is not satisfied. Substituting the values of $i\,(=1.1\times10^{-3}\,\text{A}), a\,(=3.1\times10^{-11}\,\text{m})$ and $R\,(=0)$, we get

$$B = \frac{(4\pi\times10^{-7}\,\text{Tm A}^{-1})\times(1.1\times10^{-3}\,\text{A})\times(3.1\times10^{-11}\text{m})^2}{2\times(3.1\times10^{-11}\text{m})^3} = 22\text{ T}$$

Terminal Questions

1. Let n be the number of electrons incident on the screen per second. Then $n = \frac{i}{e} = \frac{10\times10^{-6}\,\text{Cs}^{-1}}{1.6\times10^{-19}\,\text{C}} = 6.3\times10^{13}$ electrons per second. So, the charge Q striking the screen is given by

   $$Q = it = (10\,\mu\text{Cs}^{-1})\times(60\text{ s}) = 600\,\mu\text{C}$$

   Since the charges are electrons, the actual charge is: $Q = -600\,\mu\text{C}$

2. Since charge $-e$ passes a point on the orbit once every revolution, $i = e/T$, where $T = (2\pi r)/v$. So the effective current in the orbit is

   $$i = (ev)/(2\pi r)$$

   Further, the frequency is the reciprocal of time period T. So

   $$f = \frac{v}{2\pi r} = \frac{2.2\times10^{6}\,\text{ms}^{-1}}{2\pi\times(5.3\times10^{-11}\,\text{m})} = 6.6\times10^{15}\,\text{Hz}$$

   Each time the electron goes around the orbit, it carries a charge q around the loop. The charge passing a point on the loop each second, i.e., current is given as follows:

   $$i = ef = (1.6\times10^{-19}\text{C})\times(6.6\times10^{15})\,\text{s}^{-1} = 1.06\text{ mA}$$

   Note that the current flows in the direction opposite to the electron, which is negatively charged.

3. By definition, $\vec{\mathbf{E}}$, the electric field, is related to the current density $\vec{\mathbf{J}}$ through the relation [Eq. (12.23)]:

   $$|\vec{\mathbf{E}}| = \rho\,|\vec{\mathbf{J}}| = (1.72\times10^{-8}\,\Omega\text{m})\times(2.54\times10^{6}\,\text{A m}^{-2}) = 4.37\times10^{-2}\,\text{Vm}^{-1}$$

4. The magnitude of the force on a moving charge q in a magnetic field is given as

   $$F_B = qvB = evB = (1.6\times10^{-19}\text{ C})\times(10^{5}\,\text{ms}^{-1})\times(1\times10^{-5}\text{ T}) = 1.6\times10^{-19}\,\text{N}$$

   The weight of an electron near the Earth's surface

   $$F_g = mg = (9.1\times10^{-31}\text{kg})\times(9.8\,\text{ms}^{-2}) = 8.9\times10^{-30}\,\text{N}$$

   So, $(F_B/F_g) = 1.7\times10^{10}$

5. The magnetic force on a current-carrying wire kept in a magnetic field is given as [Eq. (12.29)]: $\vec{\mathbf{F}} = i\,\vec{\mathbf{L}}\times\vec{\mathbf{B}} = iLB\sin\theta$

   Since the wire is kept perpendicular to $\vec{\mathbf{B}}$, $\theta = 90°$. So, we have

   $$F = iLB \Rightarrow i = \frac{F}{LB} = \frac{2.5\times10^{-3}\,\text{N}}{(0.3\text{ m})\times(300\times10^{-3}\text{ T})} = 2.7\times10^{-2}\,\text{A}$$

6. The vertical component of $\vec{\mathbf{B}}$ is parallel to the current and does not contribute to the force. Therefore, we have [using Eq. (12.29)]

$F = iLB_H = (30\,\text{A}) \times (1\,\text{m}) \times (3.6 \times 10^{-5}\,\text{Wb m}^{-2}) = 10.8 \times 10^{-4}\,\text{N}$ due west.

7. For each straight segment, $\vec{\mathbf{F}} = i\,\vec{\mathbf{L}} \times \vec{\mathbf{B}}$, where $\vec{\mathbf{L}}$ is the directed line segment. From Fig. 12.28, we note that in sections AB and DE, $\vec{\mathbf{L}}$ and $\vec{\mathbf{B}}$ are parallel; so $\sin\theta = 0$ and $F = 0$. In section BC,
$F = iLB = (15\,\text{A}) \times (0.10\,\text{m}) \times (0.15\,\text{T}) = 0.23\,\text{N}$, into page. In section
CD, $F = (15\,\text{A}) \times (0.20\,\text{m}) \times (0.15\,\text{T}) \sin 65° = 0.408\,\text{N}$, out of page.

8. From the right-hand rule to determine the direction of magnetic field, we note that the magnetic field at any point between the two parallel wires carrying currents in the same direction will be oppositely directed. So, let the two fields balance each other at a distance x from the wire carrying current i_1. Further, the value of the magnetic field due to wire having current i_1 can be written as (Eq. (12.46)): $B_1 = \dfrac{\mu_0 i_1}{2\pi(x)}$

And that due to current $i_2 (= 2i_1)$ can be written as

$$B_2 = \frac{\mu_0 i_2}{2\pi(d-x)} = \frac{2\mu_0 i_1}{2\pi(d-x)} = \frac{\mu_0 i_1}{\pi(d-x)}$$

Since the two fields balance each other at this distance from wire having current i_1, we have

$$B_1 = B_2 \Rightarrow \frac{\mu_0 i_1}{2\pi x} = \frac{\mu_0 i_1}{\pi(d-x)} \Rightarrow 2x = (d-x) \Rightarrow x = d/3$$

9. In magnitude, the magnetic dipole moment is

$$\mu = iA = \frac{ev}{2\pi r}(\pi r^2) = \frac{e}{2m}(mvr) = \frac{e}{2m}L = -\frac{e}{2m}L$$

because the electron is negatively charged. In vector notation, we write

$$\vec{\boldsymbol{\mu}} = -\frac{e}{2m}\vec{\mathbf{L}}$$

10. The magnetic field due to a long, straight current carrying wire is given as $B = \dfrac{\mu_0 i}{2\pi R}$. So, the magnetic field due to each of the wire at point P will be

$$B_1 = B_2 \Rightarrow \frac{\mu_0 i}{2\pi(x/2)} = \frac{\mu_0 i}{\pi x} = \frac{(4 \times 10^{-7}\,\text{Tm A}^{-1}) \times (10\,\text{A})}{(0.25\,\text{m})} = 1.6 \times 10^{-5}\,\text{T}$$

Now, as per the right hand rule to determine the direction of the field at a point, we notice that the direction of field due to both current-carrying wires is towards the right. So, the total magnetic field at point P is

$$B = B_1 + B_2 = 2 \times 1.6 \times 10^{-5}\,\text{T} = 3.2 \times 10^{-5}\,\text{T},\ \text{(towards right)}$$

Similarly, the magnitude of the magnetic field due to each of the wires at point Q will be same:

$$B_1 = B_2 = \frac{\mu_0 i}{2\pi x} = \frac{(2 \times 10^{-7}\,\text{Tm A}^{-1}) \times (10\,\text{A})}{(0.25\,\text{m})} = 8.0 \times 10^{-6}\,\text{T}$$

Again, as per the right-hand rule, the direction of the magnetic field at Q due to each wire will be towards right. So, the total field at Q is

$B = B_1 + B_2 = 16 \times 10^{-6}\,\text{T}$ and it will be directed towards right.

An MRI machine applies a very strong magnetic field of about 0.2 to 3 tesla for obtaining very high resolution images of human body for medical diagnosis. In this unit you will learn Ampere's law for calculating magnetic fields. (Picture source: Wikimedia Commons)

UNIT 13

AMPERE'S LAW AND APPLICATIONS

Structure

STUDY GUIDE

The present unit is in continuation of the previous unit on magnetic field. You must have noted in the previous unit that in our discussion on magnetic field, we referred to the electric field due to static charges. We have been looking for laws and methods to calculate magnetic field due to steady currents which are similar to the laws and methods of calculating electric fields. So, to appreciate the concepts discussed in this unit, you should refer to the relevant units/sections of Block 2 on electrostatics as and when mentioned. Also, you should refresh your understanding of the concepts of divergence and curl of a vector that you have studied in Block 1. These concepts have been used in this unit to define magnetic vector potential – a concept very similar to electric potential. You should also focus on the physical significance of the mathematical expressions obtained in this unit. Try to solve the SAQs and TQs as it will give you practice in calculating the value of magnetic field and determine its direction for various steady current configurations.

"Ampere was the Newton of Electricity."

James C. Maxwell

13.1 INTRODUCTION

In the previous unit, you have learnt how steady current gives rise to magnetic field. You have learnt how to calculate the magnetic field due to a given current distribution using Biot-Savart law. The Biot-Savart law serves the same purpose for magnetic field as Coulomb's law for calculation of electric field due to static charge distribution. These laws show that both the electric field and the magnetic field exhibit inverse square dependence. You have also learnt that the divergence of the magnetic field is zero, i.e., it is solenoidal. Physically, the zero divergence of the magnetic field indicates that free magnetic charges or poles do not exist. In other words, it means that the magnetic field does not have any sources similar to the electric charges for the electric field.

In the present unit, we continue our discussion on magnetic field. Our attempt here is to explain concepts and laws obeyed by a magnetic field so that we can calculate its value for different current distributions. In doing so, we shall always seek analogy with calculation of electric field due to various charge distributions. So, the question we should ask now is: Do we have a law for magnetic field that is analogous to Gauss's law for electric field? The answer is, yes, we do have. Ampere's law, which we discuss in Sec. 13.2, enables us to calculate the magnetic field due to a symmetric current distribution just as Gauss's law enables us to calculate electric field due to a symmetric charge distribution. In Sec. 13.3, you will learn how to apply Ampere's law to calculate magnetic field due to a long, straight current-carrying wire, a solenoid, and a toroid. In Sec. 13.4, you will learn how to establish Ampere's law in differential form. You will also learn that the differential form of the Ampere's law, given as curl of $\vec{\mathbf{B}}$, enables us to define a quantity called magnetic vector potential. The magnetic vector potential simplifies the calculation of magnetic field in the same way as the electric potential simplifies the calculation of electric field. However, since magnetic vector potential is a vector quantity, its calculation is not as simple as that of electric potential.

In the next unit, you will study the magnetic properties of materials and learn that materials can be broadly classified into three categories, namely, diamagnetic, paramagnetic and ferromagnetic materials.

Expected Learning Outcomes

After studying this unit, you should be able to:

- ❖ state and explain Ampere's law;
- ❖ use Ampere's law to calculate the magnetic field due to steady current distributions having simple geometries such as straight wire, solenoid and toroid;
- ❖ obtain Ampere's law in differential form;
- ❖ define magnetic vector potential; and
- ❖ derive an expression for the torque exerted by a magnetic field on a current loops.

13.2 AMPERE'S LAW

While discussing electrostatics in Unit 5 (Block 2) of this course, we have used Coulomb's law to calculate the electric field due to an arbitrary charge distribution. In Units 6 and 7 of Block 2, you have learnt how to use Gauss's law to solve electric field problems of highly symmetric charge distributions with ease and elegance.

The situation is similar in magnetism. In Unit 12, you have learnt how to calculate the magnetic field due to current distributions using Biot-Savart law. Now, the question is: Do we have a law for magnetism which is analogous to Gauss's law for electrostatics that would help us calculate magnetic field with similar ease and elegance? You know that Gauss's law for the electric field relates the amount of charge enclosed by a surface to the flux linked with it. Is there an analogous concept or law that would prove useful in determining the magnetic field due to a current? Yes, there is such a law called Ampere's law.

According to Ampere's law, the line integral of magnetic field **around any closed loop encircling a steady straight current is proportional to the current *i* encircled by that loop and is given by** $\mu_0 i$.

Mathematically, Ampere's law is given as

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{l}} = \mu_0 i \qquad (13.1)$$

Ampere's law is true for any type of current and any closed loop, as long as the encircled current is steady (never changing in time). If the current is not in a single wire, but in a number of wires, we simply add all the currents to obtain the net current encircled by the loop. If there are currents flowing in opposite directions, then we give the opposite signs to opposite directions of the current. The **algebraic sum** of currents encircled by the loop is the net current that determines the line integral of $\vec{\mathbf{B}}$ around the loop. The loop we consider for calculating the line integral is called **Amperian loop.**

Let us elaborate the meaning of line integral of $\vec{\mathbf{B}}$ around a closed loop – the Amperian loop – and net current by considering a concrete example. Refer to Fig. 13.1 which depicts the cross-sections of three long straight wires that pierce the plane of the page at right angles. Suppose the wires carry currents i_1, i_2 and i_3. The direction of current i_1 is into the page and that of i_2 and i_3 is out of the page. The closed curve *C* is an arbitrary Amperian loop which encircles two of the currents i_1 and i_2 but it does not encircle the third current i_3.

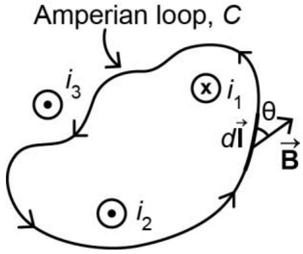


Fig. 13.1: **Amperian loop *C*, encircling two long straight wires but excludes a third wire. Note the directions of the currents.**

Let us now calculate the scalar product $\vec{\mathbf{B}} \cdot d\vec{\mathbf{l}}$ and its integral along the counter clockwise direction along the Amperian loop. To do so, we divide the loop into numerous vector element $d\vec{\mathbf{l}}$. Each vector element $d\vec{\mathbf{l}}$ is directed along the tangent to the loop in the direction of integration. Now, let us suppose that the net magnetic field $\vec{\mathbf{B}}$ due to the three currents make an angle θ with $d\vec{\mathbf{l}}$ as shown in Fig. 13.1. So, we have $\vec{\mathbf{B}} \cdot d\vec{\mathbf{l}} = B\,dl \cos\theta$. Then, Eq. (13.1) can be written as

$$\oint \vec{\mathbf{B}}.d\vec{\mathbf{l}} = \oint Bdl\cos\theta = \mu_0 i_{encircled} \tag{13.2}$$

To determine the net encircled current, $i_{encircled}$ for the given situation, we need to assign – a plus or a minus – to each of the currents i_1, i_2 and i_3. The currents are assigned sign using right hand rule: curl your right hand fingers around the Amperian loop with the fingers pointing in the direction of integration; a current in the direction of your thumb is assigned a plus sign and a current in opposite direction is assigned a minus sign.

So, as per the above sign convention, we find from Fig. 13.1 that i_1 is to be assigned a minus sign and i_2 is to be assigned a plus sign. (We need not consider the current i_3 because it is not encircled by the Amperian loop we have considered.) So, the net current encircled by the Amperian loop in Fig. 13.1 is

$$i_{encircled} = i_2 - i_1$$

So, Eq. (13.2) can be written as

$$\oint \vec{\mathbf{B}}.d\vec{\mathbf{l}} = \oint Bdl\cos\theta = \mu_0 (i_2 - i_1) \tag{13.3}$$

You may ask: Why have we not considered current i_3 on the RHS of Eq. (13.3) despite the fact that it contributes to the magnitude of B on the LHS? The contribution of i_3 to B cancels out because we take integration over the entire loop; for any given vector element $d\vec{\mathbf{l}}$ over the loop, the contribution of i_3 to B is cancelled by an oppositely located vector element $d\vec{\mathbf{l}}$ on the Amperian loop.

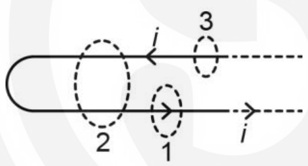


Fig. 13.2: **Diagram for SAQ 1**

To fix your understanding of Ampere's law, answer an SAQ.

SAQ 1 – **Ampere's law and Amperian loop**

Apply Ampere's law qualitatively to the three loops shown in Fig. 13.2.

From the above discussion, you must have realised that Ampere's law provides an easy method to determine $\vec{\mathbf{B}}$ due to steady currents. We need to know the net encircled current by Amperian loop and calculate the line integral on the LHS of Eqs. (13.1) or (13.3). But, how to calculate the integral of $\vec{\mathbf{B}}$. Let us find out.

In Unit 7, Block 2 of this course, you have learnt how to determine the electric field due to different types of charge distributions using Gauss's law. However, we could use Gauss's law only for certain symmetrical charge distributions by constructing suitable closed surfaces in the electric field. Ampere's law plays the same role in magnetostatics as Gauss's law plays in electrostatics. We can use Ampere's law to determine magnetic field by choosing appropriate Amperian loop for steady current distributions. Let us learn it now.

13.3 APPLICATIONS OF AMPERE'S LAW

In the following discussion, you will learn how to apply Ampere's law to determine magnetic fields due to symmetric current distributions. For this purpose, as you will see, we have to construct suitable Amperian loops in the magnetic field over which the line integral $\oint \vec{\mathbf{B}}.d\vec{\mathbf{l}}$ [see Eq. (13.1)] is to be evaluated. We shall illustrate this with a few examples.

13.3.1 Magnetic Field due to a Long Straight Wire

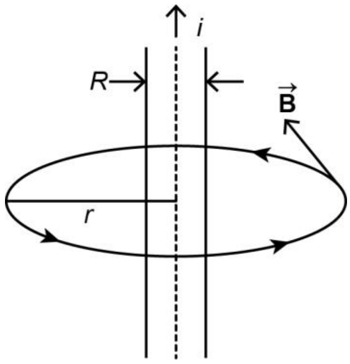


Fig. 13.3: **Magnetic field due to straight current-carrying wire.**

Let us consider a long straight wire of radius *R* carrying current *i* as shown in Fig. 13.3. The steady current in the wire produces a magnetic field. You know that magnetic field lines are closed circles concentric with the wire as shown in Fig. 13.4. Further, as per the right hand rule, the direction of the magnetic field $\vec{\mathbf{B}}$ is counter-clockwise for the given direction of current *i* (Fig. 13.3)

Now let us apply Ampere's law to determine the magnitude of $\vec{\mathbf{B}}$ at a distance *r* metres from the axis of the wire ($r >> R$). Here, we assume that *r* is much smaller in comparison with the length of the wire.

To evaluate the line integral in Ampere's law [Eq. 13.1)], we need to construct an Amperian loop. To do so, we note from Eq. (12.46) of Unit 12 that the magnetic field produced by an infinitely long current-carrying wire has same magnitude at all points which are at a distance *r* from the wire. This means that $\vec{\mathbf{B}}$ has cylindrical symmetry about the wire. We can take advantage of the cylindrical symmetry by considering an Amperian loop in the form of a circle of radius *r*. This will ensure that the magnitude of $\vec{\mathbf{B}}$ in the line integral of Eq. (13.1) is a constant and hence it can be taken out of the integral sign.

We further note from Fig. 13.3 that $\vec{\mathbf{B}}$ is tangent to the Amperian loop at every point. And, as we know, the vector element $d\vec{\mathbf{l}}$ of the loop is also tangent to the Amperian loop. So, $\vec{\mathbf{B}}$ and $d\vec{\mathbf{l}}$ are either parallel or antiparallel to each other. If we assume $\vec{\mathbf{B}}$ and $d\vec{\mathbf{l}}$ to be parallel, as is the case in Fig. (13.3), then $\theta = 0$ and we have $\vec{\mathbf{B}}.d\vec{\mathbf{l}} = Bdl\cos\theta = Bdl$.

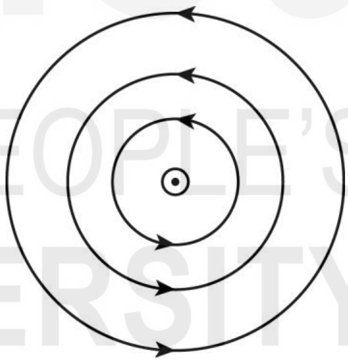


Fig. 13.4: **Top view of the magnetic field lines of a long straight current-carrying wire. The direction of current flow is out of the page.**

In view of the above, we can write the LHS of Eq. (13.1) for a long, straight current-carrying wire as

$$\oint \vec{\mathbf{B}}.d\vec{\mathbf{l}} = \oint Bdl = B\oint dl = B.2\pi r$$

Thus, from Eq. (13.1), we can write

$$B.2\pi r = \mu_0 i$$

or

$$\boxed{B = \frac{\mu_0 i}{2\pi r}} \qquad (13.4)$$

At this stage, you should pause for a moment and ask yourself: What is the advantage of using Ampere's law for determining $\vec{\mathbf{B}}$ due to a current carrying wire? We can do this using Biot-Savart law discussed in Unit 12. Well, the advantage of using Ampere's law lies in the ease and elegance of determining $\vec{\mathbf{B}}$. You just need to choose an appropriate Amperian loop. This advantage will be further illustrated when we calculate $\vec{\mathbf{B}}$ for a current-carrying solenoid in the next section. But, before that, you should go through an example.

EXAMPLE 13.1: MAGNETIC FIELD DUE TO STRAIGHT WIRE

A long cylindrical wire of radius R carries a steady current i which is uniformly distributed over its cross-sectional area. Determine the magnetic field at a distance $r(< R)$ from the axis of the wire.

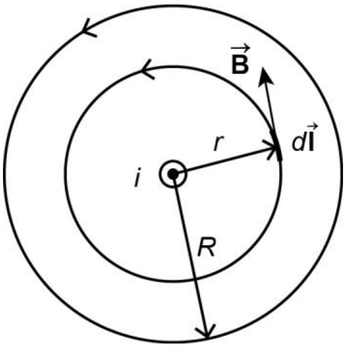


Fig. 13.5: **Diagram for Example 13.1. Note that it is the top view of a long, straight current-carrying wire and the direction of the current flow is out of the page.**

SOLUTION ■ We first notice that the point at a distance $r < R$ lies inside the wire (Fig. 13.5). However, even in this case, due to cylindrical symmetry of the magnetic field around the current-carrying wire, $\vec{\mathbf{B}}$ has a constant magnitude at all points on a path which is a circle of radius r with its centre on the axis of the wire. And the direction of $\vec{\mathbf{B}}$ at every point along this circle is along the tangent to the circle at that point.

So, we choose the circle of radius r as the path of integration for the line integral in Ampere's law [Eq. (13.1)]. Hence, we can write

$$\oint \vec{\mathbf{B}}.d\vec{\mathbf{l}} = B.2\pi r \qquad \text{(i)}$$

Now, what is the magnitude of current passing through the cross-section of the wire enclosed by the Amperian loop – the path of integration? Note that the current enclosed by this path is not i, the current through the wire; rather, it is the part of the current which passes through the cross-section of area πr^2. It is so because $r < R$. Thus, we have

Current through the Amperian loop = $\pi r^2 \times$ current per unit area of cross-section

$$= \pi r^2 \times \frac{i}{\pi R^2} = i\frac{r^2}{R^2} \qquad \text{(ii)}$$

Using Eqs. (i) and (ii) in Eq. (13.1), we can write

$$B2\pi r = \mu_0 \frac{ir^2}{R^2} \Rightarrow B = \frac{\mu_0 i}{2\pi R^2} r \qquad (13.5)$$

Now, answer the following SAQ.

SAQ 2 - Variation of magnetic field with distance from wire

Plot magnetic field B as a function of distance from the axis of the wire (of radius R) to some distance outside it.

In Unit 11, you have learnt that we can produce a uniform electric field between two closely spaced, charged conducting plates of a capacitor. You would like to know: Is there an analogous device that will produce a uniform magnetic field? Yes, the device is called solenoid. Let us now discuss how a solenoid produces a uniform magnetic field and calculate its value using Ampere's law.

13.3.2 Magnetic Field due to a Solenoid

You know that the direction of the magnetic field due to current flowing in a circular loop is given by the right hand rule (Fig. 13.6). Note that the magnetic field lines circle the wire. **A solenoid can be thought of as a cylindrical stack of current-carrying loops.**

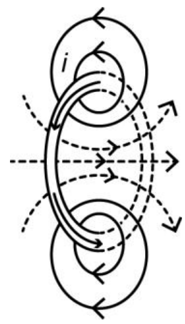


Fig. 13.6: **Magnetic field line due to current in a loop.**

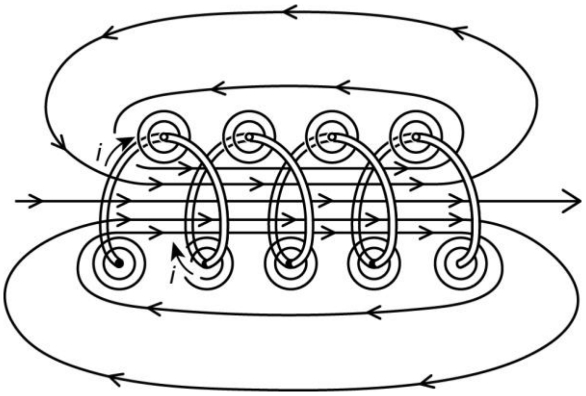


Fig. 13.7: **A loosely wound coil of wire having four loops. The magnetic field arising from a current in the wire is strongest within the coil. The field is shown only in the plane of the page.**

Refer to Fig. 13.7 which shows a solenoid having four turns. Here the turns are loosely wound compared to the common solenoids. Now, you may ask: What is the direction of $\vec{\mathbf{B}}$ due to current-carrying solenoid? Note that for any part of the wire, the magnetic field at nearby point encircles the wire. We show these field lines at the top and bottom of the coil, where the wires cross the plane of the page. But, as we move away from wire inside the coil, the fields from elements of wire at the top and bottom have a component to the right, and so they tend to reinforce each other. **The net magnetic field anywhere is the vector sum of the fields of the individual parts of the loop.**

What about the direction of $\vec{\mathbf{B}}$ outside the solenoid? Above the top of the coil, the fields arising from elements at the top all have a component to the left, while fields from elements at the bottom have a component to the right, thereby weakening the net field. A similar weakening of the field occurs below the bottom of the coil.

Hence, within the solenoid the net field is strong and points to the right and it is weaker and points to the left outside the coil, as shown in the Fig. 13.7.

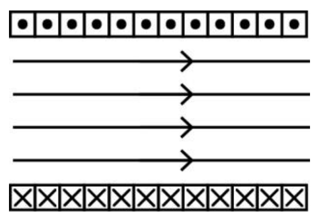


Fig. 13.8: **A longer central section of a long, more tightly wound solenoid.**

Suppose the coil is tightly wound and its length is longer than its diameter, as shown in Fig. 13.8. In such situation, the field is still strong inside the coil of the solenoid, and as the individual turns get arbitrarily close, the irregularities in the field disappear, giving straight field lines inside the solenoid.

What about the field lines outside the solenoid? The exterior field lines must connect the field lines emerging from the right of the solenoid to those going into the left because field lines cannot begin or end. The field lines close to the solenoid axis bend very gradually, and spread far from the solenoid before they return to the other end.

Now, to calculate the magnitude of $\vec{\mathbf{B}}$ inside a solenoid by applying Ampere's law, we note that: (i) the magnetic field is directed lengthwise along the axis of a tightly wound, long solenoid i.e., the length of the solenoid is very large compared to its diameter; and (ii) if the solenoid is long, the field lines emerging from the end of the solenoid will fan out widely as they come back around to enter the other end. The second point above indicates that the magnetic field outside the solenoid is many times weaker than it is inside. Consequently, we approximate the situation and consider the field outside the solenoid to be negligibly small.

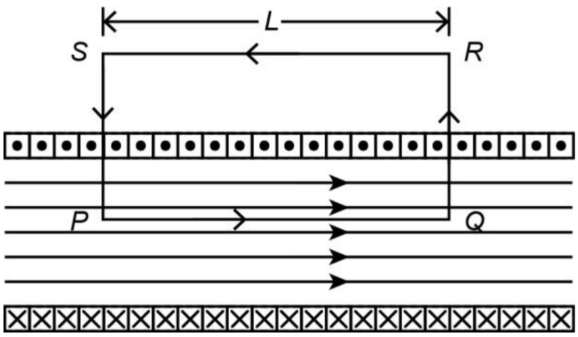


Fig. 13.9: **A long solenoid, showing a rectangular Amperian loop** ***PQRS*****.**

Now, what shape of Amperian loop, the path of integration in the line integral of Eq. (13.1), will make the calculation of *B* easier? If we take a rectangular Amperian loop such that its sides are either parallel or perpendicular to the direction of $\vec{\mathbf{B}}$, the calculation of line integral of $\vec{\mathbf{B}}$ in Eq. (13.1) will become easier because, in such cases, θ is either zero or 90°. So, let us consider a closed regtangular path *PQRS* as Amperian loop (Fig. 13.9). For this path, we can write the line integral of Eq. (13.1) as

$$\oint \vec{\mathbf{B}}.d\vec{\mathbf{l}} = \int_P^Q \vec{\mathbf{B}}.d\vec{\mathbf{l}} + \int_Q^R \vec{\mathbf{B}}.d\vec{\mathbf{l}} + \int_R^S \vec{\mathbf{B}}.d\vec{\mathbf{l}} + \int_S^P \vec{\mathbf{B}}.d\vec{\mathbf{l}} \tag{13.6}$$

The integrals over the segments *QR* and *SP* are zero because for the parts of these paths outside the solenoid, $\vec{\mathbf{B}} = 0$ and for the parts inside the solenoid, $\vec{\mathbf{B}}$ is perpendicular to $d\vec{\mathbf{l}}$. The integral over segment *RS* is zero as we have assumed that $\vec{\mathbf{B}} = 0$ outsides the solenoid. Thus, the only integral in Eq. (13.6) that is different from zero is over segment *PQ*. Hence, Eq. (13.6) reduces to

$$\oint \vec{\mathbf{B}}.d\vec{\mathbf{l}} = \int_P^Q \vec{\mathbf{B}}.d\vec{\mathbf{l}}$$

Now, as explained above, for this path, $\vec{\mathbf{B}}$ is constant and along the direction of the path. Thus,

$$\oint \vec{\mathbf{B}}.d\vec{\mathbf{l}} = \int_P^Q \vec{\mathbf{B}}.d\vec{\mathbf{l}} = \vec{\mathbf{B}} \int_P^Q d\vec{\mathbf{l}} = BL \tag{13.7}$$

where *L* is the length of the path *PQ*. If this path encloses *N* turns of wire of the solenoid each carrying a current *i*, then total current encircled by this path is *Ni*. Thus, for the right hand side of Ampere's law [Eq. (13.1)] we write $\mu_0 Ni$. Thus, using Eq. (13.7), we can write Eq. (13.1) as

$$BL = \mu_0 Ni$$

$$B = \frac{\mu_0 Ni}{L} = \mu_0 ni \qquad (13.8)$$

where n is the number of the turns per unit length of the solenoid.

Eq. (13.8), obtained for an infinitely long solenoid, holds quite well for actual solenoids, for points well inside the solenoid away from its ends. Note that $\vec{\mathbf{B}}$ does not depend upon the position of the point within the solenoid as long as we are far away from the ends of the solenoid. Therefore, we conclude that $\vec{\mathbf{B}}$ is uniform over the cross-section of the solenoid. This characteristic of a solenoid makes it a very useful electrical component to set up a known uniform magnetic field for experimental purposes.

Before proceeding further, answer an SAQ.

SAQ 3 **- Field due to solenoid**

How will the magnetic field inside a long solenoid vary if i) the number of turns per meter is doubled, ii) the current is doubled, iii) the length of the solenoid is doubled affecting the turns per meter, iv) length of the solenoid is doubled keeping the turns per meter constant, and v) the diameter of the solenoid is doubled.

13.3.3 Magnetic Field inside a Toroid

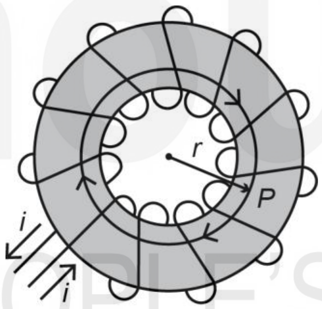


Fig. 13.10: **A toroid is solenoid bent into the form of a circle so as to join its two end. Also shown is an Amperian loop (dotted lines) for calculating the field.**

A toroid is a donut-shaped coil used as inductor in electronic circuits. If a solenoid is bent into the form of a circle so as to join its two ends, one obtains a toroid as shown in Fig. 13.10. To obtain an expression for the magnitude of $\vec{\mathbf{B}}$ due to a toroid, we note from the symmetry of its structure that it gives rise to circular and concentric field lines of $\vec{\mathbf{B}}$ inside the toroid with the centre of these fields lines coincident with the centre of the toroid. Also, the magnitude of field is constant along any field line.

Now, let us consider an Amperian loop in the form of a circle of radius r that coincides with a field line (see Fig. 13.10). So, the line integral of Eq. (13.1) for this Amperian loop can be written as

$$\oint \vec{\mathbf{B}}.d\vec{\mathbf{l}} = 2\pi rB$$

Note that the Amperian loop coincides with a field line and the magnitude of $\vec{\mathbf{B}}$ is constant at every point on the loop. As a result, the line integral of Eq. (13.1) is just the field strength times the circumference $2\pi r$ of the loop. Now, you may ask: **How much current is encircled by the loop?** If the toroid consists of N turns, and carries a current i, then an Amperian loop inside the toroid coil encircles a total current Ni. This is because each turn carries current in the same direction through the path (Amperian loop) we have chosen. Thus, substituting the value of the line integral for the Amperian loop and the total current encircled in Eq. (13.1) we get

$$2\pi rB = \mu_0 Ni$$

so that

$$B = \frac{\mu_0 Ni}{2\pi r} \qquad (13.9)$$

Eq. (13.9) holds when the Amperian loop is within the toroid itself. If the Amperian loop is inside the inner edge of the toroidal coil, there is no current encircled, and the magnetic field is zero. On the other hand, if the Amperian loop is outside the outer edge of the coil, it encircles equal but opposite currents, again giving zero field. Also, from Eq. (13.9) we note that $\vec{\mathbf{B}}$ is a function of *r*. Therefore, $\vec{\mathbf{B}}$ is not constant over the cross-section of the toroid unlike the straight solenoid.

To fix the ideas about toroid, solve a SAQ.

SAQ 4 - **Magnetic field of a toroid**

A toroid has 6000 turns upon it and carries a current of 10 A. Calculate the value of magnetic field at a point within the toroid which is located at 20 cm from its centre.

So, on the basis of the above discussion wherein we calculated $\vec{\mathbf{B}}$ using Ampere's law for different current configurations, you must have noted that calculation of $\vec{\mathbf{B}}$ is lot more easier using Ampere's law. This law for $\vec{\mathbf{B}}$ is somewhat similar to Gauss's law which enables us to calculate $\vec{\mathbf{E}}$ due to symmetric charge distributions.

However, Ampere's law is not always useful. It is because for calculating $\vec{\mathbf{B}}$, it is necessary that the current distribution is symmetric so that *B* has a constant magnitude. The constant value of $\vec{\mathbf{B}}$ enables us to take it out of the line integral $\oint \vec{\mathbf{B}}.d\vec{\mathbf{l}}$. So, the Ampere's law is useful only for the following current distributions: infinite straight wire, infinite plane, infinite solenoid and toroids. For other types of current distributions, we have to use Biot-Savart law.

Further, from Unit 8, you may recall that the concept of electric potential enables us to calculate $\vec{\mathbf{E}}$ easily. So, the question you would like to ask is: Can we define magnetic potential which enables us to calculate $\vec{\mathbf{B}}$? The answer is, yes, we can. To do that, we first need to express Ampere's law in differential form. Let us learn it now.

13.4 DIFFERENTIAL FORM OF AMPERE'S LAW

The integral form of Ampere's law is [Eq. (13.1)] :

$$\oint \vec{\mathbf{B}}.d\vec{\mathbf{l}} = \mu_0 i_{encircled}$$

To express Ampere's law in differential form, we make use of Stokes theorem which you studied in Unit 3 (Block 1) of this course. According to Stokes' theorem, the line integral of a vector field $\vec{\mathbf{F}}$ along a curve *C* is related to the surface integral of $\vec{\mathbf{F}}$ over a surface *S* bounded by curve *C:*

$$\oint_C \vec{\mathbf{F}}.d\vec{\mathbf{l}} = \oiint_S \text{curl}\,\vec{\mathbf{F}}.d\vec{\mathbf{S}} \qquad (13.10)$$

where $d\vec{\mathbf{S}}$ is the area enclosed by the closed path. Now using Eq. (13.10) we can write Eq. (13.1) as:

$$\oint_C \vec{\mathbf{B}}.d\vec{\mathbf{l}} = \mu_o i_{encircled} = \oiint_S \vec{\nabla} \times \vec{\mathbf{B}}.d\vec{\mathbf{S}} \qquad (13.11)$$

Further, you may recall from Unit 12, that current i is related to current density $\vec{\mathbf{J}}$ by Eq. (12.7):

$$i = \oiint_S \vec{\mathbf{J}}.d\vec{\mathbf{S}}$$

We can therefore write Eq. (13.11) as

$$\oiint_S \vec{\nabla} \times \vec{\mathbf{B}}.d\vec{\mathbf{S}} - \mu_0 \oiint_S \vec{\mathbf{J}}.d\vec{\mathbf{S}} = 0$$

or

$$\oiint_S \left[\vec{\nabla} \times \vec{\mathbf{B}} - \mu_0 \vec{\mathbf{J}}\right].d\vec{\mathbf{S}} = 0$$

Since $d\vec{\mathbf{S}}$ is non zero, the quantity within brackets in the above expression must be zero. Thus, we have

$$\boxed{\vec{\nabla} \times \vec{\mathbf{B}} = \mu_0 \vec{\mathbf{J}}} \qquad (13.12)$$

Eq. (13.12) is the **differential form of Ampere's law**.

At this stage, you should pause for a moment and think what inferences we can draw from the two vector relations involving magnetic field $\vec{\mathbf{B}}$. First is, div $\vec{\mathbf{B}} = 0$, the mathematical expression of Gauss's law for magnetism which you studied in Section 12.3 of Unit 12. We mentioned there that zero divergence of $\vec{\mathbf{B}}$ physically means that magnetic monopoles do not exist. In this respect, $\vec{\mathbf{B}}$ is different from electric field $\vec{\mathbf{E}}$ because $\vec{\nabla}.\vec{\mathbf{E}} \neq 0$.

The second relation is curl $\vec{\mathbf{B}} \neq 0$ [Eq. (13.12)]. The finite value of curl $\vec{\mathbf{B}}$ again distinguishes $\vec{\mathbf{B}}$ from $\vec{\mathbf{E}}$ because curl $\vec{\mathbf{E}}$ is zero. From Block 1, you may recall that the zero value of the curl of a vector field (such as $\vec{\mathbf{E}}$) implies that the vector field is a conservative field. Thus, we can say that curl $\vec{\mathbf{B}} \neq 0$ implies that $\vec{\mathbf{B}}$ is a non-conservative field.

Yet another consequence of these vector relations involving $\vec{\mathbf{B}}$ leads us to the concept of magnetic vector potential. Let us learn it now.

13.4.1 Magnetic Vector Potential

In Unit 8, you have learnt that the conservative nature of $\vec{\mathbf{E}}$ enabled us to define the concept of electric potential. To establish a relation between vector field $\vec{\mathbf{E}}$ and electric potential, V – a scalar quantity – we used the vector identity: curl of gradient of a scalar field is zero. This enabled us to write $\vec{\mathbf{E}} = -\,\text{grad}V$. This relation between $\vec{\mathbf{E}}$ and V is very useful in calculating $\vec{\mathbf{E}}$ at a point if we know V at that point due to a given charge distribution.

So, a logical question you may ask now is: Can we define a similar scalar potential for magnetic field so that calculation of $\vec{\mathbf{B}}$ becomes easier? No, we can not define a scalar potential associated with $\vec{\mathbf{B}}$ because, $\vec{\mathbf{B}}$ is a not a conservative field. In other words, since curl $\vec{\mathbf{B}} \neq 0$, we cannot express $\vec{\mathbf{B}}$ as a gradient of some scalar function as in the case of $\vec{\mathbf{E}}$.

However, the relation div $\vec{\mathbf{B}} = 0$ enables us to express $\vec{\mathbf{B}}$ as curl of vector field, say $\vec{\mathbf{A}}$ because of the vector identity:

$$\vec{\nabla}.(\vec{\nabla} \times \vec{\mathbf{A}}) = 0$$

where $\vec{\mathbf{A}}$ is a vector field. Thus, we can write

$$\vec{\mathbf{B}} = \vec{\nabla} \times \vec{\mathbf{A}} \tag{13.13}$$

If we compare Eq. (13.13) with the relation $\vec{\mathbf{E}} = -\vec{\nabla} V$ for electrostatic field, we find that we can associate a vector field $\vec{\mathbf{A}}$, with magnetic field $\vec{\mathbf{B}}$ as scalar potential V is associated with $\vec{\mathbf{E}}$. Thus, we call the vector field $\vec{\mathbf{A}}$, the **magnetic vector potential**.

Note that Eq. (13.13) is insufficient to define $\vec{\mathbf{A}}$ uniquely. It is so because we can always add a gradient of a scalar function, say β, to the vector $\vec{\mathbf{A}}$ and Eq. (13.13) will still be satisfied because curl of the gradient of a scalar functions is zero. So, we must first define $\vec{\mathbf{A}}$ uniquely. To do that, we obtain another condition to be satisfied by $\vec{\mathbf{A}}$ by using Eq. (13.12):

$$\vec{\nabla} \times \vec{\mathbf{B}} = \mu_0 \vec{\mathbf{J}}$$

Substitution Eq. (13.13) in the above relation, we get

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{\mathbf{A}}) = \mu_0 \vec{\mathbf{J}} \tag{13.14}$$

To proceed further, we use the vector identity

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{\mathbf{C}}) = \vec{\nabla}(\vec{\nabla}.\vec{\mathbf{C}}) - \nabla^2 \vec{\mathbf{C}} \tag{13.15}$$

So, using Eq. (13.15), we can write Eq. (13.14) as

$$\vec{\nabla}(\vec{\nabla}.\vec{\mathbf{A}}) - \nabla^2 (\vec{\mathbf{A}}) = \mu_0 \vec{\mathbf{J}} \tag{13.16}$$

Now, the LHS of Eq. (13.16) contains two terms: one involving divergence of $\vec{\mathbf{A}}$ and another term $\vec{\nabla}^2 \vec{\mathbf{A}}$. Since the condition given by Eq. (13.13) involves curl $\vec{\mathbf{A}}$, we are free to choose $\vec{\mathbf{A}}$ in such a way that its divergence is zero. That is, out of the many choices for the function $\vec{\mathbf{A}}$ which satisfy Eq. (13.13), we choose only those values of $\vec{\mathbf{A}}$ which makes it solenoidal, i.e.

$$\vec{\nabla}.\vec{\mathbf{A}} = 0 \tag{13.17}$$

So, Eq. (13.17) gives another condition which the magnetic vector potential $\vec{\mathbf{A}}$ must satisfy along with the condition given by Eq. (13.13). Therefore, the Ampere's law given by Eq. (13.12) can be written in terms of magnetic vector potential $\vec{\mathbf{A}}$ by putting $\vec{\nabla}.\vec{\mathbf{A}} = 0$ in Eq. (13.16):

$$\nabla^2 \vec{\mathbf{A}} = -\mu_0 \vec{\mathbf{J}} \tag{13.18}$$

Note that Eq. (13.18), being a vector equation, is actually three equations

$$\frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_x}{\partial y^2} + \frac{\partial^2 A_x}{\partial z^2} = -\mu_0 J_x \tag{13.19a}$$

$$\frac{\partial^2 A_y}{\partial x^2} + \frac{\partial^2 A_y}{\partial y^2} + \frac{\partial^2 A_y}{\partial z^2} = -\mu_0 J_y \tag{13.19b}$$

$$\frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2} + \frac{\partial^2 A_z}{\partial z^2} = -\mu_0 J_z \tag{13.19c}$$

Each of Eq. (13.19) is similar to the relation between electric potential V and volume charge density ρ:

$$\nabla^2 V = -\frac{\rho}{\varepsilon_0} \qquad (13.20)$$

Further, the electric potential at a point due to a volume charge is given as:

$$V = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho}{r} d\tau$$

where $d\tau$ is the volume element and r is the distance between the volume charge and the point where we wish to calculate V. So, by analogy, we may write magnetic vector potential $\vec{\mathbf{A}}$ in terms of current density of a volume current distribution as

$$\vec{\mathbf{A}} = \frac{\mu_0}{4\pi}\int \frac{\vec{\mathbf{J}}}{r} d\tau \qquad (13.21)$$

The magnetic vector potentials for line and surface current distributions are given as

$$\vec{\mathbf{A}} = \frac{\mu_0}{4\pi}\int \frac{\vec{\mathbf{L}}}{r} dl \qquad (13.22)$$

$$\vec{\mathbf{A}} = \frac{\mu_0}{4\pi}\int \frac{\vec{\mathbf{K}}}{r} dS \qquad (13.23)$$

where $\vec{\mathbf{L}}$ and $\vec{\mathbf{K}}$ are, respectively, current densities due to line and surface current distributions. Though magnetic vector potential enables us to calculate $\vec{\mathbf{B}}$, using Eq. (13.15), the calculation of $\vec{\mathbf{A}}$ itself is not easy (as the calculation of electric potential) because it is a vector quantity. The convenience of calculating $\vec{\mathbf{E}}$ using electric potential V, a scalar quantity, is not available for $\vec{\mathbf{B}}$. However, by defining a magnetic vector potential, we have somewhat established a symmetry between electric field and magnetic field.

Now, let us summarise what you have learnt in this unit.

13.5 SUMMARY

Concept	Description
Ampere's law	■ The line integral of magnetic field around a closed loop is equal to the current encircled: $\oint \vec{\mathbf{B}}.d\vec{\mathbf{l}} = \mu_0 i_{encircled}$
Magnetic field due to long straight wire	■ The magnetic field due to a current carrying long straight wire is given as $\lvert\vec{\mathbf{B}}\rvert = (\mu_0 i / 2\pi r)$ where r is the distance of the field point from the wire.
Solenoid	■ A solenoid is a long cylindrical coil having many turns of wire. Inside the current carrying solenoid, there is a uniform magnetic field given by $B = \mu_0 ni$ where n is the number of turns per unit length.
Toroid	■ The magnetic field inside a toroidal coil is given by $B = \frac{\mu_0 Ni}{2\pi r}$ where r is the distance from the centre of the toroid and N is the total number of turns wound on the toroid.

Differential form of Ampere's law

- Differential form of ampere's law is

$$\vec{\nabla} \times \vec{\mathbf{B}} = \mu_0 \vec{\mathbf{J}}$$

where $\vec{\mathbf{J}}$ is the current density at a given point.

Magnetic vector potential

- In terms of magnetic vector potential $\vec{\mathbf{A}}$, the magnetic field is given as

$$\vec{\mathbf{B}} = \vec{\nabla} \times \vec{\mathbf{A}}$$

because diverge of a curl is zero and divergence of $\vec{\mathbf{B}}$ is equal to zero.

- In terms of magnetic vector potential $\vec{\mathbf{A}}$, Ampere's law is written as

$$\nabla^2 \vec{\mathbf{A}} = -\mu_0 \vec{\mathbf{J}}$$

13.6 TERMINAL QUESTIONS

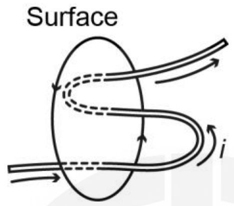


Fig. 13.11: **Diagram for TQ 2.**

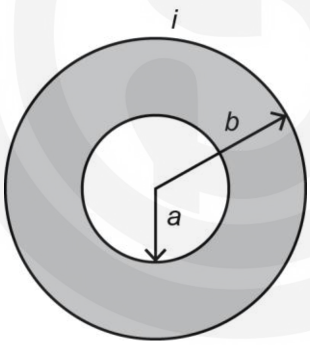


Fig. 13.12: **Diagram for TQ 4.**

1. Five very long, straight, insulated wires are closely bound together to form a small cable. Currents carried by the wires are $i_1 = 20$ A, $i_2 = -6$ A, $i_3 = 12$ A, $i_4 = -7$ A and $i_5 = 18$ A (negative currents are opposite in direction to the positive). Calculate the magnitude of $\vec{\mathbf{B}}$ at a distance of 10 cm from the cable.
2. Consider the surface bounded by the closed path shown in Fig. 13.11 with the value of *i* equal to 15 A. What is the net current passing through the surface? Calculate the value of the line integral of $\vec{\mathbf{B}}$ for this closed path.
3. A long, straight wire of diameter 4 mm carries a uniformly distributed 10 A current. At what distance from the axis of the wire the magnitude of $\vec{\mathbf{B}}$ will be maximum? Justify your answer.
4. A long, hollow conducting cylinder carries a current *i* which is uniformly distributed over the cross-section as shown in Fig. 13.12. Determine the value of magnetic field at at point a distance *r* from the axis of the cylinder for i) $r \le a$, ii) $a < r \le b$, and iii) $b \le r$.
5. A long solenoid with 900 turns per meter has a 2.6 A current. i) What is the magnitude of the magnetic field at the centre of the solenoid? ii) If the length of the solenoid is 300 mm, how many turns of wire are on the solenoid?
6. A toroid has 600 turns and a current of 200 mA is flowing in it. If the inner and outer diameters of the toroid is 80 mm and 95 mm respectively, calculate the maximum and minimum values of the magnetic field in the toroid.
7. A 15 cm long solenoid having diameter 1.5 cm carrier a current 1.5 A and the value of the magnetic field at its centre is 0.04T. If the wire used to wind the solenoid has diameter 0.6 mm, determine the number of layers in the winding and total length of the wire used.

13.7 SOLUTIONS AND ANSWERS

Self-Assessment Questions

1. The line integral $\oint \vec{\mathbf{B}}.d\vec{\mathbf{l}}$ in the Ampere's law depends on the net current encircled by the Amperian loop. The Amperian loops 1 and 3 in Fig. 13.2

encircles current i. Thus, for these loops, $\oint \vec{\mathbf{B}}.d\vec{\mathbf{l}} = \mu_0 i$. And for the Amperian loop 2 in Fig. 13.2, $\oint \vec{\mathbf{B}}.d\vec{\mathbf{l}} = 0$ because net current is zero.

2. The variation of B with distance T from the axis of the wire is shown in Fig. 13.13.

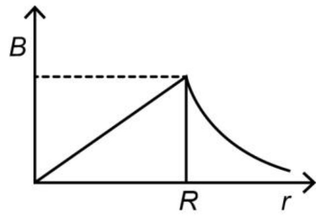


Fig. 13.13: **Diagram for answer to SAQ 2.**

3. The magnetic field inside a solenoid is given by

$$B = \mu_0 ni$$

We can answer all the questions on the basis of above relation:

i) The field will be doubled because we have made n = 2n.
ii) The field will be doubled because we have made i = 2i.
iii) The field will be halved because we have made n = n/2.
iv) The field remains unchanged because we have kept n unchanged.
v) The field remains unchanged as it is independent of the diameter of the solenoid.

4. The magnetic field within the toroid is given as $B = \dfrac{\mu_0 ni}{2\pi r}$

We have, $N = 6000, i = 10\text{ A}, r = 20\text{ cm} = 0.2\text{ m}, \mu_0 = 4\pi \times 10^{-7}\text{ T mA}^{-1}$

So, $$B = \frac{(2\times 10^{-7}\text{ T mA}^{-1})\times(6000)\times(10\text{ A})}{(0.2\text{ m})} = 0.06\text{ T}$$

Terminal Questions

1. Let us consider an Amperian loop of radius 10 cm having centre at the axis of the cable comprising five current carrying wires. Then, the magnitude of magnetic field at a distance of 10 cm from the cable is given by $B = \dfrac{\mu_0 i_{encircled}}{2\pi r}$

$$i_{encircled} = i_1 + i_2 + i_3 + i_4 + i_5 = 20\text{ A} - 6\text{ A} + 12\text{ A} - 7\text{ A} + 18\text{ A} = 37\text{ A}$$

So, $$B = [(2\times 10^{-7}\text{ T mA}^{-1})\times(37\text{ A})]/(0.1\text{m}) = 7.4\times 10^{-5}\text{ T}$$

2. Note from Fig. 13.11 that the current crosses the surface thrice; thus, the net current passing through the surface is i = 15 A. The line integral of $\vec{\mathbf{B}}$ for this closed path is given by Ampere's law as

$$\oint \vec{\mathbf{B}}.d\vec{\mathbf{l}} = \mu_0 i_{encircled} = \mu_0(15\text{ A}) = (4\pi\times 10^{-7}\text{ T mA}^{-1})\times(15\text{ A})$$

$$= 1.88\times 10^{-5}\text{ Tm}$$

3. The magnitude of the magnetic field due to current carrying wire is given by Eq. (13.4): $B = (\mu_0 i)/(2\pi r)$.

Thus, as the distance r from the axis increases, B decreases. However, from Sec. 13.3 (Example 13.1) you know that inside the wire, where distance $r < R$, the radius of the wire, B is given as

$$B = (\mu_0 i r)/(2\pi R^2)$$

This expression shows that inside the wire, B increases as r increases. And, the value of B reaches its maximum value when $r = R$. Thus, B is maximum at (4 mm/2) = 2mm.

4. i) If we take an Amperian loop of radius a around the axis of the hollow conducting cylinder, the current encircled by the loop is zero. So, $B = 0$.

   ii) For an Amperian loop having radius $r > a$ and $\leq b$, the situation is similar to the one discussed in Example 13.2. So, you can show that the magnitude of B will be

   $$B = \left(\mu_0 i (r^2 - a^2)\right)/\left(2\pi r (b^2 - a^2)\right)$$

   iii) In this case, any Amperian loop having radius $r \geq b$ will encircled the total current i. So, $B = (\mu_0 i)/(2\pi r)$

5. i) The magnitude of $\vec{\mathbf{B}}$ at the centre of a solenoid is given by Eq. (13.8): $B = \mu_0 n i$. We have, i = 2.6 A and n = 900. So,

   $$B = \mu_0 n i = (4\pi \times 10^{-7}\,\text{T mA}^{-1}) \times (900) \times (2.6\,\text{A}) = 2.9 \times 10^{-3}\,\text{T}$$

   ii) Since 1 m length has 900 turns, 300 mm will contain 270 turns.

6. From Eq. (13.9), we know that the magnetic field due to a toroid is given as $B = (\mu_0 N i)/(2\pi r)$ where r is the radial distance from the axis of the toroid. We also know that the value of B is zero inside the inner edge as well as outside the outer edge of the toroid. Further, from the above (1/r) dependence of B, we note that the value of B will be maximum just inside the toroid for which r = 80 mm. And, B will be minimum just inside the outer edge for which r = 95 mm. So,

   $$(B)_{max} = [(2 \times 10^{-7}\,\text{T mA}^{-1}) \times (600) \times (0.2\,\text{A})]/[0.08\,\text{m}] = 0.3\,\text{mT}$$

   $$(B)_{min} = [(2 \times 10^{-7}\,\text{T mA}^{-1}) \times (600) \times (0.2\,\text{A})]/[0.095\,\text{m}] = 0.25\,\text{mT}$$

7. The magnetic field due to a solenoid is given by Eq. (13.8): $B = \mu_0 n i$

   We have, $B = 0.04\,\text{T}, i = 1.5\,\text{A}$. So,

   $$n = B/(\mu_0 i) = (0.04\,\text{T})/[(4\pi \times 10^{-7}\,\text{T mA}^{-1}) \times (1.5\,\text{A})] = 2.1 \times 10^4$$

   So, the number of turns per meter is 2.1×10^4.

   The length of the solenoid is 15 cm. So, it will contain 3.15×10^3 turns. Since the length of the solenoid is 15 cm and diameter of the wire is 0.6 mm, in one layer, there would be (0.15 m)/ $(0.6 \times 10^{-3}\,\text{m}) = 2.5 \times 10^2$ turns. So, number of layers is equal to $(3.15 \times 10^3)/(2.5 \times 10^2) = 12.6$ layers.

   Further, the circumference of the solenoid is $2\pi r = 2 \times 3.14 \times 0.0075\,\text{m} = 0.047\,\text{m}$. So, in one turn, 0.047 m length of wire is used. So, total length of the wire used is approximately $(0.047\,\text{m}) \times (3.15 \times 10^3) = 148\,\text{m}$. (Note that we have neglected the gradual increase in the circumference layer after layer).

Magnetic levitation or Maglev trains work on the principle of magnetic repulsion between the cars and the track, which depend on the materials used to create high magnetic fields. You will learn about magnetic materials in this unit. (Picture source: Wikimedia Commons)

UNIT 14

MAGNETIC PROPERTIES OF MATERIALS

Structure

STUDY GUIDE

In the previous two units, you have studied the effect of magnetic field in free space. In the present unit, you will learn what happens when a magnetic field is applied to a material media. The discussion on magnetic properties of materials in the present unit will be on the lines similar to our discussion on the behaviour of dielectrics in electric field (discussed in Unit 10). So, you should go through Unit 10 once again before studying this unit. Further, to explain the magnetic properties of materials, we have introduced many new concepts such as magnetic moment, magnetisation, magnetic intensity and magnetic susceptibility. So, you should focus on the physical significance of these concepts. Also, it will help you understand the contents of this unit better if you keep looking for analogy between the concepts we introduced in Unit 10 for discussing dielectrics in electric field and the concepts introduced in the present unit. However, you should be mindful of the differences as well as the similarities between the corresponding concepts.

"Half of science is putting forth the right questions."

Francis Bacon

14.1 INTRODUCTION

In the previous two units, you have studied magnetic fields produced by steady currents. The steady currents or moving charges in conductors were considered to be placed in vacuum. Now, consider a situation where we keep a sample of material in a magnetic field. Does the sample have any effect on the magnetic field? Does the magnetic field cause some changes in the sample? Does the presence of material in magnetic field give rise to some new phenomenon? These are some of the questions which we discuss in this unit.

In Unit 10 (Block 3) of this course, you have learnt how a dielectric material responds to an electric field. We characterised the behaviour of dielectrics in terms of electric dipoles, both natural and induced, present in these materials and their response to the electric field. The macroscopic properties of these materials were explained using the concept of polarisation vector, $\vec{\mathbf{P}}$ defined as the electric dipole moment per unit volume.

The investigation of magnetic properties of materials leads us to a similar kind of explanation. However, it is a bit complicated primarily because of the absence of magnetic monopoles. The magnetic dipoles in these materials are understood in terms of the so-called Amperian current loops.

In this unit, we will try to understand, in a general way, the atomic origin of the magnetic properties of materials. Firstly, in Sec. 14.2, we give a brief description of how various substances respond to magnetic field. These experimental observations are then explained by the concepts we develop in the subsequent sections of the unit. In Sec. 14.3, you will learn the concept of magnetic dipole and how we visualise it in terms of atomic currents – currents due to motion of electrons in an atom. The response of such atomic **magnetic dipoles** to a magnetic field is understood in terms of the effect of magnetic field on a current loop. These discussions in Section 14.3 enable us to define **magnetic moment**, relate magnetic moment with the angular momentum of electrons in an atom and define **magnetisation**. The concept of magnetisation plays the same role in explaining magnetic properties of material as the concept of polarisation vector $\vec{\mathbf{P}}$ for electrical properties of dielectrics.

In Sec. 14.4, you will learn the concept of **magnetic intensity.** The relation between magnetic field $\vec{\mathbf{B}}$ and magnetic intensity $\vec{\mathbf{H}}$ is explained in Sec. 14.5. You will also learn about magnetic parameters such as magnetic susceptibility, magnetic permeability and relative permeability. These parameters are used for classification of magnetic materials into three broad categories, namely, diamagnetic, paramagnetic and ferromagnetic materials. In Sec. 14.6, you will learn the basic properties of these three types of magnetic materials. We will also discuss, qualitatively, the behaviour of ferromagnetic materials including the phenomenon of hysteresis.

In this unit, we present a simple account of magnetism based on the notions of classical physics. But, you must keep in mind that it is not possible to understand the magnetic properties of materials from the point of view of classical physics. The magnetic effects are a completely quantum mechanical phenomena. Only modern quantum physics is capable of giving a complete explanation of the magnetic properties of matter because the study requires the introduction and utilisation of quantum mechanical properties of atom. For a complete explanation, one must take recourse to quantum mechanics; however, a lot, though somewhat incomplete, information about magnetic properties of matter can be extracted by using a semi-classical approach which combines classical and some quantum concepts.

With Unit 14, we end our study of magnetism. In the next block of this course, you will study about electromagnetism.

Expected Learning Outcomes

After studying this unit, you should be able to:

- explain how a current loop can be considered as a tiny magnet having magnetic dipole moment;
- obtain an expression for torque on a magnetic dipole kept in a magnetic field;
- relate magnetic dipole moment to the angular momentum of electrons in an atom;
- explain the concept of magnetisation, magnetic susceptibility, permeability and relative permeability of materials;
- establish a relation between magnetic field and magnetic intensity;
- classify materials into diamagnetic, paramagnetic and ferromagnetic materials; and
- describe the concept of domain and explain the hysteresis curve for ferromagnetic materials.

14.2 RESPONSE OF VARIOUS SUBSTANCES TO MAGNETIC FIELD

When we speak of magnetism in everyday conversation, we almost certainly have in mind an image of a bar magnet or a compass needle. You may have observed that a magnet can be used to lift nails, tacks, safety pins, and needles (Fig. 14.1a) while, on the other hand, you cannot use a magnet to pick up a piece of wood or paper (Fig. 14.1b).

Materials such as nails, needles etc., which are influenced by a magnet, are called **magnetic materials** whereas other materials, like wood or paper, are called **non-magnetic materials**. However, this does not mean that there is no effect of magnetic field on non-magnetic materials. The difference between the behaviour of such (non-magnetic) materials and iron like magnetic materials is that the effect of magnetic field on the former is very weak.

To see how the magnetic materials respond to a magnetic field, consider a strong electromagnet, which has one sharply pointed pole piece and one flat pole piece as shown in Fig. 14.2

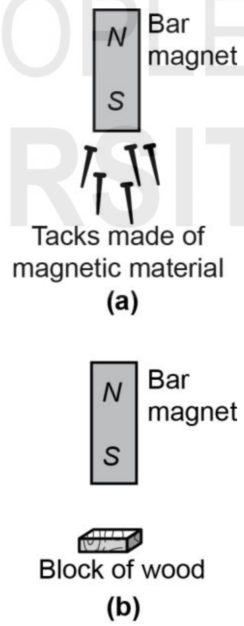


Fig. 14.1: **a) Materials that are attracted to a magnet are called magnetic materials; b) materials that do not react to a magnet are called non-magnetic materials.**

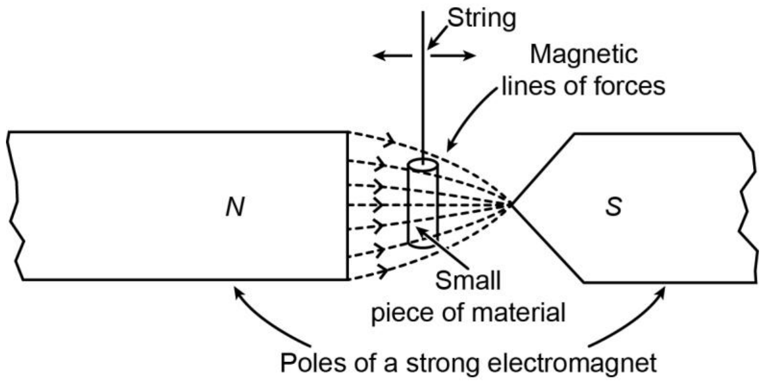


Fig. 14.2: **A small piece of material hanging from a string kept in the strong field of an electromagnet; if the piece of material is a small cylinder of bismuth, it is weakly repelled by the sharp pole (pole *S*) and if it is aluminium, it is attracted towards the sharp pole.**

The geometry of the electromagnet is such that the magnetic field is much stronger in the region near the sharply pointed pole (pole *S*) whereas near the flat pole (pole *N*), the field is weaker. When current is passed through the electromagnet (i.e., when the magnet is turned on), the hanging piece of material is slightly displaced due to the magnetic field. If specimens of various materials are used in this experiment, it is observed that some materials get displaced in the direction of increasing field, i.e., towards the pointed pole. Such materials are called **paramagnetic materials**. Examples of such materials are aluminium and liquid oxygen. On the other hand, there are materials like bismuth, which are displaced in the direction of the decreasing field, i.e. these gets repelled from the pointed pole. Such materials are called **diamagnetic**. Finally, there is a small class of materials which experience a considerably stronger force $(10^3$ to 10^5 times the force experienced by diamagnetic and paramagnetic materials) towards the pointed pole. Such substances are called **ferromagnetic materials**. Examples of such materials are iron and magnetite.

The above experimental observations may prompt you to ask many questions: Why does a substance kept in a magnetic field experience a force? Why does the force act in a particular direction for some substances and in the opposite direction for other substances? Well, you will discover answer to these and such other questions as you study this unit and understand the mechanisms of paramagnetism, diamagnetism and ferromagnetism.

In Unit 12 (Block 3) of this course you have already learnt that the source of magnetic field in free space is the electric charges in motion. In the classical picture of magnetism, this argument is extended to materials by assuming that the motion of electrons in atoms and molecules of materials give rise to tiny magnetic dipoles which interact with external magnetic field. The magnetic properties of materials arise from the magnetic moment of atomic electrons. It is this magnetic moment via which the atoms of a substance interact with the external magnetic field, and give rise to magnetic effects. Let us now discuss how a current loop can be treated like a tiny magnetic dipole, find out the value of its magnetic moment and see how magnetic moment is related to the angular momentum of the atom.

14.3 MAGNETIC MOMENT AND ANGULAR MOMENTUM OF AN ATOM

In Unit 12, you have learnt that steady currents in a conducting loop produce magnetic field. Refer to Fig. 14.3 which depicts the magnetic field lines due to a current loop and a bar magnet. The similarity of the field lines due to current loop and a bar magnet is the basis to model current loops due to the motion of electrons in the atoms and molecules of substances as tiny magnetic dipoles. The interaction of these tiny magnetic dipoles with the external magnetic field is the basis of understanding of the magnetic properties of materials in classical physics. Therefore, in the following section we first examine the response of a current-carrying loop kept in a uniform magnetic field.

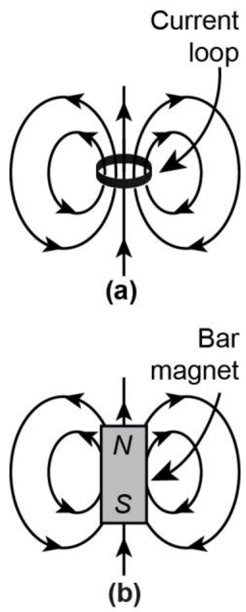


Fig. 14.3: **Magnetic field lines due to a) a current loop; b) a bar magnet.**

14.3.1 Torque on a Current Loop in a Magnetic Field

When a current loop is placed in a uniform magnetic field as shown in Fig. 14.4a, equal and opposite forces having the same line of action are exerted on it. Therefore, the net force on the current loop is zero. But, you have studied in the course BPHCT-131 entitled Mechanics that these antiparallel forces can result in a torque on such a coil which can make it rotate. You can see this in Fig. 14.4b.

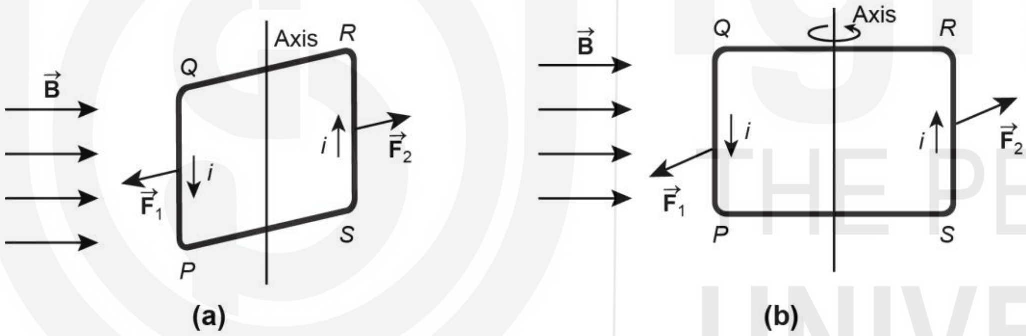


Fig. 14.4: **a) When a current-carrying coil is placed in a uniform magnetic field, equal and opposite forces are exerted on it; b) the torque due to these forces causes the loop to rotate about its axis.**

If you apply the right-hand rule to the wires of the loop shown in Fig. 14.4b, you will notice the following: the forces (not shown in the figure) on the upper and lower sides *QR* and *PS*, respectively, of the loop are parallel to the axis of rotation and are equal and opposite. Therefore, they cannot cause any rotation of the loop. However, the forces $\vec{F}_1$ and $\vec{F}_2$ that act on the sides *PQ* and *RS*, respectively, of the loop can indeed cause it to turn because their lines of action are not the same. The turning effect is zero when the coil is in the position shown in Fig. 14.4a and a torque exists for the position shown in Fig. 14.4b. Let us now find the expression for the torque.

Consider the rectangular loop *PQRS* carrying current *i* and placed in a uniform magnetic field $\vec{B}$ as shown in Fig. 14.5a. Let $PQ = RS = l$ and $QR = SP = b$. The vertical sides *PQ* and *RS* of the loop are perpendicular to the magnetic field. Therefore, the magnitude of the forces $\vec{F}_1$ and $\vec{F}_2$ on these sides is given by

$$F_1 = F_2 = i\,l\,B$$

Now, refer to Fig. 14.5b which depicts the loop when it is viewed from above. Note that the forces $\vec{\mathbf{F}}_1$ and $\vec{\mathbf{F}}_2$ are equal, parallel and directed opposite to each other and hence they form a couple. A finite torque is exerted on the loop, causing it to rotate around an axis in its plane passing through midpoint of *QR*.

You may recall from Unit 12 that the force exerted by magnetic field $\vec{\mathbf{B}}$ on a current-carrying wire of length *l* is given as

$$\vec{\mathbf{F}} = i\vec{\mathbf{l}} \times \vec{\mathbf{B}}$$
$$= i\,lB\sin\theta$$

If $\vec{\mathbf{l}}$ and $\vec{\mathbf{B}}$ are perpendicular to each other, $\theta = \pi/2$ and we have

$$F = ilB$$

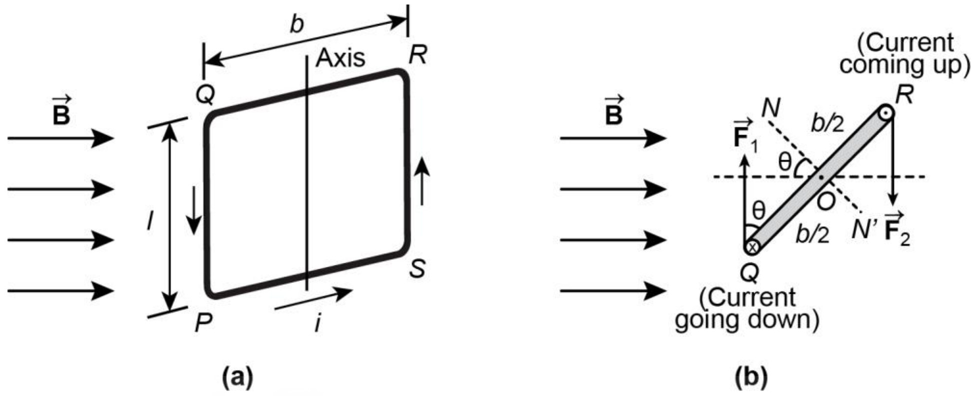


Fig. 14.5: **Torque on a current loop placed in a magnetic field: a) side view; b) top view.**

Suppose, at any instant, *NN*' – axis normal to the plane of the loop – makes an angle θ with the magnetic field as shown in Fig. 14.5b. Then, at that instant the torque τ on the loop due to forces $\vec{\mathbf{F}}_1$ and $\vec{\mathbf{F}}_2$ is given by

$$\vec{\tau}_L = \vec{\mathbf{r}} \times (\vec{\mathbf{F}}_1 + \vec{\mathbf{F}}_2)$$

$$\vec{\tau}_L = |\vec{\mathbf{r}}||\vec{\mathbf{F}}_1|\sin\theta + |\vec{\mathbf{r}}||\vec{\mathbf{F}}_2|\sin\theta$$

You know from the course Mechanics (BPHCT-131) that torque is given by

$$\vec{\tau} = \vec{\mathbf{r}} \times \vec{\mathbf{F}}$$

Since $|\vec{\mathbf{F}}_1| = |\vec{\mathbf{F}}_2| = ilB$ and $|\vec{\mathbf{r}}| = (b/2)$, we can write

$$\vec{\tau}_L = \left(ilB\frac{b}{2}\sin\theta\right) + \left(ilB\frac{b}{2}\sin\theta\right)$$
$$= ilB(b\sin\theta)$$

But $\quad l \times b = A$ (area of the loop)

$$\therefore \quad \vec{\tau}_L = iAB\sin\theta \qquad (14.1)$$

If instead of a single loop, we have a coil having *N* loops, then the net torque $\vec{\tau}_L$ is given by

$$\vec{\tau}_L = (NiA)B\sin\theta \qquad (14.2)$$

You have learnt about the area vector $\vec{\mathbf{A}}$ in Example 1.1 of Unit 1 of the course BPHCT-131. By definition $\vec{\mathbf{A}}$ is a vector such that its magnitude is the area of the loop. Its direction is perpendicular to the plane of the loop and its sense is given by the right-hand rule.

The quantities in parentheses are grouped together because they are all properties of the coil viz., its number of turns, its area and the current it carries. Eq. (14.2) tells us that a current carrying coil placed in a magnetic field will tend to rotate. We can express the toque in vector notation in terms of area vector $\vec{\mathbf{A}}$ and magnetic field $\vec{\mathbf{B}}$ as

$$\vec{\tau} = Ni\vec{\mathbf{A}} \times \vec{\mathbf{B}} \qquad (14.3)$$

where $|\vec{\mathbf{A}}| = lb$. Do you notice any similarity between Eq. (14.3) and Eq. (8.34) which gives the torque on an electric dipole kept in an electric field. Eq. (8.34) is

$$\vec{\tau} = \vec{p} \times \vec{E}$$

where $\vec{p}$ is the electric dipole moment and $\vec{E}$, the electric field. Comparison of the above expression with Eq. (14.3) suggests that a current loop in a magnetic field behaves analogously to an electric dipole in an electric field. The quantity *NiA* is called the **magnetic dipole moment** $\vec{\mu}$, of the current loop. Thus

$$\boxed{\vec{\mu} = Ni\vec{A}} \qquad (14.4)$$

Magnetic dipole moment $\vec{\mu}$ is a vector quantity and for a current loop, its direction is along the direction of $\vec{A}$ (Fig. 14.6). From Fig. 14.6, you can see that a current-carrying coil in a magnetic field behaves like a bar magnet. Using Eq. (14.4), the torque on a current loop can be written as

$$\vec{\tau} = \vec{\mu} \times \vec{B} \qquad (14.5)$$

The torque tends to align the magnetic moment with the magnetic field.

To check your understanding of the ideas discussed above, solve an SAQ.

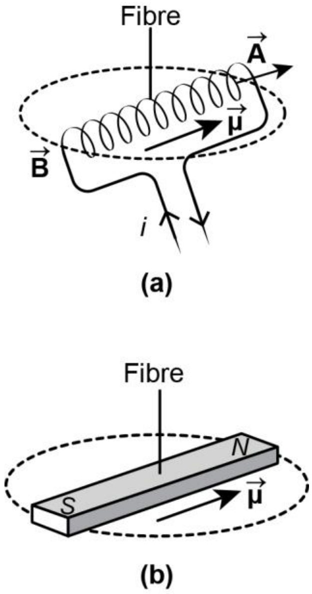


Fig. 14.6: **a) Current-carrying coil; b) bar magnet in a horizontal magnetic field.**

SAQ 1 **- Calculating torque on a current loop**

A circular loop of radius 5.0 cm consists of 10 turns of wire. A current of 3.0 A flows in the wire. What is the magnitude of the loop's magnetic moment? Suppose, initially the magnetic moment is aligned with a uniform magnetic field of 100 G. Now the loop is turned 90° from its original orientation. How much toque is required to hold the loop in its new orientation?

Let us now discuss the relation between the magnetic moment and angular momentum of an atom.

14.3.2 Electric Currents in Atoms

Electrons in an atom are in constant motion around the nucleus. To describe their motion, one needs quantum mechanics. However, in this unit we shall use only classical arguments to obtain our results.

Eq. (14.4) gives the magnetic moment of a current-carrying coil in terms of the parameters such as current, number of turns and area of the coil. But, to explain the magnetic properties of materials, we need to talk about magnetic dipoles arising due to atomic currents and define dipole moment in terms of

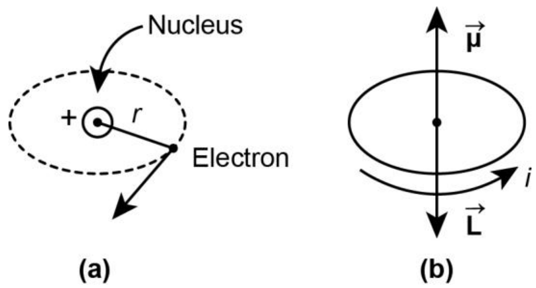


Fig. 14.7: **a) Classical model of an atom in which an electron moves with speed** ***v*** **in a circular orbit; b) the orbital angular momentum vector** $\vec{L}$ **and the magnetic moment vector** $\vec{\mu}$ **both point in opposite directions.**

the angular momentum of electrons. According to the classical model of an atom, electrons in the atom move in a circular orbit around the nucleus under the influence of a central force, known as the electrostatic force, as shown in Fig. 14.7a. The electrons in circular motion constitute a localised current loop which contributes to the magnetic dipole moment of atom. Also, due to this circular motion, the electron will have an angular momentum $\vec{\mathbf{L}}$ about the position of the nucleus.

The magnitude of the angular momentum of the electron moving in a circle of radius *r* is given by

$$L = |\vec{\mathbf{r}} \times \vec{\mathbf{p}}| = |\vec{\mathbf{r}}| \times |\vec{\mathbf{p}}| \sin\theta = mvr \quad (\text{since } \theta = 90°) \qquad (14.6)$$

where *m* is the mass of electron and *v*, its speed. Note that for circular motion, the angle between $\vec{\mathbf{r}}$ and $\vec{\mathbf{p}}$, momentum of electron, is 90°. The direction of the orbital angular momentum vector $\vec{\mathbf{L}}$ is perpendicular to the plane of the orbit.

The orbital motion of the electron constitutes an electric current. Since the charge *e* moves with speed *v*, it traverses a distance $2\pi r$ in time $(2\pi r / v)$. So, the period of rotation $2\pi r / v$. Therefore, the current due to the orbital motion of the electron is

$$i = -\frac{e}{t} = -\frac{ev}{2\pi r} \qquad (14.7)$$

The magnetic moment due to this current is the product of the current and the area of the circle in which electron moves, that is, $\mu = i\,\pi r^2$. Hence, we have

$$\mu = -\frac{evr}{2} \qquad (14.8)$$

Using Eq. (14.6) in Eq. (14.8) we get:

$$\mu = -\frac{e}{2m} L$$

In vector notation, we write

$$\boxed{\vec{\boldsymbol{\mu}} = -\frac{e}{2m}\vec{\mathbf{L}}} \qquad (14.9)$$

The negative sign in Eq. (14.9) indicates that $\vec{\boldsymbol{\mu}}$ and $\vec{\mathbf{L}}$ are in opposite directions for the electron, as shown in Fig. 14.7b. Note that $\vec{\mathbf{L}}$ is the **orbital angular momentum of the electron**. The ratio of the magnetic moment and the orbital angular momentum is called the **gyro-magnetic ratio**. It is independent of the velocity and the radius of the orbit.

EXAMPLE 14.1: ORBITAL MAGNETIC MOMENT

In the Bohr hydrogen atom, the orbital angular momentum of the electron is quantized in units of *h*, where $h = 6.626 \times 10^{34}$ Js is Planck's constant. Calculate the smallest allowed magnitude of the atomic dipole moment in JT^{-1}. (This quantity is known as **Bohr magneton**.) Mass of the electron is 9.109×10^{-31}kg.

SOLUTION ■ From Eq. (14.9), $\vec{\boldsymbol{\mu}} = -\frac{e}{2m}\vec{\mathbf{L}}$ (i)

According to the Bohr model of the hydrogen atom, the angular momentum of the orbital electrons is quantised; i.e., the angular momentum can have discrete values only. The quantised angular momentum of electron is given as

$$L = \frac{nh}{2\pi} \qquad \text{(ii)}$$

where h is Planck's constant and n is an integer.

Therefore, the minimum allowed magnitude of dipole moment is given by putting $n = 1$ in the expression for L and then using it in Eq. (i):

$$|\vec{\mu}_{min}| = \frac{e}{2m}\frac{h}{2\pi} = \frac{1.602\times10^{-19}C}{2(9.109\times10^{-31}\text{kg})}\times\frac{6.626\times10^{-34}\text{Js}}{2\pi}$$

or $$|\vec{\mu}_{min}| = 9.27\times10^{-24}\text{CJs kg}^{-1} = 9.27\times10^{-24}\text{JT}^{-1}$$

Thus, the **Bohr magneton** is given by $\frac{eh}{4\pi m} = 9.27\times10^{-24}\text{JT}^{-1}$. The Bohr magneton is a convenient unit of magnetic moment at the atomic level.

SAQ 2 - Magnetic dipole moment

What is the direction of the magnetic dipole moment of an electron relative to the direction of its orbital angular momentum $\vec{\mathbf{L}}$?

On the basis of the above discussion, you now have a fairly good idea of the magnetic field, magnetic dipole and magnetic dipole moment associated with electrons moving in circular orbits in atoms and molecules. The atoms and molecules of materials interact with the external magnetic field due to their magnetic moments. In addition, there is another way in which atomic currents and hence magnetic moments are affected by the magnetic field. In this case the magnetic moment is induced by the external field. We shall discuss this further in a later section of this unit.

In Unit 10, we discussed the macroscopic properties of dielectrics in an electric field in terms of the polarisation vector $\vec{\mathbf{P}}$. The origin of $\vec{\mathbf{P}}$ is in the dipole moments of the natural or induced electric dipoles in a dielectric material. We shall adopt a similar procedure in the study of magnetic materials by defining a quantity called magnetisation. Let us learn about it now.

14.3.3 Magnetisation

In the previous section, we restricted our discussion to isolated atoms or molecules and their magnetic dipole moment. But, a real macroscopic object comprises a large number of atoms or molecules. So, at the macroscopic level, we deal with quantities which involve averages over many atoms or molecules. **Magnetisation**, *M*, is one such quantity which is related to average dipole moment for many atoms or molecules. **It is defined as magnetic moment per unit volume.**

In view of the similarity of definitions between polarisation vector $\vec{\mathbf{P}}$ and the magnetisation vector $\vec{\mathbf{M}}$, you may be tempted to say that we should carry over all the equations in the study of dielectrics (Unit 10) to magnetic materials. One way of doing this would be to replace the electric field vector $\vec{\mathbf{E}}$ by $\vec{\mathbf{B}}$, and replace $\vec{\mathbf{P}}$ by $\vec{\mathbf{M}}$. Further, we replace the polarization charge density ρ_p by magnetic 'charge' density ρ_m (though, there are no magnetic charges or mono-poles) and write $\vec{\nabla}.\vec{\mathbf{M}} = \rho_m$ just as we had $\vec{\nabla}.\vec{\mathbf{P}} = \rho_p$. In fact, people did something like this, and they believed that magnetic charges or monopoles existed. They have built a whole theory of electromagnetism on this assumption. However, we know that magnetic 'charges' or monopoles have not yet been detected in any experiment so far, despite a long search for them. So, this approach will not do.

Now, we know the classical picture that the magnetisation of matter is due to circulating currents within the atoms of the materials. This was originally suggested by Ampere. And we call these circulating currents as 'Amperian' current loops. These currents, obviously, do not involve large scale charge transport of electrons in the magnetic materials as in the case of conduction currents. These currents are also known as **magnetisation currents**, and we shall relate these currents to the magnetisation vector $\vec{\mathbf{M}}$.

Let us consider a volume element ΔV in a material comprising a large number of atoms. Let $\vec{\boldsymbol{\mu}}_k$ be the magnetic moment of the k^{th} atom in the volume element. Then, the total magnetic dipole moment for this volume is $\sum_{\kappa} \vec{\boldsymbol{\mu}}_k$ where the vector sum is over all the atoms in the volume element.

So, the magnetisation $\vec{\mathbf{M}}$ is given as

$$\vec{\mathbf{M}} = \frac{\sum_k \vec{\boldsymbol{\mu}}_k}{\Delta V} \qquad (14.10)$$

From a macroscopic point of view, magnetisation $\vec{\mathbf{M}}$ is an adequate parameter to describe the magnetic properties of matter. Now, the question is: Can we express the magnetisation $\vec{\mathbf{M}}$ (an experimentally measurable quantity) in terms of magnetisation current (which is not measurable experimentally) in the specimen? The answer is, yes, we can. The magnetic field due to the magnetisation of the specimen can be represented by the magnetic field that would be produced by a certain distribution of atomic currents $\vec{\mathbf{J}}_m$ in the specimen. The relation between this current density $\vec{\mathbf{J}}_m$ due to the atomic currents in the specimen and $\vec{\mathbf{M}}$ is

$$\boxed{\vec{\mathbf{J}}_m = \vec{\nabla} \times \vec{\mathbf{M}}} \qquad (14.11)$$

Eq. (14.11) is general expression representing the relationship between the magnetisation of a material medium and the associated equivalent current represented by current density $\vec{\mathbf{J}}_m$. We see from Eq. (14.11) that inside a **uniformly magnetised material,** $\vec{\mathbf{M}}$ = constant and hence $\vec{\mathbf{J}}_m = 0$. **However, inside a non-uniformly magnetised material,** $\vec{\mathbf{J}}_m$ **is non-zero.**

So far we have been considering that magnetisation is due to current associated with atomic magnetic moments. Such currents are known as

bound currents or **Amperian magnetisation current**. The current density $\vec{\mathbf{J}}_m$ in Eq. (14.11) is the bound current set up within the material. However, it is not possible to measure $\vec{\mathbf{J}}_m$ experimentally. Thus, you may like to know whether we can find an expression which relates the conduction current density indicating the actual charge transport, which is experimentally measurable, and magnetisation. In other words, we are looking for an expression which relates the current density associated with the external or applied magnetic field and magnetisation. We can do that by introducing the concept of magnetic intensity which we discuss in the next section.

14.4 MAGNETIC INTENSITY

Suppose you have a piece of magnetised material. What field does this object produce? The answer is that the magnetic field produced by this object is just the magnetic field produced by the bound currents established in it. Suppose we wind a coil around this magnetised material and pass a certain current *i* through this coil. Then the magnetic field produced will be the sum of the magnetic field due to bound currents and the magnetic field due to current *i.* The current *i* is known as the **free current**. Free currents are ordinary conduction currents flowing through a macroscopic path (coil). These currents can be started and stopped with a switch and can be measured with an ammeter. (In case the magnetic material happens to be conductor, the free current will be the current flowing through the material itself.) Therefore, we can write the total current density $\vec{\mathbf{J}}$ in the material as

> Remember that free currents are the currents caused by external voltage sources, while the internal or bound currents arise due to the motion of the electrons in the atoms.

$$\vec{\mathbf{J}} = \vec{\mathbf{J}}_f + \vec{\mathbf{J}}_m \qquad (14.12)$$

where $\vec{\mathbf{J}}_f$ represents the **free current density**. Now let us use Ampere's law to calculate the magnetic field due to current density $\vec{\mathbf{J}}$. From Eq. (13.14), the differential form of Ampere's law is

$$\vec{\nabla} \times \vec{\mathbf{B}} = \mu_0 \vec{\mathbf{J}} \qquad (14.13)$$

Using Eq. (14.12) for $\vec{\mathbf{J}}$, we can write Eq. (14.13) as

$$\vec{\nabla} \times \vec{\mathbf{B}} = \mu_0 (\vec{\mathbf{J}}_f + \vec{\mathbf{J}}_m) \qquad (14.14)$$

As mentioned earlier, we have no way of measuring $\vec{\mathbf{J}}_m$ – the current density due to bound currents – experimentally. But, we have a way of expressing it in terms of a measurable quantity, the magnetisation vector $\vec{\mathbf{M}}$ through Eq. (14.11). Thus, we can write Eq. (14.14) as

$$\vec{\nabla} \times \vec{\mathbf{B}} = \mu_0 \vec{\mathbf{J}}_f + \mu_0 (\vec{\nabla} \times \vec{\mathbf{M}})$$

or

$$\vec{\nabla} \times \left(\frac{\vec{\mathbf{B}}}{\mu_0} - \vec{\mathbf{M}} \right) = \vec{\mathbf{J}}_f \qquad (14.15)$$

Eq. (14.15) is the differential equation for the vector field $\left(\frac{\vec{\mathbf{B}}}{\mu_0} - \vec{\mathbf{M}} \right)$ in terms of its source $\vec{\mathbf{J}}_f$, the **free current density**. This vector is given a new symbol $\vec{\mathbf{H}}$, i.e.,

$$\frac{\vec{\mathbf{B}}}{\mu_0} - \vec{\mathbf{M}} = \vec{\mathbf{H}} \qquad (14.16)$$

The name **magnetic intensity** has been given to $\vec{\mathbf{H}}$. Using Eq. (14.16), Eq. (14.15) becomes

$$\vec{\nabla} \times \vec{\mathbf{H}} = \vec{\mathbf{J}}_f \qquad (14.17)$$

In other words, $\vec{\mathbf{H}}$ is related to the free current in the way $\vec{\mathbf{B}}$ is related to the total current, bound plus free. This surely has made you think over the purpose of introducing the new vector field $\vec{\mathbf{H}}$. For practical reasons, the field $\vec{\mathbf{H}}$ is very useful as it can be calculated from the knowledge of free current only, whereas $\vec{\mathbf{B}}$ is related to the total current which is not known. Eq. (14.17) can also be written in the integral form as

$$\oint \vec{\mathbf{H}}.d\vec{\mathbf{l}} = i_f \qquad (14.18)$$

where i_f is the conduction current through the surface bounded by the path of the line integral on the left. Here, the line integral of $\vec{\mathbf{H}}$ is around the closed path which may or may not pass through the material. This equation can be used to calculate $\vec{\mathbf{H}}$, even in the presence of the magnetic material.

The electrical engineers working with electromagnets, transformers, etc., call the unit of $\vec{\mathbf{H}}$ as ampere turns per metre. Since 'turns', which is supposed to imply the number of turns in the coil carrying a current, is dimensionless, it need not confuse you.

SAQ 3 **- Ampere's law for $\vec{\mathrm{H}}$ field**

Derive Eq. (14.18).

In Sec. 14.3, we mentioned that Eq. (14.11) is not of much help if we want to determine magnetisation $\vec{\mathbf{M}}$ because $\vec{\mathbf{J}}_m$ cannot be measured. Further, you have learnt that magnetic intensity $\vec{\mathbf{H}}$ can be measured easily because it depends only on the free currents. So, it will be desirable to have a relation between $\vec{\mathbf{M}}$ and $\vec{\mathbf{H}}$, so that $\vec{\mathbf{M}}$ can be determined for a material. You will learn it now.

Magnetic Susceptibility and Relative Permeability

It has been observed experimentally that for most magnetic materials, the relation between $\vec{\mathbf{M}}$ and $\vec{\mathbf{H}}$ is linear, i.e.

$$\vec{\mathbf{M}} \propto \vec{\mathbf{H}}$$

or

$$\vec{\mathbf{M}} = \chi_m \vec{\mathbf{H}} \qquad (14.19)$$

The quantity χ_m is called the **magnetic susceptibility** of material. The value of magnetic susceptibility for some magnetic materials is positive and for some, negative. Magnetic materials having negative value of χ_m are called **diamagnetic** and the materials for which χ_m is positive are called **paramagnetic**. There is a third category of magnetic materials, called **ferromagnetic** materials for which the value of χ_m is very, very large and the linear relation between $\vec{\mathbf{M}}$ and $\vec{\mathbf{H}}$ [Eq. (14.19)] is only an approximation for them. Further, to obtain a relation between $\vec{\mathbf{B}}$ and $\vec{\mathbf{H}}$, we rewrite Eq. (14.16), as

$$\vec{\mathbf{B}} = \mu_0 (\vec{\mathbf{H}} + \vec{\mathbf{M}})$$

Using Eq. (14.19), we can write the above relation as

$$\vec{\mathbf{B}} = \mu_0 (\vec{\mathbf{H}} + \chi_m \vec{\mathbf{H}}) = \mu_0 (1 + \chi_m) \vec{\mathbf{H}} \qquad (14.20)$$

The quantity $(1+\chi_m)$ is defined as **relative permeability** K_m. So, in terms of K_m, Eq. (14.20) reduces to

$$\vec{\mathbf{B}} = \mu_0 K_m \vec{\mathbf{H}} \qquad (14.21)$$

The quantity $(\mu_0 K_m)$ is called **permeability**, μ of the medium. Thus, we have

$$\vec{\mathbf{B}} = \mu \vec{\mathbf{H}} \qquad (14.22)$$

Also, $\mu = \mu_0 K_m \quad \Rightarrow \quad K_m = \dfrac{\mu}{\mu_0}$ (14.23)

Note that μ has the same dimensions as μ_0, permeability of free space. Thus, K_m is a dimensionless quantity. In vacuum, $\chi_m = 0$ and $\mu = \mu_0$.

The magnetic properties of a material are completely specified if any one of the three parameters, magnetic susceptibility χ_m, relative permeability K_m or permeability μ is known.

Note that we have used the same symbol μ for two different quantities, namely, magnetic dipole moment and permeability. You have to be careful about the context of their use.

EXAMPLE 14.2: CALCULATING MAGNETIC PARAMETERS

A toroid of aluminium having length 1.0 m, is closely wound by 100 turns of wire carrying a steady current of 1.0 A. The magnetic field B in the toroid is found to be $1.2567 \times 10^{-4}\,\text{Wbm}^{-2}$. Calculate the value of (i) $\vec{\mathbf{H}}$, (ii) K_m and (iii) χ_m.

SOLUTION ■ i) To calculate the magnitude of $\vec{\mathbf{H}}$ in a toroid, we will use the integral form of Ampere's law for $\vec{\mathbf{H}}$ field [Eq. (14.18)]:

$$\oint \vec{\mathbf{H}}.d\vec{\mathbf{l}} = i_f \qquad \text{(i)}$$

To evaluate $\vec{\mathbf{H}}$ produced by the current, we consider a circular path of integration (Amperian loop) along the toroid. $\vec{\mathbf{H}}$ is constant everywhere along this path of length 1.0 m. The total free current, i_f threading this path is equal to current in the wire multiplied by the number of turns; that is $100 \times 1.0\,\text{A}$. Since $\vec{\mathbf{H}}$ is everywhere parallel to the circular path of integration, we have $\oint \vec{\mathbf{H}}.\,d\vec{\mathbf{l}} = H \times 2\pi r = H \times 1.0\,\text{m}$. Thus, we can write Eq. (i) as:

$$H \times 1.0\,\text{m} = 100 \times 1.0\,\text{A} \Rightarrow \qquad H = \frac{100 \times 1.0\,\text{A}}{1.0\,\text{m}} = 100\,\text{A}\,\text{m}^{-1}$$

ii) From Eq. (14.21), we can write the magnitude of $\vec{\mathbf{B}}$ as

$$B = \mu_0 K_m H$$

or

$$K_m = \frac{B}{\mu_0 H} = \frac{1256.7 \times 10^{-7}}{4\pi \times 10^{-7}} \times \frac{1}{100} = 1.0005$$

iii) Further, the relative permeability K_m is defined as

$$K_m = (1+\chi_m) \Rightarrow \chi_m = K_m - 1 = 1.0005 - 1 = 5 \times 10^{-4}$$

Before proceeding further, you should answer an SAQ.

***SAQ 4* - Calculating magnetic field $\vec{\mathbf{B}}$ and magnetic intensity $\vec{\mathbf{H}}$**

An air-core solenoid wound with 20 turns per centimetre carries a current of 0.18 A. Calculate the values of magnetic field $\vec{\mathbf{B}}$ and magnetic intensity $\vec{\mathbf{H}}$ at the centre of the solenoid. If an iron core of permeability $6 \times 10^{-3}\,\text{Hm}^{-1}$ is inserted in the solenoid, what will be the values of $\vec{\mathbf{H}}$ and $\vec{\mathbf{B}}$? Take $\mu_0 = 4\pi \times 10^{-7}\,\text{Hm}^{-1}$.

The magnetic parameters discussed in this section are generally determined using experimentally obtained *B-H* curve or *M-H* curve. Let us discuss the relation between $\vec{\mathbf{B}}$ and $\vec{\mathbf{H}}$ and how the *B-H* curve is obtained experimentally.

14.4.1 Relation between $\vec{\mathbf{B}}$ and $\vec{\mathbf{H}}$

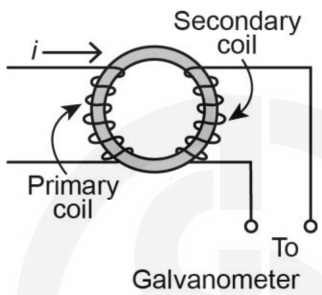


Fig. 14.8: **Arrangement for investigating the relation between *B* and *H* in a magnetic material.**

The relationship between $\vec{\mathbf{M}}$ and $\vec{\mathbf{H}}$ or equivalently, a relationship between $\vec{\mathbf{B}}$ and $\vec{\mathbf{H}}$ depends on the nature of the magnetic material, and is usually obtained form experiment. In a typical experimental arrangement such as the **magnetometer method** to obtain *M-H* curve, the magnetisation *M* of a given specimen of magnetic material is calculated on the basis of the deflection in the magnetometer for different values of applied magnetic field (that is, magnetic intensity), *H*. You will do this experiment in the laboratory course BPHCL-134 entitled Electricity and Magnetism: Laboratory.

In a typical experimental arrangement, such as the **ring method** to obtain *B-H* curve, the magnetic field *B* within the magnetised specimen is calculated for different values of magnetic intensity *H*. Refer to Fig. 14.8 which schematically depicts the experimental arrangement for the ring method. It comprises a toroid with a given magnetic material in its interior. Around the toroid, two coils – primary and secondary – are wound. If we consider the radius of the cross-section of the toroidal windings to be small in comparison with the radius of the toroid itself, the magnetic field within the toroid can be considered to be approximately uniform. A current passing through the primary coil establishes $\vec{\mathbf{H}}$. The current in the primary coil also induces an electromotive force (emf) in the secondary. By measuring the induced voltage in the secondary coil, we can determine changes in magnetic flux and hence, in $\vec{\mathbf{B}}$ inside the magnetic material. If we take $\vec{\mathbf{H}}$ as the independent variable, and if we keep the track of the change in $\vec{\mathbf{B}}$ starting from $B = 0$, we can always determine the value of $\vec{\mathbf{B}}$ for a particular value of $\vec{\mathbf{H}}$. In this way, we can obtain a *B-H* curve for different types of magnetic materials.

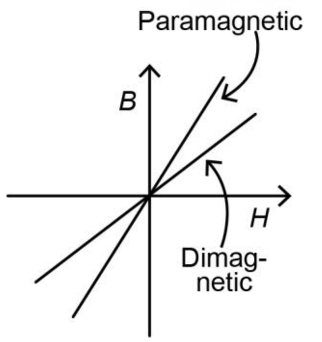


Fig. 14.9: **Magnetic field, *B* versus magnetic intensity *H* for diamagnetic and paramagnetic materials; the relationship is linear.**

The experiment described above can be carried out for diamagnetic and paramagnetic materials by commencing with $i = 0$ (that is, *H* = 0) and slowly increasing the value of *i* to obtain a series of values of *H* and *B*. A plot of *B* against *H* for these substances is shown in the Fig. 14.9. We see that the graph is a straight line as expected from Eq. (14.20):

$$B = \mu_0(1 + \chi_m)H$$

where μ_0 and χ_m are constant. The slope of the graph gives $(1 + \chi_m)$ from which χ_m can be determined using the following relation:

$$\chi_m = \frac{\text{slope}}{\mu_0} - 1$$

It is observed that, for diamagnetic materials, slope $< \mu_0$ so that $\chi_m < 0$. For paramagnetic materials slope $> \mu_0$ so that $\chi_m > 0$.

We shall discuss the *B-H* curve for a ferromagnetic material in the next section.

So far, we have discussed the concepts of magnetic moment, magnetisation, magnetic intensity, magnetic susceptibility, magnetic permeability and relative permeability. The quantities were defined on the basic premise that the magnetic behaviour of materials is caused due to atomic currents giving rise to atomic magnetic moments. These parameters enable us to classify various magnetic materials into three types, namely, diamagnetic, paramagnetic and ferromagnetic materials. Let us now briefly discuss the characteristic properties of these magnetic materials.

14.5 PROPERTIES OF MAGNETIC MATERIALS

In the beginning of this unit, we described the response of various substances to a magnetic field (Sec. 14.2). On the basis of the response to the magnetic field, magnetic materials are classified into three categories, namely, diamagnetic, paramagnetic and ferromagnetic. This classification of magnetic materials is based on experimental observations.

Let us now discuss some salient properties of these materials in terms of magnetic parameters explained in the previous sections.

14.5.1 Diamagnetism

In many materials, atoms have no permanent magnetic moment because the magnetic moments of atoms of these materials tend to cancel out. Such materials are called **diamagnetic** materials.

If a diamagnetic material is placed in a magnetic field, an emf and current is induced in their atoms in accordance with Faraday's law of electromagnetic induction. (You will study Faraday's law in detail in Unit 15.) The direction of the induced current is such that it opposes the **change** in the existing magnetic field. Hence, in such materials, the magnetic moment due to the induced currents are directed opposite to that of the external magnetic field (Fig. 14.10). Such materials are repelled by the external magnetic fields and this effect is called diamagnetism. **This effect is universal; i.e. every magnetic material exhibits diamagnetism.** However, it is a very weak effect.

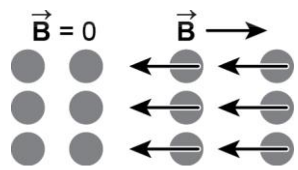


Fig. 14.10: **Alignment of magnetic moment in a diamagnet.**

The above qualitative description of diamagnetism is found to be true from experimental measurements. Refer to Fig. 14.9 which shows the *B-H* curves for diamagnetic (and paramagnetic) materials. The *B-H* curve shows that the magnetic field, *B* is directly proportional to magnetic intensity *H* which is in conformity with Eq. (14.16):

$$\vec{\mathbf{B}} = \mu_0 (1 + \chi_m) \vec{\mathbf{H}}$$

Further, the value of the slope in the *B-H* curve for diamagnets is such that the value of susceptibility χ_m for a diamagnetic material is a small negative number; i.e. $\chi_m < 0.$ Thus, it follows from the above equation that, for diamagnetic materials

$$B < \mu_0 H \qquad (14.24)$$

Further, we know that the relative permeability, K_m is given as [Eq. 14.21)]:

$$K_m = (1 + \chi_m)$$

and for diamagnetic materials, $\chi_m < 0.$ Thus, we find that the relative permeability K_m for diamagnetic materials is less than one.

Lastly, as we mentioned above, diamagnetism is a universal effect; that is, it is a common property of all magnetic materials. However, it is a very weak effect and gets masked very easily in materials which are either paramagnetic or ferromagnetic.

14.5.2 Paramagnetism

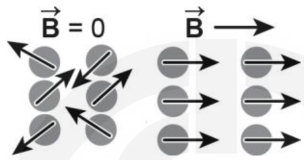


Fig. 14.11: **Alignment of magnetic moment in a paramagnet.**

In some materials, the atoms have permanent magnetic dipole moments. When such a material is placed in a magnetic field, the atomic magnetic dipoles tend to align along the direction of the magnetic field (Fig. 14.11). Thus, when such materials are placed in a magnetic field, they get attracted towards the magnet. Such materials are called paramagnetic materials. In paramagnetic materials, diamagnetism is also present, but owing to its weak nature, it gets masked.

Note that diamagnetism involves a change in the magnitude of the magnetic moment of an atom whereas paramagnetism involves change in orientation of the magnetic moment of atom.

You may note from the *B-H* curve (Fig. 14.9) of a paramagnetic substance that it conforms to the relation (Eq. 14.20):

$$\vec{\mathbf{B}} = \mu_0 (1 + \chi_m) \vec{\mathbf{H}}$$

It is found that, for paramagnetic substances, the value of susceptibility χ_m is a small positive number, i.e. $\chi_m > 0.$ Thus, from Eq. (14.20), we find that

$$B > \mu_0 H \qquad (14.25)$$

Eq. (14.25) indicates that the external magnetic field *H* produces magnetisation in its own direction. Further, we know that $K_m = (1 + \chi_m)$ and $\chi_m > 0.$ Thus, we find that the relative permeability of paramagnetic materials is greater than one.

14.5.3 Ferromagnetism

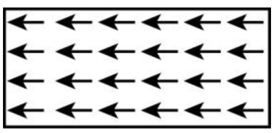


Fig. 14.12: **A domain.**

Ferromagnetic materials are those materials, which respond very strongly to the presence of magnetic fields. This unique property of ferromagnetic material can be explained using the concept of domains (Fig. 14.12). Domains are small regions in ferromagnetic materials in each of which the atomic dipole moments are aligned (that is, the atomic dipole moments are parallel to each other). However, such an alignment does not occur over the entire material; it occurs over a domain. The alignment of atomic dipole moments of one domain may be different from that of others. The domain volumes are large compared to the atomic or molecular dimensions. Such alignment takes place even in the absence of an external magnetic field. You must be wondering about the nature of forces that causes the magnetic moments of various atoms in a domain to line up parallel to each other. This can be explained only by using quantum mechanical idea of "exchange forces". We will not go into the details of exchange forces. You will study about this if you pursue higher studies in physics.

In an unmagnetised ferromagnetic material, the magnetic moments of different domains are randomly oriented as shown in Fig. 14.13, and the net magnetic moment of the materials, as a whole, is zero. However, in the presence of an external magnetic field, the magnetic moments of the domains align in such a manner as to give a net magnetic moment to the material in the direction of the field. **There are two mechanisms by which this happens.** One mechanism is that the domains with the magnetic moment in the favoured direction increase in size at the expense of the other domains, as shown in Fig. 14.14a. In the other mechanism, the magnetic moment of the entire domain can rotate and tend to align along the direction of the applied field direction as shown in Fig. 14.14b.

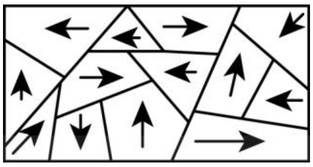


Fig. 14.13: **The domains in an unmagnetised ferromagnetic material. The arrows show the alignment direction of the magnetic moment in each domain.**

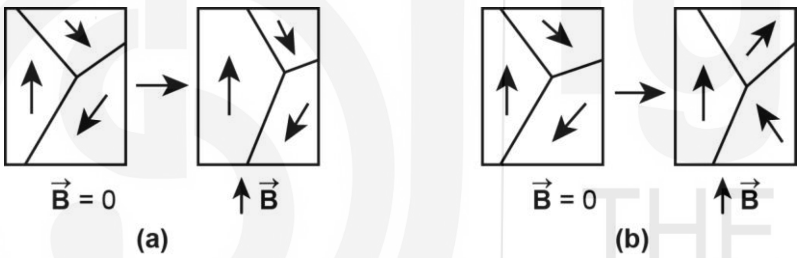


Fig. 14.14: **Domain changes in a ferromagnetic material gives rise to a net magnetic moment. The domain changes occur through a) domain growth; b) domain realignment.**

Due to either of the two mechanisms mentioned above, a sample of ferromagnetic material gets magnetised. If, after this, the external magnetic field is reduced to zero, there still remains a considerable amount of magnetisation in the material. In other words, the material gets permanently magnetised.

However, when a magnetised ferromagnetic sample is left to itself, the domains gradually tend to go back to unmagnetised state. The ferromagnetic materials are classified as **soft** and **hard** on the basis of the time required for their relaxation to unmagnetised state. In a **soft ferromagnet**, magnetisation reduces substantially as soon as the external field is removed. On the other hand, in a **hard ferromagnet**, such as many different types of steel and other alloys, magnetisation persists for years.

Further, magnetisation is generally very large in a ferromagnet. That is, magnetisation is not proportional to the applied magnetic field. Also, magnetisation reaches a saturation value which happens when all magnetic domains have the same alignment. The variation of magnetisation of an unmagnetised ferromagnet is shown in Fig. 14.15.

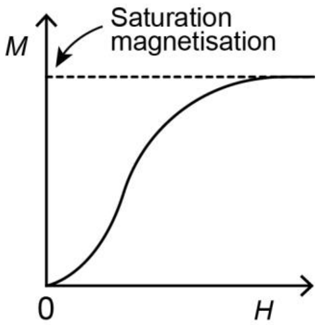


Fig. 14.15: **The magnetisation of an unmagnetised ferromagnetic material.**

The behaviour of ferromagnetic materials, under the action of changing magnetic fields, is quite different from that of diamagnetic and paramagnetic materials. Ferromagnetic material exhibits the phenomenon of **hysteresis** (which literally means **lagging behind**). Let us now discuss it in some detail.

***B-H* Curve for a Ferromagnet**

The experimentally obtained *B-H* curve for a ferromagnetic material is called **hysteresis curve** or loop. A typical *B-H* curve is shown in Fig. 14.16. The hysteresis curve contains many important information about the characteristic parameters of the ferromagnetic material. To know about them, let us first describe different segments of the curve.

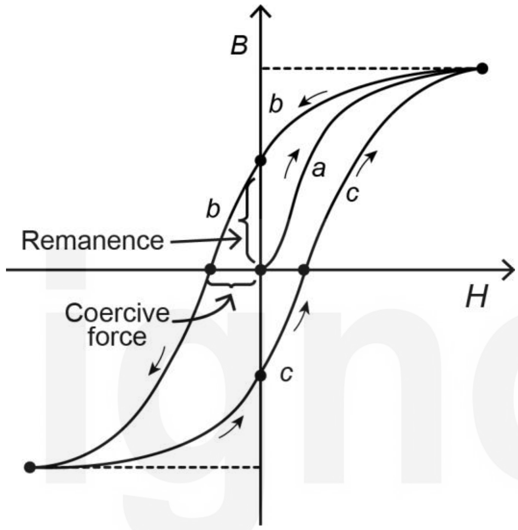


Fig. 14.16: *B-H* **curve for a ferromagnetic material.**

i) Suppose that initially, the specimen of the ferromagnetic material is unmagnetised and there is no current in the solenoid/toroid used in the experiment to obtain its *B-H* curve. That is, at $i = 0$, $H = 0$ and *B* is zero. When *i* is increased, *B* and *H* are determined for increasing values of *i*. At first, *B* increases with *H* along the curve *a* (see Fig. 14.16). The curve *a* is the initial magnetisation curve. At some high value of *H*, the curve '*a*' reaches a plateau, indicating that *M* ceases to increase, as the material has reached **saturation** with all the domain dipole moments in the same direction.

ii) If, after reaching saturation, we decrease the current in the coil to bring *H* back to zero, the *B-H* curve falls along the curve *b*. When *H* reaches zero, there is still some *B* left implying that even when *i* = 0, there is still some magnetisation *M* left in the specimen. The material is permanently magnetised. This value of *B* for *H* = 0 is called **remanence**.

iii) If the direction of current is reversed and its value is increased the *B-H* curve runs along the curve *b* until *B* becomes zero at a certain value of *H*. This value of *H* is called the **coercive force**. If we continue to increase the value of the current in the negative direction, the curve continues along path *b* until saturation is reached again.

iv) The current is now decreased until it becomes zero once again (curve *c* in the Fig. 14.16). This corresponds to *H* = 0, but *B* is not zero and has magnetisation in the opposite direction. Here we reverse the current again, so that the current in the coil is once more along the positive direction.

With the increasing current in this direction, the curve continues along the curve *c* to meet the curves *b* and *a* at saturation.

If we alternate the current between large positive and negative values, the *B*-*H* curve goes back and forth along curves *b* and *c* in a cycle. This curve is called hysteresis curve. **It shows that *B* is not a single valued function of *H*, but depends on the previous treatment of the material.**

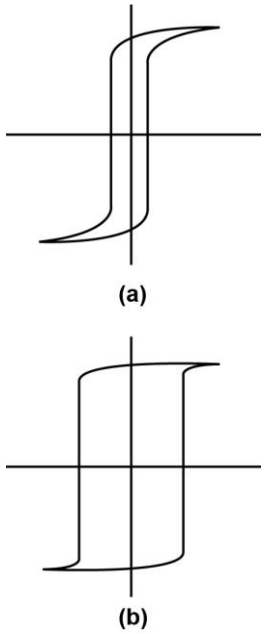


Fig. 14.17: **The hysteresis curves for specimen of a) soft iron; b) steel materials.**

The **shape** of the hysteresis loop varies very widely from one substance to another. Substances like steel, alnico, etc. from which permanent magnets are made, have a very wide hysteresis loop with a large value of coercive force (see Fig. 14.17b). However, substances like soft iron from which electromagnets (temporary magnets) are made, should have large remanance but very small coercive force. These ferromagnetic materials, used in the cores of transformers, such as iron-silicon (0.8-4.8%) alloys, have very narrow hysteresis loop (see Fig. 14.17a).

Now, let us summarise what you have learnt in this unit.

14.6 SUMMARY

Concept	Description
Types of magnetic materials	■ All materials are magnetic and respond to the presence of a magnetic field. Materials can be classified into mainly three groups: diamagnetic, paramagnetic and ferromagnetic. **Diamagnetism** is displayed by those materials in which the atoms have no permanent magnetic dipoles. **Paramagnetism** and **ferromagnetism** occurs in those materials in which the atoms have permanent magnetic dipoles.
Magnetic dipole	■ A closed current loop in a magnetic field behaves like a **magnetic dipole** and its magnetic dipole moment is given as $\vec{\boldsymbol{\mu}} = Ni\vec{\mathbf{A}}$ where *N* is the number of turns in the loop, *i* is the current in the loop and $\vec{\mathbf{A}}$ is a vector perpendicular to the plane of the loop with magnitude equal to the loop area. The torque experienced by such a current loop is given as $\vec{\boldsymbol{\tau}} = \vec{\boldsymbol{\mu}} \times \vec{\mathbf{B}}$
Magnetic moment and angular momentum of electron	■ Magnetic dipoles in magnetic materials are due to the motion of electrons in atoms or molecules. Change in the magnitude of the magnetic moment of atoms is responsible for diamagnetism whereas change in the orientation of the magnetic moment accounts for paramagnetism.

The magnetic moment due to motion of electron in a circular orbit around the nucleus in an atom is given as

$$\vec{\mu} = -\frac{e}{2m}\vec{L}$$

where $\vec{L}$ is the orbital angular momentum of the electron.

The ratio of the magnetic dipole moment to the angular momentum is called the **gyromagnetic ratio.**

Magnetisation

- Magnetisation $\vec{M}$, of a magnetic material is related to the average dipole moment for many atoms or molecules. It is defined as magnetic moment per unit volume:

$$\vec{M} = \frac{\sum_k \vec{\mu}_K}{\Delta V}$$

where $\vec{\mu}_K$ is the magnetic moment of the k^{th} atom and ΔV is the volume element containing k atoms.

Magnetisation and current density

- For non-uniform magnetisation, magnetised matter is equivalent to a current distribution $\vec{J}_m = \vec{\nabla} \times \vec{M}$, where $\vec{M}$ is magnetisation and $\vec{J}_m$ is the current density due to bound currents.

Magnetic intensity

- The magnetic field produced by the magnetised material is obtained by Ampere's law as follows:

$$\vec{\nabla} \times \vec{B} = \vec{J}_f + \vec{J}_m$$

where $\vec{J}_f$ is the **free current density** which flows through the material and $\vec{J}_m$ is the **bound current density** which is associated with magnetisation. This gives

$$\vec{\nabla} \times \left(\frac{\vec{B}}{\mu_0} - \vec{M}\right) = \vec{J}_f$$

where $\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$ is called **magnetic intensity**.

Magnetic susceptibility

- Magnetisation of a material is proportional to the magnetic intensity or magnetic field due to free currents:

$$\vec{M} \propto \vec{H} \qquad \text{or} \qquad \vec{M} = \chi_m \vec{H}$$

where χ_m is called magnetic susceptibility of the material.

Relative permeability

- Relative permeability, K_m is defined as

$$K_m = (1 + \chi_m)$$

In terms of K_m, magnetic field is expressed as

$$\vec{\mathbf{B}} = \mu_0 K_m \vec{\mathbf{H}}$$

The quantity $\mu_0 K_m$ is called permeability μ of the material.

B-H ***curve***

- For paramagnetic and diamagnetic materials, *B* or *M* and *H* are linearly related to each other. However, the *B-H* curve of ferromagnetic materials exhibit hysteresis, a non-linear behaviour.

  The **remanence** of a ferromagnetic material is the residual magnetisation in the sample of the material when the applied magnetic field has been reduced to zero.

  The **coercive force** for a ferromagnetic material is the value of the applied magnetic field which will demagnetise the sample of the material.

14.7 TERMINAL QUESTIONS

1. A uniformly charged disc having charge *q* and radius *r* is rotating with constant angular velocity of magnitude ω. Show that its magnetic dipole moment has magnitude $\frac{1}{4}(q\omega r^2)$.

2. The magnetic moment per atom for cobalt and iron are $1.6\times10^{-23}\,\text{Am}^2$ and $2.1\times10^{-23}\,\text{Am}^2$, respectively. Assume that there are 1×10^{29} atoms per cubic volume, and calculate the saturation magnetisation that can exist in these materials.

3. Calculate the magnitudes of magnetic intensity $\vec{\mathbf{H}}$ and the magnetic field $\vec{\mathbf{B}}$ at a) a point 105 mm from a long straight wire carrying a current of 15 A and b) the centre of a 2000-turn solenoid which is 0.24 m long and carries a current of $1.6\,\text{A}.(\mu_0 = 4\pi\times10^{-7}\,\text{H}\,\text{m}^{-1}.)$

4. A toroid of mean circumference 0.50 m has 500 turns, each carrying a current of 0.15 A. a) Calculate the value of $\vec{\mathbf{H}}$ and $\vec{\mathbf{B}}$ if the toroid has an air core. b) Calculate the value of $\vec{\mathbf{B}}$ and the magnetisation $\vec{\mathbf{M}}$ if the core is filled with iron of relative permeability 5000.

5. A toroid with 1500 turns is wound on an iron ring whose cross-section area is $360\,\text{mm}^2$, mean circumference is 0.75 m and relative permeability is 1500. If the windings carry 0.24 A current, calculate the value of a) the magnetic intensity $\vec{\mathbf{H}}$ and b) the magnetic field $\vec{\mathbf{B}}$.

14.8 SOLUTIONS AND ANSWERS

Self-Assessment Questions

1. The magnetic dipole moment μ due a current loop is given by Eq. (14.4):

$$\mu = NiA = (10)\times(3.0\,\text{A})\times\pi\times(0.050\,\text{m})^2 = 0.24\,\text{A}\,\text{m}^2$$

The magnitude of the torque needed to hold the loop in its new orientation can be calculated by using Eq. (14.5):

$$\tau = \mu B \sin\theta = (0.24\ \text{Am}^2) \times (0.010\ \text{T}) \times (\sin 90°) = 2.4 \times 10^{-3}\ \text{Nm}.$$

2. The direction of the magnetic dipole moment of electron is opposite to the direction of its orbital angular momentum.

3. From Eq. (14.17), we have

$$\vec{\nabla} \times \vec{\mathbf{H}} = \vec{\mathbf{J}}_f$$

We can write it as

$$\oiint_S (\vec{\nabla} \times \vec{\mathbf{H}}).d\vec{\mathbf{S}} = \oiint_S \vec{\mathbf{J}}_f .d\vec{\mathbf{S}} \qquad \text{(i)}$$

From Stoke's theorem, we can write the LHS of Eq. (i) as

$$\oiint_S (\vec{\nabla} \times \vec{\mathbf{H}}).d\vec{\mathbf{S}} = \oint_C \vec{\mathbf{H}}.d\vec{\mathbf{l}} \qquad \text{(ii)}$$

Also, from Eq. (12.7) of Unit 12, we can write the RHS of Eq. (i) as

$$\oiint_S \vec{\mathbf{J}}_f .d\vec{\mathbf{S}} = i_f \qquad \text{(iii)}$$

So, substituting Eqs. (ii) and (iii) in Eq. (i), we get

$$\oint_C \vec{\mathbf{H}}.d\vec{\mathbf{l}} = i_f$$

4. You know that, for a solenoid, the magnitude of magnetic field B is given as

$$B = \mu_0 ni$$

Since, for free space, $B = \mu_0 H,$ we can write

$$H = ni$$

where n is turns per meter. So,

$$H = ni = (2000\ \text{m}^{-1}) \times (0.18\ \text{A}) = 360\ \text{Am}^{-1}$$

The magnetic field B is given by

$$B = \mu_0 H = (4\pi \times 10^{-7}\ \text{Hm}^{-1}) \times (360\ \text{A m}^{-1}) = 0.45\ \text{mT}$$

If an iron core of absolute permeability $6 \times 10^{-3}\text{Hm}^{-1}$ is inserted in the solenoid, then H remains unchanged i.e.,

$$H = 360\ \text{Am}^{-1}$$

and $\quad B = \mu H = (6 \times 10^{-3}\text{Hm}^{-1}) \times (360\ \text{Am}^{-1}) = 2.16\ \text{T}$

Terminal Questions

1. The surface charge density of the disc is $(q / \pi r^2)$. The disc can be thought of as made up of a number of rings. Let us consider a ring of radius R and width $dR.$ The charge within this ring is given by

$$dq = \frac{q}{\pi r^2}(2\pi R dR) = \frac{2q}{r^2}(RdR)$$

The current carried by this ring is its charge divided by the rotation period:

$$di = \frac{dq}{(2\pi/\omega)} = \frac{q\omega}{\pi r^2}(RdR)$$

Thus, the magnetic moment contributed by this ring has magnitude

$$d\mu = adi$$

where *a* is the area of the ring. Therefore,

$$d\mu = \pi R^2 di = \frac{q\omega}{r^2}(R^3 dR)$$

Taking into account all the rings (radius varying from 0 to *r*), we get the magnitude of the magnetic moment as follows:

$$\mu = \int d\mu = \int_{R=0}^{R=r} \frac{q\omega}{r^2}(R^3 dR)$$

$$= \frac{q\omega}{r^2}\left[\frac{R^4}{4}\right]_0^r = \frac{q\omega}{r^2} \times \frac{1}{4} \times r^4 = \frac{1}{4} q\omega r^2$$

2. The magnetisation is defined as magnetic moment per unit volume. So, maximum or saturation magnetisation for any material is equal to the magnetic moment per atom of the material multiplied by number of atoms per cubic meter. For cobalt, therefore, we have

$$(M)_{saturation} = 1.6 \times 10^{-23}\,\text{Am}^2 \times 1 \times 10^{29}\,\text{m}^{-3} = 1.6 \times 10^6\,\text{Am}^{-1}$$

For iron, we have

$$(M)_{saturation} = 2.1 \times 10^{-23}\,\text{Am}^2 \times 1 \times 10^{29}\,\text{m}^{-3} = 2.1 \times 10^6\,\text{Am}^{-1}$$

3. a) $$H = \frac{B}{\mu_0} = \frac{\mu_0 i}{2\pi r} \times \frac{1}{\mu_0} = \frac{i}{2\pi r} = \frac{15\text{ A}}{(2\pi) \times (0.105\text{ m})} = 22.7\text{ Am}^{-1}$$

$$\left(\because \; B = \frac{\mu_0 i}{2\pi r}, \text{ see Unit 13}\right)$$

$$B = \frac{(2) \times (15)}{10^7 \times 0.105} = 28.57\ \mu\text{T}$$

b) $$H = ni = \frac{2000}{0.24\text{ m}} \times 1.6\text{ A} = 1.3 \times 10^4\,\text{Am}^{-1}$$

$$B = \mu_0 H = \frac{4\pi}{10^7} \times 1.3 \times 10^4\,\text{Am}^{-1} = 1.67 \times 10^{-2}\,\text{T}$$

4. For a toroid $H = ni$, and we use $B = (4\pi/10^7)(K_m H) = \mu H$. Thus,

a) $$H = \left(\frac{500\text{ turns}}{0.5\text{ m}}\right) \times 0.15\text{ A} = 150\text{ Am}^{-1}$$

$$B = (4\pi \times 10^{-7}\,\text{Hm}^{-1}) \times (150\ \text{Am}^{-1}) = 0.188\ \text{mT}$$

b) $$B = 5000 \times (0.188\ \text{mT}) = 0.94\ \text{T}$$

Using $(B/\mu_0) = H + M$ we can write

$$\frac{0.94}{4\pi \times 10^{-7}} = 150 + M \quad \Rightarrow \quad M = 7.5 \times 10^{5}\ \text{Am}^{-1}$$

5. a) $$H = ni = \frac{1500}{0.75\ \text{m}} \times (0.24\ \text{A}) = 480\ \text{Am}^{-1}$$

b) $$B = \frac{4\pi K_m H}{10^7} = \frac{4\pi}{10^7} \times (1500) \times (480) = 0.90\ \text{T}$$