Indira Gandhi National
Open University
School of Sciences

BPHCT-133 ELECTRICITY AND MAGNETISM

Block

4

ELECTROMAGNETISM

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The power generation and distribution systems around us are based on the phenomenon of electromagnetic induction that you will study in this unit.
(Source of picture: Wikimedia/commons)

UNIT 15

ELECTROMAGNETIC INDUCTION

Structure

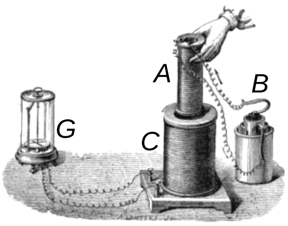
STUDY GUIDE

We hope that you have studied the concepts of vector calculus, electrostatics in free space and media, and magnetism explained in Blocks 1 to 3 of this course. You should revise them and make sure that you know all these concepts very well before studying this block. In this unit, you will learn about electromagnetic induction, Lenz's law, inductance and the energy stored in a magnetic field. You may be familiar with these concepts from your +2 physics course. However, the presentation of these concepts may be new to you. We have given many Examples and SAQs within the unit and Terminal Questions at its end to help you learn these concepts and their application well. You should study all sections thoroughly and make sure you can solve the SAQs and Terminal Questions on your own before studying the next unit.

"Nothing is too wonderful to be true, if it be consistent with the laws of nature."

Michael Faraday

15.1 INTRODUCTION



One of Faraday's 1831 experiments demonstrating induction. The battery (*B*) sends an electric current through the small coil (*A*). When it is moved in or out of the large coil (*C*), its magnetic field induces a momentary voltage in the coil *C*, which is detected by the galvanometer (*G*).

In Units 12 and 13, you have studied magnetism and learnt that electric currents (i.e., charges in motion) produce magnetic fields. In this unit, we ask: Is the reverse possible? Could magnetic fields create electric currents? This question was first asked by scientists in 1820s. For more than a decade, many scientists including the English physicist Michael Faraday did experiments to create electric currents using *static* magnetic fields but failed to do so. It was only in 1831 that Michael Faraday and the American physicist Joseph Henry independently discovered that electric currents were **induced** in circuits subjected to **changing** magnetic fields. This phenomenon, called **electromagnetic induction**, was a momentous discovery and is largely responsible for our way of life today.

When you enter a dark room and turn on the electric switch, you take it for granted that the room will be illuminated, if electric supply is available. But did you ever wonder what made this possible? If you did and sought an answer, then you might know that it is due to the discovery of electromagnetic induction by Faraday and Henry. This discovery forms the cornerstone of the entire electrical technology today. It made the generation and transmission of electric power possible as early as the end of nineteenth century. Today, it finds practical applications in thousands of electrical devices such as electric motors and electric generators in huge power plants, transformers, high speed trains, car battery chargers, electric guitars, etc. In this unit, you will learn in the space of a few hours what took Faraday and Henry many years of hard work to discover!

Michael Faraday (1791 – 1867), an English physicist, is well known for his study of electromagnetic induction, electromagnetism and electrochemistry. He discovered the principles underlying electromagnetic induction, diamagnetism and electrolysis.

In Secs. 15.2 and 15.3, you will learn the basic physics of electromagnetic induction as explained by Faraday. In Sec. 15.4, we will explain the concepts of **inductance, self-inductance** and **mutual inductance** along with many applications including the generator and the transformer. You may find it interesting to learn that apart from numerous practical applications, the discovery of electromagnetic induction also had a tremendous impact on the basic understanding of electricity and magnetism. It showed that there was a deep connection between them. This was explored further by James Clerk Maxwell and his work led to the unification of electricity and magnetism in the form of four elegant equations called Maxwell's equations. These form the basis of electromagnetic theory and you will study them in the next unit.

Expected Learning Outcomes

After studying this unit, you should be able to:

- state and apply Faraday's law of electromagnetic induction;
- state and apply Lenz's law;
- calculate the self-inductance of an inductor possessing simple geometry;
- calculate the mutual inductance of circuits in simple configurations; and
- determine the magnetic energy stored in any given circuit having an inductor and the energy stored in a magnetic field.

15.2 FARADAY'S LAW OF ELECTROMAGNETIC INDUCTION

Let us begin the discussion by briefly describing the main experiments of Michael Faraday that led to the discovery of electromagnetic induction in 1831. You may like to learn about what can be called his breakthrough experiment. In it, he wrapped two insulated coils of wire around an iron ring and observed that upon passing a current through one coil a momentary current was induced in the other coil (Fig. 15.1). This phenomenon is now known as mutual induction about which you will learn in Sec. 15.4.

The iron ring-coil apparatus is still on display at the Royal Institution of **Great Britain,** an organisation devoted to scientific education and research, based in London.

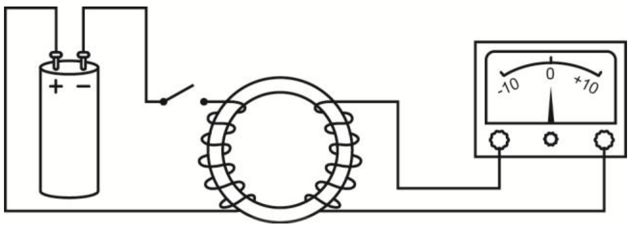


Fig. 15.1: **A schematic drawing of Faraday's iron ring-coil experiment.**

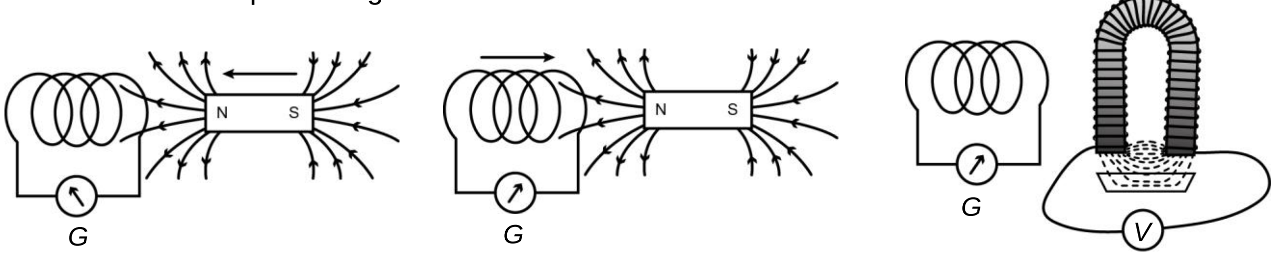
A greater breakthrough came when Faraday asked: What happens if a wire loop is kept stationary and a magnet is moved toward or away from it? Or if the magnet is kept stationary and the loop is moved toward or away from it? Faraday performed many experiments and their results led to the concept of induced current.

15.2.1 Induced Currents

We describe briefly three experiments similar to the ones performed by Faraday.

1. In one experiment, a magnet is moved through a stationary coil of wire. It is observed that an electric current flows in the coil (Fig. 15.2a). The current stops flowing when the magnet stops moving.

2. In another experiment, it is observed that current flows in the coil if it is moved over a stationary magnet (Fig. 15.2b). Again current stops flowing when the coil stops moving.



(a) **(b)** **(c)**

Fig. 15.2: **a) When a magnet is moved through a coil of wire or b) the coil of wire is moved over a magnet, electric current flows in that coil. No current flows when the coil and the magnet are stationary or the magnetic field is constant; c) current flows when the strength of the magnetic field is changed as in an electromagnet.**

3. Keeping both the coil and the magnet stationary, the magnetic field is changed. We can do this by using an electromagnet and changing the

current flowing in its coil (Fig. 15.2c). Once again, we observe that current flows in the coil.

Can you say what is common in these experiments? You can see that the magnetic field is **changing** in all cases. So, these experiments show that an electric current is 'induced' in the coil of wire by a changing magnetic field. The current flowing in the coil, when the coil and the magnet are in motion relative to each other or the magnetic field is changing, is called **induced current**. Faraday explained these results by giving a new fundamental principle in physics:

A changing magnetic field produces an electric field.

This new principle explains the origin of the induced current: It is the electric field 'induced' by the changing magnetic field that sets the charges in the wire of the coils into motion and induces electric current in the coil.

Thus, through his landmark experiments, Faraday established that a changing magnetic field produces an **induced** electric current. The work done per unit charge to produce the induced current is called the **induced emf** (**electromotive force**). The phenomenon in which an electric field, emf and electric current are induced by a time-varying magnetic field is called **electromagnetic induction**. To sum up, Faraday observed that

An induced current flowed in a circuit/coil/loop subjected to a changing magnetic field.

This is a qualitative explanation. We should also be able to give a quantitative relationship between the induced current and the changing magnetic field. Let us now arrive at the mathematical statement of **Faraday's law of electromagnetic induction**. But before you study it, you may like to answer the following SAQ to check your understanding of how current is induced in a circuit.

SAQ 1 - Electromagnetic induction

An electrical circuit is placed in the magnetic field of an electromagnet. In which of the following cases will an ammeter in the circuit register the induced current?

a) When the circuit is pulled to the right through the magnetic field.

b) When the circuit is kept at rest in the magnetic field.

c) When the electromagnet is pulled to left and the circuit is at rest.

15.2.2 Mathematical Statement of Faraday's Law

Let us begin the discussion by asking: What is it that gives rise to a current in a circuit? As you may know from school physics, we need a source of electromotive force (emf), i.e., a battery or power supply that supplies energy to the circuit. So, Faraday thought that in the same way when an induced current flows in a circuit, an induced emf must be present. Thus, on the basis of his experimental results, Faraday developed a general law that whenever (and for whatever reasons), the magnetic flux through a loop/coil/circuit changes, an emf is induced in it, and

The emf induced in any loop/coil/circuit is proportional to the negative of the rate of change of magnetic flux Φ_B through it:

$$\varepsilon = -\frac{d\Phi_B}{dt} \qquad (15.1a)$$

where ε is the induced emf in any loop/coil/circuit and Φ_B, the magnetic flux through it. We can rewrite Faraday's law given by Eq. (15.1a) so that we can omit any reference to circuits. By definition, the induced emf is equal to the work per unit charge done on a test charge that is moved around the circuit (or loop/coil) *C*. It is given by the line integral of the electric field along the closed path *C*, along the circuit/loop/coil:

$$\varepsilon = \oint_C \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}} = -\frac{d\Phi_B}{dt} \qquad (15.1b)$$

In Unit 12, you have already learnt the integral representation of magnetic flux:

$$\Phi_B = \iint_S \vec{\mathbf{B}} . d\vec{\mathbf{S}} \qquad (15.2)$$

where *S* is a closed surface whose boundary is given by *C*. Using Eqs. (15.1b) and (15.2), we can mathematically express Faraday's law as

$$\varepsilon = \oint_C \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}} = -\frac{d}{dt}\iint_S \vec{\mathbf{B}} . d\vec{\mathbf{S}} \qquad (15.3)$$

In this form of Faraday's law, there is no need to have a physical circuit or wire loop/coil. *C* can just represent a closed curve in space and *S* a surface bounded by *C*. Eq. (15.3) describes the induced electric fields that arise whenever there are changing magnetic fields. If electric circuits are present, induced currents arise as well. Using Stokes' theorem [recall Eq. (4.19) of Unit 4, Block 1 of this course], we can also express Eq. (15.3) in the differential form:

$$\oint_C \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}} = \iint_S (\vec{\nabla} \times \vec{\mathbf{E}}) . d\vec{\mathbf{S}} = -\frac{d}{dt}\iint_S \vec{\mathbf{B}} . d\vec{\mathbf{S}} = -\iint_S \frac{\partial \vec{\mathbf{B}}}{\partial t} . d\vec{\mathbf{S}} \qquad (15.4)$$

Since $\vec{\mathbf{B}}$ may depend on position as well as time, we have written $\frac{\partial \vec{\mathbf{B}}}{\partial t}$ in Eq. (15.4) instead of $\frac{d\vec{\mathbf{B}}}{dt}$ to account for only the time variation of $\vec{\mathbf{B}}$. Since the surface *S* in Eq. (15.4) is arbitrary, we must have

$$\vec{\nabla} \times \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t} \qquad (15.5)$$

Note that for constant magnetic field (static case), Eq. (15.5) reduces to $\vec{\nabla} \times \vec{\mathbf{E}} = \vec{\mathbf{0}}$ as it should.

We thus have two entirely equivalent statements of Faraday's law in integral and differential forms. Let us state them together.

Recap

FARADAY'S LAW OF ELECTROMAGNETIC INDUCTION

The emf induced in any circuit is proportional to the negative of the rate of change of magnetic flux linked to the circuit:

$$\varepsilon = -\frac{d\Phi_B}{dt} \qquad (15.1a)$$

$$\oint_C \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}} = -\frac{d}{dt}\iint_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}} \qquad \text{(Integral form)} \qquad (15.3)$$

$$\vec{\nabla} \times \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t} \qquad \text{(Differential form)} \qquad (15.5)$$

So far in this course you have learnt that electric fields are produced by static charges as well as changing magnetic fields. But is the nature of these fields the same? Try to find out yourself!

SAQ 2 - Nature of induced electric field

What is the basic difference in the nature of the electric fields produced by static charges and those induced by changing magnetic fields?

Let us now consider a technological application of Faraday's law, namely, the ac generator. It is perhaps the most important application of the law in use today. Do you know that almost all the electrical energy used by the world comes from electric generators? A generator is nothing but a system of conductors in a magnetic field. Let us discuss the working of a simple electric generator.

EXAMPLE 15.1 : THE AC GENERATOR

Mechanical energy is supplied to rotate the coil. In the power stations, the source of mechanical energy is either falling water (hydroelectric power plants), or steam from burning fossil fuels (thermal power plants), or from nuclear fission (nuclear power plants).

In a basic ac generator, a coil of area S is placed between the poles of a magnet and rotated (read the margin remark) to generate electricity (Fig. 15.3). What is the current generated by the generator?

Let B be the magnitude of the magnetic field of the magnet and θ be the angle between the axis of the coil and the magnetic field direction. Due to electromagnetic induction, rotation of the coil causes a change in the magnetic flux through the coil. The changing magnetic flux induces an emf, and an induced current flows through the coil. Let us determine the magnitudes of the induced emf and the induced current. The magnetic flux through the coil is

$$\Phi = BS\cos\theta \qquad (15.6a)$$

If the coil is rotating with a uniform angular speed ω, then $\theta = \omega t$ and

$$\varepsilon = -\frac{d\Phi}{dt} = -\frac{d}{dt}(BS\cos\omega t) \quad \text{or} \quad \varepsilon = BS\,\omega\sin\omega t \qquad (15.6b)$$

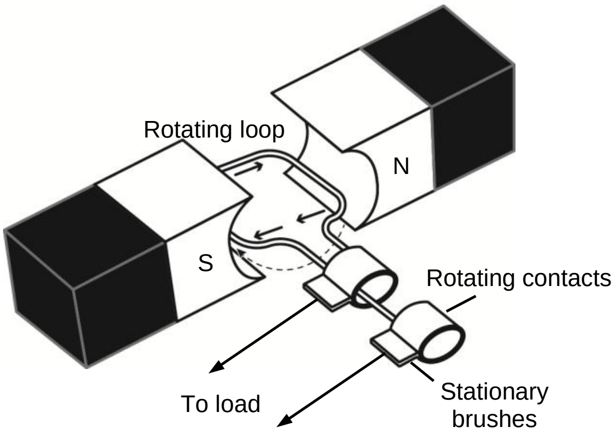


Fig.15.3: **A schematic diagram of a simple ac electric generator; as the loop rotates in the magnetic field, the changing magnetic flux induces an emf in it. Current flows through the rotating contacts and stationary brushes to an electrical load far away from the magnets.**

Now we bring the wires of the coil to a point situated very far from the rotating coil, where the magnetic field (due to the magnet) does not vary with time. Then from Eq. (15.5), the curl of the electric field in this region will be zero. Thus, it will be conservative. Then we can define an electric potential associated with this field. Let the two ends of the coil be at a potential difference V at that far-off point. If no current is being drawn from the generator, the potential difference between the two wires will be equal to the emf in the rotating coil, i.e.,

$$V = BS\omega \sin\omega t = V_0 \sin\omega t \qquad (15.7a)$$

where $V_0 = BS\omega$ is the peak output voltage of the generator. As given by Eq. (15.7a), V is an alternating voltage. If we now attach a load R to these wires, we can generate an alternating current given by

$$I = \frac{V}{R} = \frac{V_0}{R} \sin \omega t \qquad (15.7b)$$

Before studying further, you may like to work out a numerical problem on the design of an ac generator.

SAQ 3 – AC generator

An electric generator like the one shown in Fig. 15.3 consists of a 10 turn square wire coil of side 50 cm. The coil is turned at 50 revolutions per second, to produce the standard 50 Hz ac produced in India. What must the magnitude of the magnetic field be for the peak output voltage of the generator to be 300V?

So far, we have not said anything about the negative sign in Faraday's law of electromagnetic induction. The negative sign has an important purpose in the law: it gives us the direction of the induced current, which brings us to Lenz's law.

15.3 LENZ'S LAW

The direction of the induced current is determined by the Lenz's law. Let us state the law first.

> In the section on Lenz's law in some text books, you may find the following statement:
>
> **Nature abhors a change in flux.**

LENZ'S LAW

The induced current produces a magnetic field which tends to oppose the change in the magnetic flux that induces such currents.

We also state Lenz's law as follows:

The direction of the induced current (or induced emf) is such as to oppose the change giving rise to it.

Let us consider a simple example to understand how to apply Lenz's law.

EXAMPLE 15.2 : APPLYING LENZ'S LAW

Consider a bar magnet moving towards a wire loop (Fig. 15.4). Note that the north pole of the bar magnet is towards the loop. Determine the direction of the induced current in the loop.

SOLUTION ■ Let us apply Lenz's law to the situation.

Refer to Fig. 15.4. Note that the north pole of the bar magnet is on the left. As it approaches the loop, the magnetic flux through the loop increases. giving rise to an induced emf. Now the induced current in the loop should produce a magnetic field that **opposes the increase** in the magnetic flux through it due to the bar magnet. So, the magnetic field due to it should be directed opposite to the magnetic field of the bar magnet, i.e., to the right in Fig. 15.4. So, the loop should behave as if it presents a north pole towards the approaching bar magnet. Using the right-hand rule, you can see that the induced current in the loop must flow in the counter-clockwise direction.

> We can use the right-hand rule to find the direction of current in a loop if we know the direction of the magnetic field. Point the thumb of your right-hand in the direction of magnetic field and curl your fingers as you would for right-hand rule. Then the direction in which your fingers curl gives the direction of the flow of current in the loop. In this case, the loop presents a north pole towards the bar magnet. So, the magnetic field is directed to the right. So, if you follow the right-hand rule, you can see that the current flows in counter-clockwise direction in the loop.

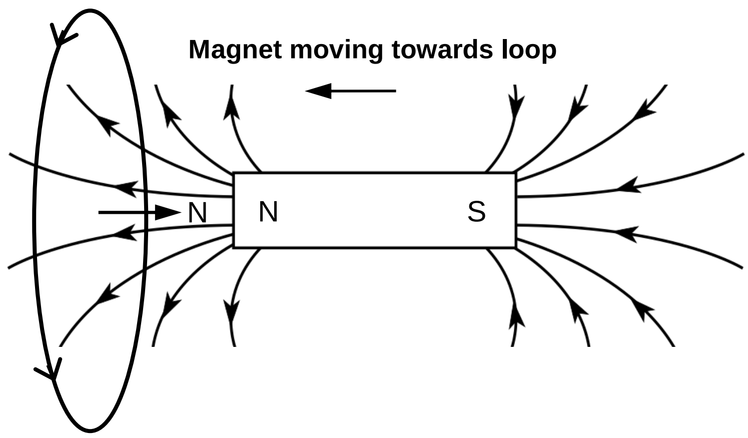


Fig. 15.4: **As the bar magnet moves towards the loop, the direction of the induced current is such that the magnetic field due to it is opposite to the magnetic field of the bar magnet. The loop presents a north pole to the bar magnet that opposes its motion (towards the loop).**

What happens when we pull the magnet away from the loop? Refer to Fig. 15.5. Note that as the bar magnet moves away from the loop, the magnetic flux through the loop due to it decreases. In this case, the induced current in the loop should produce a magnetic field that **opposes the decrease** in the magnetic flux through it due to the bar magnet. So, the magnetic field due to the induced current in the loop should be in the same direction as the magnetic field of the bar magnet, i.e., to the left as shown in Fig. 15.5. The loop now behaves as if it presents a south pole towards the bar magnet moving away from it. Using the right-hand rule, you can see that now the induced current in the loop must flow in the clockwise direction.

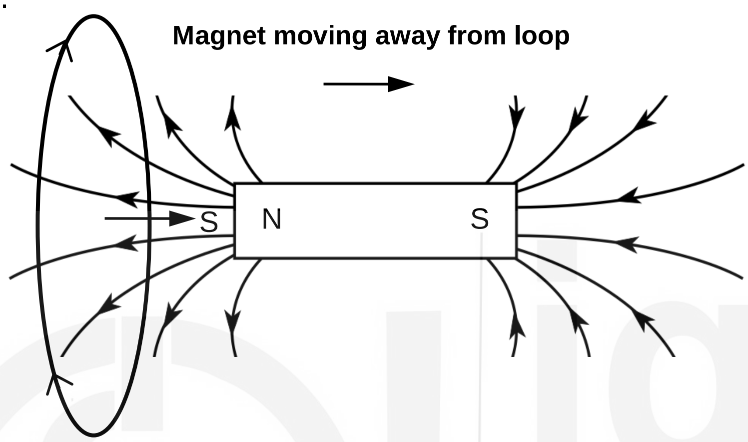


Fig. 15.5: **When we pull away the bar magnet from the loop, the direction of the induced current is such that the magnetic field due to the loop is in the same direction as the magnetic field of the bar magnet. It opposes the withdrawal of the bar magnet.**

Note that the increase in the magnetic field $\vec{\mathbf{B}}$ could be due to any reason: A bar magnet moving towards the coil as in Example 15.2 or increasing current in an electromagnet, etc. The **point to remember** is that *when a loop is placed in an increasing magnetic field*, **the magnetic flux linked with it increases**. Then the direction of the induced current is such as to oppose this increase in the magnetic flux. Therefore, $\vec{\mathbf{B}}_L$ is directed opposite to $\vec{\mathbf{B}}$ when $\vec{\mathbf{B}}$ increases.

The same argument holds for decreasing $\vec{\mathbf{B}}$ but now the magnetic flux through the loop will decrease. Therefore, the direction of the induced current is such that $\vec{\mathbf{B}}_L$ is in the same direction as $\vec{\mathbf{B}}$. **Note that the arrows in Fig. 15.6 are just to show the directions, their lengths do not represent the magnitudes of the fields**.

In Example 15.2, you should note that the magnetic field produced by the induced current in the loop, $\vec{\mathbf{B}}_L$, always opposes the **<u>change</u>** in the magnetic field $\vec{\mathbf{B}}$ inducing the current. This does not mean that $\vec{\mathbf{B}}_L$ is always opposite to $\vec{\mathbf{B}}$. When $\vec{\mathbf{B}}$ is increasing, the magnetic flux through the loop increases and the magnetic field of the loop $\vec{\mathbf{B}}_L$ is directed to *oppose this increase*. So, it is directed opposite to $\vec{\mathbf{B}}$. When $\vec{\mathbf{B}}$ is decreasing, $\vec{\mathbf{B}}_L$ is directed so as to oppose the decrease in magnetic flux through the loop. So, $\vec{\mathbf{B}}_L$ is in the same direction as $\vec{\mathbf{B}}$ (see Fig. 15.6). **<u>Remember</u>**: In all cases, the direction of the induced current is determined by the right-hand rule.

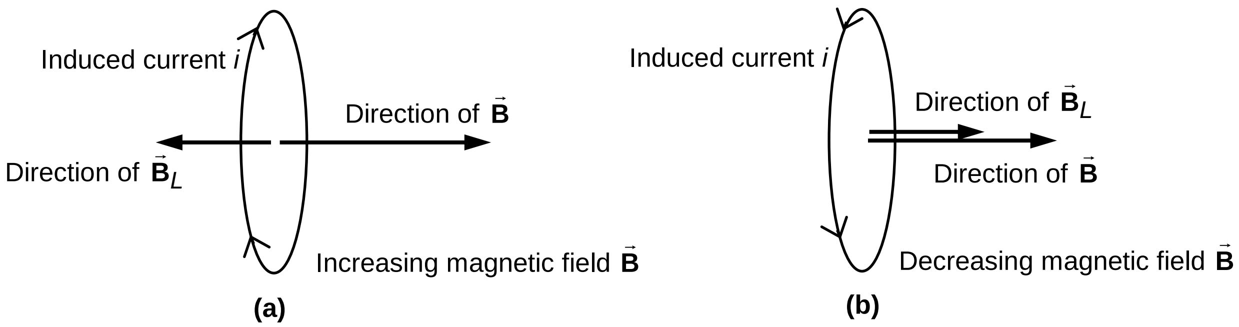


Fig. 15.6: **The direction of the induced current is such that the magnetic field $\vec{\mathbf{B}}_L$ due to it opposes the change in the magnetic field $\vec{\mathbf{B}}$ that induced the current. a) $\vec{\mathbf{B}}_L$ is always directed opposite to an increasing $\vec{\mathbf{B}}$; b) $\vec{\mathbf{B}}_L$ is always in the same direction as decreasing $\vec{\mathbf{B}}$.**

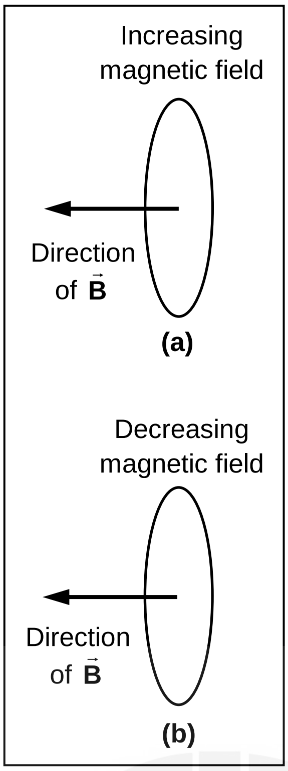


Fig. 15.7: **Diagram for SAQ 4.**

You may like to check if you have understood Lenz's law before studying further. Try an SAQ.

SAQ 4 – Applying Lenz's law

In each of Figs. 15.7a and b, what is the direction of the magnetic field produced by the induced current and the direction of the induced current in the loop? Draw them on each figure.

As we have said earlier, Lenz's law is reflected mathematically in the minus sign of the right-hand side of Faraday's law given by Eq. (15.3) or Eq. (15.5).

Also Lenz's law is a consequence of conservation of energy.

To understand how it is so, think of what would happen if the direction of the induced current aided the change in the magnetic flux. For instance, suppose that the loop presented a south pole to the magnet in Fig. 15.4, instead of north pole so that the bar magnet was attracted to the loop. What would happen then? Then you would need to push the magnet only slightly to get it moving and the action would carry on forever. The magnet would accelerate toward the loop, gaining kinetic energy in the process. At the same time thermal energy would appear in the loop due to its resistance. Thus, we would have created energy from practically nothing.

Needless to say, this violates energy conservation and does not happen.

While applying Lenz's law you should **always remember that the magnetic field of the induced current does not per se oppose the magnetic field that induces it, but the <u>change</u> in this magnetic field**. For example, if the magnetic flux through a loop decreases, the induced current flows in the loop so that its magnetic field adds to the original magnetic flux; if the magnetic flux is increasing, the current will flow in the opposite direction. This is a sort of an "inertial" phenomenon: A conducting loop 'likes' to keep a constant magnetic flux through it; if we try to change the magnetic flux, the loop responds by sending a current in such a direction as to counter our efforts.

You may now like to apply Lenz's law to a simple situation and determine the direction of the induced current.

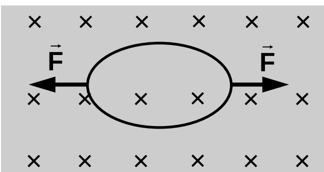


Fig. 15.8: **Diagram for SAQ 5.**

SAQ 5 – Applying Lenz's law

In Fig. 15.8, what is the direction of the induced current in the loop when the area of the loop is decreased by pulling on it with the forces labelled $\vec{F}$? The magnetic field is directed into the page and perpendicular to it.

So far, you have studied Faraday's law of electromagnetic induction, which relates the changing magnetic flux through a circuit/loop/coil to the emf induced in it. You have also learnt Lenz's law that gives the direction of the induced current in the circuit/loop/coil. In the next section, we consider two situations: when the change in magnetic flux through an electrical circuit is caused by a changing current (a) in the same circuit or (b) in a circuit nearby.

We have to introduce a physical quantity called **inductance** of the circuit to describe these phenomena.

15.4 INDUCTANCE

You know that when a current changes in a circuit, the magnetic field around it also changes. If a part of this changing magnetic field passes through the circuit itself, then an emf is induced in it. If there is another circuit in its neighbourhood, then the magnetic flux through that circuit changes, resulting in an induced emf in that other circuit. Thus, induced emf or induced currents in circuits can occur in two ways:

i) When current in a coil of wire with one or more loops changes, an emf is induced in the **same** coil. You know that the induced emf is produced as a result of the change in magnetic flux through the coil. This process is known as **self-induction**.

ii) When two coils are situated near enough, so that the magnetic flux associated with one coil passes through the other, a changing current in one coil induces an emf in the other. This process is known as **mutual induction**.

In the first case we associate a property called **self-inductance** of the coil; in the second we speak of **mutual inductance of the two coils**. Let us consider these effects separately.

15.4.1 Self-inductance

Consider a circular loop carrying a current *I* (Fig. 15.9). A magnetic field is set up by the current in this loop, so there is a magnetic flux through it. As long as the current is steady, the magnetic flux does not change and there is no induced current. But if we change the current in the loop, the magnetic flux through it changes and an emf is induced in it. An induced current flows in it for as long as the magnetic flux through it is changing. The more rapidly we change the current in the loop, the greater is the rate of change of magnetic flux, and the induced emf which opposes the change in current in the loop.

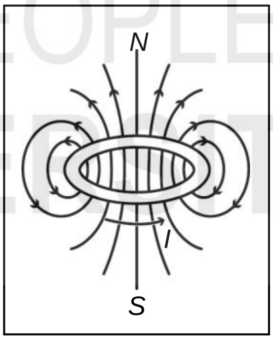
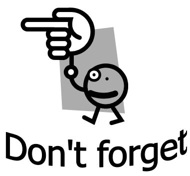


Fig. 15.9: **Self-inductance of a loop.**

In general, a self-induced emf appears in any coil when the current flowing in it is changing.



Let us now deduce a mathematical expression for the self-inductance of a coil having a single loop. You have learnt in Unit 13 of Block 3 that the magnetic field is proportional to the current that produces it. Therefore, the magnetic flux would also be proportional to the current. Thus, we can write:

$$\Phi_B \propto i \quad \text{or} \quad \Phi_B = Li \tag{15.8}$$

The constant of proportionality L is called the **self-inductance** of the coil also known as an **inductor**.

So, self-inductance of a coil is defined as follows:

$$L = \frac{\Phi_B}{i} \qquad (15.9a)$$

where Φ_B is the magnetic flux linked to the coil when a current i flows in the coil. The unit of self-inductance is the henry (H), named after Joseph Henry, an American scientist. Since the unit of magnetic flux is tesla square metre, one henry is defined as:

$$1\,\text{henry} = 1\,\text{H} = 1\,\text{tesla}-\text{m}^2/\text{ampere}$$

For a coil having N turns

$$L = \frac{N\Phi_B}{i} \qquad (15.9b)$$

If the current flowing in a coil changes by an amount di in a time interval dt, then the magnetic flux linked to the coil changes by an amount $d\Phi_B = L\,di$ in the same time interval. From Faraday's law, the emf induced in the coil is

$$\varepsilon = -\frac{d\Phi_B}{dt} \qquad (15.10)$$

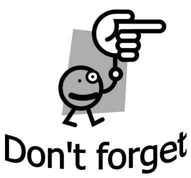
or

$$\varepsilon = -L\frac{di}{dt} \qquad (\text{since } d\Phi = L\,di) \qquad (15.11)$$

where L is always positive. Thus, the emf induced in a coil due to changes in the coil current is directly proportional to the rate at which the current changes. What is the direction of the induced current? From Lenz's law, the direction of the induced emf is such that it opposes the change in the current in the coil. That is, the induced current flows opposite to the change in current in the coil. So, if the current i in the coil is increasing, the induced current flows opposite to it; if i is decreasing, the induced current flows in the same direction.

The emf induced in the coil/inductor is also called the **back emf** because it is opposite to the emf in the coil/inductor. Eq. (15.11) tells us that the **back emf in an inductor depends on the rate of change of the inductor current and acts to oppose the change in current**.

When $dt = 0$ in Eq. (15.11), the induced emf is infinite. Since an infinite emf is impossible, therefore, from Eq. (15.11), $dt = 0$ is not possible. This means that an instantaneous change in the inductor current cannot occur. Thus, always remember that



The current through an inductor cannot change instantaneously.

All coils/circuits whether in the form of straight wires or coiled ones, possess self-inductance. However, the effect of self-inductance is important only when the magnetic flux through the circuit is large or when current changes very rapidly. For example, a 1 cm length of straight wire has an inductance of about 5×10^{-9} H and it exhibits very little opposition to current changes in the 50 Hz ac flowing through it. But in TV sets, high speed computers or in high frequency communications, such as satellite communication, current changes on time scales of the order of 10^{-9} s. Then the self-inductance of the wires themselves must be taken into account. There are devices, called **inductors** (coils of wire), designed specifically to exhibit self-inductance. These are useful in circuits whenever it is required to stabilize currents.

You have just studied that the self-inductance of an inductor is a measure of the opposition to the change in current through it. Now we ask: **How do we determine the magnitude of the self-inductance of an inductor**?

The inductance of an inductor depends on its geometry. In principle, we can calculate the self-inductance of any inductor, but in practice it is difficult unless the geometry is pretty simple. A typical inductor consists of a wire that is coiled into a large number of turns/loops around a hollow cardboard cylinder or a rod. A common example of inductors is the **solenoid**. It is a device that consists of a wound helical coil and is widely used in electrical circuits. Let us, therefore, determine the self-inductance of a solenoid in the following example.

EXAMPLE 15.3: SELF-INDUCTANCE OF A SOLENOID

A long solenoid of cross-sectional area *A* and length ℓ consists of *N* turns of wire (see Fig. 15.10). Determine its self-inductance.

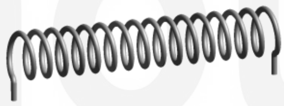


Fig. 15.10: **A solenoid.**

SOLUTION ■ We use Eq. (15.9a) to determine the self-inductance of the solenoid. For this, we must relate the current in the solenoid to the magnetic flux through it. In Unit 13, you have used Ampere's law to determine the magnetic field of a long solenoid, which is given by

$$B = \mu_0 n i \qquad \text{(i)}$$

where *n* is the number of turns per unit length of the solenoid and *i*, the current through it. For our problem $n = N/\ell$, which gives

$$B = \frac{\mu_0 N i}{\ell} \qquad \text{(ii)}$$

The total flux through the *N* turns of the solenoid is

$$\Phi = N \iint_{1turn} \vec{\mathbf{B}}.d\vec{\mathbf{S}}$$

Since the magnetic field of the solenoid is uniform and perpendicular to the cross-section of the individual turns, we have

$$\vec{\mathbf{B}}.\,d\vec{\mathbf{S}} = BdS \quad \text{and} \quad \iint_{1turn} \vec{\mathbf{B}}.d\vec{\mathbf{S}} = B \iint_{1turn} dS \qquad \text{(iii)}$$

Therefore, the surface integral is simply equal to the area of the cross-section of one turn of the solenoid, which is *A*. Using Eqs. (ii) and (iii), we get

$$\Phi = NB \iint_{1turn} dS = NBA = \frac{\mu_0 N^2 A i}{\ell}$$

Thus, the self-inductance of the solenoid having N turns is

$$L = \frac{\Phi}{i} = \frac{\mu_0 N^2 A}{\ell} \tag{15.12}$$

You may now like to determine the self-inductance and the back emf for a typical solenoid to get an idea of the magnitudes.

SAQ 6 – Self-inductance

A solenoid of length 1m and diameter 20 cm has 10000 turns of wire. A current of 2.5 A flowing in it is reduced steadily to zero in 1.0 ms. What is the magnitude of the back emf of the solenoid while the current is being switched off? Take $\mu_0 = 1.26 \times 10^{-6}$ Hm^{-1}.

You have learnt in this section that the back emf in an inductor opposes the change in current in the circuit and its magnitude depends on how rapidly the current changes. If we try to stop current in a very short time, $\frac{di}{dt}$ is very large and a very large back emf appears.

This is why switching off inductive devices, such as solenoids, can result in the destruction of delicate electronic devices by induced currents. Having worked out SAQ 6, you would realize that you have to be extremely cautious in closing switches in circuits containing large inductors. Even in your day-to-day experience, you may have seen that you often draw a spark when you unplug an iron. Why does this happen? This is due to electromagnetic induction which tries to keep the current going, even if it has to jump the gap in the circuit.

Let us now consider the second situation wherein two coils are placed close to each other. The changing current in one coil induces a current in the coil placed nearby. This is the phenomenon of **mutual induction** and the property associated with the circuits is called **mutual inductance**. It also forms the basis of another technological cornerstone of the power distribution systems, namely, the transformer.

15.4.2 Mutual Inductance

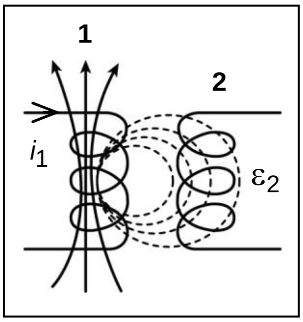


Fig. 15.11: **Mutual inductance of two coils.**

Consider two wire coils 1 and 2 situated close together and at rest with respect to each other (Fig. 15.11). Suppose when current i_1 flows through coil 1, it produces the magnetic field B_1. Since the two loops are nearby, magnetic field B_1 passes through coil 2 as well. Let Φ_2 be the magnetic flux of B_1 through coil 2. If we change i_1, B_1 and hence, Φ_2 will change and an induced emf ε_2 will appear in the coil 2. This induced emf will drive an induced current in coil 2.

Thus, every time the current in coil 1 changes, an induced current will flow in coil 2. Since the magnitude of the magnetic field $\vec{\mathbf{B}}_1$ is proportional to the current i_1, the magnetic flux of $\vec{\mathbf{B}}_1$ through coil 2 is also proportional to i_1. From Biot-Savart's law (Unit 12), it is given by

$$\vec{\mathbf{B}}_1 = \frac{\mu_0 i_1}{4\pi} \oint \frac{d\vec{\mathbf{l}}_1 \times \hat{\mathbf{r}}}{r^2} \qquad (15.13a)$$

and

$$\Phi_2 = \iint_S \vec{\mathbf{B}}_1 . d\vec{\mathbf{S}}_2 \qquad (15.13b)$$

Thus, from Eqs. (15.13a and b), we can write that

$$\Phi_2 \propto i_1 \qquad (15.13c)$$

The constant of proportionality is called the **mutual inductance** of the two coils and denoted by M. So, we have

$$\Phi_2 = M i_1 \qquad (15.14)$$

or

$$\frac{d\Phi_2}{dt} = M \frac{di_1}{dt} \qquad (15.15)$$

From Faraday's law, the induced emf in coil 2 is

$$\varepsilon_2 = -\frac{d\Phi_2}{dt} = -M \frac{di_1}{dt} \qquad (15.16)$$

The mutual inductance of the two coils/circuits is a purely geometrical quantity which depends on their sizes, shapes and relative arrangement. The unit of mutual inductance is also henry (H).

Now suppose we had started with the change in the magnetic flux linked with coil 1 due to the changing magnetic field in coil 2 (resulting from change in the current flowing in it). We would have got a similar result as in Eq. (15.16) for the emf induced in coil 1:

$$\varepsilon_1 = -M \frac{di_2}{dt} \qquad (15.17)$$

Mutual inductances found in common electronic circuits range from micro henries (μH) to several henries.

An extremely important application of the phenomenon of mutual inductance is found in the transformer. Let us study it in some detail.

15.4.3 Transformer

You have just studied that a changing current in one coil induces an emf in another coil. And the emf induced in the second coil is given by the same law: that it is equal to the rate of change of the magnetic flux through the coil. Suppose we take two coils and connect one of them to an ac generator. The continuously changing current produces a changing magnetic flux in the second coil. This varying flux generates an alternating emf in the second coil, which has the same frequency as the generating current in the first coil. Now, the induced emf in the second coil can be made much larger than that in the first coil if we **increase the number of turns in the second coil**. This is because in a given magnetic field, the magnetic flux through the coil is proportional to the number of turns. This is the basic principle of a **step-up transformer**.

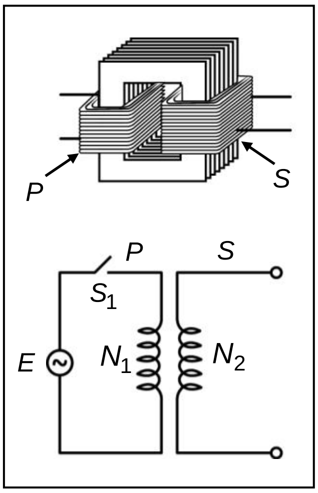


Fig. 15.12: **Schematic diagrams of a transformer.**

In the same way, the emf in the second coil can be made much smaller, if the number of turns in it is much less than the first coil, which is the underlying principle of a **step-down transformer** such as the ones used in the power distribution networks.

Let us compute the magnitude of the voltage in the second coil (also known as the **secondary coil**) vis-à-vis the voltage in the first coil (known as the **primary coil**). Fig. 15.12 shows a schematic diagram of a transformer. It has a primary coil (P) with N_1 turns. When the switch S_1 is closed, electric current starts flowing in the primary coil. As the current increases, it generates an increasing magnetic flux in the circuit, which induces a back emf ε_1 in the primary coil. The back emf exactly balances the applied voltage E if the resistance of the coil can be neglected. According to Faraday's law, the magnitude of the back emf ε_1 is given as

$$\varepsilon_1 = N_1 \frac{d\Phi}{dt} = E \tag{15.18a}$$

Now the changing magnetic flux in the primary coil is linked to the secondary coil S and generates an emf ε_2 whose magnitude is given by

$$\varepsilon_2 = N_2 \frac{d\Phi}{dt} \tag{15.18b}$$

where N_2 is the number of turns in the secondary coil. Eliminating $\frac{d\Phi}{dt}$ from Eqs. (15.18a and b), we get

$$\frac{\varepsilon_1}{\varepsilon_2} = \frac{N_1}{N_2} \tag{15.19}$$

Thus, the instantaneous emfs or voltages in the two coils are in the ratio of the number of turns on the coils. By setting the ratio of turns in the primary and secondary coils, we can make a step-up or step-down transformer that transforms a given ac voltage to any level we want. For example, if we want a step-down transformer to convert the high grid voltage of 22000 V to the low mains voltage of 220 V, we must have

$$\frac{N_1}{N_2} = \frac{22000\,\text{V}}{220\,\text{V}} = \frac{1000}{1}$$

Thus, for every turn on the secondary, there must be 1000 turns on the primary. In practice, the high grid voltage is reduced to the low mains voltage via a series of substations rather than via a single transformer. As you may have seen in your town or a nearby town, transformers are used in the entire power distribution network, to transform high voltages to low voltages.

You may now like to pause for a while and review what you have learnt so far. So far, you have studied the phenomenon of electromagnetic induction and learnt about Faraday's law, Lenz's law, self-inductance, mutual inductance and some of their applications. We now turn our attention to another important aspect associated with this phenomenon, viz. the storage of energy in magnetic field.

Recall from Block 2 of this course that when we move two unlike charges away from each other, we have to do work against the attractive Coulomb force between them. The resulting potential energy is stored in the electric field of the charges. In the same way, we can consider energy to be stored in the magnetic field. Let us find its expression.

15.5 ENERGY STORED IN A MAGNETIC FIELD

While studying self-inductance, you have learnt the concept of back emf of an inductor. Now work needs to be done **against the back emf to get the current going in a circuit**. This means that it takes a certain amount of energy to get current to flow in the circuit. This energy can be regarded as energy stored in the magnetic field of the current. In this section of the unit, we will determine the magnitude of the energy stored in a current-carrying circuit having an inductor, and then the energy stored in a magnetic field.

15.5.1 Energy Stored in a Current-carrying Circuit having an Inductor

In order to build a current in an electric circuit or loop of self-inductance L, work has to be done against the back emf of the inductor. This is equal to the work required to build up a current i in it. This work is stored as magnetic energy of the circuit in which an inductor having finite self-inductance is connected. This magnetic energy has a fixed value, which can be recovered. We get it back when the current in the circuit is turned off or reduced to zero. We can calculate this magnetic energy with the help of Faraday's law of electromagnetic induction.

Suppose a source of voltage V is connected in a circuit having an inductor. When current builds up in a circuit, back emf is induced and it opposes the flow of current. Suppose the back emf at some instant is ε. It is given by:

$$\varepsilon = -\frac{d\Phi}{dt} \qquad (15.20a)$$

Let us determine the work done by charge q against the back emf ε. Now if i is the current in the circuit and R, the resistance, from Kirchoff's voltage law, we have:

$$V = \frac{d\Phi}{dt} + Ri \qquad (15.20b)$$

Now the charge dq passing through the wire in a small interval of time dt is $i\,dt$. The work dW done by the voltage V in moving the charge dq $(= i\,dt)$ through the circuit in the time interval dt is (read the margin remark):

$$dW = V\,dq = V\,i\,dt = i\left(\frac{d\Phi}{dt}\right)dt + R\,i^2\,dt = i\,d\Phi + R\,i^2\,dt \qquad (15.20c)$$

Remember that in arriving at Eq. (15.20c), we have substituted V from Eq. (15.20b) in it and used the result

$$\left(\frac{d\Phi}{dt}\right)dt = d\Phi$$

Do you recognise the term $R\,i^2\,dt$ in Eq. (15.20c)? It represents the irreversible Joule heating loss due to the resistance in the circuit. The term $i\,d\Phi$ is equal to the work done against the back emf in the circuit. At this point, we are not concerned with the Joule loss. So we do not consider the term $R\,i^2\,dt$ in our discussion. Then the work done by *V* gives the increase in magnetic energy *dU* of the circuit. Thus,

$$dU = i\,d\Phi = L\,i\,di \tag{15.21a}$$

since $\Phi = L\,i$. When the current is increased from zero to a final value *i*, the magnetic energy stored in the circuit becomes

$$U = L\int_0^i i\,di \tag{15.21b}$$

or
$$U = \frac{1}{2}L\,i^2 \tag{15.22}$$

This was a specific example of storage of magnetic energy in a circuit having an inductor. We can generalize Eq. (15.22) to surface and volume currents. Then we can show how this energy can be regarded as being the energy of the magnetic field produced by the steady current. This is what we do now.

15.5.2 Magnetic Field Energy

You know that the magnetic flux through a single loop is equal to $L\,i$ where *L* is its inductance and *i*, the current though the loop:

$$\Phi = Li \tag{15.23a}$$

You also know that
$$\Phi = \iint_S \vec{\mathbf{B}}.\,d\vec{\mathbf{S}} \tag{15.23b}$$

From Unit 13 of Block 3 of this course, you know that the divergence of $\vec{\mathbf{B}}$ is zero. So, we can use the vector identity $\vec{\nabla}.\,(\vec{\nabla}\times\vec{\mathbf{A}}) = 0$ [recall Eq. (2.10f) of Unit 2, Block 1 of this course]. Here, as you know, $\vec{\mathbf{A}}$ is a vector field. Then we can express $\vec{\mathbf{B}}$ in terms of $\vec{\mathbf{A}}$ as:

$$\vec{\mathbf{B}} = \vec{\nabla}\times\vec{\mathbf{A}} \tag{15.23c}$$

Here $\vec{\mathbf{A}}$ is termed the **vector potential** associated with the magnetic field $\vec{\mathbf{B}}$. Therefore, substituting Eq. (15.23c) in Eq. (15.23b), we get

$$\Phi = \iint_S (\vec{\nabla}\times\vec{\mathbf{A}}).d\vec{\mathbf{S}} \tag{15.23d}$$

Using Stokes' theorem [Eq. (4.19) of Unit 4, Block 1 of this course] in Eq. (15.23d), we can write

$$\Phi = \oint_C \vec{\mathbf{A}}.d\vec{\mathbf{l}} \tag{15.23e}$$

Thus, from Eqs. (15.23a and e), we get

$$L\,i = \oint_C \vec{\mathbf{A}}.d\vec{\mathbf{l}} \qquad (15.24)$$

Therefore, from Eq. (15.22), the energy of this circuit is

$$U = \frac{1}{2}L\,i^2 = \frac{1}{2}\,i\oint_C \vec{\mathbf{A}}.d\vec{\mathbf{l}} \qquad (15.25)$$

Now, to generalize this expression, let us suppose that we do not have a current circuit defined by a wire. Instead, let the 'circuit' be a closed path that follows a line of current density. Then *U* given by Eq. (15.25) can approximate this situation very closely if we replace

$$i\,d\vec{\mathbf{l}} \quad \text{by} \quad \vec{\mathbf{J}}dV \quad \text{and} \quad \oint_C \quad \text{by} \quad \iiint_V$$

where *V* is the volume occupied by the current. Hence, we can write Eq. (15.25) as

$$U = \frac{1}{2}\iiint_V \vec{\mathbf{A}}.\vec{\mathbf{J}}\,dV \qquad (15.26a)$$

Using Ampere's law $(\vec{\nabla}\times\vec{\mathbf{B}} = \mu_0\vec{\mathbf{J}})$, [refer to Unit 13, Block 3 of this course], we get

$$U = \frac{1}{2\mu_0}\iiint_V \vec{\mathbf{A}}.(\vec{\nabla}\times\vec{\mathbf{B}})dV \qquad (15.26b)$$

We now use the vector identity given by Eq. (2.9e) of Unit 2, Block 1 of this course:

$$\vec{\nabla}.(\vec{\mathbf{A}}\times\vec{\mathbf{B}}) = \vec{\mathbf{B}}.(\vec{\nabla}\times\vec{\mathbf{A}}) - \vec{\mathbf{A}}.(\vec{\nabla}\times\vec{\mathbf{B}})$$

and write

$$\vec{\mathbf{A}}.(\vec{\nabla}\times\vec{\mathbf{B}}) = \vec{\mathbf{B}}.(\vec{\nabla}\times\vec{\mathbf{A}}) - \vec{\nabla}.(\vec{\mathbf{A}}\times\vec{\mathbf{B}})$$

$$= \vec{\mathbf{B}}.\vec{\mathbf{B}} - \vec{\nabla}.(\vec{\mathbf{A}}\times\vec{\mathbf{B}}) \quad [\because\ \vec{\mathbf{B}} = (\vec{\nabla}\times\vec{\mathbf{A}})]$$

As a result we get

$$U = \frac{1}{2\mu_0}\left(\iiint_V \vec{\mathbf{B}}.\vec{\mathbf{B}}dV - \iiint_V \vec{\nabla}.(\vec{\mathbf{A}}\times\vec{\mathbf{B}})dV\right)$$

or

$$U = \frac{1}{2\mu_0}\left(\iiint_V \vec{\mathbf{B}}.\vec{\mathbf{B}}dV - \oiint_S (\vec{\mathbf{A}}\times\vec{\mathbf{B}}).d\vec{\mathbf{S}}\right) \qquad (15.26c)$$

where we have used the divergence theorem [Eq. (4.40) of Unit 4, Block 1 of this course] in the second term and *S* is the closed surface that bounds *V*. The integration is to be taken over the entire volume occupied by the current. However, we can even choose a larger region for integration without altering the result, since current density $\vec{\mathbf{J}}$ corresponding to the current *i* will be zero beyond the volume occupied by it. We now extend the volume integral to include all space. In such an event, the contribution from the surface integral will go to zero, since the farther the surface is from the current, the smaller $\vec{\mathbf{B}}$ and $\vec{\mathbf{A}}$ are in those regions of the surface. Thus, we are left with

$$U = \frac{1}{2\mu_0}\iiint_V \vec{\mathbf{B}}.\vec{\mathbf{B}}\,dV = \frac{1}{2\mu_0}\iiint_V B^2\,dV \tag{15.27}$$

In view of this result we say that energy of the current-carrying circuits can be regarded as stored in the magnetic field produced by these currents, in the amount $\frac{B^2}{2\mu_0}$ per unit volume. Thus, there are two ways to think about the energy stored in magnetic fields, which are entirely equivalent: i.e., energy stored per unit volume is either $\frac{1}{2\mu_0}(\vec{\mathbf{A}}.\vec{\mathbf{J}})$ or $\frac{B^2}{2\mu_0}$.

Does it appear strange to you that it takes work to set up a magnetic field? After all, magnetic fields themselves do no work. The point is that setting up a magnetic field, where previously there was none, requires changing the magnetic field. And, as you know from Faraday's law of electromagnetic induction, a changing *magnetic* field induces an *electric* field. The electric field *can* do work. So, in the beginning and at the end there is no electric field. But, in between, when the magnetic field is building up, there **is** an electric field. It is against this electric field that the work is done. This work done appears as the energy stored in the magnetic field and is called the **magnetic field energy**.

We now summarise the concepts you have studied in this unit.

15.6 SUMMARY

Concept	Description
Electromagnetic induction and Faraday's law	■ **Electromagnetic induction** is a phenomenon in which emf and electric current are **induced** in a loop/coil/circuit subjected to a **changing** magnetic field. Its explanation requires the introduction of a new fundamental principle called Faraday's law: **A changing magnetic field gives rise to an induced electric field**. Mathematically, Faraday's law gives the relation between the emf ε induced in a loop/coil/circuit with the changing magnetic flux Φ_B through the loop/coil/circuit as $$\varepsilon = -\frac{d\Phi_B}{dt}$$ Faraday's law can be expressed in equivalent integral and differential forms, which relate the induced electric field and the changing magnetic field: $$\oint_C \vec{\mathbf{E}}\,.\,d\vec{\mathbf{l}} = -\frac{d}{dt}\iint_S \vec{\mathbf{B}}.d\vec{\mathbf{S}} \quad \text{and} \quad \vec{\nabla}\times\vec{\mathbf{E}} = -\frac{\partial\vec{\mathbf{B}}}{\partial t}$$ The **induced electric field is not conservative** unlike the conservative electrostatic field of a stationary charge. Thus, it can do work on charges as they move around a closed loop.

Lenz's law

- The direction of an induced current is specified by **Lenz's law**: The induced current produces magnetic field which tends to oppose the **change** in the magnetic flux that induces such a current. We also state Lenz's law as follows: The direction of the induced current (or induced emf) is such as to oppose the change giving rise to it. This law is reflected mathematically in the minus sign on the right hand side of Faraday's law. Lenz's law is a consequence of conservation of energy.

Back emf

- A changing current in a coil or circuit gives rise to a changing magnetic flux through the same coil or circuit, which induces a **back emf** in it. The **back emf in an inductor** depends on the rate of change of the inductor current and acts to oppose the change in current.

Self-inductance

- For a coil of wire in which current changes, the changing magnetic field and magnetic flux through it induces an emf in the same coil, which opposes the change in the current through the coil. In such cases, we associate a property called **self-inductance** of the coil, also called an **inductor**. The self-inductance L of an inductor is the ratio of the magnetic flux Φ to the current i through it:

$$L = \frac{\Phi}{i}$$

An inductor opposes instantaneous change in current. Faraday's law relates the emf in an inductor to the rate of change of current in it as:

$$\varepsilon = -L\frac{di}{dt}$$

The self-inductance of a solenoid of length ℓ, cross-section A, and having N turns is given by

$$L = \frac{\mu_0 N^2 A}{\ell}$$

Mutual inductance

- When two coils are placed close to each other so that the magnetic flux of one coil is linked with the other coil, a changing current in one coil induces an emf in the other. This property of the coils is called **mutual induction**. The **mutual inductance** of a pair of coils is defined as the ratio of the total magnetic flux in the second coil to the current in the first:

$$M = \frac{\Phi_2}{i_1}$$

Faraday's law relates the emf in the second coil to the rate of change of current in the first

$$\varepsilon_2 = -M\frac{di_1}{dt}$$

The same mutual inductance M describes the emf induced in the first coil as a result of changing current in the second coil.

Transformer

- The instantaneous emfs or voltages in the primary (ε_1) and secondary (ε_2) coils in a transformer are in the ratio of the number of turns in the coils:

$$\frac{\varepsilon_1}{\varepsilon_2} = \frac{N_1}{N_2}$$

where N_1 and N_2 are the number of turns in the primary and secondary coils, respectively.

Energy stored in a circuit having an inductor

- Work needs to be done to build up current in a circuit having an inductor and, therefore, magnetic field in it. This work ends up as stored energy in the circuit. The energy stored in a circuit having an inductor is given by

$$U = \frac{1}{2} L i^2$$

where L is the self-inductance of the inductor carrying current i.

Magnetic field energy

- The energy stored in the magnetic field $\vec{\mathbf{B}}$ is given by

$$U = \frac{1}{2\mu_0} \iiint_V \vec{\mathbf{B}}.\vec{\mathbf{B}}\, dV$$

This expression is very general and applies to a single inductor, two or more inductors, and surface and volume distributions of currents.

15.7 TERMINAL QUESTIONS

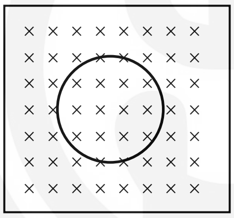


Fig.15.13: **Diagram for TQ 1.**

1. A wire loop of radius 20 cm and resistance $5.0\,\Omega$ is kept in a uniform magnetic field $\vec{\mathbf{B}}$ at right angles to it (Fig. 15.13). The magnetic field points into the page and is increasing at the rate of 0.10 T s^{-1}. Determine the magnitude and direction of the induced current in the loop.

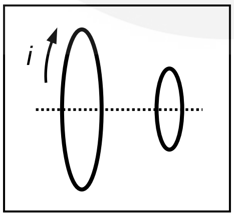


Fig.15.14: **Diagram for TQ 2.**

2. What is the direction of the induced current in the smaller loop of Fig. 15.14 when a clockwise current as seen from the left is suddenly established in the larger loop, by a battery not shown?

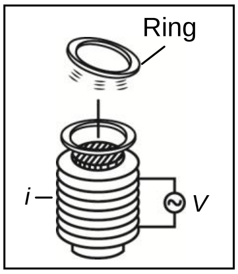


Fig.15.15: **Diagram for TQ 3.**

3. A metal ring placed on top of a solenoid jumps when current through the solenoid is switched on (Fig. 15.15). Explain why.

4. Two coils are arranged as shown in Fig. 15.16. If the resistance of the variable resistor is being increased, what is the direction of the induced current in the fixed resistor R?

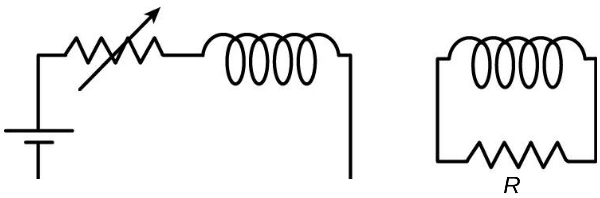


Fig.15.16: **Diagram for TQ 4.**

5. A horizontal metallic ring is placed in a uniform magnetic field $\vec{B}$ pointing up as seen from above the ring (see Fig. 15.17). In which direction will the induced current flow in the ring when the magnetic field $\vec{B}$ is turned off?

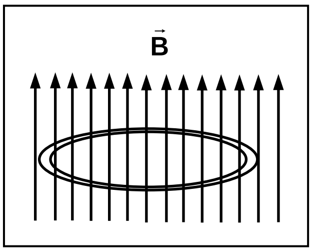


Fig.15.17: **Diagram for TQ 5.**

6. A sheet of copper is placed in a magnetic field as shown in Fig. 15.18. If the sheet is pulled out of the field as shown in the figure, a resisting force appears. Explain its origin.

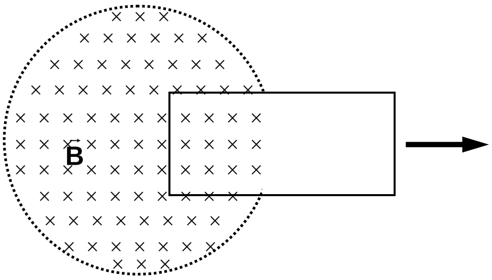


Fig.15.18: **Diagram for TQ 6.**

7. What rate of change of current in a solenoid having self-inductance 9.7 mH produces a self-induced emf of 35 mV in it?

8. A typical ignition coil (made up of two coils) draws a current of 3.0 A, and supplies an emf of 24 kV to the spark plugs. If the current in the two coils is interrupted every 0.10 ms, what is their mutual inductance?

9. A solenoid is 0.90 m in diameter and 2.2 m long. The magnetic field at its centre is 0.40 T. Estimate the energy stored in the magnetic field of the solenoid.

10. A long coaxial cable carries current i which flows down the surface of the inner cylinder of radius a and back along the outer cylinder of radius b (Fig. 15.19). Determine the energy stored in a section of length ℓ of the cable. It is given that the magnitude of the magnetic field between the cylinders is

$$B = \frac{\mu_0 i}{2\pi r} \quad \text{and zero elsewhere.}$$

Hence, determine the self-inductance per unit length of the cable.

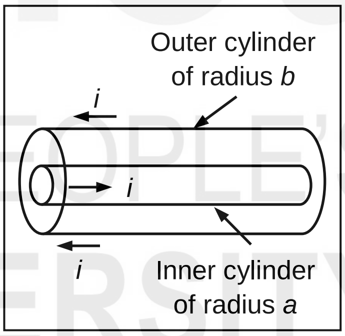


Fig.15.19: **Diagram for TQ 10.**

15.8 SOLUTIONS AND ANSWERS

Self-Assessment Questions

1. The ammeter will register an induced current in the cases (a) and (c) because the magnetic flux through the circuit changes only when there is relative motion between the circuit and the electromagnet, not when they are at rest with respect to each other.

2. From Unit 5, you may recall that the electrostatic force field due to static charges is conservative and therefore, the curl of the electrostatic force field and the electric field due to static charges is zero. However, from Eq. (15.5), you can see that the curl of the electric field induced by changing magnetic fields is not zero. Hence, the electric field given by

Eq. (15.5) is not conservative. This is the basic difference between the two electric fields. As a result, the force field corresponding to the electric field given by Eq. (15.5) can do work on charges as they move around closed paths. Moreover, we cannot associate a scalar potential with this field.

3. The induced emf is given by Eq. (15.1a): $\varepsilon = -\dfrac{d\Phi_B}{dt}$.

From Example 15.1, for a uniform magnetic field, the magnetic flux through one turn of the coil is given by

$$\Phi_B = BS\cos\omega t$$

where $\omega = 2\pi f$, f is the frequency at which the coil rotates and S, the area of one turn given by L^2 (L being the side of the square coil). Thus, the induced emf for a coil consisting of N turns is

$$\varepsilon = -\frac{d\Phi_B}{dt} = -\frac{d(NBS\cos\omega t)}{dt} = -NBL^2[-\omega\sin\omega t]$$

$$= -NBL^2[-2\pi f\sin(2\pi ft)] \quad \text{or} \quad \varepsilon = 2\pi NBL^2 f\sin(2\pi ft)$$

The peak emf is then $\varepsilon_{peak} = 2\pi NBL^2 f$ and it is given to be equal to 300V. Hence,

$$300\,\text{V} = (2\pi)\times(10)\times(0.50\,\text{m})^2\times(50\,\text{Hz})B$$

or $$B = \frac{300\,\text{V}}{(2\pi)\times(10)\times(0.50\,\text{m})^2\times(50\,\text{Hz})} = 0.38\,\text{T}$$

This is the typical magnetic field strength near the poles of a strong permanent magnet.

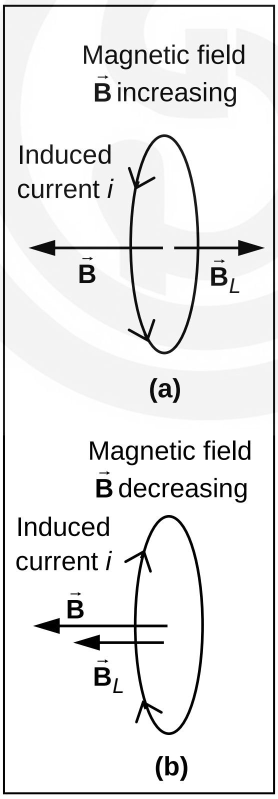


Fig. 15.20: **Diagram for the answer of SAQ 4.**

4. See Fig. 15.20. In part (a), the magnetic field is increasing and, hence, the magnetic flux through the loop is also increasing. Therefore, the direction of the induced current would be such as to produce a magnetic field $\vec{\mathbf{B}}_L$ in the direction opposite to $\vec{\mathbf{B}}$ as shown in Fig. 15.20a. Therefore, from the right-hand rule, the induced current will flow in the counter-clockwise direction to produce $\vec{\mathbf{B}}_L$ opposite to $\vec{\mathbf{B}}$. In part (b), the magnetic field is decreasing and, hence, the magnetic flux through the loop is also decreasing. Therefore, the direction of the induced current would be such as to produce a magnetic field $\vec{\mathbf{B}}_L$ in the same direction as $\vec{\mathbf{B}}$ (Fig. 15.20b). So, from the right-hand rule, the induced current will flow in the clockwise direction to produce $\vec{\mathbf{B}}_L$. Note that the arrows in the figure are just to show the directions, their lengths do not represent the magnitudes of the fields.

5. Since the area of the loop decreases when it is pulled by the forces, the magnetic flux through it also decreases. The direction of the induced current is such as to oppose this decrease, i.e., the magnetic field due to the induced current in the loop should be in the direction of the existing magnetic field. It is given that the existing magnetic field is directed into the page and perpendicular to it. So, the induced current should flow in the

clockwise direction in the loop to give rise to a magnetic field directed into the page as we view it from top.

6. We use Eq. (15.11) to determine the back emf. So we will first determine the self-inductance of the solenoid. From Eq. (15.12), it is given by

$$L = \frac{\mu_0 N^2 A}{\ell} = \frac{(1.26\times10^{-6}\,\text{Hm}^{-1})\times(10000)^2\times\pi\times(0.10\text{ m})^2}{1\text{ m}} = 3.96\,\text{H}$$

Since the current in the solenoid changes steadily, the magnitude of its rate of change is given by $\frac{di}{dt} = \frac{2.5\,\text{A}}{1.0\,\text{ms}} = 2500\ \text{As}^{-1}$

The magnitude of the back emf is

$$|\varepsilon| = L\,\frac{di}{dt} = (3.96\,\text{H})\times(2500\ \text{As}^{-1}) = 9900\,\text{V} = 9.9\times10^3\,\text{V}$$

Terminal Questions

1. The magnitude of the induced current in the loop is given by Ohm's law as: $I = \frac{\varepsilon}{R}$

   So, we have to determine the emf induced in the wire loop. From Faraday's law, we have $\varepsilon = -\frac{d\Phi_B}{dt}$

   The magnetic flux Φ_B through the loop is $\Phi_B = \iint_S \vec{\mathbf{B}}.d\vec{\mathbf{S}}$

   Since the magnetic field $\vec{\mathbf{B}}$ is uniform in space and the loop is at right angles to it, $\vec{\mathbf{B}}.d\vec{\mathbf{S}} = BdS$ and we get

$$\Phi_B = \iint_S B\,dS = B\iint_S dS = B\pi R^2,$$

   where R is the radius of the loop. Since the area of the loop is constant, the magnitude of the emf is given by:

$$|\varepsilon| = \frac{d\Phi_B}{dt} = \pi R^2\frac{dB}{dt} = \pi(0.20\,\text{m})^2(0.10\,\text{Ts}^{-1}) = 1.26\times10^{-2}\,\text{V}$$

   and therefore, the magnitude of the induced current through the loop is:

$$I = \frac{\varepsilon}{R} = \frac{1.26\times10^{-2}\,\text{V}}{5.0\,\Omega} = 2.5\,\text{mA}$$

   Since the magnetic field, pointing into the page is increasing, the direction of the induced current will be such as to oppose this increase. Thus, the induced current's magnetic field will be in the opposite direction, i.e., the induced current will flow in the counter-clockwise direction in the loop as seen from top of the page.

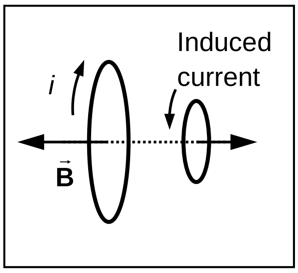


Fig.15.21: **Diagram for the answer of TQ 2. $\vec{\mathbf{B}}$ is increasing in the direction shown.**

2. When a clockwise current is established in the bigger loop, it sets up an increasing magnetic field pointing to its left as shown in Fig. 15.21. The

direction of induced current in the smaller loop has to be such that it opposes the increasing magnetic field. So, the magnetic field due to the induced current flowing in the smaller loop has to point towards right. This will happen only if the induced current flows in the counter-clockwise direction in the smaller loop as per the right-hand rule.

3. Before the current in the solenoid is switched on, the magnetic flux through the ring is zero. When the current in the solenoid is switched on, a magnetic flux appears in the upward direction in Fig. 15.15. Due to the change in magnetic flux linked with the metal ring, an emf and current are induced in it.

   The direction of the induced current is such that the magnetic field due to the induced current flowing in the ring is directed opposite to the change in the magnetic field due to the solenoid. Therefore, the current in the ring is opposite to the current in the solenoid. In Unit 12, you have learnt that the force between two conductors carrying currents in the opposite direction is repulsive. This repulsive force causes the ring to jump.

4. The current in the coil on the left (coil 1) is flowing clockwise in it. So, the magnetic field due to it points away from the second coil (coil 2). As the resistance in coil 1 increases, the current in it decreases, causing a decrease in its magnetic field. This causes a decrease in the magnetic flux linked with coil 2. The induced current in coil 2 flows in a direction such that it opposes this decrease. Therefore, the magnetic field due to coil 2 points towards its left. So, from the right-hand rule, the induced current flows in coil 2 in the clockwise direction. So, it flows from right to left in the resistor R.

5. When the magnetic field is turned off, it changes and as a result, a current and emf are induced in the horizontal metallic ring. Since the magnetic field is decreasing, the direction of the induced current will be such as to oppose this decrease. So, the magnetic field due to the induced current flowing in the loop will be in the same direction as the uniform magnetic field $\vec{\mathbf{B}}$ pointing up. Therefore, from the right-hand rule, the induced current will flow in the counter-clockwise direction in the ring.

6. As we try to pull the copper sheet out of the magnetic field, current is induced in it. Since the magnetic flux through the sheet is decreasing, the direction of the induced current in it is such that it gives rise to a magnetic field in the same direction as the original magnetic field, pointing into the page. Thus, from the right-hand rule, the direction of the current induced in the sheet is clockwise. The magnetic force due to the induced current is given by $\vec{\mathbf{F}} = i d\vec{\mathbf{l}} \times \vec{\mathbf{B}}$. From the right-hand rule, the direction of the magnetic force will be towards the left of the loop, i.e., the force will oppose the motion of the sheet.

   When we push the sheet in, the magnetic flux linked with it increases. So, the direction of the induced current is counter-clockwise to oppose this increase. The magnetic force due to the induced current and the associated magnetic field points towards right opposing the motion of the sheet.

The currents induced in solid conductors placed in changing magnetic fields are called **eddy currents**. As you have learnt while solving this TQ, **eddy currents can make it difficult to move a conductor through a magnetic field**.

7. The rate of change of current in an inductor is related to the self-induced emf by Eq. (15.11): $\varepsilon = -L\,\dfrac{di}{dt}$. For the magnitudes of ε and L given in the problem, we have:

$$\left|\frac{di}{dt}\right| = \left|\frac{\varepsilon}{L}\right| = \frac{35\text{ mV}}{9.7\text{ mH}} = 3.6\,\text{As}^{-1}$$

8. The rate of change of current is

$$\frac{di}{dt} = \frac{3.0\text{ A}}{0.1\text{ms}} = 3.0\times10^{4}\text{ As}^{-1}$$

Therefore, from Eq. (15.16 or 15.17), the mutual inductance of the two coils is:

$$M = \frac{|\varepsilon|}{(di/dt)} = \frac{24\text{ kV}}{3.0\times10^{4}\text{ As}^{-1}} = 0.8\text{H}$$

9. From Eq. (15.22), the energy stored in the solenoid is $U = \dfrac{1}{2}Li^2$. The magnitude of the magnetic field of the solenoid of length ℓ (Unit 13) is

$$B = \frac{\mu_0 N i}{\ell}$$

Thus, the current through the solenoid is $i = \dfrac{B\ell}{\mu_0 N}$

From Eq. (15.12), the inductance of a long solenoid is

$$L = \frac{\mu_0 N^2 A}{\ell}$$

Thus, $$U = \frac{1}{2}Li^2 = \frac{1}{2}\left(\frac{\mu_0 N^2 A}{\ell}\right)\times\left(\frac{B\ell}{\mu_0 N}\right)^2 = \frac{1}{2\mu_0}\,\ell B^2 A$$

Substituting the numerical values in the above expression, we get

$$U = \frac{(2.2\text{m})\times(0.40\text{T})^2\times\pi\,(0.45\text{m})^2}{2\times1.26\times10^{-6}\text{ Hm}^{-1}} = 8.9\times10^{4}\text{ J}$$

10. From Eq. (15.27), the energy stored per unit volume is given by

$$\frac{B^2}{2\mu_0} = \frac{1}{2\mu_0}\left(\frac{\mu_0 i}{2\pi r}\right)^2 = \frac{\mu_0 i^2}{8\pi^2 r^2}$$

The volume V of the cylindrical shell of length ℓ, radius r, thickness dr is given by $V = 2\pi r\,dr\,\ell$.

Therefore, the energy stored in the cylindrical shell of volume V is given by

$$U = \frac{B^2}{2\mu_0}V = \left(\frac{\mu_0 i^2}{8\pi^2 r^2}\right) \times (2\pi r\, dr\, \ell) = \frac{\mu_0 i^2 \ell}{4\pi}\frac{dr}{r}$$

Now we integrate from *a* to *b* for the given cylindrical shell of the coaxial cable to obtain *U* as follows:

$$U = \left(\frac{\mu_0 i^2 \ell}{4\pi}\right)\int_a^b \frac{dr}{r} = \left(\frac{\mu_0 i^2 \ell}{4\pi}\right)\ln\left(\frac{b}{a}\right)$$

Since $U = \frac{1}{2}Li^2$, we get an expression for *L* from the above equation as follows:

$$L = \frac{2U}{i^2} = \left(\frac{\mu_0 \ell}{2\pi}\right)\ln\left(\frac{b}{a}\right)$$

Thus, the self-inductance per unit length of the cable is given by:

$$\frac{L}{\ell} = \left(\frac{\mu_0}{2\pi}\right)\ln\left(\frac{b}{a}\right)$$

Electromagnetism plays a vital role in our lives. From mobile phones, microwave ovens, computers, bullet trains to MRI machines, we are surrounded by devices around us that are based on the principles of electromagnetism, set out in the form of Maxwell's equations. (Source of pictures: Wikipedia/commons)

UNIT 16

MAXWELL'S EQUATIONS

Structure

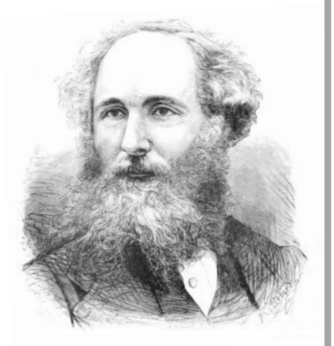
STUDY GUIDE

We hope that you have studied Gauss's laws for electric and magnetic fields, Ampere's law and Faraday's law explained in Blocks 2 and 3 and Unit 15 of this course. You should revise them and make sure that you know these laws and how to apply them before studying this unit. In this unit, you will learn about Maxwell's equations deduced from these laws. To be able to learn the concepts of this unit well, you should revise Blocks 2, 3 and Unit 15 before studying it. You should keep at hand the Block 1 of this course for reference as you will be using vector differential and integral calculus, and vector identities extensively in this unit. For revising the concepts related to wave equation that you may have learnt in your school physics, you should revise Unit 19, Block 4 of the course BPHCT-131 entitled Mechanics. This unit is quite mathematical in its presentation. Therefore, keep a pencil/pen and paper at hand and work out all steps given in it. We also advise you to work out all Examples, SAQs and Terminal Questions given in the unit on your own. These will help you learn Maxwell's equations thoroughly.

"Thoroughly conscious ignorance is the prelude to every real advance in science."	***James C. Maxwell***

16.1 INTRODUCTION

At this point of the course, you know the four fundamental laws that govern electric and magnetic phenomena, namely Gauss' law for electric fields, Gauss' law for magnetic fields, Ampere's law and Faraday's law. All these laws together explain the electric and magnetic interactions that make matter act as it does. In this unit, we will explain how James Clerk Maxwell put together these laws into a single set of equations, called **Maxwell's equations**.



James Clerk Maxwell (1831 – 1879), a Scottish physicist, is well known for his electromagnetic theory, which is the classical theory of electromagnetic radiation encapsulated in Maxwell's equations. His work unified the three seemingly different phenomena – electricity, magnetism and light. He showed that these three were different manifestations of the same phenomenon.

Maxwell's equations govern the behaviour of electric and magnetic fields everywhere and describe all electromagnetic phenomena. For example, they help us explain why a compass needle points north, why light bends when it enters water, why thunderstorms occur, why we see aurora in the polar regions, and many other natural phenomena. These equations also form the basis for the operation of a large number of devices in use today, e.g., electric motors, television, television transmitters and receivers, microwave ovens, telephones and mobiles, computers of all kind, radars, cyclotrons, MRI scanners, etc. Understanding Maxwell's equations is a truly rewarding experience.

In Sec. 16.2 of the unit, we state four fundamental laws of electricity and magnetism together and point out the underlying asymmetry of these laws that led to the formulation of Maxwell's equations. In Sec. 16.3, you will learn how the concept of displacement current proposed by Maxwell led to the removal of asymmetry. In Sec. 16.4, we put together Maxwell's equations and explain their significance. Finally, in Sec. 16.5, you will learn about how the existence of electromagnetic waves was predicted theoretically by Maxwell's equations and how it was established that light is an electromagnetic wave. We will derive the electromagnetic wave equation from Maxwell's equations and explain the transverse nature of electromagnetic waves.

Maxwell's contribution may well be regarded as the greatest achievement of nineteenth century physics. Maxwell's equations lead us into a fundamental insight into the nature of light and other electromagnetic radiations. This is what you will learn in the next unit, which is the last unit of this course.

Expected Learning Outcomes

After studying this unit, you should be able to:

- ❖ discuss the symmetry considerations, which led to Maxwell's equations;
- ❖ explain the concept of displacement current and how Ampere's law was modified by Maxwell;
- ❖ write Maxwell's equations in charge-free and current-free regions, and in regions containing charges and currents; and
- ❖ derive the wave equation for electromagnetic fields from Maxwell's equations and explain the nature of electromagnetic waves.

16.2 FUNDAMENTAL LAWS OF ELECTRICITY AND MAGNETISM

Recall all the laws governing electric and magnetic phenomena that you have studied so far in this course. Write them down in the margin of this page.

Now think: **Which ones amongst these laws can be thought of as fundamental**?

Do you recall Gauss's law for electric fields that you have studied in Sec. 6.3 of Unit 6? It is given by Eq. (6.16) as

$$\oint\!\!\oint_S \vec{\mathbf{E}}.d\vec{\mathbf{S}} = \frac{q}{\varepsilon_0} \tag{16.1a}$$

where q is the charge enclosed by the closed surface S. Go through the section again. In its differential form, it is given by Eq. (6.18) as

$$\vec{\nabla}.\vec{\mathbf{E}} = \frac{\rho}{\varepsilon_0} \tag{16.1b}$$

You can recall that Gauss's law contains all the information you can get from Coulomb's law and it has wider applicability. Therefore, we write it as the first fundamental law of electricity.

Next, recall Gauss's law for magnetic fields from Unit 13 of Block 3 of this course. It states that magnetic field lines do not begin or end at any point. They close on themselves. The integral and differential forms of Gauss's law for magnetic field are:

$$\oint\!\!\oint_S \vec{\mathbf{B}}.d\vec{\mathbf{S}} = 0 \tag{16.2a}$$

and $$\vec{\nabla}.\vec{\mathbf{B}} = 0 \tag{16.2b}$$

Faraday's law of electromagnetic induction connects changing magnetic fields with induced electric fields. It tells us that an electric field is induced in a circuit/loop/coil when the magnetic field linked to that circuit/loop/coil changes. In integral and differential forms, this law is given by:

$$\oint_C \vec{\mathbf{E}}.d\vec{\mathbf{l}} = -\frac{d}{dt}\oint\!\!\oint_S \vec{\mathbf{B}}.d\vec{\mathbf{S}} \tag{16.3a}$$

and $$\vec{\nabla}\times\vec{\mathbf{E}} = -\frac{\partial\vec{\mathbf{B}}}{\partial t} \tag{16.3b}$$

Finally, we write Ampere's law for steady currents, which states that steady electric currents give rise to magnetic fields (recall Unit 13) in integral and differential forms:

$$\oint_C \vec{\mathbf{B}}.d\vec{\mathbf{l}} = \mu_0 i \tag{16.4a}$$

and $$\vec{\nabla}\times\vec{\mathbf{B}} = \mu_0\vec{\mathbf{J}} \tag{16.4b}$$

You may be wondering: **Why have we not taken the law of Biot and Savart as a fundamental law**? This is because Biot-Savart's law follows from Gauss's law for magnetic fields and Ampere's law.

Thus, both Coulomb's law and Biot-Savart's law can be obtained from a combination of two of the four laws listed above. All the other equations that you have studied in this course so far apply to special situations and are incorporated in the four laws stated so far. In this sense, these laws may be regarded as the fundamental laws of electricity and magnetism

Let us put them all together at one place in Table 16.1.

Table 16.1: A tentative list of the fundamental laws governing electric and magnetic phenomena

S.No.	Law	What the law says	Mathematical statement
1.	Gauss's law for electric field	Electric flux through a closed surface is proportional to the charge enclosed by that surface. That is, charges give rise to electric field.	$\oint\!\!\oint_S \vec{\mathbf{E}}.d\vec{\mathbf{S}} = \frac{q}{\varepsilon_0}$ (16.1a) or $\vec{\nabla}.\vec{\mathbf{E}} = \frac{\rho}{\varepsilon_0}$ (16.1b)
2.	Gauss's law for magnetic field	The total magnetic flux through a closed surface is zero. This implies that magnetic field lines close on themselves. Isolated magnetic charge does not exist.	$\oint\!\!\oint_S \vec{\mathbf{B}}.d\vec{\mathbf{S}} = 0$ (16.2a) or $\vec{\nabla}.\vec{\mathbf{B}} = 0$ (16.2b)
3.	Faraday's law of electromagnetic induction	Changing magnetic flux gives rise to an electric field.	$\oint_C \vec{\mathbf{E}} \, . \, d\vec{\mathbf{l}} = -\frac{d}{dt}\oint\!\!\oint_S \vec{\mathbf{B}}.d\vec{\mathbf{S}}$ (16.3a) or $\vec{\nabla}\times\vec{\mathbf{E}} = -\frac{\partial\vec{\mathbf{B}}}{\partial t}$ (16.3b)
4.	Ampere's law (for steady currents only)	Steady electric current gives rise to a magnetic field.	$\oint_C \vec{\mathbf{B}} \, . \, d\vec{\mathbf{l}} = \mu_0 i$ (16.4a) or $\vec{\nabla}\times\vec{\mathbf{B}} = \mu_0\vec{\mathbf{J}}$ (16.4b)

Let us now examine the four laws we have put together in Table 16.1.

Do you see some similarities in them? Notice that the left-hand sides of Eqs. (16.1a, b and 16.2a, b) and Eqs. (16.3a, b and 16.4a, b), respectively, are completely identical except for the interchanging of $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$. The LHS of Eqs. (16.1a and 16.2a) have surface integrals over closed surfaces. The LHS of Eqs. (16.3a and 16.4a) have line integrals around closed paths. All these pairs of equations differ only in the interchange of $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ on the left-hand side. This means that in all these pairs of equations, we can interchange $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ and their left-hand sides would look the same.

So we can say that **the left-hand sides of the equations in Table 16.1 are symmetrical in the interchange of $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$, <u>in pairs</u>**.

What about the right-hand sides of these laws? Do these exhibit a similar symmetry? These do not seem to be symmetrical at all in the interchange of $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$. You may like to know: **What is the asymmetry in these laws**? Let us explain.

16.2.1 Asymmetry in the Fundamental Laws of Electricity and Magnetism

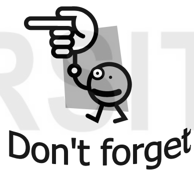
We can identify two kinds of asymmetry in the right-hand side of these laws.

1. The first kind of asymmetry is to be seen in the right-hand sides of Eqs. (16.1a, b and 16.2a, b).

   Note that the right-hand side of Gauss's law for electric field [Eqs. (16.1a, b)] has a charge q enclosed by a surface. But Gauss's law for magnetic field [Eqs. (16.2a, b)] has zero on the right-hand side. There is no equivalent magnetic charge.

   This asymmetry arises from the following fact, which you should always remember:

   > **Isolated electric charges exist in nature, but there is no evidence so far that isolated magnetic charges exist**.



   Therefore, the magnetic charge on the right-hand side of the second law is zero. If and when magnetic monopoles are discovered, the right-hand side of this law would be non-zero for any surface enclosing a net magnetic charge.

   In the same way, the terms $\mu_0 i\ (=\mu_0\, dq/dt)$, $\mu_0 \vec{\mathbf{J}}$ representing the flow of electric charges appear on the right-hand sides of Eqs. (16.4a, b). But no similar terms (representing a current of magnetic monopoles) appear on the right-hand sides of Eqs. (16.3a, b).

   This is one kind of asymmetry in these laws, which could be resolved if we knew for sure that magnetic monopoles existed. Current theories of elementary particles suggesting the existence of magnetic monopoles have prompted an earnest search for them.

2. There is another asymmetry in these laws. On the right-hand side of Faraday's law (Eq. 16.3b), we have the term $-\partial \vec{\mathbf{B}}/dt$. Recall that we interpreted this law by saying that **changing magnetic field produces**

electric field. We find no similar term (representing changing electric fields) in Ampere's law given by Eq. (16.4b). Are we missing something? From symmetry considerations, could we suggest the following?

Changing electric field produces magnetic field.

This was the line of thought followed by Maxwell. Showing remarkable insight into the symmetry of electric and magnetic phenomena, he introduced the concepts of **induced magnetic fields due to changing electric fields** and **displacement current**. Thus, he removed the asymmetry in the fundamental laws of electricity and magnetism. For this, he generalized Ampere's law to arrive at the symmetrical counterpart of Faraday's law. In the next section, you will learn how Maxwell did this. But before studying further, you may like to fix the concepts of this section in your mind.

SAQ 1 - Fundamental laws of electricity and magnetism

a) Write the four fundamental laws of electricity and magnetism. Why are these laws called the fundamental laws and not any other equations that you have studied in this course?

b) State the symmetries in these laws.

c) What are the asymmetries in the fundamental laws of electricity and magnetism?

16.3 MAXWELL'S GENERALISATION OF AMPERE'S LAW

Let us reconsider Ampere's law for steady currents. For mathematical convenience, we use its differential form in our discussion.

$$\vec{\nabla} \times \vec{\mathbf{B}} = \mu_0 \vec{\mathbf{J}} \qquad (16.5)$$

where $\vec{\mathbf{J}}$ is the current density associated with the electric current *i*. The relation of current with current density is given as

$$i = \iint_S \vec{\mathbf{J}}.d\vec{\mathbf{S}} \qquad (16.6)$$

Let us see if we can use Eq. (16.5) for fields that vary with time. If we take the divergence of both sides of Eq. (16.5), we get

$$\vec{\nabla}.(\vec{\nabla} \times \vec{\mathbf{B}}) = \mu_0 (\vec{\nabla}.\vec{\mathbf{J}}) \qquad (16.7)$$

The left-hand side of Eq. (16.7) is zero because for any vector field, the divergence of its curl is always zero: $\vec{\nabla}.(\vec{\nabla} \times \vec{\mathbf{A}}) = 0$ [recall Eq. (2.10f) of Unit 2, Block 1 of this course). Thus, Eq. (16.7) becomes

$$\vec{\nabla} \cdot \vec{\mathbf{J}} = 0 \qquad (16.8)$$

Note that Eq. (16.8) is true only for steady currents. It does not contain any term for time-varying fields.

To understand this point, you need to know the equation of continuity for the flow of charge in a region. You have learnt this equation in Sec. 12.2 of Unit 12 [recall Eq. (12.5)]. Let us state the equation of continuity in its integral and differential forms:

$$\oint\!\!\oint_S \vec{\mathbf{J}}.d\vec{\mathbf{S}} = -\frac{d}{dt}\iiint_V \rho\, dV \qquad (16.9a)$$

$$\vec{\nabla}.\vec{\mathbf{J}} + \frac{\partial \rho}{\partial t} = 0 \qquad (16.9b)$$

Eqs. (16.9a or b) tell us that the **net outflow of current density $\vec{\mathbf{J}}$ from some region in space is equal to the rate of decrease of charge contained in that region**.

For steady currents, $\frac{\partial \rho}{\partial t} = 0$ and we get $\vec{\nabla}.\vec{\mathbf{J}} = 0$

This is just Eq. (12.16) of Unit 12. So, you should note that Eq. (16.5) is true only for steady currents. This is how the equation of continuity has helped us understand that a term for time-varying fields is missing in Ampere's law [Eq. (16.5)].

How did Maxwell resolve this problem of the missing term in Ampere's law? How did he generalise the law? He used the symmetry considerations. Let us see how he did that.

16.3.1 Generalisation of Ampere's Law

In physics, **symmetry** is often a very powerful notion. From the discussion so far, you may already have formed the idea that from symmetry considerations Eq. (16.5) should also have a time-varying term.

The question is: **What should the form of such a term be**? We can say that this term should be the **time derivative** of some vector field so that for static fields, the generalised equation would reduce to Eq. (16.5). Perhaps, Maxwell's most important contribution was the determination of this missing term. He asked:

Could a changing electric field/electric flux induce a magnetic field?

Maxwell modified Ampere's law [Eq. (16.5)] by adding a term $\mu_0 \varepsilon_0 \frac{\partial \vec{\mathbf{E}}}{\partial t}$ to it, which was rewritten in the differential form as

$$\vec{\nabla} \times \vec{\mathbf{B}} = \mu_0 \vec{\mathbf{J}} + \mu_0 \varepsilon_0 \frac{\partial \vec{\mathbf{E}}}{\partial t} \qquad (16.10)$$

The integral form of Eq. (16.10) can be obtained by integrating both sides of the equation over some open surface *S* and applying Stokes' theorem. Let us do this before moving on for the sake of completeness. Integrating both sides of Eq. (16.10) over an open surface *S*, we get

$$\iint_S (\vec{\nabla}\times\vec{\mathbf{B}}).d\vec{\mathbf{S}} = \mu_0 \iint_S \vec{\mathbf{J}}.d\vec{\mathbf{S}} + \mu_0\,\varepsilon_0 \iint_S \frac{\partial}{\partial t}\vec{\mathbf{E}}.d\vec{\mathbf{S}} \qquad (16.11a)$$

Applying Stokes' theorem to the LHS of Eq. (16.11a), we get

$$\oint_C \vec{\mathbf{B}}.d\vec{\mathbf{l}} = \mu_0 \iint_S \vec{\mathbf{J}}.d\vec{\mathbf{S}} + \mu_0\,\varepsilon_0 \iint_S \frac{\partial}{\partial t}\vec{\mathbf{E}}.d\vec{\mathbf{S}} \qquad (16.11b)$$

Now, note that from Eq. (16.6),

$$i = \iint_S \vec{\mathbf{J}}.d\vec{\mathbf{S}} \qquad (16.11c)$$

and recall the definition of electric flux from Unit 5:

$$\Phi_E = \iint_S \vec{\mathbf{E}}.d\vec{\mathbf{S}} \qquad (16.11d)$$

Substituting these expressions for i and Φ_E from Eqs. (16.11c and d) in Eq. (16.11b), we can write it as

$$\oint_C \vec{\mathbf{B}}.d\vec{\mathbf{l}} = \mu_0\, i + \mu_0\,\varepsilon_0 \frac{d\Phi_E}{dt} \qquad (16.12)$$

This is the **generalization of Ampere's law as carried out by Maxwell**.

You should note that in writing Eq. (16.12), the minus sign in Faraday's law has been replaced by a plus sign. This is dictated by experiment and considerations of symmetry. The factor $\mu_0\,\varepsilon_0$ is inserted to express the equation in SI units.

What do Eqs. (16.10 and 16.12) tell us? These equations tell us that **there are two ways of setting up a magnetic field**:

1) by an **electric current**, and
2) by a **time-varying electric field**.

Eqs. (16.10 and 16.12) express the generalised differential and integral forms of Ampere's law, respectively. This generalised Ampere's law is also called the **Ampere-Maxwell law**.

Always remember that Maxwell did not derive this law from any empirical considerations. He was motivated by symmetry considerations and he deduced the additional term by requiring that Ampere's law be consistent with the law of conservation of electric charge (or equation of continuity). Since Maxwell's time many experiments including direct measurements of the magnetic field associated with a huge capacitor, have confirmed this remarkable insight of Maxwell.

In the next section, we will study Eqs. (16.10 and 16.12) further and understand their meaning. In particular, we will interpret the second term on the RHS of Eqs. (16.10 and 16.12). But you should verify that the term $\varepsilon_0\, d\Phi_E / dt$ has the dimensions of current. Do so before studying further.

SAQ 2 - **Generalisation of Ampere's law**

a) Explain the symmetry consideration that led to the generalisation of Ampere's law.

b) Show that the term $\varepsilon_0 \, d\Phi_E / dt$ has the dimensions of current.

So, on solving SAQ 2b, you have learnt that the generalised Ampere-Maxwell's law contains a term that has the dimensions of current. Let us now examine the nature of this term.

16.3.2 Displacement Current

Note that the changing electric flux in Eq. (16.12) is **not** an electric current as no charge actually flows but it has the same effect as a current in producing magnetic fields. For this reason Maxwell called this term the **displacement current**. Historically, it has been treated as a fictitious current and the name 'displacement current' given by Maxwell has stuck. It is given as

$$i_d = \varepsilon_0 \frac{d\Phi_E}{dt} \tag{16.13a}$$

To understand what this term represents, let us recall the expression for electric flux as a surface integral of the electric field:

$$\Phi_E = \iint_S \vec{\mathbf{E}} \,.\, d\vec{\mathbf{S}} \tag{16.13b}$$

Using Eq. (16.13b), we can write Eq. (16.13a) as

$$i_d = \varepsilon_0 \frac{d\Phi_E}{dt} = \varepsilon_0 \frac{d}{dt} \iint_S \vec{\mathbf{E}}.d\vec{\mathbf{S}} = \varepsilon_0 \iint_S \frac{\partial \vec{\mathbf{E}}}{\partial t}.d\vec{\mathbf{S}} \tag{16.13c}$$

Notice that in moving the time derivative of Φ_E inside the integral in Eq. (16.13c), we have written it as a partial derivative with respect to time. This is because the electric flux/electric field is a function of both space and time coordinates. The **displacement current density** is defined as

$$\vec{\mathbf{J}}_d = \varepsilon_0 \frac{\partial \vec{\mathbf{E}}}{\partial t} \tag{16.14}$$

Then, using Eq. (16.14), we can write Eq. (16.13c) as

$$i_d = \iint_S \vec{\mathbf{J}}_d.d\vec{\mathbf{S}} \tag{16.15}$$

The word 'displacement' does not have any physical meaning. But the word 'current' is relevant in the sense that the effect of the displacement current cannot be distinguished from that of a real current in producing magnetic fields. So, we can say that a magnetic field can be set up by a conduction current *i* or by a displacement current i_d.

Thus, we can express Ampere-Maxwell law [Eq. (16.12)] as:

$$\oint_C \vec{\mathbf{B}}.d\vec{\mathbf{l}} = \mu_0 \iint_S \left(\vec{\mathbf{J}} + \varepsilon_0 \frac{\partial \vec{\mathbf{E}}}{\partial t} \right).d\vec{\mathbf{S}} = \mu_0 (i + i_d) \tag{16.16}$$

To better understand the role of the displacement current, let us determine the displacement current in a circuit containing a parallel plate capacitor.

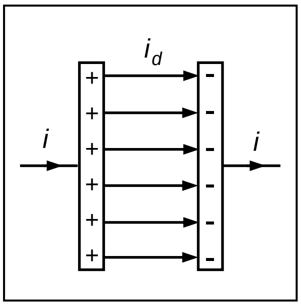


Fig. 16.1: **Parallel plate capacitor being charged by a constant current *i*.**

EXAMPLE 16.1: DISPLACEMENT CURRENT IN A CIRCUIT WITH A PARALLEL-PLATE CAPACITOR

Consider a parallel plate capacitor being charged by a constant current i as shown in Fig. 16.1. The real current i changes the electric field $\vec{\mathbf{E}}$ between the capacitor plates. The fictitious current i_d between the plates is associated with that changing electric field $\vec{\mathbf{E}}$. Determine the displacement current in the circuit.

SOLUTION ■ We assume that the plates are large in comparison with their separation. Then, there will be an electric field $\vec{\mathbf{E}}$ only between the plates and to a good approximation, it will be uniform over most of the area of the plates. Under these conditions we shall use Eq. (16.15) to determine i_d for the parallel plate capacitor.

We can relate the amount of excess charge $|Q|$ on each of the plates at any given time to the magnitude $|\vec{\mathbf{E}}|$ of the electric field between the plates at that time. To do so, we use Gauss's law [Eq. (6.16), Unit 6, Block 2]:

$$\oiint_S \vec{\mathbf{E}}.d\vec{\mathbf{S}} = \frac{Q_{encl}}{\varepsilon_0} \qquad \text{(i)}$$

You have learnt in Unit 6 that symmetry of the geometry allows us to choose the Gaussian surface in such a way that the magnitude $|\vec{\mathbf{E}}|$ of the electric field is uniform and $\vec{\mathbf{E}}$ and $d\vec{\mathbf{S}}$ are parallel.

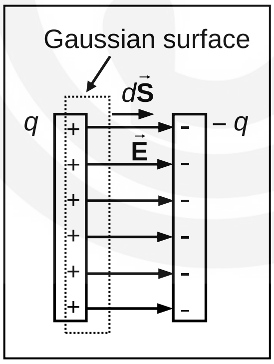


Fig. 16.2: **A charged parallel plate capacitor. The dotted Gaussian surface encloses the charge on the positive plate.**

In this case, we choose the Gaussian surface that encloses the charge on the positive plate (see Fig. 16.2). Then $\vec{\mathbf{E}}$ and $d\vec{\mathbf{S}}$ are parallel and $\vec{\mathbf{E}}.d\vec{\mathbf{S}}$ is equal to $E\,dS$. Since E is constant, from Eq. (i), we have

$$|Q| = \varepsilon_0\,\Phi_E = \varepsilon_0 \oiint_S \vec{\mathbf{E}}.d\vec{\mathbf{S}} = \varepsilon_0 \oiint_S E\,dS = \varepsilon_0 E \oiint_S dS = \varepsilon_0 EA \qquad \text{(ii)}$$

where A is the area of the plate. We can determine the real current i by differentiating Eq. (ii):

$$|i| = \left|\frac{dQ}{dt}\right| = \varepsilon_0 A \left|\frac{dE}{dt}\right| \qquad \text{(iii)}$$

Next we determine the displacement current. To do so, we use Eq. (16.13a). From Eq. (ii), we have $\Phi_E = EA$ and we can write

$$|i_d| = \varepsilon_0 \left|\frac{d\Phi_E}{dt}\right| = \varepsilon_0 \left|\frac{dE\,A}{dt}\right| = \varepsilon_0 A \left|\frac{dE}{dt}\right| \qquad \text{(iv)}$$

Comparing Eqs. (iii) and (iv), we find that the real current i charging the capacitor and the fictitious displacement current i_d between the plates have the same value:

$$i_d = i \qquad \text{(v)}$$

How do we interpret this result? It tells us that we may consider the fictitious displacement current i_d to be a continuation of the real current *i*. While the real current flows in the circuit, the displacement current is its continuation from one plate, across the gap in the capacitor, to the other plate (see Fig. 16.1). This displacement current is uniformly spread between the plates because the electric field is uniform over the plates of the capacitor. This is shown in Fig. 16.1 by the uniform spread of current arrows between the plates of the capacitor.

This example of a parallel plate capacitor shows concretely the necessity for the displacement current term in the fourth Maxwell's equation. To get an idea of the importance of displacement current we would like you to work out the following SAQ.

SAQ 3 - Displacement current

Obtain the maximum value of the displacement current in a parallel plate capacitor made up of plates of area $1.0\,\text{m}^2$. It is given that the electric field between the plates is $E = E_0 \sin\omega t$ with $E_0 = 10\,\text{V}$ and frequency 10 MHz.

It was indeed Maxwell's genius to recognize that Ampere's law should be modified to reflect the symmetry suggested by Faraday's law. To honour Maxwell, the four complete laws of electromagnetism are given the name **Maxwell's equations**. Maxwell's equations belong to the category of the fundamental laws of nature. As you have seen, they are not derived from any fundamental precepts by logical reasoning and mathematical calculations. Fundamental laws of nature are generalizations of our knowledge and they are discovered, found or ascertained. In the next section, we present Maxwell's equations.

16.4 PUTTING MAXWELL'S EQUATIONS TOGETHER

We first list the four equations known as Maxwell's equations.

Table 16.2: Maxwell's equations

Eq. No.	Differential form	Integral form	
1.	$\vec{\nabla}.\vec{\mathbf{E}} = \dfrac{\rho}{\varepsilon_0}$	$\oiint_S \vec{\mathbf{E}}.d\vec{\mathbf{S}} = \dfrac{q}{\varepsilon_0}$	(16.17)
2.	$\vec{\nabla}\times\vec{\mathbf{E}} = -\dfrac{\partial\vec{\mathbf{B}}}{\partial t}$	$\oint_C \vec{\mathbf{E}}.d\vec{\mathbf{l}} = -\dfrac{d\Phi_B}{dt}$	(16.18)
3.	$\vec{\nabla}.\vec{\mathbf{B}} = 0$	$\oiint_S \vec{\mathbf{B}}.d\vec{\mathbf{S}} = 0$	(16.19)
4.	$\vec{\nabla}\times\vec{\mathbf{B}} = \mu_0\left(\vec{\mathbf{J}} + \varepsilon_0\dfrac{\partial\vec{\mathbf{E}}}{\partial t}\right)$	$\oint_C \vec{\mathbf{B}}.d\vec{\mathbf{l}} = \mu_0 i + \mu_0 i_d$ where $i_d = \varepsilon_0\dfrac{d\Phi_E}{dt}$	(16.20)

This set of equations, first published by Maxwell in 1864, governs the behaviour of electric and magnetic fields everywhere. The equations in Table 16.2 are written for the fields in the presence of electric charges and electric currents. Notice that the lack of symmetry in these equations, with respect to $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ fields is entirely due to the absence of magnetic charge and its corresponding current.

Maxwell's equations are also written for the fields in vacuum, and in material media. We now take the special case of Maxwell's equations in vacuum, that is, regions in space that are free of charges and currents. As you can see, in such regions, the terms containing *q* and *i* or ρ and $\vec{\mathbf{J}}$ in Eqs. (16.17) to (16.20) are zero. Then Maxwell's equations take the following form.

Table 16.3: Maxwell's equations in vacuum with no source charges or currents

Differential form	Integral form	
$\vec{\nabla}.\vec{\mathbf{E}} = 0$	$\oiint_S \vec{\mathbf{E}}.d\vec{\mathbf{S}} = 0$	(16.21)
$\vec{\nabla}\times\vec{\mathbf{E}} = -\frac{\partial\vec{\mathbf{B}}}{\partial t}$	$\oint_C \vec{\mathbf{E}}.d\vec{\mathbf{l}} = -\frac{d\Phi_B}{dt}$	(16.22)
$\vec{\nabla}.\vec{\mathbf{B}} = 0$	$\oiint_S \vec{\mathbf{B}}.d\vec{\mathbf{S}} = 0$	(16.23)
$\vec{\nabla}\times\vec{\mathbf{B}} = \mu_0\varepsilon_0\frac{\partial\vec{\mathbf{E}}}{\partial t}$	$\oint_C \vec{\mathbf{B}}.d\vec{\mathbf{l}} = \mu_0\varepsilon_0\frac{d\Phi_E}{dt}$	(16.24)

You can see that in charge-free and current-free regions, the symmetry in these equations is complete. The electric and magnetic fields appear on an equal footing in Maxwell's equations in vacuum. The constants ε_0 and μ_0 appear in Ampere-Maxwell law due to our choice of SI units.

You could be wondering about the discrepancy in sign. The difference of signs in Eqs. (16.18) and (16.20) or in Eqs. (16.22) and (16.24) is actually due to symmetry: it reflects the complementary way in which electric and magnetic fields give rise to each other.

Maxwell arrived at four compact and elegant equations that reveal the inherent symmetry between electric and magnetic fields.

What do Maxwell's equations tell us?

In a nutshell,

The first two Maxwell's equations (in Table 16.2) tell us that an electric field is set up in two ways:

i) by electric charges and
ii) by a changing (variable) magnetic field.

The last two equations tell us that

i) a magnetic field has no sources (there are no magnetic charges), and

ii) a magnetic field is set up by electric currents and a variable electric field.

Maxwell's equations also indicate that a **time-varying magnetic field cannot exist without a variable electric field, and a time-varying electric field, without a variable magnetic field**.

This is why the two fields are not regarded as separate. Thus arose the concept of **electromagnetic field as a single entity**. By combining Faraday's law and the generalised Ampere's law, Maxwell's equations tell us how electricity and magnetism can be treated as complementary aspects of the same phenomenon – electromagnetism. We can say that electricity and magnetism are two sides of the same coin. In this manner, Maxwell succeeded in formulating mathematically a unified theory of electricity and magnetism.

Maxwell's equations are also important because of their ability to predict a wide range of new phenomena. The consequences of Maxwell's formulations are legion – all of electrical and radio engineering is contained in these equations. The design of the entire range of communication systems (such as radio, TV, cell phones, Internet) is based on Maxwell's equations. Further, the presence of the displacement current term in Eq. (16.20) alongwith Eq. (16.18) implies the existence of electromagnetic waves and helps explain the nature of light. This forms the discussion of the next section.

For these reasons, Maxwell's equations are considered to be a path-breaking achievement of the nineteenth century physics and places Maxwell on the same footing as Newton and Einstein.

You may now like to stop for a while and apply Maxwell's equations to time-varying electric and magnetic fields before studying the next section on electromagnetic waves.

SAQ 4 - Maxwell's equations

Under what conditions do the following time-varying electric and magnetic fields satisfy Maxwell's equations [Eqs. 16.21 to 16.24)]?

$$\vec{\mathbf{E}} = \hat{\mathbf{j}}\, E_0 \sin(z - vt)$$

$$\vec{\mathbf{B}} = \hat{\mathbf{i}}\, B_0 \sin(z - vt)$$

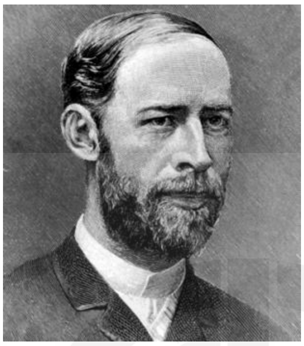
where E_0 and B_0 are constants.

Maxwell's equations were a culmination of more than half a century of work on electricity and magnetism by a galaxy of scientists like Benjamin Franklin (American polymath, 1706 – 1790), Charles-Augustin de Coulomb (French physicist, 1736 – 1806), Carl Friedrich Gauss (German mathematician and physicist, 1777 – 1855), Hans Christian Oersted (Danish physicist, 1777 – 1851), Jean-Baptiste Biot (French physicist, 1774 –1862), Felix Savart (French physicist, 1791 – 1841), Hendrik Antoon Lorentz (Dutch physicist 1853 – 1928), Andre-Marie Ampere (French physicist, 1775 – 1836), Joseph

Henry (American scientist, 1797–1878), Michael Faraday (English physicist, 1791 – 1867) and James Clerk Maxwell (Scottish scientist, 1831 – 1879).

But as you have learnt in Sec. 16.4, it was Maxwell who reformulated the fundamental laws of electricity and magnetism and presented them as four compact equations given in Tables 16.2 and 16.3 that explain all electromagnetic phenomena. You will discover in the next section, what is perhaps the most exciting and profound scientific outcome of Maxwell's equations. It is their ability to predict electromagnetic waves and provide the basis for our understanding of the nature of electromagnetic waves.

16.5 ELECTROMAGNETIC WAVES



Heinrich Rudolf Hertz (1857 – 1894) was a German physicist who proved the existence of electromagnetic waves through his experiments. The unit of frequency, cycle per second, was named 'hertz' in his honour.

One of the great successes of Maxwell's equations was that they predicted the existence of electromagnetic waves in 1864 long before these were generated or detected in experiments. Almost 25 years later, it was Heinrich Hertz who first generated and detected these waves experimentally in 1887. Maxwell also predicted that all electromagnetic waves would travel at a speed that was very close to the speed of visible light in the air. And Maxwell correctly asserted that visible light was an electromagnetic wave. Now we know that radio waves, infrared, visible, ultraviolet, *X*-rays and gamma rays are all electromagnetic waves differing only in frequency.

In this section, we will see how Maxwell's equations led to the prediction of electromagnetic waves and that of visible light being an electromagnetic wave. From Maxwell's equations, we will derive an equation which is just the wave equation and understand its physical meaning.

16.5.1 The Wave Equations for $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ Fields

We shall first derive the wave equation from Maxwell's equations in vacuum, i.e., a region of space where there are no charges or currents using Eqs. (16.21 to 16.24) given in Table 16.3. As you can see, these equations are coupled, first order partial differential equations. But we can uncouple these equations.

For the sake of mathematical convenience, we will use their differential form and first derive the equation for the electric field $\vec{\mathbf{E}}$. Taking the curl of Eq. (16.22), we get

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{\mathbf{E}}) = \vec{\nabla} \times \left(-\frac{\partial \vec{\mathbf{B}}}{\partial t}\right) = -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{\mathbf{B}}) \qquad (16.25a)$$

We now make use of the following vector identity [see Eq. (2.10g) of Unit 2, Block 1 of this course] for any vector field $\vec{\mathbf{F}}$:

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{\mathbf{F}}) = \vec{\nabla}(\vec{\nabla} . \vec{\mathbf{F}}) - \nabla^2 \vec{\mathbf{F}}$$

Thus, using this vector identity in Eq. (16.25), we can write it as

$$\vec{\nabla}(\vec{\nabla} . \vec{\mathbf{E}}) - \nabla^2 \vec{\mathbf{E}} = -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{\mathbf{B}}) \qquad (16.25b)$$

Then using $\vec{\nabla} \times \vec{\mathbf{B}} = \varepsilon_0 \frac{\partial \vec{\mathbf{E}}}{\partial t}$ from Eq. (16.24), we can write Eq. (16.25b) as

$$\vec{\nabla}(\vec{\nabla}.\vec{\mathbf{E}}) - \nabla^2 \vec{\mathbf{E}} = -\mu_0 \varepsilon_0 \frac{\partial^2 \vec{\mathbf{E}}}{\partial t^2} \tag{16.25c}$$

Since $\vec{\nabla}.\vec{\mathbf{E}} = \vec{\mathbf{0}}$ from Eq. (16.21), Eq. (16.25c) becomes

$$\nabla^2 \vec{\mathbf{E}} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{\mathbf{E}}}{\partial t^2} \tag{16.26}$$

We can follow similar steps and show that

$$\nabla^2 \vec{\mathbf{B}} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{\mathbf{B}}}{\partial t^2} \tag{16.27}$$

In fact, you should practice the steps and deduce Eq. (16.27) in SAQ 5.

SAQ 5 **- Deriving wave equation**

Derive Eq. (16.27).

You should practice the mathematical steps in the derivations of Eqs. (16.26 and 16.27) so that you are completely comfortable with them. Thus, from Maxwell's equations, we get two uncoupled second order partial differential equations for the time-varying $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ fields in vacuum in the absence of charges or currents. Let us put them together.

Equations for Time-varying Electric and Magnetic Fields

Recap

The time-varying electric and magnetic fields in Maxwell's equations obey second order partial differential equations given by:

$$\nabla^2 \vec{\mathbf{E}} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{\mathbf{E}}}{\partial t^2} \tag{16.26}$$

$$\nabla^2 \vec{\mathbf{B}} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{\mathbf{B}}}{\partial t^2} \tag{16.27}$$

Let us now consider Eqs. (16.26 and 16.27) in detail. You will see in the following discussion that Eqs. (16.26 and 16.27) are of the form of the wave equation.

Recall the classical wave equation that you have learnt in school physics (or revise the basic concepts of wave motion from Unit 19 of the course BPHCT-131 entitled Mechanics). It is given as

$$\nabla^2 f = \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2} \tag{16.28}$$

This equation describes a wave travelling with speed *v*. Now compare Eq. (16.28) with Eqs. (16.26 and 16.27). You can see that Eqs. (16.26 and 16.27) are wave equations for time-varying electric and magnetic fields which propagate like waves in space.

Note that Maxwell's equations tell us that a **changing electric field gives rise to a magnetic field, which itself may be changing with time**. Also **a changing magnetic field gives rise to an electric field**, **which itself may be changing with time**.

Taken together with Maxwell's equations, Eqs. (16.26 and 16.27) suggest the following:

Time-varying electric and magnetic fields could continuously generate each other and propagate in space transporting electromagnetic energy with them.

In this way, these equations suggest the possibility of self-sustaining travelling electromagnetic fields, which travel in space like waves. This is how the concept of an 'electromagnetic wave' was born. Now we think of electromagnetic waves as structures consisting of electric and magnetic fields that travel freely through vacuum. Comparing Eqs. (16.26 and 16.27) with Eq. (16.28) gives us the speed of the electromagnetic waves. It is

Maxwell predicted that all electromagnetic waves would move at a speed that was about the same as the measured value of the speed of light in air. That is why he asserted right from the beginning that light was an electromagnetic wave.

$$v = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} \quad (16.29)$$

When we substitute the values of ε_0 and μ_0 in Eq. (16.29), we get the value of *v* as

$$v = 3.0 \times 10^8 \, \text{ms}^{-1} \quad (16.30)$$

Do you recognise this value? It is just the **speed of light in vacuum**! The implication of this result is extremely exciting:

Light is an electromagnetic wave.

In a nutshell, Maxwell discovered that the speed of the electromagnetic wave depended on the familiar constants ε_0 and μ_0 measured in static electric and magnetic experiments.

He also discovered that a calculation of the speed of the electromagnetic wave in vacuum gave a value that was the same as the speed of light. This led him to make the additional bold hypothesis that light was an electromagnetic wave.

This conclusion would not surprise you today. But imagine what a triumph it was in Maxwell's times. Let us explain this point.

Do you recall where ε_0 and μ_0 came in the theory in the first place? They appeared as constants in Coulomb's law and Biot-Savart law. We can measure them in experiments involving charged pith balls, batteries, and wires – *experiments which have nothing to do with light*. And yet, in Maxwell's theory these two are related in a beautifully simple manner to the speed of light!

Notice also the crucial role of the displacement current term in Ampere-Maxwell law. Without this term, the wave equation would not have emerged. Thus, according to Maxwell's equations,

Vacuum supports the propagation of electromagnetic waves at a speed given by Eq. (16.29).

We would like you to understand clearly the nature of these waves. This is what you will learn in the next section.

16.5.2 The Nature of Electromagnetic Waves

We have suggested in Sec. 16.5.1 that an electromagnetic wave is constituted of time-varying electric and magnetic fields. Now we ask: **How are we to visualize such a travelling electromagnetic wave**?

To do so, let us consider an electromagnetic wave travelling through a region in vacuum (i.e., charge-free and current-free region).

You may like to watch the animation of a travelling electromagnetic wave at the url https://www.youtube.com/watch?v=aCTRjVEmeC0 to visualize the nature of electromagnetic waves and their propagation in space.

As the electromagnetic wave travels through the region, the changing electric field in the wave produces a changing magnetic field and vice-versa. The induced electric field is, in fact, the **electric component of the electromagnetic wave** and the **induced magnetic field is magnetic component of the electromagnetic wave**.

The induced electric and magnetic fields, which satisfy the respective wave equations, travel through space and constitute the electromagnetic wave.

So we have a self-perpetuating electromagnetic wave whose $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ fields exist and change without the need for charged matter.

In this way, Maxwell's equations teach us that a beam of sunlight is a configuration of changing electric and magnetic fields travelling through space. The same is true for radio waves, microwaves, infrared rays, ultraviolet rays, X-rays and γ-rays. In relation to electromagnetic waves, we would like to stress on the following very important point:

It is not enough that an electromagnetic field satisfies the wave Eqs. (16.26 and 16.27). **It must also satisfy Maxwell's equations**.

Maxwell's equations impose **extra constraints** on the electric and magnetic fields that satisfy them. Whereas every solution of Maxwell's equations (in vacuum) must obey the wave equation, the **converse is not true**.

Every solution of a wave equation need not obey Maxwell's equations, i.e., it need not represent an electromagnetic wave.

Thus, in solving wave equations for electromagnetic waves, you must take special care to see whether the solutions satisfy Maxwell's equations. Only then would they represent an electromagnetic wave. Let us consider an example to determine the extra constraints imposed by Maxwell's equations. That would also give us a clue about the transverse nature of electromagnetic waves.

Proof of Eq. (iv) in Example 16.2:

Let us substitute Eq. (i) of Example 16.2 in the relation $\vec{\nabla}\cdot\vec{\mathbf{E}} = 0$:

Then we have

$$\vec{\nabla}\cdot\vec{\mathbf{E}} = 0$$

$$= \frac{\partial}{\partial x}(E_{0x})\sin(z-ct)$$

$$+ \frac{\partial}{\partial y}(E_{0y})\sin(z-ct)$$

$$+ \frac{\partial}{\partial z}[E_{0z}\sin(z-ct)]$$

Now, in order that the equation $\vec{\nabla}\cdot\vec{\mathbf{E}} = 0$ be satisfied by $\vec{\mathbf{E}}$, **all partial derivatives in the above equation should be equal to zero**. The partial derivative with respect to z will be zero if and only if E_{0z} is zero. This is because the partial derivative of the sine function with respect to z is non-zero.

EXAMPLE 16.2: TRANSVERSE NATURE OF AN ELECTROMAGNETIC WAVE

For keeping the mathematics simple, we suppose that the electromagnetic waves are travelling in the z-direction with the speed c. So the electric and magnetic fields in the waves will have the form (recall Unit 19 of the course BPHCT-131):

$$\vec{\mathbf{E}} = \vec{\mathbf{E}}_0 \sin(z-ct) \qquad \text{(i)}$$

$$\text{and} \quad \vec{\mathbf{B}} = \vec{\mathbf{B}}_0 \sin(z-ct) \qquad \text{(ii)}$$

where $\vec{\mathbf{E}}_0$ and $\vec{\mathbf{B}}_0$ are the respective amplitudes of the electric and magnetic fields associated with the electromagnetic wave. Show that the electromagnetic wave is a transverse wave.

SOLUTION ■ From Maxwell's equations (16.21 and 16.23) in vacuum, we have

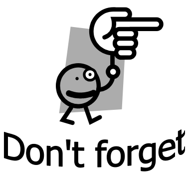
$$\vec{\nabla}\cdot\vec{\mathbf{E}} = 0 \quad \text{and} \quad \vec{\nabla}\cdot\vec{\mathbf{B}} = 0 \qquad \text{(iii)}$$

If we substitute Eqs. (i) and (ii) in both equations of Eq. (iii), we find that these will be satisfied if only if

$$E_{0z} = \left(\vec{\mathbf{E}}_0\right)_z = 0 \quad \text{and} \quad B_{0z} = \left(\vec{\mathbf{B}}_0\right)_z = 0 \qquad \text{(iv)}$$

Eq. (iv) tells us that the components of the electric and magnetic fields of the electromagnetic wave in the direction of propagation are zero. This implies that the direction of propagation of an electromagnetic wave is perpendicular to the directions of the electric and magnetic fields that constitute it. You have learnt in your school physics and Unit 19 of the course BPHCT-131 that such waves are transverse waves. Thus, **electromagnetic waves are transverse waves**.

You have just learnt that electromagnetic waves are transverse in nature. It means that at any given instant of time, the direction of propagation of electromagnetic waves is perpendicular to the directions of the electric and magnetic fields that constitute it. Also, **the directions of the electric field component and magnetic field component of the electromagnetic wave are perpendicular to each other**. So,



An electromagnetic wave comprises mutually perpendicular electric and magnetic fields propagating in a direction perpendicular to both fields.

SAQ 6 - Conditions for an electromagnetic wave

For the electric and magnetic fields of SAQ 4, you have verified that these satisfy Maxwell's equations provided that $E_0 = vB_0$ and $B_0 = \dfrac{vE_0}{c^2}$, where c is the speed of light given by $c = 1/\sqrt{\mu_0\varepsilon_0}$. Show that together these conditions require that $v = \pm c$ and $|B_0|c = |E_0|$.

So, the $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ fields of SAQ 4/SAQ 6 describe an electromagnetic wave under certain conditions. Let us further understand how the electromagnetic wave described by the $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ fields of SAQ 4/SAQ 6 travels in space. You have verified in SAQ 6 that the electromagnetic field of SAQ 4 satisfies Maxwell's equations provided that

$$E_0 = B_0 c \tag{16.31a}$$

or

$$\frac{E_0}{B_0} = c \tag{16.31b}$$

where c is the speed of light given by $c = 1/\sqrt{\mu_0 \varepsilon_0}$. You can also verify that these $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ fields satisfy the respective wave equations [Eqs. (16.26 and 16.27)].

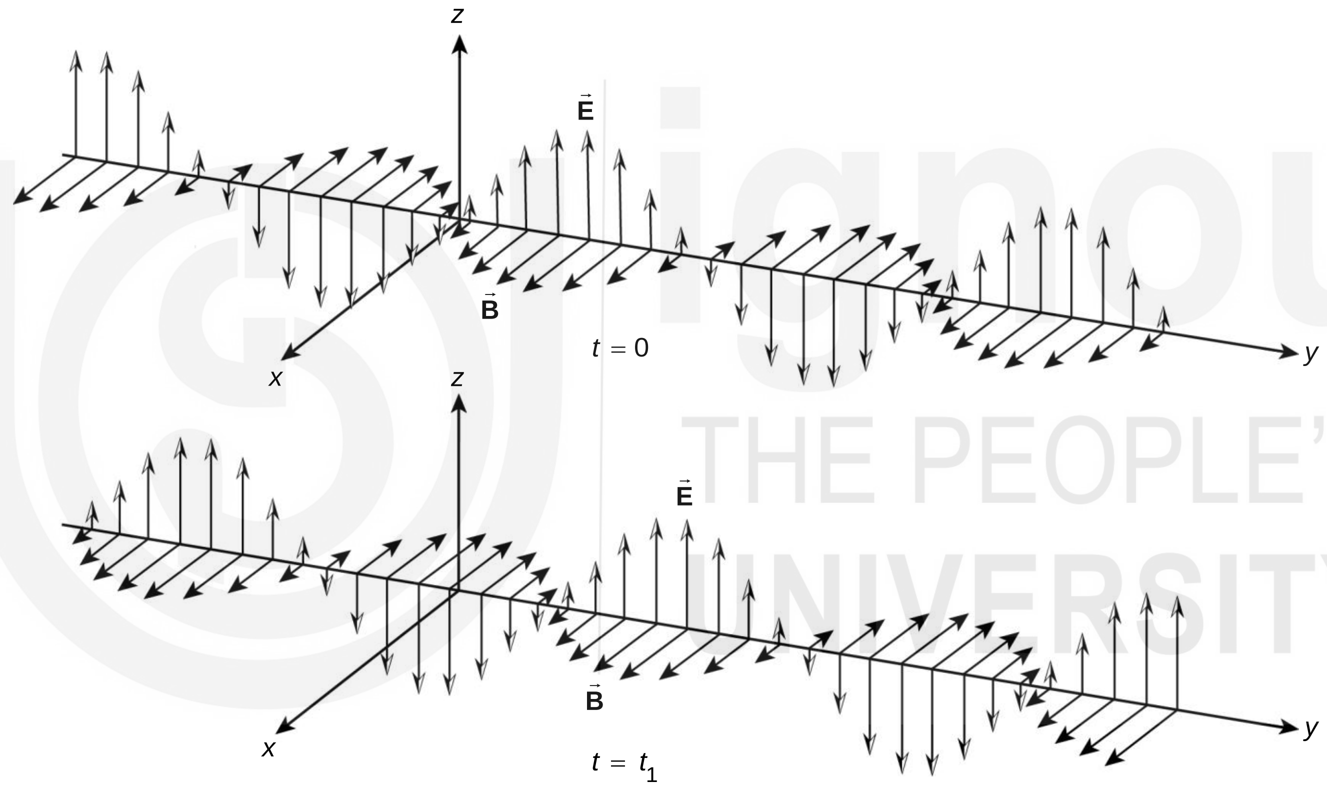


Fig. 16.3: **Travelling electromagnetic wave.**

Study Fig. 16.3. The electromagnetic wave is shown at two different times. Note that as time passes, the entire pattern slides to the right because $(y - ct)$ has the same value at $(y + \Delta y)$ and $t + \Delta t$ as it had at y and t, provided that $\Delta y = c\,\Delta t$. This is an example of a **plane electromagnetic wave**.

In other words, we have a plane electromagnetic wave travelling with a constant speed c in the y-direction.

Did you notice that we have introduced a new term: **plane electromagnetic wave**?

You may well ask: **What is a plane electromagnetic wave**?

By definition, **a plane electromagnetic wave is a wave for which the $\vec{E}$ and $\vec{B}$ fields are constants at a given instant of time at all points in a plane normal to the direction of propagation**.

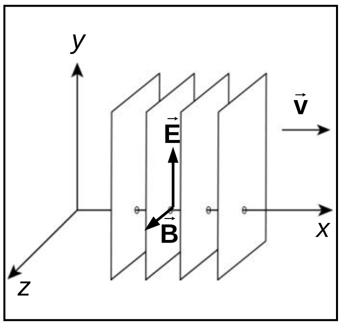


Fig. 16.4: **In a plane electromagnetic wave, the planes in which the field vectors $\vec{E}$ and $\vec{B}$ lie at any two different points are parallel to each other.**

You can visualise a plane electromagnetic wave by picturing the field vectors $\vec{E}$ and $\vec{B}$ at each point in space as lying in a plane. Also the planes at any two different points are parallel to each other (study Fig. 16.4).

In the next unit of this block, we shall be mainly concerned with propagation of plane electromagnetic waves in vacuum and material media, as these are found to be very useful in various areas of physics, engineering and technology.

As you have seen, the electromagnetic fields given in SAQ 4 describe an electromagnetic wave which is a specific example of a plane electromagnetic wave. Our interest now is to find the plane wave solutions of the wave equations, which also satisfy Maxwell's equations. This is what you will learn in Unit 17, which is the last unit of this block. We now summarise the concepts you have studied in this unit.

16.6 SUMMARY

Concept	Description
Fundamental laws of electricity and magnetism	■ The four fundamental laws of electricity and magnetism are: Gauss's law for electric field: $\oiint_S \vec{\mathbf{E}}.d\vec{\mathbf{S}} = \frac{q}{\varepsilon_0}$ or $\vec{\nabla}.\vec{\mathbf{E}} = \frac{\rho}{\varepsilon_0}$ Faraday's law of electromagnetic induction: $\oint_C \vec{\mathbf{E}}.d\vec{\mathbf{l}} = -\frac{d}{dt}\oiint_S \vec{\mathbf{B}}.d\vec{\mathbf{S}}$ or $\vec{\nabla}\times\vec{\mathbf{E}} = -\frac{\partial\vec{\mathbf{B}}}{\partial t}$ Gauss's law for magnetic field: $\oiint_S \vec{\mathbf{B}}.d\vec{\mathbf{S}} = 0$ or $\vec{\nabla}.\vec{\mathbf{B}} = 0$ Ampere's law (for steady currents only): $\oint_C \vec{\mathbf{B}}.d\vec{\mathbf{l}} = \mu_0 i$ or $\vec{\nabla}\times\vec{\mathbf{B}} = \mu_0\vec{\mathbf{J}}$
Asymmetry in the fundamental laws of electricity and magnetism	■ There are two kinds of asymmetry in the fundamental laws of electricity and magnetism: The first kind of asymmetry is in the Gauss's laws for electric and magnetic fields. The RHS of Gauss's law for electric field has a charge q enclosed by a surface. But Gauss's law for magnetic field has zero on its RHS. There is no equivalent magnetic charge. This asymmetry arises because while isolated electric charges exist in nature, there is no evidence so far that isolated magnetic charges exist. The second kind of asymmetry in these laws is that on the RHS of Faraday's law there is a term $-\partial\vec{\mathbf{B}}/dt$. But there is no similar term in Ampere's law. Faraday's law tells us that **changing**

magnetic fields produce an electric field. From symmetry considerations, it should follow that **changing electric fields produce a magnetic field**. But there is no such term in Ampere's law.

Maxwell's Generalisation of Ampere's Law

- From symmetry considerations, Maxwell added a term containing what he called the **displacement current** $i_d = \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}$ in Ampere's law and generalized it as follows:

$$\vec{\nabla} \times \vec{\mathbf{B}} = \mu_0 \vec{\mathbf{J}} + \varepsilon_0 \frac{\partial \vec{\mathbf{E}}}{\partial t} \quad \text{or} \quad \oint_C \vec{\mathbf{B}}.d\vec{\mathbf{l}} = \mu_0 \iint_S \left(\vec{\mathbf{J}} + \varepsilon_0 \frac{\partial \vec{\mathbf{E}}}{\partial t} \right).d\vec{\mathbf{S}} = \mu_0 (i + i_d)$$

This equation is now known as **Ampere-Maxwell law**. This law tells us that **there are two ways of setting up a magnetic field**: by an **electric current**, and by a **time-varying electric field**.

Maxwell's equations

- **Maxwell's equations** constitute the fundamental set of differential equations describing electric and magnetic fields. These equations in their integral and differential forms are given in the two tables below in the presence of charges and currents and in vacuum, i.e., charge-free and current-free regions:

In the presence of charges and currents

$$\vec{\nabla}.\vec{\mathbf{E}} = \frac{\rho}{\varepsilon_0} \qquad \oiint_S \vec{\mathbf{E}}.d\vec{\mathbf{S}} = \frac{q}{\varepsilon_0}$$

$$\vec{\nabla} \times \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t} \qquad \oint_C \vec{\mathbf{E}}.d\vec{\mathbf{l}} = -\frac{d\Phi_B}{dt}$$

$$\vec{\nabla}.\vec{\mathbf{B}} = 0 \qquad \oiint_S \vec{\mathbf{B}}.d\vec{\mathbf{S}} = 0$$

$$\vec{\nabla} \times \vec{\mathbf{B}} = \mu_0 \vec{\mathbf{J}} + \varepsilon_0 \frac{\partial \vec{\mathbf{E}}}{\partial t} \qquad \oint_C \vec{\mathbf{B}}.d\vec{\mathbf{l}} = \mu_0 (i + i_d) \ \text{where} \ i_d = \varepsilon_0 \frac{d\Phi_E}{dt}$$

In vacuum, i.e., charge-free and current-free regions,

$$\vec{\nabla}.\vec{\mathbf{E}} = 0 \qquad \oiint_S \vec{\mathbf{E}}.d\vec{\mathbf{S}} = 0$$

$$\vec{\nabla} \times \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t} \qquad \oint_C \vec{\mathbf{E}}.d\vec{\mathbf{l}} = -\frac{d\Phi_B}{dt}$$

$$\vec{\nabla}.\vec{\mathbf{B}} = 0 \qquad \oiint_S \vec{\mathbf{B}}.d\vec{\mathbf{S}} = 0$$

$$\vec{\nabla} \times \vec{\mathbf{B}} = \mu_0 \varepsilon_0 \frac{\partial \vec{\mathbf{E}}}{\partial t} \qquad \oint_C \vec{\mathbf{B}}.d\vec{\mathbf{l}} = \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}$$

Maxwell's equations indicate that a **time-varying magnetic field cannot exist without a variable electric field, and a time-varying electric field, without a variable magnetic field**. So, the two fields are not to be regarded as separate.

Thus arose the concept of the **electromagnetic field as a single entity**. Maxwell's equations are a mathematical formulation of a unified theory of electricity and magnetism. Maxwell's equations predicted the existence of electromagnetic waves and provided the basis for an understanding of the nature of electromagnetic waves.

Electromagnetic wave equation

- Maxwell's equations predict the existence of electromagnetic waves in vacuum. The wave equations for electric field and magnetic field are:

$$\nabla^2 \vec{\mathbf{E}} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{\mathbf{E}}}{\partial t^2} \quad \text{and} \quad \nabla^2 \vec{\mathbf{B}} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{\mathbf{B}}}{\partial t^2}$$

These two equations model **transverse electromagnetic waves**, as constituted of **time-varying, self-perpetuating, electric and magnetic fields**, which are **mutually perpendicular** and **perpendicular to the direction of propagation**. The speed of electromagnetic waves in vacuum is

$$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$$

Thus, electromagnetic waves are transverse waves that propagate in vacuum at the speed of light:

$c = 3.0\times10^8\,\text{ms}^{-1}$ establishing that **light is an electromagnetic wave**.

16.7 TERMINAL QUESTIONS

1. Prove that the displacement current in a parallel plate capacitor having capacitance C with a potential difference of V across its plates is given by $C(dV/dt)$. (Hint: Use the definition of the capacitance of a parallel plate capacitor that you have learnt in Unit 11 of Block 3.) The capacitance of a parallel plate capacitor is 5.0 nF. If a displacement current of 1.0 A is to be produced across the capacitor, what should be the rate of change of the potential difference applied across the plates?

2. Show that the electromagnetic field described by

$$\vec{\mathbf{E}} = E_0\,\hat{\mathbf{z}}\cos kx \cos ky \cos \omega t$$

and $$\vec{\mathbf{B}} = B_0(\hat{\mathbf{x}}\cos kx \sin ky - \hat{\mathbf{y}}\sin kx \cos ky)\sin \omega t$$

will satisfy Maxwell's equations in charge-free and current-free space if $E_0 = \sqrt{2}\,cB_0$ and $\omega = \sqrt{2}\,ck$.

Note that in TQs 2 and 4 and their answers, we have denoted the unit vectors along the x, y, z-axes by $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$, respectively instead of the more frequently used symbols $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$, respectively.

3. The magnitude of the maximum electric field associated with an electromagnetic wave travelling in vacuum is $600\,\text{Vm}^{-1}$. Determine the magnitude of the maximum magnetic field associated with the wave.

4. Consider the following electromagnetic waves travelling in the $\hat{\mathbf{y}}$ and $-\hat{\mathbf{y}}$ directions, respectively:

$$\vec{\mathbf{E}}_1 = \hat{\mathbf{z}}E_0 \sin\frac{2\pi}{\lambda}(y-ct), \quad \vec{\mathbf{B}}_1 = \hat{\mathbf{x}}B_0 \sin\frac{2\pi}{\lambda}(y-ct)$$

$$\vec{\mathbf{E}}_2 = \hat{\mathbf{z}}E_0 \sin\frac{2\pi}{\lambda}(y+ct), \quad \mathbf{B}_2 = -\hat{\mathbf{x}}B_0 \sin\frac{2\pi}{\lambda}(y+ct)$$

Show that if $E_0 = B_0 c$, the following resultant electric and magnetic fields of the two waves satisfy Maxwell's equations in vacuum:

$$\vec{\mathbf{E}} = \vec{\mathbf{E}}_1 + \vec{\mathbf{E}}_2, \quad \vec{\mathbf{B}} = \vec{\mathbf{B}}_1 + \vec{\mathbf{B}}_2$$

5. The electric field of an electromagnetic wave travelling in vacuum is:

$$E_x = 0, \; E_y = 0, \; E_z = (3.0\ \text{Vm}^{-1}) \sin(x - 10^8 t)$$

where t is in seconds and x, in metres. It is given that $c = 3.0 \times 10^8\ \text{ms}^{-1}$. What is the magnitude of the magnetic field associated with the wave?

16.8 SOLUTIONS AND ANSWERS

Self-Assessment Questions

1. a) The four fundamental laws of electricity and magnetism are the ones given in Table 16.1. You should be able to write them on your own. These laws are called fundamental laws because all other laws/equations can be deduced from them.

   b) The symmetries in these laws are that the left hand sides of the pairs of equations are symmetrical in the interchange of electric and magnetic fields.

   c) There are two kinds of asymmetries in the laws: One kind of asymmetry is in the Gauss's law for electric and magnetic fields in which the electric charges are sources of electric fields but magnetic fields have no sources; isolated magnetic charges do not exist. The second asymmetry is seen in Faraday's law and Ampere's law for steady currents: whereas changing magnetic fields produce electric fields, there is no term in Ampere's law representing the situation that changing electric fields produce magnetic fields.

2. a) Ampere's law is stated for only steady currents. From consideration of symmetry with Faraday's law, Ampere's law should contain a term representing the fact that changing electric fields produce magnetic fields. Maxwell modified Ampere's law by adding the term $\mu_0 \varepsilon_0 \dfrac{\partial \vec{\mathbf{E}}}{\partial t}$ to represent this fact. Adding this term led to the generalisation of Ampere's law to include both steady currents and time-varying electric fields.

   b) From Coulomb's law, the dimension of the constant ε_0 is $\dfrac{(\text{charge})^2}{(\text{length})^2 (\text{force})}$. From the definition of electric flux, the dimension of $(d\Phi_E / dt)$ is $\dfrac{(\text{force})(\text{length})^2}{(\text{charge})(\text{time})}$. Therefore, the dimension of $(\varepsilon_0\, d\Phi_E / dt)$ is:

$$\frac{(\text{charge})^2}{(\text{length})^2 (\text{force})} \times \frac{(\text{force})(\text{length})^2}{(\text{charge})(\text{time})} = \frac{\text{charge}}{\text{time}}$$

   This is the dimension of current.

3. From Eq. (iv) of Example 16.1, we can write the displacement current in a parallel plate capacitor as

$$|i_d| = \varepsilon_0 A \left|\frac{dE}{dt}\right|$$

Since $E = E_0 \sin\omega t$, $\dfrac{dE}{dt} = \omega E_0 \cos\omega t$ and $i_d = \varepsilon_0 A \omega E_0 \cos\omega t$. So, the maximum value of displacement current is

$$i_d = \varepsilon_0 E_0 A\omega = \varepsilon_0 E_0 A(2\pi f) \qquad (\because \omega = 2\pi f)$$

Substituting the numerical values, we get

$$i_d = (8.85\times10^{-12}\,\text{C}^2\text{N}^{-1}\text{m}^{-2})\times(10\,\text{V})\times(1.0\,\text{m}^2)\times 2\pi\times(10^7\,\text{Hz}) = 5.6\,\text{mA}$$

4. Substituting the expressions for $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ given in the problem in the differential form of Maxwell's equations (16.21 to 16.24), we get

i) $$\vec{\nabla}.\vec{\mathbf{E}} = \vec{\nabla}.[\hat{\mathbf{j}}\,E_0\sin(z-vt)] = (\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}).[\hat{\mathbf{j}}\,E_0\sin(z-vt)]$$

$$= \frac{\partial}{\partial y}[E_0\sin(z-vt)] = 0$$

So, Eq. (16.21) is an identity.

ii) $$\vec{\nabla}\times\vec{\mathbf{E}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0 & E_0\sin(z-vt) & 0 \end{vmatrix} = \hat{\mathbf{i}}\frac{\partial}{\partial z}[E_0\sin(z-vt)] = \hat{\mathbf{i}}\,E_0\cos(z-vt)$$

$$\frac{\partial\vec{\mathbf{B}}}{\partial t} = \hat{\mathbf{i}}\frac{\partial}{\partial t}[B_0\sin(z-vt)] = -\hat{\mathbf{i}}\,vB_0\cos(z-vt)]$$

So, Eq. (16.22) gives the condition that $E_0 = vB_0$

iii) $$\vec{\nabla}.\vec{\mathbf{B}} = (\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}).[\hat{\mathbf{i}}\,B_0\sin(z-vt)] = \frac{\partial}{\partial x}[B_0\sin(z-vt)] = 0$$

$$= \frac{\partial}{\partial y}[E_0\sin(z-vt)] = 0$$

So, Eq. (16.23) is also an identity.

iv) $\vec{\nabla}\times\vec{\mathbf{B}} = \dfrac{1}{c^2}\dfrac{\partial\vec{\mathbf{E}}}{\partial t}$ where $c^2 = \dfrac{1}{\mu_0\varepsilon_0}$ and

$$\vec{\nabla}\times\vec{\mathbf{B}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ B_0\sin(z-vt) & 0 & 0 \end{vmatrix}$$

$$= -\hat{\mathbf{j}}\frac{\partial}{\partial z}[B_0\sin(z-vt)] = -\hat{\mathbf{j}}\,B_0\cos(z-vt)$$

and $$\frac{\partial\vec{\mathbf{E}}}{\partial t} = \hat{\mathbf{j}}\frac{\partial}{\partial t}[E_0\sin(z-vt)] = -\hat{\mathbf{j}}\,vE_0\cos(z-vt)$$

So, Eq. (16.24) gives the condition that $B_0 = \dfrac{vE_0}{c^2}$

5. We take the curl of curl $\vec{\mathbf{B}}$ and get

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{\mathbf{B}}) = \vec{\nabla}\times\,\mu_0\,\varepsilon_0\left(\frac{\partial\vec{\mathbf{E}}}{\partial t}\right) = \mu_0\,\varepsilon_0\,\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{\mathbf{E}}) \qquad \text{(i)}$$

We now make use of the following vector identity for a vector field $\vec{\mathbf{F}}$:

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{\mathbf{F}}) = \vec{\nabla}(\vec{\nabla}.\vec{\mathbf{F}}) - \nabla^2\vec{\mathbf{F}}$$

Using this vector identity in Eq. (i), we can write it as

$$\vec{\nabla}(\vec{\nabla}.\vec{\mathbf{B}}) - \nabla^2\vec{\mathbf{B}} = \mu_0\,\varepsilon_0\,\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{\mathbf{E}}) \qquad \text{(ii)}$$

Then using $\vec{\nabla}\times\vec{\mathbf{E}} = -\dfrac{\partial\vec{\mathbf{B}}}{\partial t}$ from Eq. (16.22), we can write Eq. (ii) as

$$\vec{\nabla}(\vec{\nabla}.\vec{\mathbf{B}}) - \nabla^2\vec{\mathbf{B}} = -\mu_0\,\varepsilon_0\,\frac{\partial^2\vec{\mathbf{B}}}{\partial t^2} \qquad \text{(iii)}$$

Since $\vec{\nabla}.\,\vec{\mathbf{B}} = 0$ from Eq. (16.23), Eq. (iii) becomes

$$\nabla^2\vec{\mathbf{B}} = \mu_0\,\varepsilon_0\,\frac{\partial^2\vec{\mathbf{B}}}{\partial t^2}$$

6. We have obtained the conditions $E_0 = vB_0$ and $B_0 = \dfrac{vE_0}{c^2}$ for the electric and magnetic fields of SAQ 4.

Since both these conditions are satisfied at the same time, we substitute the expression $E_0 = vB_0$ in the second condition and get

$$B_0 = \frac{vE_0}{c^2} = \frac{v^2\,B_0}{c^2} \quad\Rightarrow\quad \frac{v^2}{c^2} = 1 \quad\Rightarrow\quad v = \pm c$$

If we substitute this result in the first condition, we get $|B_0|c = |E_0|$.

Terminal Questions

1. From the definition of the capacitance of a parallel plate capacitor given in Unit 11 of Block 3, we have

$$C = \frac{q}{V} \quad \text{or} \quad q = CV$$

where q is the charge on the capacitor plates and V, the potential difference across them. Now from Example 16.1, we know that

$$i_d = i = \frac{dq}{dt} \qquad \text{or} \qquad i_d = C\frac{dV}{dt}$$

Substituting the numerical values of the capacitance of the parallel plate capacitor and displacement current in the above expression, we get

$$\frac{dV}{dt} = \frac{i_d}{C} = \frac{1.0\,\text{A}}{5.0\,\text{nF}} = 2.0\times10^8\ \text{Vs}^{-1}$$

2. Substituting the expressions for $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ given in the problem in Maxwell's equations (16.21 to 16.24), we get

i) $\vec{\nabla}.\vec{\mathbf{E}} = \vec{\nabla}.(E_0\,\hat{\mathbf{z}}\cos kx\cos ky\cos\omega t) = E_0\dfrac{\partial}{\partial z}\cos kx\cos ky\cos\omega t = 0$

ii) $$\vec{\nabla}\times\vec{\mathbf{E}} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0 & 0 & E_0\cos kx\cos ky\cos\omega t \end{vmatrix}$$

$$= E_0\left(\hat{\mathbf{x}}\frac{\partial}{\partial y}\cos kx\cos ky\cos\omega t - \hat{\mathbf{y}}\frac{\partial}{\partial x}\cos kx\cos ky\cos\omega t\right)$$

$$= E_0(-\hat{\mathbf{x}}\,k\cos kx\sin ky\cos\omega t) + \hat{\mathbf{y}}\,k\sin kx\cos ky\cos\omega t)$$

$$\frac{\partial\vec{\mathbf{B}}}{\partial t} = \frac{\partial}{\partial t}[B_0(\hat{\mathbf{x}}\cos kx\sin ky - \hat{\mathbf{y}}\sin kx\cos ky)\sin\omega t]$$

$$= \omega B_0(\hat{\mathbf{x}}\cos kx\sin ky - \hat{\mathbf{y}}\sin kx\cos ky)\cos\omega t$$

From Eq. (16.22), we have

$$-kE_0(\hat{\mathbf{x}}\cos kx\sin ky - \hat{\mathbf{y}}\sin kx\cos ky)\cos\omega t$$

$$= -\omega B_0(\hat{\mathbf{x}}\cos kx\sin ky - \hat{\mathbf{y}}\sin kx\cos ky)\cos\omega t$$

from where we get the condition that

$$kE_0 = \omega B_0 \quad \text{or} \quad \frac{E_0}{B_0} = \frac{\omega}{k} \qquad \text{(i)}$$

iii) $\vec{\nabla}.\vec{\mathbf{B}} = \vec{\nabla}.[B_0(\hat{\mathbf{x}}\cos kx\sin ky - \hat{\mathbf{y}}\sin kx\cos ky)\sin\omega t]$

$$= B_0\left(\frac{\partial}{\partial x}\cos kx\sin ky - \frac{\partial}{\partial y}\sin kx\cos ky\right)\sin\omega t$$

$$= B_0(-k\sin kx\sin ky + k\sin kx\sin ky)\sin\omega t = 0$$

iv) $$\vec{\nabla}\times\vec{\mathbf{B}} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ B_0\cos kx\sin ky\sin\omega t & -B_0\sin kx\cos ky\sin\omega t & 0 \end{vmatrix}$$

or $$B_0\,\hat{\mathbf{z}}\left(-\frac{\partial}{\partial x}\sin kx\cos ky - \frac{\partial}{\partial y}\cos kx\sin ky\right)\sin\omega t$$

$$= \frac{E_0}{c^2}\hat{\mathbf{z}}\frac{\partial}{\partial \mathrm{t}}(\cos kx\cos ky\cos\omega t)$$

or $$-B_0\,\hat{\mathbf{z}}\,k(\cos kx\cos ky + \cos kx\cos ky)\sin\omega t$$

$$= -\frac{E_0\,\omega}{c^2}\hat{\mathbf{z}}(\cos kx\cos ky\sin\omega t)$$

or $$2kB_0 = \frac{E_0\,\omega}{c^2}$$

or $$B_0 = \frac{E_0}{2c^2}\frac{\omega}{k} = \frac{B_0}{2c^2}\frac{\omega^2}{k^2} \qquad \left(\because \frac{E_0}{B_0} = \frac{\omega}{k}\right)$$

So, Eq. (16.24) gives $\dfrac{\omega^2}{k^2} = 2c^2$

or $\dfrac{\omega}{k} = \sqrt{2}\,c \quad \Rightarrow \quad \omega = \sqrt{2}\,ck \quad$ and $\quad E_0 = \dfrac{\omega}{k} B_0 = \sqrt{2}\,c B_0$

3. Since the electromagnetic wave is travelling in vacuum, the maximum electric and magnetic fields associated with the wave satisfy the relation $E_0 = cB_0$, where c is the speed of light. Therefore, the magnitude of the maximum magnetic field is

$$B_0 = \frac{E_0}{c} = \frac{600\,\text{Vm}^{-1}}{3.0 \times 10^8\,\text{ms}^{-1}} = 2.0 \times 10^{-6}\,\text{T}$$

4. Let us first write the expressions of the resultant electric and magnetic fields:

$$\vec{\mathbf{E}} = \vec{\mathbf{E}}_1 + \vec{\mathbf{E}}_2, \quad \vec{\mathbf{B}} = \vec{\mathbf{B}}_1 + \vec{\mathbf{B}}_2$$

$$\vec{\mathbf{E}} = \hat{\mathbf{z}} E_0 \sin\frac{2\pi}{\lambda}(y - ct) + \hat{\mathbf{z}} E_0 \sin\frac{2\pi}{\lambda}(y + ct)$$

$$= \hat{\mathbf{z}} E_0 \left[\sin\frac{2\pi}{\lambda}(y - ct) + \sin\frac{2\pi}{\lambda}(y + ct)\right]$$

$$= 2\hat{\mathbf{z}} E_0 \sin\frac{2\pi y}{\lambda}\cos\frac{2\pi ct}{\lambda} \quad \left[\because \sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}\right]$$

$$\vec{\mathbf{B}} = \hat{\mathbf{x}} B_0 \sin\frac{2\pi}{\lambda}(y - ct) - \hat{\mathbf{x}} B_0 \sin\frac{2\pi}{\lambda}(y + ct) = -2\hat{\mathbf{x}} B_0 \cos\frac{2\pi y}{\lambda}\sin\frac{2\pi ct}{\lambda}$$

$$\left[\because \sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}\right]$$

We have to show that the fields $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ satisfy Maxwell's equations in vacuum:

i) $\vec{\nabla}.\vec{\mathbf{E}} = 2E_0 \dfrac{\partial}{\partial z}\left(\sin\dfrac{2\pi y}{\lambda}\cos\dfrac{2\pi ct}{\lambda}\right) = 0$

ii) $\vec{\nabla}\times\vec{\mathbf{E}} = \hat{\mathbf{x}}\, 2E_0 \left(\dfrac{\partial}{\partial y}\sin\dfrac{2\pi y}{\lambda}\cos\dfrac{2\pi ct}{\lambda}\right) = \hat{\mathbf{x}}\, 2E_0 \dfrac{2\pi}{\lambda}\cos\dfrac{2\pi y}{\lambda}\cos\dfrac{2\pi ct}{\lambda}$

$$\frac{\partial\vec{\mathbf{B}}}{\partial t} = -2\hat{\mathbf{x}} B_0 \left(\frac{\partial}{\partial t}\cos\frac{2\pi y}{\lambda}\sin\frac{2\pi ct}{\lambda}\right) = -2\hat{\mathbf{x}} B_0 \frac{2\pi c}{\lambda}\cos\frac{2\pi y}{\lambda}\cos\frac{2\pi ct}{\lambda}$$

From Eq. (16.22), we have

$$\hat{\mathbf{x}}\, 2E_0 \frac{2\pi}{\lambda}\cos\frac{2\pi y}{\lambda}\cos\frac{2\pi ct}{\lambda} = \hat{\mathbf{x}}\, 2B_0 \frac{2\pi c}{\lambda}\cos\frac{2\pi y}{\lambda}\cos\frac{2\pi ct}{\lambda}$$

or $\quad E_0 = B_0 c \qquad$ (i)

iii) $\vec{\nabla}.\vec{\mathbf{B}} = -2B_0 \dfrac{\partial}{\partial x}\left(\cos\dfrac{2\pi y}{\lambda}\sin\dfrac{2\pi ct}{\lambda}\right) = 0$

iv) $$\vec{\nabla}\times\vec{\mathbf{B}} = 2B_0\,\hat{\mathbf{z}}\left(\frac{\partial}{\partial y}\cos\frac{2\pi y}{\lambda}\,\sin\frac{2\pi ct}{\lambda}\right) = -\hat{\mathbf{z}}\,2B_0\,\frac{2\pi}{\lambda}\sin\frac{2\pi y}{\lambda}\,\sin\frac{2\pi ct}{\lambda}$$

and $$\frac{\partial\vec{\mathbf{E}}}{\partial t} = -2\,\hat{\mathbf{z}}\,E_0\,\frac{2\pi c}{\lambda}\sin\frac{2\pi y}{\lambda}\,\sin\frac{2\pi ct}{\lambda}$$

From Eq. (16.24), we have

$$-\hat{\mathbf{z}}\,2B_0\,\frac{2\pi}{\lambda}\sin\frac{2\pi y}{\lambda}\,\sin\frac{2\pi ct}{\lambda} = -\frac{1}{c^2}\,2\,\hat{\mathbf{z}}\,E_0\,\frac{2\pi c}{\lambda}\sin\frac{2\pi y}{\lambda}\,\sin\frac{2\pi ct}{\lambda}$$

or $E_0 = B_0 c$

So, the fields $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ satisfy Maxwell's equations in vacuum if $E_0 = B_0 c$.

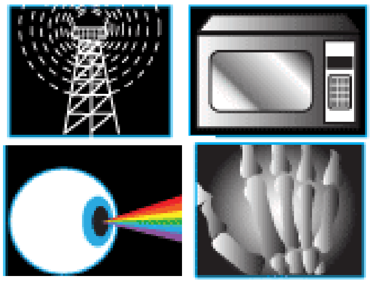
5. We can write the electric field in vector form as

$$\vec{\mathbf{E}} = \hat{\mathbf{z}}\,(3.0\ \text{Vm}^{-1})\sin(x - 10^8\,t)$$

The magnitude of the magnetic field is obtained from Eq. (16.31a): $E_0 = B_0 c$, where $E_0 = 3.0\ \text{Vm}^{-1}$. So,

$$B_0 = \frac{E_0}{c} = \frac{3.0\,\text{Vm}^{-1}}{3.0\times 10^8\,\text{m s}^{-1}} = 1.0\times 10^{-8}\ \text{T}$$

and $$|\vec{\mathbf{B}}| = (1.0\times 10^{-8}\ \text{T})\sin(x - 10^8\,t)$$



Electromagnetic waves of different frequencies have numerous applications in our lives, some of which are shown here!
(Source of pictures: Wikipedia/commons)

UNIT 17

ELECTROMAGNETIC WAVE PROPAGATION

Structure

STUDY GUIDE

This last unit on propagation of electromagnetic waves is a culmination of what you have studied so far in the course. While studying Unit 16, you would have realised that you were using the concepts discussed in all previous units of this course. You also used the wave equation and the fundamental quantities that describe a wave about which you studied in school physics. You may like to revisit Unit 19 of the course BPHCT-131entitled Mechanics to revise the wave equation and its solutions. Knowledge of all these concepts is also required to study the propagation of electromagnetic waves and the energy carried by the waves that we discuss in this unit. To have a sound understanding of these concepts, you should always work out all steps in the text and the Examples given in the unit. You should also make sure you can solve the SAQs and Terminal Questions on your own.

"All the mathematical sciences are founded on relations between physical laws and laws of numbers."

James C. Maxwell

17.1 INTRODUCTION

In Unit 16, you have learnt that Maxwell's equations govern the behaviour of electric and magnetic fields everywhere and describe all natural electromagnetic phenomena. These equations also form the basis for the operation of a large number of practical devices in use today. You have also learnt about the fundamental insight into the nature of light and other electromagnetic waves provided by Maxwell's equations.

You will now study the propagation of plane electromagnetic waves in vacuum (Sec. 17.2) and in dielectric media (Sec. 17.3). In Sec. 17.3, we will first write Maxwell's equations in dielectric media. Then we will deduce the electromagnetic wave equation in an isotropic dielectric medium before discussing electromagnetic wave propagation in such media. In Sec. 17.4, we extend the idea of the energy carried by waves to electromagnetic waves. We also explain the Poynting theorem that gives us the energy transported per unit time by electromagnetic waves across a surface. Finally, we determine the energy density of an electromagnetic field.

Expected Learning Outcomes

After studying this unit, you should be able to:

- describe mathematically the propagation of plane electromagnetic waves in vacuum and calculate the wave parameters for plane electromagnetic waves;
- deduce Maxwell's equations and the electromagnetic wave equations in dielectric media (with plane wave solutions); and
- state Poynting's theorem, compute the Poynting vector and the energy density of electromagnetic fields.

17.2 ELECTROMAGNETIC WAVE PROPAGATION IN VACUUM

In this section, we discuss the propagation of electromagnetic waves and shall be mainly concerned with **plane** electromagnetic waves. Recall from Unit 16 the concept of a plane electromagnetic wave. You should be able to explain: **What does the term 'plane' electromagnetic wave mean**?

Recall that the term 'plane' is meant to indicate that field vectors $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ at each point in space lie in a plane. Also the planes at any two different points are parallel to each other (revisit Fig. 16.4).

Plane electromagnetic waves are found to be very useful in various areas of physics, engineering and technology. As you have seen, the electromagnetic field given in SAQ 4 of Unit 16 is a specific example of a plane electromagnetic wave. Our interest now is to find the general plane wave solutions of the wave equations in vacuum (charge-free and current-free regions of space), which also satisfy Maxwell's equations. You have learnt about the general solutions of the wave equation in school physics.

You can verify that the sinusoidal scalar functions of the form

$$u(z,t) = A\cos(z - vt) \quad \text{or} \quad u(z,t) = A\sin(z - vt) \qquad (17.1a)$$

are general solutions of the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} \qquad (17.1b)$$

for a wave travelling in the **positive** *z*-direction. Here *A* is the amplitude of the wave and *v*, its velocity. These solutions can also be expressed in terms of the angular frequency ω of the wave and its wave number *k* as follows:

$$u(z,t) = A\cos(kz - \omega t) \quad \text{or} \quad u(z,t) = A\sin(kz - \omega t) \qquad (17.1c)$$

You know that we can represent waves either by the sine function or the cosine function or their linear combination as these are linearly independent solutions. For a wave travelling in the **negative** *z*-direction, the solutions would be:

$$u(z,t) = A\cos(z + vt) \quad \text{or} \quad u(z,t) = A\sin(z + vt) \qquad (17.1d)$$

$$\text{or} \quad u(z,t) = A\cos(kz + \omega t) \quad \text{or} \quad u(z,t) = A\sin(kz + \omega t) \qquad (17.1e)$$

You have learnt a slightly different form of the expressions for the arguments of sine and cosine functions in Unit 19 of BPHCT-131 from the ones given in Eqs. (17.1a, c, d and e). Do not worry. All these expressions represent solutions of the wave equation. Also recall from school physics and Unit 19 of BPHCT-131 that

$\omega = 2\pi f,\ f = \frac{1}{T},$

$k = \frac{2\pi}{\lambda}$ and

$v = f\lambda$

In a similar way, we can write the plane wave solutions of the electromagnetic wave equations for the electric field and the magnetic field as:

$$\vec{\mathbf{E}}(z,t) = \vec{\mathbf{E}}_0 \cos(kz - \omega t) \qquad (17.2a)$$

$$\vec{\mathbf{B}}(z,t) = \vec{\mathbf{B}}_0 \cos(kz - \omega t) \qquad (17.2b)$$

Note that in Eqs. (17.2a and b), the vectors $\vec{\mathbf{E}}_0$ and $\vec{\mathbf{B}}_0$ have constant magnitudes. In this unit, we will take the magnitudes of the electric and magnetic fields to be constant.

Recall from Unit 16 that for these solutions to represent electromagnetic waves propagating in vacuum, they should satisfy Maxwell's equations for charge-free and current-free regions. Let us check if indeed it is so. From what you have learnt in Example 16.2 of Unit 16, you can show that the Maxwell's equations (16.21 and 16.23)

$$\vec{\nabla}\cdot\vec{\mathbf{E}} = 0 \quad \text{and} \quad \vec{\nabla}\cdot\vec{\mathbf{B}} = 0 \qquad (17.3a)$$

are satisfied by equations (17.2a and b) if and only if

$$E_{0z} = \left(\vec{\mathbf{E}}_0\right)_z = 0 \quad \text{and} \quad B_{0z} = \left(\vec{\mathbf{B}}_0\right)_z = 0 \qquad (17.3b)$$

You should satisfy yourself that this is true before studying further. Try SAQ 1.

SAQ 1 – Solution of Maxwell's equations

Verify the condition given by Eq. (17.3b) under which Eqs. (17.2a and b) satisfy Maxwell equations (17.3a).

Next, we verify if Eqs. (17.2a and b) satisfy the remaining Maxwell's equations [Eqs. (16.22 and 16.24)] in vacuum. To do so, let us substitute them first in the Maxwell's equation (16.22):

$$\vec{\nabla} \times \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t} \tag{17.4}$$

Let us first see if the solution for the electric field given by Eq. (17.2a) satisfies Eq. (17.4). For this, we use the definition of the curl of a vector field that you have learnt in Unit 2.

Note that in this unit, we have denoted the unit vectors along the x, y, z-axes by $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$, respectively instead of the more frequently used symbols $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$, respectively so as not to confuse with the notation of the wave number k and the corresponding wave vector $\hat{\mathbf{k}}$.

Then putting $E_z = 0$ from Eq. (17.3b), we can write the left-hand side of Eq. (17.4) as follows:

$$\vec{\nabla} \times \vec{\mathbf{E}} = \begin{bmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & 0 \end{bmatrix} \tag{17.5a}$$

Now you can see from Eq. (17.2a) that

$$E_x = E_{0x} \cos(kz - \omega t) \quad \text{and} \quad E_y = E_{0y} \cos(kz - \omega t) \tag{17.5b}$$

Using Eq. (17.5b) in Eq. (17.5a), you can verify that

$$\vec{\nabla} \times \vec{\mathbf{E}} = \hat{\mathbf{x}}\,[-\frac{\partial}{\partial z} E_{0y} \cos(kz - \omega t)] + \hat{\mathbf{y}}\,[\frac{\partial}{\partial z} E_{0x} \cos(kz - \omega t)] \tag{17.6a}$$

$$\text{or } \vec{\nabla} \times \vec{\mathbf{E}} = \hat{\mathbf{x}}\,[k E_{0y} \sin(kz - \omega t)] + \hat{\mathbf{y}}\,[-k E_{0x} \sin(kz - \omega t)] \tag{17.6b}$$

Using Eq. (17.2b) with $B_z = 0$, the right-hand side of Eq. (17.4) becomes

$$-\frac{\partial \vec{\mathbf{B}}}{\partial t} = -\frac{\partial}{\partial t}\left[\hat{\mathbf{x}} B_{0x} \cos(kz - \omega t) + \hat{\mathbf{y}} B_{0y} \cos(kz - \omega t)\right] \tag{17.7a}$$

$$\text{or} \quad -\frac{\partial \vec{\mathbf{B}}}{\partial t} = -\omega\, \hat{\mathbf{x}} B_{0x} \sin(kz - \omega t) - \omega \hat{\mathbf{y}} B_{0y} \sin(kz - \omega t) \tag{17.7b}$$

Comparing Eqs. (17.6b and 17.7b), we get the following results:

$$k E_{0y} = -\omega B_{0x} \tag{17.8a}$$

$$\text{and} \qquad k E_{0x} = \omega B_{0y} \tag{17.8b}$$

You can verify that Eqs. (17.8a and b) can be expressed as

$$\vec{\mathbf{B}}_0 = \frac{k}{\omega} (\hat{\mathbf{z}} \times \vec{\mathbf{E}}_0) \tag{17.9a}$$

We can generalise Eq. (17.9a) to any arbitrary direction of propagation represented by the wave vector $\vec{\mathbf{k}}$. Here we state, without proof, the generalised Eq. (17.9a) for an electromagnetic wave propagating in vacuum in an arbitrary direction represented by the wave vector $\vec{\mathbf{k}}$:

$$\vec{\mathbf{B}} = \frac{k}{\omega} (\hat{\mathbf{k}} \times \vec{\mathbf{E}}) \quad \text{or} \quad \hat{\mathbf{k}} \times \vec{\mathbf{E}} = c\vec{\mathbf{B}} \quad (\because\ c = \frac{\omega}{k}) \tag{17.9b}$$

where $\hat{\mathbf{k}}$ is the unit vector along the direction of propagation.

You should verify Eq. (17.9a) before studying further.

SAQ 2 - Solution of Maxwell's equations

Verify that Eq. (17.9a) and Eqs. (17.8a and b) are equivalent.

What do Eqs. (17.9a and b) tell us? These equations express the $\vec{\mathbf{B}}$ field as a vector product of the unit vector in the direction of propagation (*z*-axis or, in general, $\hat{\mathbf{k}}$) and the $\vec{\mathbf{E}}$ field. This means that the magnetic field vector is perpendicular to both the electric field vector and the unit vector along the direction of propagation of the electromagnetic wave. Eqs. (17.9a and b) give us the rule for determining the directions of the electric field, magnetic field and the direction of wave propagation of an electromagnetic wave. So, we find that the requirement that the $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ fields of Eqs. (17.2a and b) should satisfy Maxwell's equations in vacuum leads us to the following conclusion:

The $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ fields of Eqs. (17.2a and b) are perpendicular to each other (or mutually perpendicular) and also perpendicular to the direction of propagation. These satisfy the wave equation and represent the electric and magnetic fields associated with an electromagnetic wave.

The $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ fields of Eqs. (17.2a and b) represent **plane wave solutions** of the electromagnetic wave equation as explained in Unit 16. Such waves are also called **monochromatic sinusoidal electromagnetic plane waves**. Monochromatic means single colour. Since frequency corresponds to colour, especially for light, sinusoidal **waves of a single frequency** are called **monochromatic**.

Note that the $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ fields of Eqs. (17.2a and b) are in phase. Since the angle between $\hat{\mathbf{z}}$ and $\vec{\mathbf{E}}_0$ is 90°, the definition of vector product in Eq. (17.9a) yields the following relation between their amplitudes:

$$B_0 = \frac{k}{\omega} E_0 = \frac{E_0}{c} \quad \text{or} \quad \frac{|\vec{\mathbf{E}}|}{|\vec{\mathbf{B}}|} = c \qquad (17.10)$$

The fourth Maxwell equation gives us the same result. You may like to verify this yourself. Solve SAQ 3. It will also give you some practice with the mathematical steps you have just studied.

SAQ 3 - Solution of Maxwell's equations

Show that the $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ fields of Eqs. (17.2a and b) satisfy the Maxwell equation

$$\vec{\nabla} \times \vec{\mathbf{B}} = \mu_0 \varepsilon_0 \frac{\partial \vec{\mathbf{E}}}{\partial t} \quad \text{if} \quad \hat{\mathbf{k}} \times \vec{\mathbf{B}}_0 = -\frac{\vec{\mathbf{E}}_0}{c}$$

So far you have learnt that the plane wave solutions [Eqs. (17.2a and b)] of the wave equations for the electric and magnetic fields, satisfy Maxwell's equations in vacuum (charge-free and current-free regions). Eqs. (17.9a, b) tell us that the $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ fields of an electromagnetic wave are perpendicular to each other and to the direction of propagation of the electromagnetic wave given by $\hat{\mathbf{k}}$, in general. An electromagnetic wave is, thus, a transverse wave as you have also learnt in Unit 16 (see Fig. 17.1).

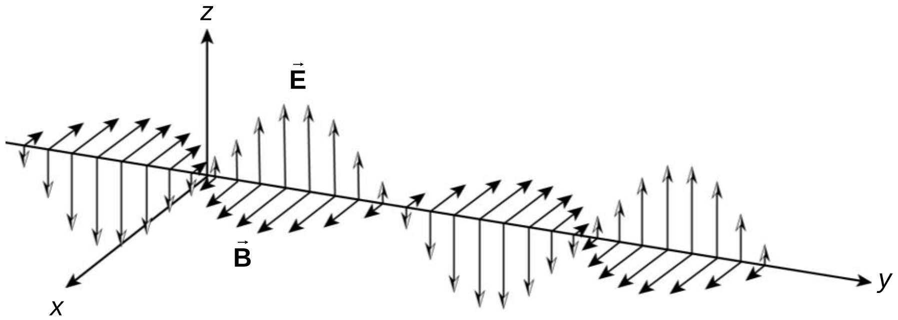


Fig. 17.1: **Electromagnetic wave is a transverse wave. The electromagnetic wave shown in this figure is propagating in the *y*-direction. The $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ fields are in the *z* and *x*-directions, respectively.**

In general, $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ can have any functional dependence on the space coordinates *x*, *y*, *z* and the time coordinate *t*. However, that discussion is beyond the scope of this course. Let us sum up what you have learnt about the propagation of a plane electromagnetic wave in vacuum.

Recap

Propagation of a Plane Electromagnetic Wave in Vacuum

1. At every point in an electromagnetic wave travelling in vacuum at any instant of time, the electric and magnetic fields are related by Eq. (17.9b). Their magnitudes are related by Eq. (17.10): $\frac{|\vec{\mathbf{E}}|}{|\vec{\mathbf{B}}|} = c$

2. The electric field and the magnetic field in the electromagnetic wave are perpendicular to one another and also to the direction of wave propagation, i.e., the wave is transverse.

3. The electromagnetic wave travels with speed *c* in vacuum.

As you know from school physics, radio waves, microwaves, infrared, light waves, ultraviolet rays, *X*-rays and gamma rays are all electromagnetic waves. These constitute what is called the **electromagnetic spectrum**. What distinguishes each of these waves is their frequency or wavelength (see Fig. 17.2).

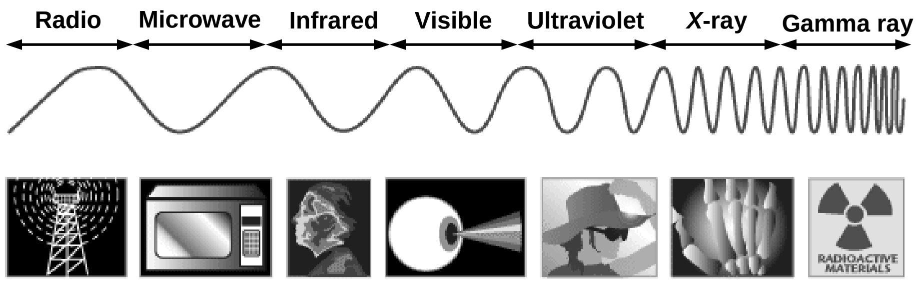


Fig. 17.2: **The electromagnetic spectrum.**

You know that the wavelength of the wave is given by

$$\lambda = \frac{2\pi}{k} \qquad (17.11a)$$

and its frequency is given by

$$f = \frac{c}{\lambda} \qquad (17.11b)$$

As you know, k is the **wave number** of the wave (as there are $k/2\pi$ wavelengths per unit distance), and $\vec{\mathbf{k}}$ is the **wave vector**.

You may now like to pause and recapitulate what you have studied so far in this section about electromagnetic waves as predicted by Maxwell's equations, and their propagation.

Recap

Maxwell's Equations and Electromagnetic Waves

1. Maxwell's equations predict the existence of electromagnetic waves in vacuum, i.e., charge-free and current-free region. Maxwell's equations model a travelling electromagnetic wave as being constituted of time-varying electric and magnetic fields.

2. The **electromagnetic waves are self-perpetuating, which travel with the speed of light** c **in vacuum (charge-free and current-free regions)**.

3. The electric field and the magnetic field in an electromagnetic wave are perpendicular to one another and also to the direction of wave propagation. Thus, electromagnetic waves are transverse waves. Even amongst plane electromagnetic waves, sinusoidal waves of single frequency are used very frequently in physics.

4. The $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ fields in an electromagnetic wave must satisfy the respective wave equations as well as Maxwell's equations.

5. Plane electromagnetic waves are of special interest in physics. Such waves have the property that their $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ field vectors at each point in space lie in a plane with the planes at two different points being parallel to each other.

We will now illustrate these ideas with the help of an example.

EXAMPLE 17.1: PLANE ELECTROMAGNETIC WAVES

Consider an electromagnetic wave in vacuum whose electric field is given by

$$\vec{\mathbf{E}} = (60\ \text{Vm}^{-1})\,\hat{\mathbf{x}}\cos(10^8\,t + kz)$$

Determine the direction of propagation, the wave number, the frequency and the magnetic field of the wave.

SOLUTION ■ We will make use of Eqs. (17.1d, 17.2a, 17.9b, 17.11b and 17.10) to solve this problem.

Let us first ask: What information can we obtain from the expression of the given electric field? Comparing the expression with Eq. (17.2a), you can immediately see that for this wave

$$\omega = 10^8\,\text{s}^{-1} \qquad \text{and} \qquad E_0 = 60\,\text{V}\,\text{m}^{-1}$$

To determine the direction of propagation of the wave, we compare the argument of the cosine function in the expression of the electric field of the wave with Eq. (17.1d). You can see that the direction of propagation is in the negative z-direction. We obtain the wave number k from the relation

$$c = \frac{\omega}{k} \quad \Rightarrow \quad 3\times 10^8\,\text{m}\,\text{s}^{-1} = \frac{10^8\,\text{s}^{-1}}{k} \quad \text{or} \quad k = \frac{1}{3}\text{m}^{-1}$$

From Eq. (17.11b), the frequency of the wave $f = \frac{10^8}{2\pi}\text{Hz} = 1.67\times 10^7\,\text{Hz}$

The magnetic field is of a form similar to the $\vec{\mathbf{E}}$ field:

$$\vec{\mathbf{B}} = \vec{\mathbf{B}}_0 \cos(10^8 t + kz)$$

To evaluate $\vec{\mathbf{B}}$, we use Eqs. (17.9b and 17.10). So, the direction of the magnetic field is

$$\hat{\mathbf{B}} = \hat{\mathbf{k}} \times \hat{\mathbf{E}} = -\hat{\mathbf{z}}\times\hat{\mathbf{x}} = -\hat{\mathbf{y}}$$

From Eq. (17.10), the magnitude $\vec{\mathbf{B}}_0$ of the magnetic field of the wave is $\left|\vec{\mathbf{B}}_0\right| = \left|\vec{\mathbf{E}}_0\right|/c = (60/c)\,\text{T}$. Thus, the magnetic field associated with the electromagnetic wave is

$$\vec{\mathbf{B}} = -\left(\frac{60}{c}\,\text{T}\right)\hat{\mathbf{y}}\cos(10^8 t + \frac{1}{3}z)$$

You must now work out an SAQ to ensure that you have grasped the ideas of this subsection.

SAQ 4 - Electromagnetic waves

The electric field given by $\vec{\mathbf{E}} = (1000\ \text{Vm}^{-1})\,\hat{\mathbf{y}}\left[\sin\left(\frac{\pi x}{25} - \omega t\right)\right]$ represents the $\vec{\mathbf{E}}$ field of a plane electromagnetic wave in a charge-free and current-free region (or vacuum). Calculate the wavelength and frequency of the wave, its direction of propagation and the associated magnetic field.

So far in Unit 16 and in this unit, you have learnt about Maxwell's equations and the propagation of plane electromagnetic waves in vacuum. We now turn our attention to Maxwell's equations and propagation of electromagnetic waves in dielectric media. This study is important as it helps us understand many phenomena, e.g., propagation of light in glass and water, propagation of X-rays and gamma rays in human body, and so on.

So, in the next section, we first deduce Maxwell's equations in a dielectric medium and then derive the wave equation for it.

17.3 ELECTROMAGNETIC WAVE PROPAGATION IN DIELECTRIC MEDIA

We will first deduce Maxwell's equations in a dielectric medium. To keep the discussion simple, we consider only linear, isotropic, homogeneous dielectric media about which you have studied in Unit 10 of Block 3 of this course.

17.3.1 Maxwell's Equations in Dielectric Media

Recall from the discussion in Unit 10 that dielectric materials are subject to electric polarisation and magnetisation. In such materials there is an accumulation of 'bound' charge and current. You have learnt that in a dielectric, for the static case, an electric polarisation $\vec{\mathbf{P}}$ results in an accumulation of bound charge given by

From now on in this unit, wherever we mention the phrase 'dielectric media', you should take it that we are considering only **linear**, **isotropic** and **homogeneous** dielectric media.

$$\rho_P = -\vec{\nabla}.\vec{\mathbf{P}} \tag{17.12a}$$

You have learnt that, by definition, $\vec{\mathbf{P}}$ is the electric dipole moment per unit volume. For the non-static case, any change in the electric polarisation involves a flow of charge, which yields a polarisation current. The corresponding polarisation current density $\vec{\mathbf{J}}_P$, which arises because ρ_P changes with time, is given by $\frac{\partial \vec{\mathbf{P}}}{\partial t}$. Therefore,

$$\vec{\mathbf{J}}_P = \frac{\partial \vec{\mathbf{P}}}{\partial t} \tag{17.12b}$$

Similarly, you have learnt in Unit 14 that the magnetisation $\vec{\mathbf{M}}$ in any material results in a bound current

$$\vec{\mathbf{J}}_M = \vec{\nabla} \times \vec{\mathbf{M}} \tag{17.12c}$$

As you know, magnetisation is just the magnetic dipole moment per unit volume. You must note, however, that the polarisation current $\vec{\mathbf{J}}_P$ has nothing to do with the bound current $\vec{\mathbf{J}}_M$. The bound current is associated with the magnetisation of the material (including dielectric materials) and involves the spin and orbital motion of electrons. By contrast, $\vec{\mathbf{J}}_P$ is the result of linear motion of charge when the electric polarisation changes. Now suppose $\vec{\mathbf{P}}$ points to the right and is increasing. Then each positive charge moves to the right and each negative charge to the left, resulting in the polarisation current. Keeping this discussion in mind, we can write the total charge density as

$$\rho = \rho_f + \rho_P = \rho_f - \vec{\nabla}.\vec{\mathbf{P}} \tag{17.13a}$$

where ρ_f is the free charge density. The total current density is

$$\vec{\mathbf{J}} = \vec{\mathbf{J}}_f + \vec{\mathbf{J}}_P + \vec{\mathbf{J}}_M = \vec{\mathbf{J}}_f + \frac{\partial \vec{\mathbf{P}}}{\partial t} + \vec{\nabla} \times \vec{\mathbf{M}} \tag{17.13b}$$

where $\vec{\mathbf{J}}_f$ is the free current density.

We substitute these expressions for ρ and $\vec{\mathbf{J}}$ from Eqs. (17.13a and b) in Maxwell's equations [(Eqs. 16.17 to 16.20)] and write Maxwell's equations for isotropic dielectric media in differential form as follows:

$$\vec{\nabla}.\vec{\mathbf{E}} = \frac{1}{\varepsilon_0}\left(\rho_f - \vec{\nabla}.\vec{\mathbf{P}}\right) \quad (17.14)$$

$$\vec{\nabla}\times\vec{\mathbf{E}} = -\frac{\partial\vec{\mathbf{B}}}{\partial t} \quad (17.15)$$

$$\vec{\nabla}.\vec{\mathbf{B}} = 0 \quad (17.16)$$

$$\vec{\nabla}\times\vec{\mathbf{B}} = \mu_0\left(\varepsilon_0\frac{\partial\vec{\mathbf{E}}}{\partial t} + \vec{\mathbf{J}}_f + \frac{\partial\vec{\mathbf{P}}}{\partial t} + \vec{\nabla}\times\vec{\mathbf{M}}\right) \quad (17.17)$$

From Eq. (17.14), we can write

$\varepsilon_0\vec{\nabla}.\vec{\mathbf{E}} = \rho_f - \vec{\nabla}.\vec{\mathbf{P}}$

or $\vec{\nabla}.\varepsilon_0\vec{\mathbf{E}} + \vec{\nabla}.\vec{\mathbf{P}} = \rho_f$

or $\vec{\nabla}.(\varepsilon_0\vec{\mathbf{E}} + \vec{\mathbf{P}}) = \rho_f$

Using Eq. (17.18), we get Eq. (17.20):

$\vec{\nabla}.\vec{\mathbf{D}} = \rho_f$

Let us recast Eqs. (17.14 to 17.17) in a form similar to Eqs. (16.17 to 16.20). For this we introduce the fields $\vec{\mathbf{D}}$ and $\vec{\mathbf{H}}$ about which you have studied in Units 10 and 14 of Block 3:

$$\vec{\mathbf{D}} = \varepsilon_0\vec{\mathbf{E}} + \vec{\mathbf{P}} \quad (17.18)$$

$$\vec{\mathbf{H}} = \frac{1}{\mu_0}\vec{\mathbf{B}} - \vec{\mathbf{M}} \quad (17.19)$$

Then we can rewrite Eqs. (17.14 and 17.17) as follows (read the margin remarks):

$$\vec{\nabla}.\vec{\mathbf{D}} = \rho_f \quad (17.20)$$

$$\vec{\nabla}\times\vec{\mathbf{H}} = \vec{\mathbf{J}}_f + \frac{\partial\vec{\mathbf{D}}}{\partial t} \quad (17.21)$$

From Eq. (17.18), we have

$\vec{\mathbf{D}} = (\varepsilon_0\vec{\mathbf{E}} + \vec{\mathbf{P}})$

Using $\vec{\mathbf{P}} = \varepsilon_0\chi_e\vec{\mathbf{E}}$, we can write

$\vec{\mathbf{D}} = \varepsilon_0\vec{\mathbf{E}} + \varepsilon_0\chi_e\vec{\mathbf{E}}$

or

$\vec{\mathbf{D}} = \varepsilon_0\vec{\mathbf{E}}(1+\chi_e)$

Using $\varepsilon = \varepsilon_0(1+\chi_e)$, we get Eq. (17.22):

$\vec{\mathbf{D}} = \varepsilon\vec{\mathbf{E}}$

From Eq. (17.19), we have

$\mu_0\vec{\mathbf{H}} = \vec{\mathbf{B}} - \mu_0\vec{\mathbf{M}}$

Using $(\vec{\mathbf{M}} = \chi_M\vec{\mathbf{H}})$, we can write

$\mu_0\vec{\mathbf{H}} = \vec{\mathbf{B}} - \mu_0\chi_M\vec{\mathbf{H}}$

or $\vec{\mathbf{H}} = \dfrac{\vec{\mathbf{B}}}{\mu_0 + \mu_0\chi_M}$

Using $\mu = \mu_0(1+\chi_M)$, we get Eq. (17.23):

$\vec{\mathbf{H}} = \dfrac{\vec{\mathbf{B}}}{\mu}$

You have learnt in Unit 10 that in a linear dielectric medium, $\vec{\mathbf{P}}$ is parallel to $\vec{\mathbf{E}}$ $(\vec{\mathbf{P}} = \varepsilon_0\chi_e\vec{\mathbf{E}})$, where χ_e is the electric susceptibility of the medium. You have also learnt in Unit 14 that $\vec{\mathbf{M}}$ is parallel to $\vec{\mathbf{H}}$ $(\vec{\mathbf{M}} = \chi_M\vec{\mathbf{H}})$, where χ_M is the magnetic susceptibility of the medium. For such materials, we can write Eqs. (17.18 and 17.19) as

$$\vec{\mathbf{D}} = \varepsilon\vec{\mathbf{E}} \quad (17.22)$$

and $$\vec{\mathbf{H}} = \frac{1}{\mu}\vec{\mathbf{B}} \quad (17.23)$$

Here $$\varepsilon = \varepsilon_0(1+\chi_e) \quad (17.24)$$

and $$\mu = \mu_0(1+\chi_M) \quad (17.25)$$

In general, ε (electric permittivity of the medium) and μ (magnetic permeability of the medium) are frequency dependent. If the medium is homogeneous, ε and μ are constants, i.e., they do not vary from point to point. Given this information, we can recast the Maxwell's equations inside a dielectric. We present Maxwell's equations for a linear, isotropic and homogeneous dielectric medium in Table 17.1.

Table 17.1: Maxwell's equations for linear, isotropic, homogeneous dielectric media.

Differential form	Integral form
$\vec{\nabla}.\vec{\mathbf{E}} = \dfrac{\rho_f}{\varepsilon}$	$\oiint_S \vec{\mathbf{E}}.d\vec{\mathbf{S}} = \dfrac{q_f}{\varepsilon}$ (17.26)
$\vec{\nabla}\times\vec{\mathbf{E}} = -\dfrac{\partial\vec{\mathbf{B}}}{\partial t}$	$\oint_C \vec{\mathbf{E}}.d\vec{\mathbf{l}} = -\dfrac{d\Phi_B}{dt}$ (17.27)
$\vec{\nabla}.\vec{\mathbf{B}} = 0$	$\oiint_S \vec{\mathbf{B}}.d\vec{\mathbf{S}} = 0$ (17.28)
$\vec{\nabla}\times\vec{\mathbf{B}} = \mu\left(\vec{\mathbf{J}}_f + \vec{\mathbf{J}}_d\right)$ where $\vec{\mathbf{J}}_d = \varepsilon\dfrac{\partial\vec{\mathbf{E}}}{\partial t}$	$\oint_C \vec{\mathbf{B}}\,.\,d\vec{\mathbf{l}} = \mu(i + i_d)$ where $i_d = \varepsilon\dfrac{d\Phi_E}{dt}$ (17.29)

You may like to prove Maxwell's equations (17.26 to 17.29) yourself to get some practice. Solve the following SAQ.

SAQ 5 **- Maxwell's equations in dielectric media**

Prove Eqs. (17.26 to 17.29).

Eqs. (17.26 to 17.29) [with Eqs. (17.18 and 17.19) and Eqs. (17.22 and 17.23)] are the fundamental laws of electromagnetism inside linear, isotropic, homogeneous dielectric media. We can now consider plane wave propagation in such dielectric media.

17.3.2 Plane Wave Propagation in Dielectrics

Let us first consider regions inside a linear, isotropic and homogeneous dielectric medium where there is no free charge or free current. Then Maxwell's equations (17.26 to 17.29) for regions inside such dielectric media become

$$\vec{\nabla}.\vec{\mathbf{E}} = 0 \tag{17.30}$$

$$\vec{\nabla}\times\vec{\mathbf{E}} = -\frac{\partial\vec{\mathbf{B}}}{\partial t} \tag{17.31}$$

$$\vec{\nabla}.\vec{\mathbf{B}} = 0 \tag{17.32}$$

$$\vec{\nabla}\times\vec{\mathbf{B}} = \mu\varepsilon\frac{\partial\vec{\mathbf{E}}}{\partial t} \tag{17.33}$$

You can see that these equations are similar to Maxwell's equations in vacuum [Eqs. (16.21 to 16.24)]. Once again, you can follow the procedure adopted in Sec. 16.5 and derive the wave equation for electromagnetic waves propagating through charge-free and current-free isotropic dielectric media. You will get the results given ahead.

$$\nabla^2 \vec{\mathbf{E}} = \frac{1}{\mu\varepsilon}\frac{\partial^2 \vec{\mathbf{E}}}{\partial t^2} \quad (17.34a)$$

and $$\nabla^2 \vec{\mathbf{B}} = \frac{1}{\mu\varepsilon}\frac{\partial^2 \vec{\mathbf{B}}}{\partial t^2} \quad (17.34b)$$

You may like to prove these equations yourself for practice.

SAQ 6 - Wave equations in dielectric media

Prove Eqs. (17.34a and b).

Thus, we find that in an isotropic dielectric medium, electromagnetic waves propagate at a speed

$$v = \frac{1}{\sqrt{\mu\varepsilon}} \quad (17.35)$$

Now a well known result from optics tells us that the speed of light in a transparent medium is reduced by a factor of *n*, the refractive index of the medium:

$$v = \frac{c}{n} \quad (17.36)$$

From Eqs. (17.35 and 17.36), it follows that *n* is related to the electric and magnetic properties of materials by the equation

$$n = \sqrt{\frac{\varepsilon\mu}{\varepsilon_0\mu_0}} \quad (17.37a)$$

where we have used the result $c = \frac{1}{\sqrt{\varepsilon_0\mu_0}}$ in arriving at Eq. (17.37a). For a dielectric $\mu \approx \mu_0$, and therefore,

$$n = \sqrt{\frac{\varepsilon}{\varepsilon_0}} = \sqrt{\kappa} \quad (17.37b)$$

where $$\kappa = \frac{\varepsilon}{\varepsilon_0} \quad (17.37c)$$

is called the **dielectric constant** of the material.

You should realise that to be able to relate the expression of the **index of refraction of a material with its electric and magnetic properties** was another triumph of Maxwell's theory of electromagnetism. Note that the index of refraction as measured optically could be calculated easily from the dielectric constant measured electrically. This was a convincing piece of evidence for Maxwell's identification of light with electromagnetic waves.

Once again, we can write plane wave solutions similar to Eqs. (i and ii) in Example 16.2 of the wave equations (17.34a and b) for a linear, isotropic, homogeneous dielectric medium with no free charge and no free current. The only difference is that now the waves travel with a speed v given by Eq. (17.35):

$$\vec{\mathbf{E}} = \vec{\mathbf{E}}_0 \cos(z - vt) \quad \text{or} \quad \vec{\mathbf{E}} = \vec{\mathbf{E}}_0 \cos(kz - \omega t) \qquad (17.38a)$$

$$\vec{\mathbf{B}} = \vec{\mathbf{B}}_0 \cos(z - vt) \quad \text{or} \quad \vec{\mathbf{B}} = \vec{\mathbf{B}}_0 \cos(kz - \omega t) \qquad (17.38b)$$

where $v = \frac{\omega}{k}$. You can verify that monochromatic sinusoidal plane waves of the form of Eqs. (17.38a and b) satisfy the wave equation and Maxwell's equations inside charge-free and current-free isotropic dielectric media, with the conditions that

$$E_{0z} = \left(\vec{\mathbf{E}}_0\right)_z = 0 \quad \text{and} \quad B_{0z} = \left(\vec{\mathbf{B}}_0\right)_z = 0 \qquad (17.39)$$

Also $$\vec{\mathbf{B}}_0 = \frac{k}{\omega}(\hat{\mathbf{z}} \times \vec{\mathbf{E}}_0) \quad \text{and} \quad B_0 = \frac{k}{\omega}E_0 = \frac{E_0}{v} \qquad (17.40)$$

or in general,

$$\vec{\mathbf{B}} = \frac{k}{\omega}(\hat{\mathbf{k}} \times \vec{\mathbf{E}}) \quad \text{or} \quad \hat{\mathbf{k}} \times \vec{\mathbf{E}} = v\vec{\mathbf{B}} \quad \left(\because v = \frac{\omega}{k}\right) \qquad (17.41)$$

where, from Eq. (17.35), $v = \frac{1}{\sqrt{\mu\varepsilon}}$

These equations show that the plane wave solutions of the wave equation in an isotropic dielectric, having no free charge and no free current, resemble the plane wave solutions of the wave equation in vacuum: $\vec{\mathbf{B}}$ is perpendicular to $\vec{\mathbf{E}}$ and the wave travels in the direction of $\hat{\mathbf{k}}$ which is perpendicular to both $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$.

What is the difference between the two sets of equations? Note that the speed at which the wave travels in the dielectric is different from c, the speed of light in vacuum, by a constant factor, termed the **index of refraction** (in optics).

Thus, if we compare a wave in a dielectric with a wave of the same frequency in vacuum, **the wavelength in the dielectric will be less than the vacuum wavelength by a factor** $(1/n)$, since (frequency × wavelength = wave speed).

Now in the last section (Sec. 17.4) of this unit, we shall discuss another interesting aspect of electromagnetic waves. Consider the following situation:

When you sit outdoors on a cold winter morning in bright sunshine, you feel warm after a while. Why does this happen? Obviously, it is the energy carried by the sunlight which gets transferred to you and gives you this pleasant sensation. You already know from school physics that waves transport energy from one region of space to another.

In the next section, we will determine the amount of energy carried by electromagnetic waves across the space.

17.4 ENERGY CARRIED BY ELECTROMAGNETIC WAVES

Recall that in Sec. 9.4 of Unit 9 in Block 2, we have expressed the work necessary to assemble a continuous static charge distribution (against the Coulomb repulsion of like charges) as

$$W_E = \frac{1}{2}\iiint_V \rho\phi\, dV \tag{17.42}$$

We can rewrite this expression in terms of the electric field. We first use Gauss's law or Maxwell's equation (16.17) to express ρ in terms of $\vec{\mathbf{E}}$ as

$$\rho = \varepsilon_0 \vec{\nabla}.\vec{\mathbf{E}} \tag{17.43a}$$

Thus,
$$W_E = \frac{\varepsilon_0}{2}\iiint_V (\vec{\nabla}.\vec{\mathbf{E}})\phi\, dV \tag{17.43b}$$

Now we use the following vector identity [refer to Eq. (2.6c), Unit 2, Block 1]

$$\vec{\nabla}.(\phi\vec{\mathbf{E}}) = \phi(\vec{\nabla}.\vec{\mathbf{E}}) + \vec{\mathbf{E}}.(\vec{\nabla}\phi)$$

Recall the divergence theorem from Unit 4, Block 1 of this course:

$\iiint_V (\vec{\nabla}.\vec{\mathbf{A}})dV = \oiint_S \vec{\mathbf{A}}.d\vec{\mathbf{S}}$

We substitute $\vec{\mathbf{A}} = \phi\vec{\mathbf{E}}$ in the divergence theorem and get Eq. (17.43d).

Using the above vector identity and substituting $\vec{\mathbf{E}} = -\vec{\nabla}\phi$ [from Eq. (9.13), Unit 9, Block 2], we can write Eq. (17.43b) as

$$W_E = \frac{\varepsilon_0}{2}\left(\iiint_V \vec{\nabla}.(\phi\vec{\mathbf{E}})dV - \iiint_V \vec{\mathbf{E}}.(\vec{\nabla}\phi)dV\right)$$

or
$$W_E = \frac{\varepsilon_0}{2}\left(\iiint_V \vec{\nabla}.(\phi\vec{\mathbf{E}})dV + \iiint_V E^2\, dV\right) \tag{17.43c}$$

Applying the divergence theorem to the first term in Eq. (17.43c), we obtain

$$W_E = \frac{\varepsilon_0}{2}\left[\iint_S \phi\vec{\mathbf{E}}.d\vec{\mathbf{S}} + \iiint_V E^2\, dV\right] \tag{17.43d}$$

Now suppose we enlarge the volume. The extra volume will not contribute to the term W_E, since $\rho = 0$ in that volume. But as we enlarge the volume, the second integral in Eq. (17.43d) can only increase, since the integrand E^2 is positive. Therefore, the surface integral must decrease so that the sum remains the same so that the work done is conserved (read the margin remark).

Note that in our derivation, W_E is the work done to assemble a continuous static charge distribution (against the Coulomb repulsion of like charges). Since the Coulomb force is conservative, the electrostatic energy of the system will be conserved. So, the work done by this force will be constant.

If we integrate over all space, the surface integral goes to zero, and we are left with

$$W_E = \frac{\varepsilon_0}{2}\iiint_V E^2\, dV \tag{17.44a}$$

where $\vec{\mathbf{E}}$ is the resulting electric field. Likewise, we have shown in Unit 15 [Eq. (15.27)] that the work required to build a current (against the back emf) is given by

$$W_{EB} = \frac{1}{2\mu_0}\iiint_V B^2\, dV \tag{17.44b}$$

where $\vec{\mathbf{B}}$ is the resulting magnetic field. So, we can express the total energy stored in a current and charge distribution in terms of electric and magnetic fields produced by this distribution as follows:

$$W_{EB} = \frac{1}{2}\iiint_V \left(\varepsilon_0 E^2 + \frac{B^2}{\mu_0}\right) dV \qquad (17.44c)$$

We would now like to derive Eq. (17.44c) more generally, keeping in view the law of conservation of energy. Thus, we will arrive at **Poynting's theorem**, which expresses conservation of energy in electromagnetism.

17.4.1 Poynting's Theorem

Suppose some charge and current configuration produces fields $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ at time *t*. Suppose the charges move around in space with velocity $\vec{\mathbf{v}}$. We would like to know: **How much work**, *dW*, **is done by the electromagnetic forces on these charges in the small time interval** *dt*? According to the Lorentz force law, the work done on an element of charge *dq* is

$$dW = \vec{\mathbf{F}}.d\vec{\mathbf{l}} = dq(\vec{\mathbf{E}} + \vec{\mathbf{v}}\times\vec{\mathbf{B}}).\vec{\mathbf{v}}\,dt = \vec{\mathbf{E}}.\vec{\mathbf{v}}\,dq\,dt \quad [\because (\vec{\mathbf{a}}.(\vec{\mathbf{a}}\times\vec{\mathbf{b}}) = 0] \qquad (17.45a)$$

Now, we can write $dq = \rho dV$ and $\rho\vec{\mathbf{v}} = \vec{\mathbf{J}}$ (read the margin remark). So, we can write

$$dW = \vec{\mathbf{E}}.\frac{\vec{\mathbf{J}}}{\rho}\rho\, dV\, dt = \vec{\mathbf{E}}.\vec{\mathbf{J}}\, dV\, dt \qquad (17.45b)$$

The power delivered on the charges in volume *dV* is given by

$$\frac{dW}{dt} = \vec{\mathbf{E}}.\vec{\mathbf{J}}\,dV \qquad (17.45c)$$

So, the total power delivered on all the charges in some volume *V* is obtained by integrating Eq. (17.45c) over the entire volume:

$$P = \iiint_V (\vec{\mathbf{E}}.\vec{\mathbf{J}})\,dV \qquad (17.46)$$

Let us express Eq. (17.46) in terms of the fields alone. We use Eq. (17.17) to eliminate $\vec{\mathbf{J}}$ as follows:

$$\vec{\mathbf{E}}.\vec{\mathbf{J}} = \frac{1}{\mu_0}\vec{\mathbf{E}}.(\vec{\nabla}\times\vec{\mathbf{B}}) - \varepsilon_0\vec{\mathbf{E}}.\frac{\partial\vec{\mathbf{E}}}{\partial t} \qquad (17.47a)$$

We use the vector identity given by Eq. (2.9e) of Unit 2, Block 1 and write

$$\vec{\nabla}.(\vec{\mathbf{E}}\times\vec{\mathbf{B}}) = \vec{\mathbf{B}}.(\vec{\nabla}\times\vec{\mathbf{E}}) - \vec{\mathbf{E}}.(\vec{\nabla}\times\vec{\mathbf{B}}) \qquad (17.47b)$$

Combining this result with Faraday's law, $\vec{\nabla}\times\vec{\mathbf{E}} = -\frac{\partial\vec{\mathbf{B}}}{\partial t}$, we get

$$\vec{\mathbf{E}}.(\vec{\nabla}\times\vec{\mathbf{B}}) = -\vec{\mathbf{B}}.\frac{\partial\vec{\mathbf{B}}}{\partial t} - \vec{\nabla}.(\vec{\mathbf{E}}\times\vec{\mathbf{B}}) \qquad (17.47c)$$

We can also write (read the second margin remark):

$$\vec{\mathbf{B}}.\frac{\partial\vec{\mathbf{B}}}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}(B^2) \quad \text{and} \quad \vec{\mathbf{E}}.\frac{\partial\vec{\mathbf{E}}}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}(E^2) \qquad (17.47d)$$

Let us quickly deduce the relation $\rho\vec{\mathbf{v}} = \vec{\mathbf{J}}$. To do so, consider a small cylinder of area of cross-section *A* in which all charged particles, each of charge *q*, move with the same speed *v*. In time *dt*, the charged particles move a distance *vdt*. So, the volume of the cylinder is *Avdt*. Then if *n* is the number of charged particles per unit volume, the number of charged particles in the cylinder is *nAvdt* and the total charge in it is *qnAvdt*. The current through the cylinder is given by

$$i = \frac{qnAvdt}{dt} = qnAv$$

Since current density is defined as current per unit area normal to the velocity of charged particles, we have

$$J = \frac{i}{A} = qnv$$

Since the direction of *J* is that of the actual flow of charges, it is parallel to the velocity $\vec{\mathbf{v}}$, and we can write $\vec{\mathbf{J}} = qn\vec{\mathbf{v}}$

Note that the product *nq* is the volume charge density of the charges carrying the current. So, we have

$$\vec{\mathbf{J}} = \rho\vec{\mathbf{v}}$$

$$\frac{1}{2}\frac{\partial}{\partial t}(B^2) = \frac{1}{2}\frac{\partial}{\partial t}(\vec{\mathbf{B}}.\vec{\mathbf{B}})$$
$$= \frac{1}{2}\frac{\partial\vec{\mathbf{B}}}{\partial t}.\vec{\mathbf{B}} + \frac{1}{2}\vec{\mathbf{B}}.\frac{\partial\vec{\mathbf{B}}}{\partial t}$$
$$= \vec{\mathbf{B}}.\frac{\partial\vec{\mathbf{B}}}{\partial t}$$

since scalar product is commutative.

Using Eqs. (17.47c and d) in Eq. (17.47a), we get

$$\vec{\mathbf{E}}.\vec{\mathbf{J}} = \frac{1}{\mu_0}[-\vec{\mathbf{B}}\,.\frac{\partial\vec{\mathbf{B}}}{\partial t} - \vec{\nabla}.(\vec{\mathbf{E}}\times\vec{\mathbf{B}})] - \frac{\varepsilon_0}{2}\frac{\partial}{\partial t}(E^2)$$

or
$$\vec{\mathbf{E}}.\vec{\mathbf{J}} = \frac{1}{\mu_0}[-\frac{1}{2}\frac{\partial}{\partial t}(B^2) - \vec{\nabla}.(\vec{\mathbf{E}}\times\vec{\mathbf{B}})] - \frac{\varepsilon_0}{2}\frac{\partial}{\partial t}(E^2) \qquad (17.48)$$

Thus, we can write

$$\vec{\mathbf{E}}.\vec{\mathbf{J}} = -\frac{1}{2}\frac{\partial}{\partial t}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) - \frac{1}{\mu_0}\vec{\nabla}.(\vec{\mathbf{E}}\times\vec{\mathbf{B}}) \qquad (17.49)$$

Putting Eq. (17.49) in Eq. (17.46), we get the expression for power as

Notice that in Eq. (17.50), we have replaced $\frac{\partial}{\partial t}$ by $\frac{d}{dt}$ as we have taken it out of the integral.

$$P = -\frac{d}{dt}\iiint_V \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)dV - \frac{1}{\mu_0}\iiint_V \vec{\nabla}.(\vec{\mathbf{E}}\times\vec{\mathbf{B}})dV \qquad (17.50)$$

Now applying divergence theorem to the second term on the RHS of Eq. (17.50), we have

$$P = -\frac{d}{dt}\iiint_V \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)dV - \frac{1}{\mu_0}\iint_S (\vec{\mathbf{E}}\times\vec{\mathbf{B}}).d\vec{\mathbf{a}} \qquad (17.51)$$

where S is the surface bounding the volume V and $d\vec{\mathbf{a}}$ is the element of surface area. Using Eq. (17.44c) in Eq. (17.51), we can write

$$P = -\frac{d}{dt}W_{EB} - \frac{1}{\mu_0}\iint_S (\vec{\mathbf{E}}\times\vec{\mathbf{B}}).d\vec{\mathbf{a}} \qquad (17.52)$$

Eq. (17.52) is the mathematical statement of **Poynting's theorem**. It expresses **conservation of energy in electromagnetism**.

The first integral on the RHS of Eq. (17.51) represents the rate at which energy stored in a current/charge distribution is carried out of the volume V, across its boundary surface, by the electromagnetic fields. The second term represents the electromagnetic field energy that flows out through the surface. Poynting's theorem says, then, that,

The rate of work done on the charges by the electromagnetic force is equal to the decrease in energy stored in the field, minus the electromagnetic field energy which flows out through the surface.

The quantity
$$\vec{\mathbf{S}} = \frac{1}{\mu_0}(\vec{\mathbf{E}}\times\vec{\mathbf{B}}) \qquad (17.53a)$$

in Eq. (17.52) is called **Poynting's vector**. It represents the energy flux density, i.e., $\vec{\mathbf{S}}.d\vec{\mathbf{a}}$ is the energy per unit time, transported by the electromagnetic field across a surface of area $d\vec{\mathbf{a}}$. Note that the direction of Poynting's vector of an electromagnetic wave at any point is the same as the direction of wave propagation. Its magnitude is given by

$$S = \frac{1}{\mu_0}EB = \frac{1}{c\mu_0}E^2 = \frac{c}{\mu_0}B^2 \quad \text{since } E = cB \qquad (17.53b)$$

The magnitude of the Poynting vector gives the rate of flow of energy at any given instant. We can state Poynting's theorem more compactly in terms of the Poynting vector $\vec{\mathbf{S}}$ and W_{EB}. Let us do so.

Substituting $\vec{\mathbf{S}}$ from Eq. (17.53a) in Eq. (17.52), we have

$$P = -\frac{dW_{EB}}{dt} - \iint_S \vec{\mathbf{S}}.d\vec{\mathbf{a}} \qquad (17.54)$$

Of course, the work W done on the charges will increase their mechanical energy (kinetic, potential, or some other form of energy). Let U_M denote the mechanical energy density, so that

$$P = \frac{d}{dt}\iiint_V U_M \, dV \qquad (17.55)$$

We define U_{EB} as the energy density of the electromagnetic field as follows:

$$U_{EB} = \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0} B^2\right) \qquad (17.56)$$

Then we can write Eq. (17.54) as

$$\frac{d}{dt}\iiint_V (U_M + U_{EB})dV = -\iint_S \vec{\mathbf{S}}.d\vec{\mathbf{a}} = -\iiint_V (\vec{\nabla}.\vec{\mathbf{S}})dV \qquad (17.57a)$$

where we have used the divergence theorem. Thus, we get

$$\iiint_V \frac{\partial}{\partial t}(U_M + U_{EB})dV = -\iiint_V (\vec{\nabla}.\vec{\mathbf{S}})dV \qquad (17.57b)$$

or

$$\iiint_V [(\vec{\nabla}.\vec{\mathbf{S}}) + \frac{\partial}{\partial t}(U_M + U_{EB})]dV = 0 \qquad (17.57c)$$

Since the volume V is arbitrary, for the volume integral in Eq. (17.57c) to be zero, its integrand must be zero. Thus, we get

$$\vec{\nabla}.\vec{\mathbf{S}} = -\frac{\partial}{\partial t}(U_M + U_{EB}) \qquad (17.58)$$

This is the **differential form of Poynting's theorem**. Compare this with the continuity equation that expresses the conservation of charge:

$$\vec{\nabla}.\vec{\mathbf{J}} = -\frac{\partial \rho}{\partial t}$$

Note that in Eq. (17.58), the charge density is replaced by the total energy density (mechanical plus electromagnetic), while the current density is replaced by the Poynting vector. Thus,

The Poynting vector $\vec{\mathbf{S}}$ describes the flow of energy in the same way that $\vec{\mathbf{J}}$ describes the flow of charge.

Let us now summarise what you have learnt in this unit.

17.5 SUMMARY

Concept **Description**

Plane wave solutions of the electromagnetic wave equations for the electric and magnetic fields

- The wave equations for the electric fields and magnetic fields associated with an electromagnetic wave in vacuum are given by

$$\nabla^2 \vec{\mathbf{E}} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{\mathbf{E}}}{\partial t^2} \quad \text{and} \quad \nabla^2 \vec{\mathbf{B}} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{\mathbf{B}}}{\partial t^2}$$

where the speed of the electromagnetic wave in vacuum is given by

$$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$$

Here c is the speed of light. The monochromatic plane wave solutions of the electromagnetic wave equations for the electric field and the magnetic field are given by

$$\vec{\mathbf{E}}(z,t) = \vec{\mathbf{E}}_0 \cos(kz - \omega t) \text{ and } \vec{\mathbf{B}}(z,t) = \vec{\mathbf{B}}_0 \cos(kz - \omega t)$$

provided these equations also satisfy Maxwell's equations. Such electromagnetic waves are described by their frequency f or angular frequency ω, the speed of propagation c (in vacuum), the wave number k, the wavelength λ.

The requirement that these solutions satisfy Maxwell's equations for charge-free and current-free regions in order to be able to represent electromagnetic waves in vacuum, yields the set of equations given below:

$$(\vec{\mathbf{E}}_0)_z = 0, \qquad (\vec{\mathbf{B}}_0)_z = 0,$$

$$\hat{\mathbf{k}} \cdot \vec{\mathbf{E}} = 0, \qquad \hat{\mathbf{k}} \times \vec{\mathbf{E}} = c\vec{\mathbf{B}},$$

$$\hat{\mathbf{k}} \cdot \vec{\mathbf{B}} = 0, \qquad \hat{\mathbf{k}} \times \vec{\mathbf{B}} = -\frac{\vec{\mathbf{E}}}{c}$$

where $\hat{\mathbf{k}}$ is the unit vector in the direction of wave propagation. Also

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = \frac{\omega}{k} \quad \text{where} \quad \omega = 2\pi f \text{ and } \lambda = \frac{2\pi}{k}$$

The magnitudes of the electric field and magnetic field are related by

$$|\vec{\mathbf{E}}| = c|\vec{\mathbf{B}}|$$

Maxwell's equations for linear, isotropic, homogeneous dielectric media

- Maxwell's equations for linear, isotropic and homogeneous dielectric media are given as

$$\vec{\nabla} \cdot \vec{\mathbf{E}} = \frac{\rho_f}{\varepsilon} \qquad \oiint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{S}} = \frac{q_f}{\varepsilon}$$

$$\vec{\nabla} \times \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t} \qquad \oint_C \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}} = -\frac{d\Phi_B}{dt}$$

$$\vec{\nabla} \cdot \vec{\mathbf{B}} = 0 \qquad \oiint_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}} = 0$$

$$\vec{\nabla} \times \vec{\mathbf{B}} = \mu\left(\vec{\mathbf{J}}_f + \varepsilon \frac{\partial \vec{\mathbf{E}}}{\partial \mathbf{t}}\right) \qquad \oint_C \vec{\mathbf{B}} \cdot d\vec{\mathbf{l}} = \mu(i + i_d) \text{ where } i_d = \varepsilon \frac{d\Phi_E}{dt}$$

Wave equations for electromagnetic waves in charge-free and current-free linear, isotropic, homogeneous dielectric media

- The wave equations for electromagnetic waves propagating through charge-free and current-free linear, isotropic, homogeneous dielectric media are given as

$$\nabla^2 \vec{\mathbf{E}} = \frac{1}{\mu\varepsilon}\frac{\partial^2 \vec{\mathbf{E}}}{\partial t^2} \quad \text{and} \quad \nabla^2 \vec{\mathbf{B}} = \frac{1}{\mu\varepsilon}\frac{\partial^2 \vec{\mathbf{B}}}{\partial t^2}$$

where ε is the electric permittivity of the medium and μ, its magnetic permeability. In such dielectric media, the solutions of the wave equations for electric and magnetic fields that satisfy Maxwell's equations are electromagnetic waves, which propagate at a speed $v = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{c}{n}$, where n is the refractive index of the medium. For a dielectric material,

$$n = \sqrt{\frac{\varepsilon}{\varepsilon_0}} = \sqrt{\kappa} \quad \text{where} \quad \kappa = \frac{\varepsilon}{\varepsilon_0}$$

is called the **dielectric constant** of the material.

Plane wave solutions of the electromagnetic wave equations for linear, isotropic, homogeneous dielectric media

- The plane wave solutions for the wave equations for a linear, isotropic, homogeneous dielectric medium with no free charge and no free current are:

$$\vec{\mathbf{E}}(z,t) = \vec{\mathbf{E}}_0 \cos(z - vt) \quad \text{and} \quad \vec{\mathbf{B}}(z,t) = \vec{\mathbf{B}}_0 \cos(z - vt)$$

where v is the speed at which the wave travels in the medium. These waves satisfy Maxwell's equations in such dielectric media provided

$$\hat{\mathbf{k}} \,.\, \vec{\mathbf{E}} = 0, \quad \hat{\mathbf{k}} \,.\, \vec{\mathbf{B}} = 0, \quad \hat{\mathbf{k}} \times \vec{\mathbf{E}} = v\vec{\mathbf{B}}, \quad \frac{|\vec{\mathbf{E}}|}{|\vec{\mathbf{B}}|} = v$$

where $v = \frac{1}{\sqrt{\mu\varepsilon}}$

Energy carried by electromagnetic waves: Poynting's theorem and Poynting vector

- **Poynting's theorem** expresses the conservation of energy in electromagnetism. According to Poynting's theorem,

The rate of work done on the charges by the electromagnetic force is equal to the decrease in energy stored in the field, minus the electromagnetic field energy which flowed out through the surface.

The mathematical statement of Poynting's theorem is:

$$\frac{dW}{dt} = -\frac{d}{dt}\iiint_V \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)dV - \frac{1}{\mu_0}\iint_S (\vec{\mathbf{E}} \times \vec{\mathbf{B}}).d\vec{\mathbf{a}}$$

The **Poynting vector** given by

$$\vec{\mathbf{S}} = \frac{1}{\mu_0}(\vec{\mathbf{E}} \times \vec{\mathbf{B}})$$

represents the energy flux density, i.e., $\vec{\mathbf{S}}.d\vec{\mathbf{a}}$ is the energy per unit time, transported by the electromagnetic field across a surface of area $d\vec{\mathbf{a}}$. The Poynting vector $\vec{\mathbf{S}}$ describes the flow of energy in the same way that $\vec{\mathbf{J}}$ describes the flow of charge.

The differential form of Poynting's theorem is given by

$$\vec{\nabla}.\vec{\mathbf{S}} = -\frac{\partial}{\partial t}(U_M + U_{EB})$$

where U_M is the mechanical energy density and U_{EB}, the energy density of the electromagnetic field given by

$$U_{EB} = \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0} B^2\right)$$

17.6 TERMINAL QUESTIONS

1. A plane electromagnetic sinusoidal wave is characterized by these parameters: the wave is travelling in the direction $-\hat{\mathbf{x}}$, its frequency is 100 MHz, the associated electric field is perpendicular to the direction of $\hat{\mathbf{z}}$. The amplitude of the electric field is $100\,\text{Vm}^{-1}$. Write down the expressions for the $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ fields that specify this wave.

2. The electric field of an electromagnetic wave in vacuum is given by

$$E_x = 0,\ E_y = 30\ \cos[(2\pi/3)\,x - 2\pi\times 10^7 t],\ E_z = 0$$

where E is in Vm^{-1}, t in s and x in m. Determine the frequency ν, wavelength λ, the direction of propagation of the wave and the direction of the associated magnetic field.

3. The electric field given by

$$\vec{\mathbf{E}} = (1000\,\text{Vm}^{-1})\,\hat{\mathbf{x}}\left[\cos\left(50y - \omega t\right)\right]$$

represents the electric field of a plane electromagnetic wave in a charge-free and current-free region. Determine the wavelength and frequency of the wave. Calculate the associated magnetic field.

4. When an electromagnetic wave of a given frequency travels from vacuum into a dielectric medium, which quantities associated with the wave change: wave speed, wavelength or wave frequency?

5. The speed of light in a dielectric medium is two-thirds its value in vacuum. Calculate the refractive index and the dielectric constant of the medium.

6. A plane electromagnetic wave whose electric field is given by

$$\vec{\mathbf{E}} = (100\,\text{Vm}^{-1})\,\hat{\mathbf{z}}\cos\left(6\pi x - \omega t\right)$$

is travelling in a dielectric medium for which $\varepsilon = 9\varepsilon_0$ and $\mu = 4\mu_0$. What is the speed of the wave in the medium? Determine the wavelength and frequency of the wave. What are the refractive index and dielectric constant of the medium? Write the expression for the associated magnetic field of the wave in the dielectric medium.

7. The electric and magnetic fields of an electromagnetic wave are directed along the positive x and y-axes, respectively, at a certain instant of time. What is the direction of the Poynting vector? In which direction does the wave transport energy at that instant of time?

8. The electric field of an electromagnetic wave is directed along the positive x-axis at a certain instant of time. The wave transports energy in the negative y-direction. Determine the direction of the magnetic field associated with the wave at that instant of time.

9. Determine the magnitude and direction of the Poynting vector for the plane electromagnetic wave having the following electric and magnetic fields at an instant of time:

$$\vec{\mathbf{E}} = E_0 \cos(kx - \omega t)\hat{\mathbf{z}} \quad \text{and} \quad \vec{\mathbf{B}} = B_0 \cos(kx - \omega t)\hat{\mathbf{y}}$$

10. Determine the energy density of the electromagnetic field of Terminal Question 9.

17.7 SOLUTIONS AND ANSWERS

Self-Assessment Questions

1. Substituting Eqs. (17.2a and 17.2b) in the first equation of (17.3a), we get

$$\vec{\nabla}\cdot\vec{\mathbf{E}} = \vec{\nabla}\cdot[\vec{\mathbf{E}}_0 \cos(kz - \omega t)]$$

$$= [\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}]\cdot[E_{0x}\hat{\mathbf{i}} + E_{0y}\hat{\mathbf{j}} + E_{0z}\hat{\mathbf{k}}]\cos(kz - \omega t)$$

$$= \frac{\partial}{\partial z}[E_{0z}\cos(kz - \omega t)]$$

For $\vec{\nabla}\cdot\vec{\mathbf{E}} = 0,$ we must have $E_{0z} = 0$ because the derivative of the function $\cos(kz - \omega t)$ with respect to z is non-zero.

Similarly, we can show that

$$\vec{\nabla}\cdot\vec{\mathbf{B}} = \vec{\nabla}\cdot[\vec{\mathbf{B}}_0 \cos(kz - \omega t)]$$

$$= [\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}]\cdot[B_{0x}\hat{\mathbf{i}} + B_{0y}\hat{\mathbf{j}} + B_{0z}\hat{\mathbf{k}}]\cos(kz - \omega t)$$

$$= \frac{\partial}{\partial z}[B_{0z}\cos(kz - \omega t)]$$

For $\vec{\nabla}\cdot\vec{\mathbf{B}} = 0,$ we must have $B_{0z} = 0,$ since the derivative of the function $\cos(kz - \omega t)$ is non-zero.

2. We can write the equation $\vec{\mathbf{B}}_0 = \frac{k}{\omega}(\hat{\mathbf{z}} \times \vec{\mathbf{E}}_0)$ as

$$(B_{0x}\hat{\mathbf{x}} + B_{0y}\hat{\mathbf{y}} + B_{0z}\hat{\mathbf{z}}) = \frac{k}{\omega}\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 0 & 0 & 1 \\ E_{0x} & E_{0y} & E_{0z} \end{vmatrix} = \frac{k}{\omega}(-E_{0y})\hat{\mathbf{x}} + \frac{k}{\omega}(E_{0x})\hat{\mathbf{y}}$$

Comparing the left-hand and right-hand sides of the above equation, we get

$$k E_{0y} = -\omega B_{0x} \qquad \text{and} \qquad k E_{0x} = \omega B_{0y}$$

Therefore, Eq. (17.9a) and Eqs. (17.8a and b) are equivalent.

3. We substitute Eqs. (17.2a and b) in the equation $\vec{\nabla}\times\vec{\mathbf{B}} = \mu_0 \varepsilon_0 \frac{\partial\vec{\mathbf{E}}}{\partial t}$:

$$\vec{\nabla}\times[\vec{\mathbf{B}}_0 \cos(kz - \omega t)] = \mu_0 \varepsilon_0 \frac{\partial}{\partial t}[\vec{\mathbf{E}}_0 \cos(kz - \omega t)]$$

or $\quad \vec{\nabla}\times[(B_{0x}\hat{\mathbf{x}} + B_{0y}\hat{\mathbf{y}} + B_{0z}\hat{\mathbf{z}})\cos(kz - \omega t)] = \mu_0 \varepsilon_0 \omega \vec{\mathbf{E}}_0 \sin(kz - \omega t)$

Let us first solve the LHS of this equation:

$$\left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right)\times(B_{0x}\hat{\mathbf{x}} + B_{0y}\hat{\mathbf{y}} + B_{0z}\hat{\mathbf{z}})\cos(kz - \omega t)$$

$$= -\hat{\mathbf{x}}\frac{\partial}{\partial z}[B_{0y}\cos(kz - \omega t)] + \hat{\mathbf{y}}\frac{\partial}{\partial z}[B_{0x}\cos(kz - \omega t)]$$

$$= k(\hat{\mathbf{x}}\,B_{0y} - \hat{\mathbf{y}}B_{0x})\sin(kz - \omega t)$$

So, we have

$$k(\hat{\mathbf{x}}\,B_{0y} - \hat{\mathbf{y}}B_{0x})\sin(kz - \omega t) = \mu_0\,\varepsilon_0\omega\vec{\mathbf{E}}_0\sin(kz - \omega t)$$

or $$k(\hat{\mathbf{x}}\,B_{0y} - \hat{\mathbf{y}}B_{0x}) = \mu_0\,\varepsilon_0\omega\vec{\mathbf{E}}_0 \qquad \text{(i)}$$

You can verify that the LHS of Eq. (i) is just $-(\vec{\mathbf{k}}\times\vec{\mathbf{B}}_0)$ as follows:

Since $\vec{\mathbf{k}}$ is in the *z*-direction, we can write $\vec{\mathbf{k}} = k\hat{\mathbf{z}}$ so that

$$\vec{\mathbf{k}}\times\vec{\mathbf{B}}_0 = k\hat{\mathbf{z}}\times(B_{0x}\hat{\mathbf{x}} + B_{0y}\hat{\mathbf{y}} + B_{0z}\hat{\mathbf{z}}) = \hat{\mathbf{y}}kB_{0x} - \hat{\mathbf{x}}kB_{0y} \qquad \text{(ii)}$$

Substituting Eq. (ii) in Eq. (i) and using $\vec{\mathbf{k}} = k\hat{\mathbf{k}}$, we get

$$-k(\hat{\mathbf{k}}\times\vec{\mathbf{B}}_0) = \mu_0\,\varepsilon_0\omega\vec{\mathbf{E}}_0 = \frac{1}{c^2}\omega\vec{\mathbf{E}}_0 \qquad \left(\because c = \frac{1}{\sqrt{\mu_0\,\varepsilon_0}}\right)$$

or $$(\hat{\mathbf{k}}\times\vec{\mathbf{B}}_0) = -\frac{1}{c^2}\frac{\omega}{k}\vec{\mathbf{E}}_0 = -\frac{1}{c^2}c\,\vec{\mathbf{E}}_0 = -\frac{1}{c}\vec{\mathbf{E}}_0$$

Thus, we get $\hat{\mathbf{k}}\times\vec{\mathbf{B}}_0 = -\dfrac{\vec{\mathbf{E}}_0}{c}$

4. Comparing the electric field given by $\vec{\mathbf{E}} = (1000\ \text{Vm}^{-1})\hat{\mathbf{y}}\sin\left(\frac{\pi x}{25} - \omega t\right)$ with the expression $\vec{\mathbf{E}} = \hat{\mathbf{y}}E_0\sin(kx - \omega t)$, we can see that

$$k = \frac{2\pi}{50}\text{m}^{-1} \Rightarrow \lambda = \frac{2\pi}{k} = \frac{(2\pi)\times 50}{2\pi}\text{m} = 50\,\text{m}$$

In vacuum, $f = \dfrac{c}{\lambda} \Rightarrow f = \dfrac{3\times10^8\ \text{ms}^{-1}}{50\,\text{m}} = 6\times10^6\,\text{Hz}$

Comparing the expression of the given electric field with Eq. (17.1c), we get the direction of propagation of the wave as positive *x*-direction. For the expression of the associated magnetic field, we need to find its amplitude and direction and ω: $\omega = 2\pi f = 12\pi\times10^6\,\text{Hz}$

From Eq. (17.10), the amplitude of the associated magnetic field is

$$B_0 = \frac{E_0}{c} = \frac{1000\,\text{Vm}^{-1}}{3\times10^8\ \text{ms}^{-1}} = 3\times10^{-6}\ \text{T}$$

Since the direction of propagation of the wave is along the positive *x*-direction and the electric field is along the positive *y*-direction, from Eq. (17.9b), the magnetic field is along the positive *z*-direction.

$$\vec{\mathbf{B}} = \hat{\mathbf{z}}(3\times10^{-6}\ \text{T})\sin\left(\frac{\pi x}{25} - \omega t\right)$$

5. Let us first prove Eq. (17.26), $\vec{\nabla}.\vec{\mathbf{E}} = \dfrac{\rho_f}{\varepsilon}$

From Eq. (17.14), we have $\vec{\nabla}.\vec{\mathbf{E}} = \dfrac{1}{\varepsilon_0}\left(\rho_f - \vec{\nabla}.\vec{\mathbf{P}}\right)$ or $\vec{\nabla}.(\varepsilon_0\vec{\mathbf{E}} + \vec{\mathbf{P}}) = \rho_f$

or $\quad \vec{\nabla}.\vec{\mathbf{D}} = \rho_f$ or $\varepsilon\vec{\nabla}.\vec{\mathbf{E}} = \rho_f$ or $\vec{\nabla}.\vec{\mathbf{E}} = \dfrac{\rho_f}{\varepsilon}$ $\quad (\because \vec{\mathbf{D}} = \varepsilon\vec{\mathbf{E}})$

To cast the equation in its integral form, we integrate both sides over a volume *V*:

$$\iiint_V \vec{\nabla}.\vec{\mathbf{E}}\, dV = \frac{1}{\varepsilon}\iiint_V \rho_f\, dV$$

We apply the divergence theorem to the LHS of the above equation and note that the volume integral in its RHS is equal to just the net charge q_f enclosed in the volume. So,

$$\oiint_S \vec{\mathbf{E}}.d\vec{\mathbf{S}} = \frac{q_f}{\varepsilon},$$ where *S* is a closed surface that bounds volume *V*.

Eqs. (17.27 and 17.28) are the same as Eqs. (16.18 and 16.19).

To prove Eq. (17.29), $\vec{\nabla}\times\vec{\mathbf{B}} = \mu\left(\vec{\mathbf{J}}_f + \vec{\mathbf{J}}_d\right)$ we write it in the form of Eq. (17.17) as

$$\vec{\nabla}\times\vec{\mathbf{B}} = \mu_0\left(\varepsilon_0\frac{\partial\vec{\mathbf{E}}}{\partial t} + \vec{\mathbf{J}}_f + \frac{\partial\vec{\mathbf{P}}}{\partial t} + \vec{\nabla}\times\vec{\mathbf{M}}\right)$$

or $$\vec{\nabla}\times(\vec{\mathbf{B}} - \mu_0\vec{\mathbf{M}}) = \frac{\partial}{\partial t}(\mu_0\varepsilon_0\vec{\mathbf{E}} + \mu_0\vec{\mathbf{P}}) + \mu_0\vec{\mathbf{J}}_f$$

From Eq. (17.19), we have $\vec{\mathbf{H}} = \dfrac{1}{\mu_0}\vec{\mathbf{B}} - \vec{\mathbf{M}}$ and using $\vec{\mathbf{D}} = \varepsilon_0\vec{\mathbf{E}} + \vec{\mathbf{P}}$ from Eq. (17.18), we can write the above equation as

$$\mu_0\vec{\nabla}\times\vec{\mathbf{H}} = \mu_0\frac{\partial\vec{\mathbf{D}}}{\partial t} + \mu_0\vec{\mathbf{J}}_f \quad \text{or} \quad \vec{\nabla}\times\vec{\mathbf{H}} = \frac{\partial\vec{\mathbf{D}}}{\partial t} + \vec{\mathbf{J}}_f$$

Using Eqs. (17.22 and 17.23), we can write $\vec{\nabla}\times\vec{\mathbf{B}} = \mu\varepsilon\dfrac{\partial\vec{\mathbf{E}}}{\partial t} + \mu\vec{\mathbf{J}}_f$

or $\quad \vec{\nabla}\times\vec{\mathbf{B}} = \mu\left(\vec{\mathbf{J}}_f + \vec{\mathbf{J}}_d\right)$ where $\vec{\mathbf{J}}_d = \varepsilon\dfrac{\partial\vec{\mathbf{E}}}{\partial t}$. This is Eq. (17.29).

To cast the equation in its integral form, we integrate both sides on an open surface *S*: $\displaystyle\iint_S (\vec{\nabla}\times\vec{\mathbf{B}}).d\vec{\mathbf{S}} = \mu\varepsilon\iint_S \frac{\partial\vec{\mathbf{E}}}{\partial t}.d\vec{\mathbf{S}} + \mu\iint_S \vec{\mathbf{J}}_f.d\vec{\mathbf{S}}$

Now you know that electric flux $\Phi_E = \displaystyle\iint_S \vec{\mathbf{E}}.d\vec{\mathbf{S}}$ and current $i = \displaystyle\iint_S \vec{\mathbf{J}}_f.d\vec{\mathbf{S}}$

So the RHS becomes $\mu\varepsilon\dfrac{d\Phi_E}{dt} + \mu i$

We next apply Stokes' theorem to the LHS and get

$$\oint_C \vec{\mathbf{B}}.d\vec{\mathbf{l}} = \mu\varepsilon\frac{d\Phi_E}{dt} + \mu i = \mu(i + i_d) \qquad \left(\because i_d = \varepsilon\frac{d\Phi_E}{dt}\right)$$

where the open surface *S* is bounded by the closed curve *C*. This is Eq. (17.29) in integral form.

6. Once again we take the curl of $\vec{\nabla}\times\vec{\mathbf{E}}$ in Eq. (17.31):

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{\mathbf{E}}) = -\frac{\partial(\vec{\nabla}\times\vec{\mathbf{B}})}{\partial t}$$

or $$\vec{\nabla}(\vec{\nabla}.\vec{\mathbf{E}}) - \nabla^2\vec{\mathbf{E}} = -\mu\varepsilon\frac{\partial^2\vec{\mathbf{E}}}{\partial t^2} \qquad \left(\because\ \vec{\nabla}\times\vec{\mathbf{B}} = \mu\varepsilon\frac{\partial\vec{\mathbf{E}}}{\partial t}\right)$$

or $$\nabla^2\vec{\mathbf{E}} = \mu\varepsilon\frac{\partial^2\vec{\mathbf{E}}}{\partial t^2} \qquad [\because\ \vec{\nabla}.\vec{\mathbf{E}} = 0 \text{ from Eq. (17.30)}]$$

Similarly, we take the curl of $\vec{\nabla}\times\vec{\mathbf{B}}$ in Eq. (17.33):

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{\mathbf{B}}) = \mu\varepsilon\frac{\partial(\vec{\nabla}\times\vec{\mathbf{E}})}{\partial t}$$

or $$\vec{\nabla}(\vec{\nabla}.\vec{\mathbf{B}}) - \nabla^2\vec{\mathbf{B}} = -\mu\varepsilon\frac{\partial^2\vec{\mathbf{B}}}{\partial t^2} \qquad \left(\because\ \vec{\nabla}\times\vec{\mathbf{E}} = -\frac{\partial\vec{\mathbf{B}}}{\partial t}\right)$$

or $$\nabla^2\vec{\mathbf{B}} = \mu\varepsilon\frac{\partial^2\vec{\mathbf{B}}}{\partial t^2} \qquad [\because\ \vec{\nabla}.\vec{\mathbf{B}} = 0 \text{ from Eq. (17.32)}]$$

Terminal Questions

1. Since the wave is travelling in the direction $-\hat{\mathbf{x}}$, its $\vec{\mathbf{E}}$ field will be perpendicular to $\hat{\mathbf{x}}$. It is given that $\vec{\mathbf{E}}$ is perpendicular to the direction of $\hat{\mathbf{z}}$. Then $\vec{\mathbf{E}}$ could be either in the direction of $\hat{\mathbf{y}}$ or $-\hat{\mathbf{y}}$. Suppose, we take it in the direction of $\hat{\mathbf{y}}$. Then from Eq. (17.41), the magnetic field $\vec{\mathbf{B}}$ is in the direction of $-\hat{\mathbf{z}}$. It is given that the wave frequency is 100 MHz. Therefore,

$$\omega = 2\pi f = 2\pi\times10^8\,\text{Hz} \quad \text{and} \quad k = \frac{2\pi}{\lambda} = \frac{2\pi f}{c} = \frac{2\pi\times10^8\,\text{Hz}}{3\times10^8\,\text{ms}^{-1}} = \frac{2\pi}{3}\,\text{m}^{-1}$$

Thus, $$\vec{\mathbf{E}} = \hat{\mathbf{y}}(100\,\text{Vm}^{-1})\cos 2\pi\left(\frac{x}{3} + 10^8\,t\right)$$

The corresponding $\vec{\mathbf{B}}$ field is given by

$$\vec{\mathbf{B}} = -\hat{\mathbf{z}}\frac{100\,\text{Vm}^{-1}}{3\times10^8\,\text{ms}^{-1}}\cos 2\pi\left(\frac{x}{3} + 10^8\,t\right)$$

$$= -\hat{\mathbf{z}}\,(3\times10^{-7}\,\text{T})\cos 2\pi\left(\frac{x}{3} + 10^8\,t\right)$$

2. Comparing the given electric field with the expression $\vec{\mathbf{E}} = \hat{\mathbf{y}}E_0\cos(kx-\omega t)$, we can see that the frequency of the wave is

$$\nu = 10^7\,\text{Hz} \text{ and the wavelength is } \lambda = \frac{2\pi}{k} = \frac{2\pi}{(2\pi/3)}\text{m} = 3\,\text{m}$$

Comparing the expression of the electric field with Eq. (17.1c), we get the direction of propagation of the wave as positive *x*-direction. Since $\vec{\mathbf{E}}$ is along the positive *y*-direction and $\hat{\mathbf{k}}$ is in the positive *x*-direction, from Eq. (17.9b), the magnetic field is along the positive *z*-direction.

3. We follow the steps of SAQ 4. Comparing the electric field given by

$$\vec{\mathbf{E}} = (1000\,\text{Vm}^{-1})\,\hat{\mathbf{x}}\cos(50y - \omega t) \qquad \text{(i)}$$

with the expression $\vec{\mathbf{E}} = \hat{\mathbf{x}} E_0 \cos(ky - \omega t)$, we can see that

$$k = 50\,\text{m}^{-1} \Rightarrow \lambda = \frac{2\pi}{k} = \frac{(2\pi)}{50}\text{m} = \frac{\pi}{25}\text{m} \quad \text{and}$$

in vacuum, $f = \frac{c}{\lambda} \Rightarrow f = \frac{3\times 10^8 \times 25\,\text{ms}^{-1}}{\pi\,\text{m}} = 2.4\times 10^9\,\text{Hz}$

Comparing Eq. (i) with Eq. (17.1c), we get the direction of propagation of the wave as positive *y*-direction. For the expression of the associated magnetic field, we need to find its amplitude and direction, and ω:

$$\omega = 2\pi f = 1.5\times 10^{10}\,\text{rad s}^{-1}$$

From Eq. (17.10), the amplitude of the associated magnetic field is

$$B_0 = \frac{E_0}{c} = \frac{1000\,\text{Vm}^{-1}}{3\times 10^8\,\text{ms}^{-1}} = 3\times 10^{-6}\,\text{T}$$

Since the direction of propagation of the wave is along the positive *y*-direction and the electric field is along the positive *x*-direction, from Eq. (17.9b), the magnetic field is along the negative *z*-direction. It is given by

$$\vec{\mathbf{B}} = -\hat{\mathbf{z}}(3\times 10^{-6}\,\text{T})\cos\left(50y - 1.5\times 10^{10}\,t\right)$$

4. When an electromagnetic wave travels from vacuum into a dielectric medium, the wave speed and wavelength change.

5. From Eq. (17.36), the refractive index is

$$n = \frac{c}{v} = \frac{c}{(2/3)c} = 1.5$$

From Eq. (17.37b), the dielectric constant of the medium is

$$\kappa = n^2 = 2.25$$

6. The speed of the wave in the medium is

$$v = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{1}{\sqrt{36\mu_0\varepsilon_0}} = \frac{1}{6\sqrt{\mu_0\varepsilon_0}} = \frac{c}{6} = \frac{3\times 10^8\,\text{ms}^{-1}}{6} = 0.5\times 10^8\,\text{ms}^{-1}$$

Comparing the given electric field with the expression $\vec{\mathbf{E}} = \hat{\mathbf{z}} E_0 \cos(kx - \omega t)$, we can see that the wave number of the wave is

$$k = 6\pi\,\text{m}^{-1} \Rightarrow \lambda = \frac{2\pi}{6\pi} = \frac{1}{3}\text{m} \text{ and the frequency}$$

$$f = \frac{v}{\lambda} = \frac{0.5\times 10^8\,\text{ms}^{-1}}{(1/3)\,\text{m}} = 1.5\times 10^8\,\text{Hz}$$

The refractive index of the medium is $n = \frac{c}{v} = \frac{c}{c/6} = 6$

From Eq. (17.37b), the dielectric constant of the medium is $\kappa = n^2 = 36$

Since $\hat{\mathbf{k}}$ is along the positive *x*-direction and $\vec{\mathbf{E}}$ is along the positive *z*-direction, from Eq. (17.9b), $\vec{\mathbf{B}}$ will be along the negative *y*-direction. So, the associated magnetic field of the wave in the dielectric medium is given by

$$\vec{\mathbf{B}} = -\hat{\mathbf{y}}\left(\frac{100\,\mathrm{V m^{-1}}}{v}\right)\cos(6\pi x - 3\pi\times 10^8\, t)$$

or
$$\vec{\mathbf{B}} = -\hat{\mathbf{y}}\left(\frac{100}{0.5\times 10^8}\,\mathrm{T}\right)\cos(6\pi x - 3\pi\times 10^8\, t)$$

$$= -\hat{\mathbf{y}}(2\times 10^{-6}\,\mathrm{T})\cos(6\pi x - 3\pi\times 10^8\, t)$$

7. From Eq. (17.53a), $\vec{\mathbf{S}} = \frac{1}{\mu_0}(\vec{\mathbf{E}}\times\vec{\mathbf{B}})$. Since the electric field of the electromagnetic wave is directed along the positive *x*-axis and its magnetic field along the positive *y*-axis at a certain instant of time, from Eq. (17.53a), the direction of the Poynting vector will be along the positive *z*-axis at that instant of time. The wave transports energy along the positive *z*-axis at that instant of time.

8. We use Eq. (17.53a) again. It is given that the electric field of the electromagnetic wave is directed along the positive *x*-axis at a certain instant of time. The wave transports energy in the negative *y*-direction. So, the Poynting vector is directed along the negative *y*-axis at that instant of time. Hence, from Eq. (17.53a), the direction of the magnetic field at that instant of time is along positive *z*-axis.

9. From Eq. (17.53a),

$$\vec{\mathbf{S}} = \frac{1}{\mu_0}(\vec{\mathbf{E}}\times\vec{\mathbf{B}}) = \frac{1}{\mu_0}E_0 B_0\cos^2(kx - \omega t)(\hat{\mathbf{z}}\times\hat{\mathbf{y}})$$

$$= -\hat{\mathbf{x}}\frac{E_0 B_0}{\mu_0}\cos^2(kx - \omega t)$$

The magnitude of the Poynting vector for the given plane electromagnetic wave at the given instant of time is $\frac{E_0 B_0}{\mu_0}\cos^2(kx - \omega t)$.

Its direction is along the negative *x*-axis at that instant of time.

10. From Eq. (17.56), the energy density of the electromagnetic field of Terminal Question 9 is given by

$$U_{EB} = \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)$$

$$= \frac{1}{2}\cos^2(kx - \omega t)\left(\varepsilon_0 E_0^2 + \frac{1}{\mu_0}B_0^2\right)$$

FURTHER READINGS

1. **Introduction to Electrodynamics**; David J. Griffiths, Pearson New International Edition (2014).

2. **Electricity and Magnetism (Berkeley Physics Course. Volume 2)**; Edward M. Purcell, International Student Edition, McGraw Hill (1986).

TABLE OF PHYSICAL CONSTANTS

Symbol	Quantity	Value
c	Speed of light in vacuum	3.00×10^{8} ms^{-1}
μ_0	Permeability of free space	1.26×10^{-6} NA^{-2}
ε_0	Permittivity of free space	8.85×10^{-12} C^{2}N^{-1}m^{-2}
$1/4\pi\varepsilon_0$		8.99×10^{9} Nm^{2}C^{-2}
e	Charge of the proton	1.60×10^{-19} C
$-e$	Charge of the electron	-1.60×10^{-19} C
h	Planck's constant	6.63×10^{-34} Js
$\hbar$	$h / 2\pi$	1.05×10^{-34} Js
m_e	Electron rest mass	9.11×10^{-31} kg
$-e/m_e$	Electron charge to mass ratio	-1.76×10^{11} Ckg^{-1}
m_p	Proton rest mass	1.67×10^{-27} kg (1 amu)
m_n	Neutron rest mass	1.68×10^{-27} kg
a_0	Bohr radius	5.29×10^{-11} m
N_A	Avogadro constant	6.02×10^{23} mol^{-1}
R	Universal gas constant	8.31 Jmol^{-1}K^{-1}
k_B	Boltzmann constant	1.38×10^{-23} J K^{-1}
G	Universal gravitational constant	6.67×10^{-11} Nm^{2}kg^{-2}

Astrophysical Data

Celestial Body	Mass (kg)	Mean radius (m)	Mean distance from the centre of Earth (m)
Sun	1.99×10^{30}	6.96×10^{8}	1.50×10^{11}
Moon	7.35×10^{22}	1.74×10^{6}	3.84×10^{8}
Earth	5.97×10^{24}	6.37×10^{6}	0

LIST OF BLOCKS AND UNITS: BPHCT-133

SYLLABUS: ELECTRICITY AND MAGNETISM (BPHCT-133) **4 Credits**

Vector Analysis: Brief review of vector algebra (scalar and vector products). Scalar fields and their gradient and its significance. Vector fields, divergence and curl of vector field and their significance. Vector integration, line and surface integrals of vector fields, volume integrals. Vector integral theorems, divergence theorem and Stoke's theorem (statement only).

Electrostatics: Electrostatic force and electric field, electric flux, Gauss's law of electrostatics. Applications of Gauss's law – electric field due to point charge, uniformly charged spherical shell and solid sphere, infinite line of charge, plane charged sheet, charged conductor. Electric potential, electric potential as line integral of electric field, potential due to a point charge, potential due to a system of charges, relation between electric field and potential, electric dipole, electric dipole in an electric field. Potential for continuous charge distributions, line charge, uniformly charged spherical shell and uniformly charged non-conducting solid sphere, equipotential surfaces, electrostatic potential energy.

Electrostatics in Medium: Dielectric medium, dielectric in electric field, polarisation, displacement vector, Gauss's law in dielectrics. Capacitors, capacitance of an isolated spherical conductor, parallel plate, spherical and cylindrical capacitors, parallel plate capacitor completely filled with dielectric, energy per unit volume in electrostatic field.

Magnetism: Magnetic field, Gauss's law for magnetism, Biot-Savart's law and its applications – straight conductor, circular coil, current carrying solenoid. Ampere's law, divergence and curl of magnetic field, magnetic vector potential. Magnetic properties of materials – magnetic intensity, magnetic induction, permeability, magnetic susceptibility, brief introduction of diamagnetic, paramagnetic and ferromagnetic materials.

Electromagnetic Induction: Faraday's laws of electromagnetic induction, Lenz's law, self and mutual inductance, self-inductance of single coil, mutual inductance of two coils. Energy stored in magnetic field. Equation of continuity for current, displacement current, Maxwell's equations, electromagnetic waves, transverse nature of electromagnetic waves. Electromagnetic wave propagation through vacuum and isotropic dielectric medium, Poynting vector, energy density in electromagnetic field.