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Indira Gandhi National Open University
School of Sciences

BPHCT-131
MECHANICS

Block

1

MATHEMATICAL PRELIMINARIES

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MECHANICS : COURSE INTRODUCTION

In our everyday life we come across a wide variety of objects in motion. The branch of physics dealing with the motion of bodies and bodies at rest in equilibrium is called **mechanics**. You use the laws of mechanics when you ride a bicycle, lift heavy loads, play football or build a house. Many fascinating developments of the space age, such as launching of space probes and artificial satellites are direct applications of the laws of mechanics.

Where do we use the laws of mechanics?

Today, mechanics is regarded as the most fundamental area of physics. In order to study other areas of physics, such as waves, thermal physics, electromagnetism, optics, etc., you need to have a sound knowledge of mechanics.

Therefore, Mechanics is being offered as the very first course in physics in B.Sc. In this 4 credit course, you will learn the basic concepts and laws of mechanics in detail and apply them to objects in motion. We will discuss translational motion, angular/rotational motion and oscillatory/vibrational motion of a variety of objects. The course consists of 4 blocks.

You have studied many of these concepts in your school physics courses. You know that the concepts and laws in physics are expressed very efficiently in the language of mathematics. Therefore, in **Block 1** of this course entitled '**Mathematical Preliminaries**', we explain the preliminary concepts of mathematics needed for studying mechanics. In this block, you will study the elementary concepts of vector algebra and learn how to differentiate vector functions with respect to a scalar. You will also learn how to solve first and second order ordinary differential equations.

In **Block 2** entitled '**Basic Concepts of Mechanics**', you will revise the concepts of **kinematics** and **dynamics** that you have studied in school physics. These include **Newton's laws of motion** and the concepts of **force, linear momentum, impulse, work** and **energy**. Here we will present them in greater detail and apply these laws and concepts to a variety of simple physical situations involving translational motion of objects. For example, you will study the motion of a parachutist falling under the force of gravity and air resistance, the motion of a cart/box being pulled up or pushed down on an inclined plane or on the floor under the force of friction, the change in the velocity of a rocket as gas is ejected from it, and so on.

Block 3 entitled '**Rotational Motion and Many-particle Systems**' deals with the concepts of **angular/rotational motion**, **torque** and **angular momentum** as well as the three important **conservation laws of linear momentum**, **energy** and **angular momentum**. We apply all these concepts and laws to many simple and complex physical situations involving the motion of single particle, two-particle and many-particle systems. For example, you will study the motion of cars on curved roads, the Moon or geostationary satellites orbiting the Earth, the Earth and other planets in orbits around the Sun, dumb-bells, children riding giant wheels or merry-go-rounds, collisions of particles, etc.

The subject matter of **Block 4** is **Harmonic Oscillations**, which is very common in nature. Examples of oscillatory motion are an oscillating pendulum, vibrating strings of a guitar or *veena*, atoms vibrating around their equilibrium position in a crystal lattice, our heartbeat, etc. A proper understanding of oscillatory motion is important for two reasons: Firstly, a large variety of mechanical as well as non-mechanical systems execute oscillatory motion. Secondly, and perhaps more importantly, **a thorough understanding of oscillations is an essential background for the study of wave phenomena**. We begin the block by discussing simple harmonic motion (SHM) and the effect of damping on a harmonic oscillator. Since oscillatory motion and the wave phenomenon are interconnected, we end our discussion on oscillations in Block 4 with a brief introduction to wave phenomenon.

One last word about how to study the course material.

In Block 1 of this course, you will learn the mathematics necessary to understand the contents of Blocks 2, 3, and 4. To use mathematics effectively in applications you need not just knowledge, but *skill*. Skill comes only through practice. For acquiring the necessary skill you will need to work through the text and examples, and solve problems. So always study with a paper and pencil in hand. This is true for the remaining blocks as well. Physics, as you know, cannot be learnt passively. You have to not only understand concepts but acquire the abilities of reasoning and problem solving. Work through all steps in the derivations given in the text yourself.

This course has been designed with a large number of worked out examples along with SAQs and TQs. Don't just read through solved examples or answers to SAQs (Self Assessment Questions) and TQs (Terminal Questions) given at the end of each unit. Try to do them yourself! In the course material you will find many problems intended for drill and a few challenging ones as well. Do not feel satisfied with your study until you can solve a reasonable number of these problems.

Our best wishes are with you for happy problem solving and a good understanding of the course. We wish you success.

BLOCK 1: MATHEMATICAL PRELIMINARIES

The first block of this course deals with the preliminary concepts of vector algebra (Units 1 and 2) and ordinary differential equations (Units 3 and 4) that you will be using in this course.

In your school physics and mathematics courses, you have learnt about scalar and vector quantities. For example, you know that length, mass, density and temperature are scalar quantities. You also know that displacement, velocity, acceleration, force and linear momentum are vector quantities.

In **Unit 1** entitled **'Vector Algebra-I'**, you will revise the definitions of scalars and vectors and the geometrical representation of vectors that you have learnt in school. You will also revise the addition and subtraction of vectors, and their scalar and vector products using the geometrical representation. Then you will study vectors and vector algebra in greater detail. In **Unit 2** entitled **'Vector Algebra-II'**, you will learn how to express vectors algebraically in terms of their components with reference to a given coordinate system. You will also learn how to add, subtract and multiply vectors in their component form. It is essential that you study vector algebra in the algebraic form as you will be using these results very often in physics courses. In Unit 2, you will also study how to determine the derivatives of vector functions (i.e., vectors which are functions of one or more scalar in a given region) and their products.

In Units 3 and 4, we discuss the methods of solving ordinary differential equations. In **Unit 3** entitled **'First Order Ordinary Differential Equations'**, we first present the basic definitions and classification of ordinary differential equations as well as the concepts of general solution and particular solution. Then we discuss some methods of solving these equations along with examples from mechanics, radioactive decay and electrical circuits. Unit 3 contains an Appendix on Partial Derivatives of a function of two or more variables. You should study it carefully before studying Unit 3.

In **Unit 4** entitled **'Second Order Ordinary Differential Equations with Constant Coefficients'**, you will study the methods of solving such equations along with their applications in oscillating systems. Units 3 and 4 may be entirely new for you. You should study them carefully and work out the examples, SAQs and Terminal Questions given in them.

We hope you enjoy studying the block and once again wish you success.

What is the velocity of the bird with respect to ground? Solve TQ 3a to find an answer!

UNIT 1

VECTOR ALGEBRA – I

Structure

STUDY GUIDE

We hope that you have studied physics and mathematics at the senior secondary (+ 2) level. We shall take it for granted that you know the basic concepts of vector algebra presented in this unit. Therefore, we shall quickly revise those concepts in this unit. You have to make sure that you know these concepts about vectors very well and then study the remaining course. We have given a set of problems as pre-test in the beginning of this unit and an SAQ in each section of the unit. Each of these should take you at most 5 to 10 minutes to solve. If you are able to solve them, then you know these basic vector concepts. You can skip studying these sections. Otherwise, you should study the sections thoroughly and make sure you can solve the problems before studying the next unit. Of course, you should try to solve the problems on your own without first looking at the solutions and answers!

> ***"Do not worry about your difficulties in Mathematics. I can assure you mine are still greater."*** — ***Albert Einstein***

1.1 INTRODUCTION

In your school physics, you have studied about many physical quantities and have learnt to classify them as scalars and vectors. For example, you know that mass, temperature and time are scalars. You also know that velocity, momentum, acceleration, electric field, etc. are vectors. You must understand that **vectors are mathematical concepts used to describe the real physical properties of the world we live in**. Vectors as we use them today were developed mainly in the 19^{th} and 20^{th} centuries though they have a longer history in a slightly different form.

You may like to read the history of vectors at **http://www.math.mcgill.ca/labute/courses/133f03/VectorHistory.html**

At the beginning of this unit, we shall **revise the elementary concepts of vector algebra**. In Sec. 1.2, you will revise how to classify physical quantities as scalars and vectors. You will also revisit the way of **representing vector quantities geometrically** without any reference to the system of coordinates.

In Sec. 1.3, you will solve problems on how to **add**, **subtract** or **multiply** vectors with scalars using this representation. You will revise the concepts of **scalar** and **vector products** of vectors in Sec. 1.4 and represent various physical quantities as products of vectors. In each section, you will also solve problems applying the concept presented there.

In the next unit, you will revise the algebraic representation of vectors.

Expected Learning Outcomes

After studying this unit, you should be able to:

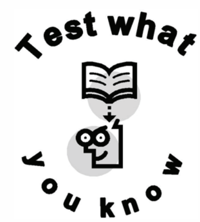
- classify physical quantities as scalars and vectors;
- express a vector in its geometric representation;
- define unit vectors;
- determine the component of a vector in any direction;
- add and subtract vectors using the geometric representation;
- compute the scalar and vector products of two vectors; and
- solve simple physics problems based on the application of vector algebra using the geometric representation.

IN YOUR WRITTEN WORK, ALWAYS USE AN ARROW ABOVE THE LETTER YOU USE TO DENOTE A VECTOR, E.G., $\vec{r}$. USE A CAP ABOVE THE LETTER YOU USE TO DENOTE A UNIT VECTOR, E.G., $\hat{r}$.

1.2 SCALARS AND VECTORS

Do you recall the definitions of scalars and vectors? If so, use those definitions to classify some physical quantities as scalars and vectors in the pre-test given below. Otherwise revise these definitions given in Secs. 1.2.1 and 1.2.2 and then try to solve these problems again.

PRE-TEST



1. Classify the physical quantities in the following statements as scalars and vectors:

   a) The maximum temperature today was 42° C.

   b) The lift's acceleration is 2 ms^{-2} in the upward direction.

   c) The monsoon clouds are moving at a speed of 2 kmh^{-1}.

   d) The density of iron is 7.9×10^{-3} kgm^{-3}.

   e) A stone thrown into a pond sinks at a velocity of 0.5 ms^{-1}.

   f) A sodium vapour lamp produces monochromatic light of wavelength 5893 Å.

   g) The mass of the Earth is 5.9742×10^{24} kg.

   h) The displacement of a train with respect to Delhi is 270 km due north.

   i) The charge of an electron is 1.6×10^{-19} C.

   j) The melting point of iron is 1538° C.

2. Identify the vector quantities from among the following: Charge, force, momentum, speed, distance, impulse, electric field, electric potential, melting point, moment of inertia, velocity, energy, displacement, magnetic field, pressure, weight. Represent each vector graphically in any direction of your choice. Use appropriate notation to denote each vector.

If you have solved these problems correctly, you know how to classify physical quantities as scalars and vectors and represent vectors geometrically. You can skip Secs. 1.2.1 and 1.2.2. Otherwise, study them and try the pre-test again.

1.2.1 Scalars

In your school science courses you have studied about many physical quantities like mass, length, charge, temperature. These quantities are described by a number followed by an appropriate unit of measurement. You know that such quantities are called scalars. Let us first revise the definition of scalars and state some of their properties.

SCALARS

Physical quantities that are **scalars**, are quantities that can be completely described by a number followed by an appropriate unit of measurement.

We represent a scalar in a diagram or an equation, by a letter or a symbol, which represents both the unit of measurement and the number of units for the particular scalar quantity. For example, we can represent the mass of an object by the letter M, where M would be a number with a unit, e.g., 60 kg. Similarly, the temperature at the core of the Earth is T where $T = 300°$ C.

Scalar quantities are added, subtracted, multiplied and divided exactly like ordinary numbers. In fact, all the rules of elementary arithmetic operations apply to the values of a scalar quantity. Thus, if a, b and c are three values of a scalar quantity, for example, the mass of an object, then these satisfy the following properties:

$$a + b = b + a \tag{1.1a}$$

$$ab = ba \tag{1.1b}$$

$$a + 0 = a \tag{1.1c}$$

$$a \times 1 = 1 \times a \tag{1.1d}$$

$$a(bc) = (ab)c \tag{1.1e}$$

$$a(b + c) = ab + ac \tag{1.1f}$$

Scalar quantities have another important property:

A scalar is a quantity whose value does not depend on a coordinate system. **It remains the same in all coordinate systems**. This property is called **invariance.**

Scalar quantities remain invariant (unchanged) under any transformation of coordinate systems.

1.2.2 Vectors

In your school physics, you have also studied about many physical quantities that are vectors. For example, the displacement, velocity and acceleration of objects, the force being exerted on an object and the electric field of a charge are vector quantities.

We revisit the definition of vectors and the way we represent them geometrically.

VECTORS AND THEIR REPRESENTATION

Physical quantities that are **vectors**, are quantities that can be completely described by a **magnitude** which is a non-negative scalar quantity, that is, a positive number along with an appropriate unit, and a **direction** in space. However, if the dimension of the vector quantity is a pure number then we do not specify a unit, e.g., in the case of unit vectors.

Geometric/Graphical Representation of Vectors

To represent a vector geometrically or graphically we need to specify both its magnitude and a direction. So a vector is represented by a **directed line segment** or an arrow, that is, a straight line with an arrowhead. The length of the arrow represents the **magnitude** of the vector quantity, which is a positive scalar quantity and the **arrowhead points along the direction of the vector**. The arrowhead is placed either at the end or somewhere along the line segment (see Fig. 1.1). In Fig. 1.1a, the point *A* is called the **tail** (the starting point) of the vector and the point *B* is called the **head** (terminal point) of the vector. The **direction** of the vector is **from** *A* **to** *B*. The line along which the vector is directed is called the **line of action** of the vector. In Fig. 1.1, this is the line *AB*. It is at an angle θ with respect to the reference line, which is the + *x*-axis in this case (Fig. 1.1b).

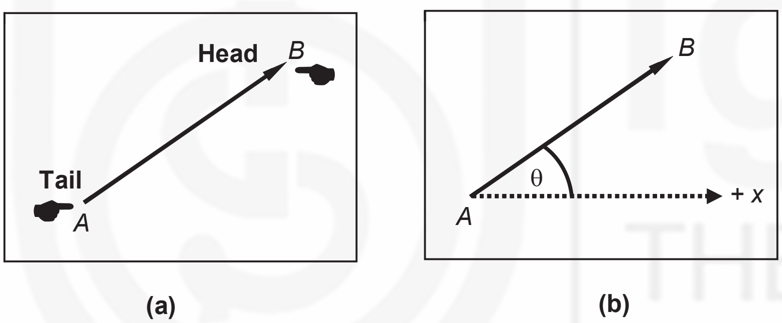


Fig. 1.1: **Geometric representation of a vector. a) A vector is a directed line segment having both magnitude and direction; b) the angle θ specifies the direction of the vector.**

We usually denote a vector in a diagram or an equation by a letter or a symbol with an arrow on top. In the printed text, we will denote vectors by **boldface** letters, with an arrow on top, e.g., $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$, $\vec{\mathbf{c}}$, etc. In your written work, you should denote vectors by drawing arrows above the letter denoting them, e.g., $\vec{a}$, $\vec{b}$, $\vec{c}$, etc. In print, we will denote the magnitude of a vector $\vec{\mathbf{a}}$ by $|\vec{\mathbf{a}}|$, called the modulus of $\vec{\mathbf{a}}$ or by *a*, a light letter in italics.

NOTE

To specify the direction of a vector we need to specify two characteristics, namely the **orientation** and the **sense** of the vector quantity. The *orientation* of the vector is the relationship between the vector and any reference line or plane in space. The *sense* of the vector is determined by the **order** of two points on a line parallel to the vector. It is represented by the arrowhead.

You may now like to go back to the pre-test problems and work them out to test your understanding. In the next section, we briefly revise some concepts about vectors, which you have learnt in your school, namely, the equality of vectors, unit vectors and null vectors. If you know these concepts well, you should first try to solve SAQ 1. Skip the discussion in Sec. 1.2.3 if you answer it correctly.

1.2.3 Equality of Vectors, Unit Vectors and Null Vector

We explain each of these concepts in boxes and give some examples. Then you can try SAQ 1.

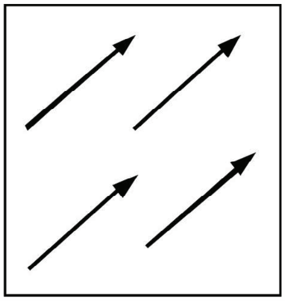


Fig. 1.2: **Equality of vectors. All vectors shown here are equal.**

EQUALITY OF VECTORS

Two vectors are said to be **equal** if they have the same magnitude and the same direction. We denote the equality of two vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ by writing

$$\vec{\mathbf{A}} = \vec{\mathbf{B}} \tag{1.2a}$$

For example, the four vectors shown in Fig. 1.2 are equal even though they are drawn at different places on the page. This is because their magnitudes are equal and they are all in the same direction. **REMEMBER that the locations and starting points of equal vectors do not matter but they should be parallel to each other and represent the same quantity**. Such vectors are also called **free vectors**. By definition, **a free vector** remains the same when translated parallel to itself in space.

In physics, sometimes **the line of action of a vector remains fixed**. For example, the line of action of the acceleration due to gravity ($\vec{\mathbf{g}}$) of a falling body is fixed. Similarly, a force exerted on a rigid body may be applied at any point on the body on its line of action. In such cases two vectors are equal only if they have the same magnitude and direction, and the same line of action. Such vectors are often called **sliding vectors**. Unlike a free vector which can be translated parallel to itself anywhere in space, a **sliding vector can be translated only along its line of action**.

Sometimes even the **initial point of a vector** is fixed. For example, the initial point of the force applied on an elastic body (or the point of application of the force) is fixed. The deformation caused by the force $\vec{\mathbf{F}}$ applied at some point *A* on an elastic body is different from the deformation caused when the same force is applied to a different point *B* on the body. Thus the effect of the force depends on the point of application. Such a vector then has a fixed magnitude, direction and point of application, and is called a **bound vector**. In this case, **two vectors are equal only if they are identical**.

If a vector $\vec{\mathbf{a}}$ has the same magnitude as any other vector $\vec{\mathbf{b}}$ but is in the opposite direction to $\vec{\mathbf{b}}$, then we have

$$\vec{\mathbf{a}} = -\vec{\mathbf{b}} \tag{1.2b}$$

We now define a **unit vector**.

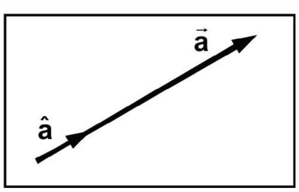


Fig. 1.3: **A unit vector $\hat{\mathbf{a}}$ in the direction of a vector $\vec{\mathbf{a}}$. Its magnitude is 1. We pronounce $\hat{\mathbf{a}}$ as "a cap" or "a hat"**.

UNIT VECTOR

A vector of length or magnitude 1 is called a **unit vector**. By convention, unit vectors are taken to be dimensionless. **A unit vector is used to denote a direction in space**. Any vector $\vec{\mathbf{a}}$ can be represented as the product of its magnitude (a) and a unit vector along its direction denoted by $\hat{\mathbf{a}}$ (see Fig. 1.3). Then we write:

$$\vec{\mathbf{a}} = a\,\hat{\mathbf{a}} \tag{1.3a}$$

or

$$\hat{\mathbf{a}} = \frac{\vec{\mathbf{a}}}{a} = \frac{\vec{\mathbf{a}}}{|\vec{\mathbf{a}}|} \tag{1.3b}$$

A unit vector specifies a direction. Once we define the unit vector in a given direction, we can express any vector in that direction as the product of its magnitude with this unit vector.

A unit vector does not have a dimension or a unit.

Let us explain why we need this concept. Suppose a person P travels 5 m eastwards and another person Q travels 10 m in the same direction from the same point O (see Fig. 1.4). We can define a unit vector having magnitude 1 m pointing towards the eastern direction and denote it by $\hat{\mathbf{i}}$. Then, with respect to O, the displacements of P (vector $\vec{\mathbf{d}}$) and Q (vector $\vec{\mathbf{D}}$) are given as

$$\vec{\mathbf{d}} = (5\text{ m})\hat{\mathbf{i}} \quad \text{and} \quad \vec{\mathbf{D}} = (10\text{ m})\hat{\mathbf{i}}$$

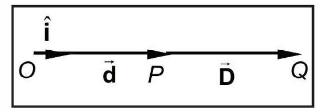


Fig. 1.4: **Using a unit vector, we can write** $\vec{\mathbf{d}} = (5\text{ m})\hat{\mathbf{i}}$ **and** $\vec{\mathbf{D}} = (10\text{ m})\hat{\mathbf{i}}$.

Now, **to represent any new vector in that direction we only need to multiply its magnitude by** $\hat{\mathbf{i}}$.

Let us now define the **null vector** or the **zero vector**.

NULL VECTOR OR ZERO VECTOR

A **null vector** or **zero vector** is a vector which has zero magnitude and no definite direction. It is denoted by the symbol $\vec{\mathbf{0}}$.

Why do we need to define a null vector? Consider this example from physics. Suppose a girl walks 1 km due north and then turns around and walks 1 km due south to return to the starting point. What is her displacement? Its magnitude is zero but since displacement is a vector, it has to be represented as one. We say that the girl's displacement is $\vec{\mathbf{0}}$.

Similarly, when two equal and opposite forces are exerted on a body, the net force on it is the zero vector $\vec{\mathbf{0}}$. When we multiply a vector by a scalar $m = 0$, the result is a zero or null vector $\vec{\mathbf{0}}$.

You may now like to attempt an SAQ. Try SAQ 1!

SAQ 1 – Equality of vectors, unit vector and zero vector

a) In Fig. 1.5, identify the vector equal to the vector $\vec{\mathbf{a}}$ shown there.

b) Let $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ be the unit vectors along the x and y-axes shown in Fig. 1.5. Draw the vectors 2.5 $\hat{\mathbf{i}}$ and 4.0 $\hat{\mathbf{j}}$.

c) Represent each vector shown in Fig. 1.5 in the unit vector notation of Eq. (1.3a).

d) A bus starts from its depot in the morning and returns to the same position in the evening. What is the displacement of the bus during this time period?

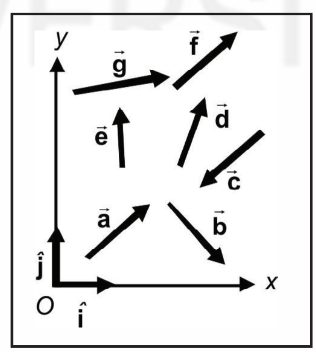


Fig. 1.5: Diagram for SAQ 1c.

So far you have revised the definition of vectors and their geometric representation. You have also revised the concepts of equal vectors, unit vectors and zero vector/null vector. You know that in physics while expressing physical quantities and laws mathematically, we need to perform mathematical operations on vectors. These operations follow specific rules, different from those for mathematical operations on scalars. In Sec.1.3, you will learn some elementary mathematical operations on vectors such as **vector addition**, **subtraction** and **multiplication of a vector by a scalar**.

1.3 VECTOR ALGEBRA

Vectors may be added and subtracted as well as multiplied by regular numbers (scalars). In Secs. 1.3.1 and 1.3.2, you will revise addition and subtraction of vectors using the geometric representation of vectors. If you can solve the SAQs 2 and 3 in this section, then you know these concepts and you can skip the discussion. Otherwise solve them after studying Secs. 1.3.1 and 1.3.2.

1.3.1 Addition of Vectors

Suppose two forces act on an object and we wish to find the net force on it. We can find it by performing the operation of vector addition. In this operation, we add two vector quantities, say, $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ which are of the same type (for example, two displacements or two forces) to produce another vector $\vec{\mathbf{c}}$ of the same type. We then write

$$\vec{\mathbf{c}} = \vec{\mathbf{a}} + \vec{\mathbf{b}} \tag{1.4}$$

Always think of the symbols + and = in the equations showing vector addition as 'combined with' and 'equivalent to'. The meanings of these symbols are different from their meanings in ordinary algebra, viz., '*added to*' and '*equal to*'.

We use special methods, namely, the **triangle law of vector addition** and the **parallelogram law of vector addition** to determine $\vec{\mathbf{c}}$. These are equivalent methods and you have studied them in your school physics. Let us now state these laws (see Fig. 1.6a).

VECTOR ADDITION: TRIANGLE LAW

To add two vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ and determine their sum $\vec{\mathbf{c}} = \vec{\mathbf{a}} + \vec{\mathbf{b}}$:

- Draw them in such a way that the head of the first vector, say $\vec{\mathbf{a}}$, is joined with the tail of the second vector, say $\vec{\mathbf{b}}$ (Fig. 1.6b).
- Draw the arrow from the tail of the first vector ($\vec{\mathbf{a}}$) to the head of the second vector ($\vec{\mathbf{b}}$). This represents the resultant vector

$$\vec{\mathbf{c}} = \vec{\mathbf{a}} + \vec{\mathbf{b}} \tag{1.4}$$

Fig. 1.6: The triangle law of vector addition for two vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$.

It is better to use the **parallelogram law of vector addition** when the two vectors we want to add have their tails at a common point. For example, this method is useful when we want to find the resultant of two forces acting at the same point on an object. Using this law, we can calculate algebraically, the magnitude and direction of the resultant vector.

PARALLELOGRAM LAW OF VECTOR ADDITION

The sum $\vec{\mathbf{c}} = \vec{\mathbf{a}} + \vec{\mathbf{b}}$ of vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ with their tails at a common point *O* is represented by the diagonal *OC* of the parallelogram through *O* with $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ as its sides (Fig. 1.7).

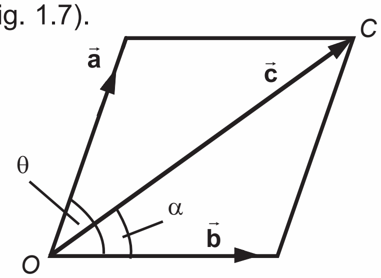


Fig. 1.7: **The parallelogram law of vector addition**.

You may like to know: What are the magnitude and direction of the resultant $\vec{\mathbf{c}}$? The expressions for the magnitude and direction of the resultant $\vec{\mathbf{c}}$ for two vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ having the angle θ between them are given as follows:

$$c = \sqrt{b^2 + 2ab\cos\theta + a^2} \qquad (1.5a)$$

$$\alpha = \tan^{-1}\left[\frac{a\sin\theta}{b + a\cos\theta}\right] \qquad (1.5b)$$

Here *a*, *b* and *c* are the magnitudes of the vectors $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$ and $\vec{\mathbf{c}}$, respectively, and the angle α between the vectors $\vec{\mathbf{b}}$ and $\vec{\mathbf{c}}$ gives the direction of the vector $\vec{\mathbf{c}}$ (see Fig. 1.7).

You can prove Eqs. (1.5a and b) yourself. This is given as terminal question 8.

Note that vector addition is not an algebraic sum. We cannot add the magnitudes of the vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ to get the magnitude of vector $\vec{\mathbf{c}}$. The two methods of vector addition described above are called the graphical methods where we use the geometric representation of vectors. What happens when you wish to **add more than two vectors**? Vector addition is binary, which means that, just like numbers, you must add vectors two at a time. Before you revise how to do that, you may like to recall two important properties of the sum of vectors.

PROPERTIES OF THE SUM OF VECTORS

1. **Commutative property of vector addition:** The sum of two vectors is the same whatever be the order in which the two vectors are added. You can verify from Fig. 1.8 that

$$\vec{\mathbf{a}} + \vec{\mathbf{b}} = \vec{\mathbf{b}} + \vec{\mathbf{a}} \qquad (1.6)$$

2. **Associative property of vector addition:** If more than two vectors are added, it does not matter how they are grouped:

$$(\vec{\mathbf{a}} + \vec{\mathbf{b}}) + \vec{\mathbf{c}} = \vec{\mathbf{a}} + (\vec{\mathbf{b}} + \vec{\mathbf{c}}) \qquad (1.7)$$

NOTE

To find the sum $\vec{\mathbf{a}} + \vec{\mathbf{b}}$ of two vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ situated at different points:

Shift the second vector $\vec{\mathbf{b}}$ parallel to itself so that its tail joins **the head of the first vector $\vec{\mathbf{a}}$. Draw the vector from the tail of the first to the head of the second.**

Note that the vector sum $\vec{\mathbf{c}}$ lies in the plane containing vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$, which is the plane of the page you are reading.

You can think of adding two vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ as taking **two successive walks**: their vector sum is the vector from the beginning point to the end point. **Note that here we are using the concept of equality of vectors.**

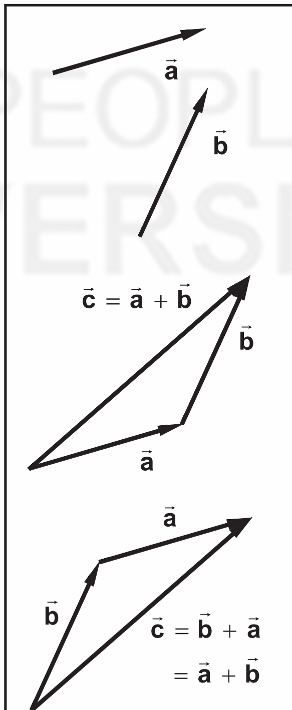


Fig. 1.8: **Vector addition is commutative**.

You may like to verify the associative property of vector addition before studying further. For this you will need to apply the triangle law of addition twice. Study Fig. 1.9 and then attempt SAQ 2 for adding more than two vectors.

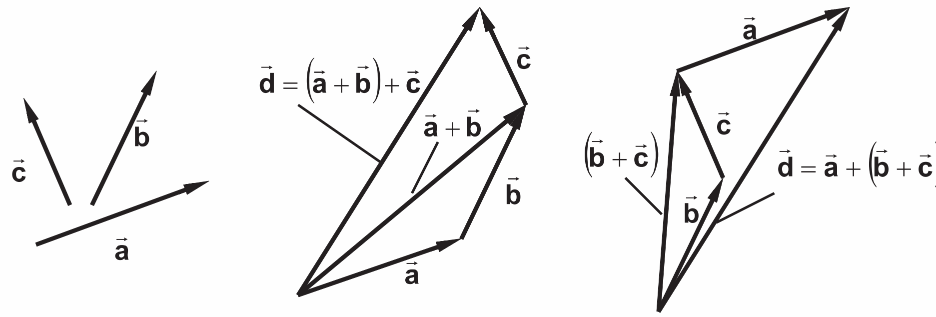


Fig. 1.9: **Vector addition is associative:** $\vec{d} = (\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$.

SAQ 2 – Adding more than two vectors

a) Determine the sum of the vectors shown in Fig. 1.10a.

b) Three forces $\vec{F}_1$, $\vec{F}_2$ and $\vec{F}_3$, all in the same plane are exerted on an object (Fig. 1.10b). What force should be applied to it, so that it does not move under the action of these three forces?

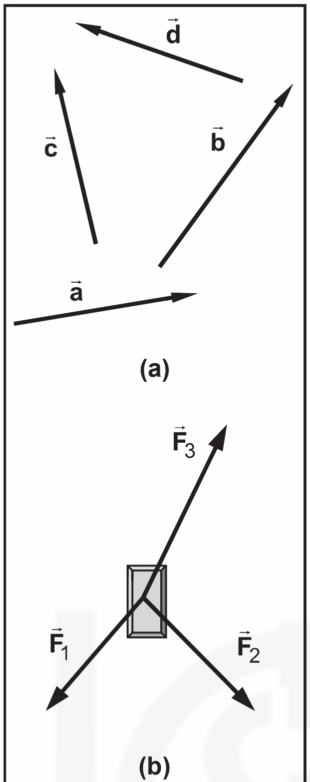


Fig. 1.10: **Adding more than two vectors**.

While solving SAQ 2a, did you note that the resultant is the vector drawn from the tail of the first vector in the sum to the head of the last vector in the sum? This is the **polygon law of vector addition** (read the margin remark).

POLYGON LAW OF VECTOR ADDITION

If a number of vectors are represented in magnitude and direction, by the sides of a polygon, taken in order, then the resultant vector is represented in magnitude and direction by the closing side of the polygon taken in the opposite order, that is, from the tail of the first vector to the head of the last vector (see Fig. 1.11).

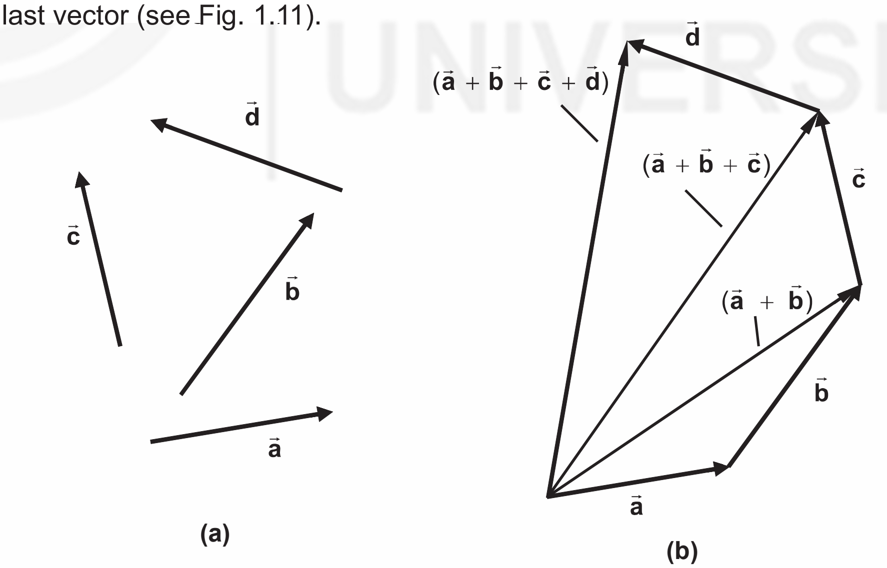


Fig. 1.11: **Polygon law of vector addition applied for determining the resultant** $(\vec{a} + \vec{b} + \vec{c} + \vec{d})$ **of four vectors** $\vec{a}$, $\vec{b}$, $\vec{c}$ **and** $\vec{d}$.

It may seem from Figs. 1.8 to 1.11 that all vectors lie in the same plane, which is the plane of the paper. This may not always be true. For example, in Fig. 1.9, the vector $\vec{c}$ may not lie in the plane of the vector $(\vec{a} + \vec{b})$. Then the geometric representation of vectors is not convenient. If all the vectors do not lie in the same plane, you will have to draw each step of Eq. (1.7) separately.

You would have noted in your answer to SAQ 2a that **even if we use a different sequence of vectors for addition, the end result is the same**.

Now suppose you wish to add a vector $\vec{\mathbf{a}}$ three times to determine $(\vec{\mathbf{a}} + \vec{\mathbf{a}} + \vec{\mathbf{a}})$. From vector addition, you can see that the sum is a vector with three times the magnitude of $\vec{\mathbf{a}}$ and in the same direction as $\vec{\mathbf{a}}$. We can extend this idea to the product of a vector with a scalar m (Fig. 1.12). Let us give a formal definition.

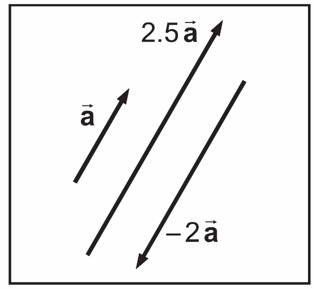


Fig. 1.12: **Multiplying a vector by a scalar.**

MULTIPLICATION OF A VECTOR BY A SCALAR

A vector $\vec{\mathbf{a}}$ when multiplied by a scalar quantity m, is equal to the vector $m\,\vec{\mathbf{a}}$. It has magnitude $|m||\vec{\mathbf{a}}|$. Also,

1. If $\vec{\mathbf{a}} \neq \vec{\mathbf{0}}$ and $m > 0$, then $m\vec{\mathbf{a}}$ is in the same direction as $\vec{\mathbf{a}}$.
2. If $\vec{\mathbf{a}} \neq \vec{\mathbf{0}}$ and $m < 0$, then $m\vec{\mathbf{a}}$ is in the direction opposite to $\vec{\mathbf{a}}$.
3. If $\vec{\mathbf{a}} \neq \vec{\mathbf{0}}$ and $m = 0$ or $\vec{\mathbf{a}} = \vec{\mathbf{0}}$ then $m\vec{\mathbf{a}} = \vec{\mathbf{0}}$.

For a scalar m, $|m|$ is always a positive quantity.

$|m| = m$ if $m > 0$ and

$|m| = -m$ if $m < 0$.

In physics, there are many quantities which involve the product of a vector with a scalar. For example, the linear momentum $\vec{\mathbf{p}}$ of a particle of mass m moving with a velocity $\vec{\mathbf{v}}$ is $\vec{\mathbf{p}} = m\vec{\mathbf{v}}$. As per Newton's second law of motion, for a particle of constant mass, the force $\vec{\mathbf{F}}$ is the product of its mass and acceleration $\vec{\mathbf{a}}$: $\vec{\mathbf{F}} = m\,\vec{\mathbf{a}}$. So far, you have learnt how to add vectors geometrically. Let us now revise the concept of subtraction of vectors.

$\vec{\mathbf{a}} + \vec{\mathbf{0}} = \vec{\mathbf{a}}$

$\vec{\mathbf{a}} + (-\vec{\mathbf{a}}) = \vec{\mathbf{0}}$

$m(\vec{\mathbf{a}} + \vec{\mathbf{b}}) = m\vec{\mathbf{a}} + m\vec{\mathbf{b}}$

$(m+n)\vec{\mathbf{a}} = m\vec{\mathbf{a}} + n\vec{\mathbf{a}}$

$m(n\vec{\mathbf{a}}) = (mn)\vec{\mathbf{a}}$

$1(\vec{\mathbf{a}}) = \vec{\mathbf{a}}$

$0(\vec{\mathbf{a}}) = \vec{\mathbf{0}}$

$(-1)\vec{\mathbf{a}} = -\vec{\mathbf{a}}$

1.3.2 Subtraction of Vectors

To subtract vector $\vec{\mathbf{b}}$ from vector $\vec{\mathbf{a}}$ of the same type, we add the vectors $\vec{\mathbf{a}}$ and $(-\vec{\mathbf{b}})$:

$$\vec{\mathbf{a}} - \vec{\mathbf{b}} = \vec{\mathbf{a}} + (-\vec{\mathbf{b}}) \tag{1.8}$$

This is also called the difference of the two vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$. Recall that the vector $-\vec{\mathbf{b}}$ **is a vector of the same magnitude as $\vec{\mathbf{b}}$ but opposite in direction**. **If you interchange the head and tail of any vector $\vec{\mathbf{a}}$, you get the vector** $-\vec{\mathbf{a}}$. Note that, unlike vector addition, vector subtraction is not commutative because

$$\vec{\mathbf{b}} - \vec{\mathbf{a}} = -(\vec{\mathbf{a}} - \vec{\mathbf{b}}) \tag{1.9}$$

Thus, the vectors $(\vec{\mathbf{b}} - \vec{\mathbf{a}})$ and $(\vec{\mathbf{a}} - \vec{\mathbf{b}})$ are equal in magnitude but exactly opposite in direction. We now show you how to subtract vectors.

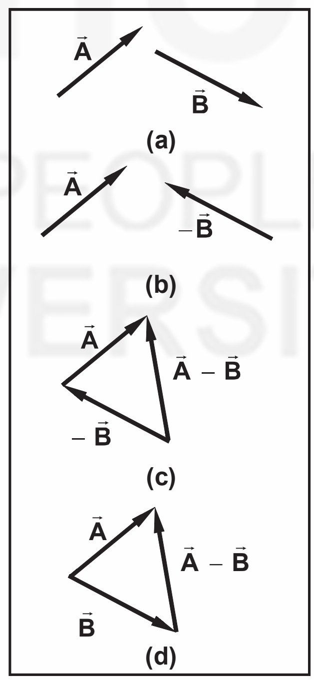


Fig. 1.13: **a) Subtraction of vector $\vec{\mathbf{B}}$ from vector $\vec{\mathbf{A}}$; b) reverse the direction of $\vec{\mathbf{B}}$ to get $-\vec{\mathbf{B}}$; c) add $\vec{\mathbf{A}}$ and $-\vec{\mathbf{B}}$; d) the alternative method.**

HOW TO SUBTRACT VECTORS

Suppose you have to subtract vector $\vec{\mathbf{B}}$ from vector $\vec{\mathbf{A}}$ shown in Fig. 1.13a. **Since $\vec{\mathbf{A}} - \vec{\mathbf{B}}$ is the vector $\vec{\mathbf{A}} + (-\vec{\mathbf{B}})$, we can reverse the direction of vector $\vec{\mathbf{B}}$** (as shown in Fig. 1.13b) **and then add it to $\vec{\mathbf{A}}$ to get $\vec{\mathbf{A}} - \vec{\mathbf{B}}$** (Fig. 1.13c).

Alternately, **when** $\vec{\mathbf{A}} - \vec{\mathbf{B}}$ is added to $\vec{\mathbf{B}}$, it gives $\vec{\mathbf{A}}$. Hence, we can **move $\vec{\mathbf{B}}$ parallel to itself so that the tails of $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ are placed together. Then $\vec{\mathbf{A}} - \vec{\mathbf{B}}$ is the vector from the head of $\vec{\mathbf{B}}$ to the head of $\vec{\mathbf{A}}$** (Fig. 1.13d).

You may now like to solve a problem on vector addition and subtraction.

SAQ 3 **– Addition and subtraction of vectors**

Using Eqs. (1.5a and b), obtain the vectors $\vec{\mathbf{A}}+\vec{\mathbf{B}}$ and $\vec{\mathbf{A}}-\vec{\mathbf{B}}$ given that $|\vec{\mathbf{A}}| = 3.0\,\mathrm{ms}^{-1}$ with $\vec{\mathbf{A}}$ directed towards east and $|\vec{\mathbf{B}}| = 4.0\ \mathrm{ms}^{-1}$ with $\vec{\mathbf{B}}$ directed 45° west of north. Choose appropriate scale and draw vector diagrams.

1.4 PRODUCTS OF VECTORS

Vectors can be multiplied in two different ways to give either a scalar or a vector. Depending on whether the result is a scalar or a vector, the product is called either a **scalar product** or a **vector product**. Again, if you know these concepts well enough, just try the SAQs given in this section. If you are not able to solve them, go through this section carefully and then try the SAQs.

Note that unlike vector addition, in the scalar product of vectors, the vectors need not represent the same physical quantity.

1.4.1 Scalar Product

Let us first define the scalar product and state its properties.

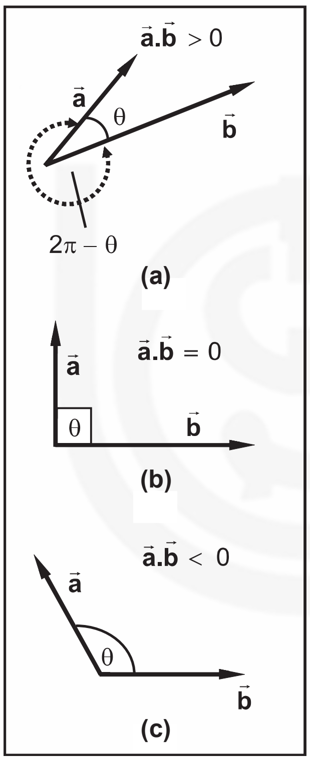


Fig. 1.14: **a) Definition of the scalar product. The scalar product of vectors at acute angles is positive**; **b) scalar product of perpendicular vectors is zero**; **c) scalar product of vectors at obtuse angles is negative**.

SCALAR PRODUCT AND ITS PROPERTIES

The **scalar** or **dot product** of any two non-zero vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ is written as $\vec{\mathbf{a}}.\vec{\mathbf{b}}$ and is a scalar quantity defined as:

$$\vec{\mathbf{a}}.\vec{\mathbf{b}} = |\vec{\mathbf{a}}|\,|\vec{\mathbf{b}}|\cos\theta = ab\cos\theta \qquad (1.10)$$

The angle θ is the angle between $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ (or more properly, between the directions of $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ as shown in Fig. 1.14a) when they are placed tail to tail. There are actually two such angles: θ and $(360^o - \theta)$ or $(2\pi - \theta)$. However, either of these can be used as their cosines have the same values. The product $\vec{\mathbf{a}}.\vec{\mathbf{b}}$ is pronounced as 'a dot b'. We now state some properties of the scalar product, which follow from its definition.

- If the two vectors are **parallel**, their scalar product is maximum:

$$\vec{\mathbf{a}}\,.\,\vec{\mathbf{b}} = ab \quad \text{for } \theta = 0^\circ \quad \text{since } \cos 0^\circ = 1 \qquad (1.11a)$$

- If the two vectors are **perpendicular** to each other, we have:

$$\vec{\mathbf{a}}\,.\,\vec{\mathbf{b}} = 0 \quad \text{for } \theta = 90^\circ \quad \text{since } \cos 90^\circ = 0 \qquad (1.11b)$$

- The scalar product of a vector $\vec{\mathbf{b}}$ with itself is given by:

$$\vec{\mathbf{b}}\,.\,\vec{\mathbf{b}} = b^2 \quad \text{or} \quad b = \sqrt{\vec{\mathbf{b}}\,.\,\vec{\mathbf{b}}} \qquad (1.11c)$$

- The angle between the vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ (when these are placed tail to tail) is given by:

$$\cos\theta = \frac{\vec{\mathbf{a}}.\vec{\mathbf{b}}}{ab} \qquad (1.11d)$$

- **Commutative property**: Scalar product is commutative as it is a scalar quantity and does not depend on the order in which the vectors appear:

$$\vec{\mathbf{a}}\,.\,\vec{\mathbf{b}} = \vec{\mathbf{b}}\,.\,\vec{\mathbf{a}} \qquad (1.11e)$$

- **Distributive property**: Scalar product also obeys the **distributive law**:

$$\vec{\mathbf{a}}\,.\,(\vec{\mathbf{b}} + \vec{\mathbf{c}}) = \vec{\mathbf{a}}\,.\,\vec{\mathbf{b}} + \vec{\mathbf{a}}\,.\vec{\mathbf{c}} \qquad (1.11f)$$

There are many scalar quantities in physics that are expressed as the scalar product of vectors. For example,

- The work (W) done on an object by a constant force $\vec{\mathbf{F}}$ during its displacement $\vec{\mathbf{d}}$ is the scalar product of $\vec{\mathbf{F}}$ and $\vec{\mathbf{d}}$: $W = \vec{\mathbf{F}} \,.\, \vec{\mathbf{d}}$ (see Fig. 1.15a).
- Power (P) is defined as the rate at which work is done by a force on an object and is expressed as the scalar product of the force $\vec{\mathbf{F}}$ applied on it and its velocity $\vec{\mathbf{v}}$: $P = \vec{\mathbf{F}} \,.\, \vec{\mathbf{v}}$
- The potential energy (U) of an electric dipole having dipole moment $\vec{\mathbf{p}}$ placed in an electric field $\vec{\mathbf{E}}$ depends on the angle which the dipole makes with the field. It is expressed as the scalar product of $\vec{\mathbf{p}}$ and $\vec{\mathbf{E}}$: $U = \vec{\mathbf{p}} \,.\, \vec{\mathbf{E}}$ (Fig. 1.15b).
- The potential energy (U) of a magnetic dipole moment $\vec{\boldsymbol{\mu}}$ in a magnetic field $\vec{\mathbf{B}}$ depends on the angle between the magnetic dipole and the field $U = \vec{\boldsymbol{\mu}}.\vec{\mathbf{B}}$ (Fig. 1.15c).

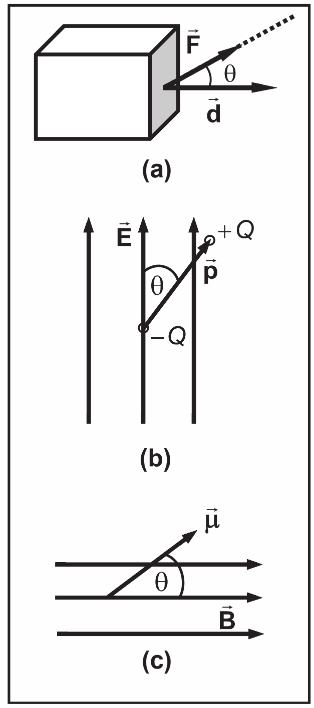


Fig. 1.15: **Examples of scalar product in physics**.

We can use the scalar product to determine the **projection of one vector on another vector**. The **projection** of a vector $\vec{\mathbf{a}}$ on another vector $\vec{\mathbf{b}}$ is defined as the component of $\vec{\mathbf{a}}$ **along** $\vec{\mathbf{b}}$ (Fig. 1.16). It is $|\vec{\mathbf{a}}|\cos\theta$, where θ is the angle between $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$. Also the **component of the vector** $\vec{\mathbf{a}}$ in the direction **perpendicular to the direction of** $\vec{\mathbf{b}}$ is $|\vec{\mathbf{a}}|\cos(90° - \theta)$ or $|\vec{\mathbf{a}}|\sin\theta$.

Thus, the component of $\vec{\mathbf{a}}$ **parallel** to $\vec{\mathbf{b}}$ $= \dfrac{\vec{\mathbf{a}}.\vec{\mathbf{b}}}{|\vec{\mathbf{b}}|} = |\vec{\mathbf{a}}|\cos\theta$ (1.11g)

The component of $\vec{\mathbf{a}}$ perpendicular to $\vec{\mathbf{b}}$ $= |\vec{\mathbf{a}}|\sin\theta$ (1.11h)

You may now like to solve an SAQ on the concept of scalar product.

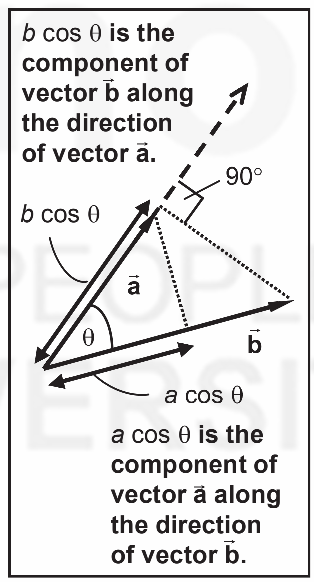


Fig. 1.16: **Projection or the component of a vector along the direction of another vector**.

SAQ 4 – Scalar product of vectors

a) For each pair of vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ given below, calculate $\vec{\mathbf{a}} \,.\, \vec{\mathbf{b}}$:

  i) $a = 4$ units, $b = 5$ units, $\theta = 30^\circ$

  ii) $a = 5$ units, $b = 5$ units, $\theta = 150^\circ$

  iii) $a = 2$ units, $b = 3$ units, $\theta = 90^\circ$

  iv) $a = 2$ units, $b = 3$ units, $\theta = 0^\circ$

b) The scalar product of two non-zero vectors is zero. Are the vectors parallel or perpendicular to each other?

Let us now study the vector product.

1.4.2 Vector Product

In many cases in physics, the product of two vectors is a **vector**. We, therefore, introduce another useful product of vectors called the **vector product** or **cross product** and devise a special notation for it.

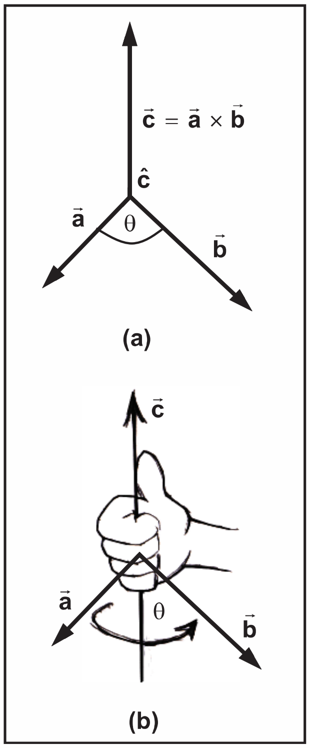


Fig. 1.17: **a) Definition of the vector product**; **b) right-hand rule for the direction of the vector product**.

INTRODUCING THE VECTOR PRODUCT

The **vector product** of the vectors $\vec{a}$ and $\vec{b}$ is the vector $\vec{c} = \vec{a} \times \vec{b}$ **given by**

$$\vec{c} = \vec{a} \times \vec{b} = ab \sin\theta \, \hat{c} \quad \text{with magnitude} \quad c = ab \sin\theta \qquad (1.12)$$

Here θ is the angle between $\vec{a}$ and $\vec{b}$ (or more properly, between the directions of $\vec{a}$ and $\vec{b}$) when they are placed tail-to-tail. There are actually two such angles: θ and $(360^o - \theta)$. Since the sines of these angles have different values, we take **the smaller of the two angles** in the calculations. Thus, $0 \leq \theta \leq \pi$ (Fig. 1.17a). $\vec{a} \times \vec{b}$ **is pronounced as "a cross b"**.

The **direction of the vector product** is given by the unit vector $\hat{c}$, which is a **unit vector perpendicular to the plane containing the vectors** $\vec{a}$ and $\vec{b}$. We determine the **sense** of $\hat{c}$ from the **right hand rule**: Curl the fingers of your right hand so that your fingertips point in the direction of rotation of $\vec{a}$ towards $\vec{b}$. Then the extended thumb as shown in Fig. 1.17b gives the direction of $\hat{c}$. Defined in this way, the vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are said to form a **right-handed triad**.

To understand the point about the order of vectors, can you tell: What is $\vec{b} \times \vec{a}$? Follow the definition of the vector product given above. If you curl your fingers around the line so that your fingertips point in the direction of rotation of $\vec{b}$ to $\vec{a}$, then your thumb points in a direction opposite to that of $\vec{a} \times \vec{b}$. Thus, the direction of the vector $\vec{b} \times \vec{a}$ is opposite to that of the vector $\vec{a} \times \vec{b}$ but the magnitudes of both vectors are equal. Thus, we have

$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a} \qquad (1.13)$$

Therefore, the vector product is **not commutative**. We also say that the vector product is **anti-commutative**.

The order of the vectors is important in the vector product. Vector product is <u>not commutative</u>.

We can express many physical quantities as vector products. Here we give some examples.

Torque: You may have learnt in your school courses about torque. When a net external torque is exerted on an object, it brings about a change in its rotational motion. The torque on an object is defined as

$$\vec{\tau} = \vec{r} \times \vec{F} \qquad (1.14a)$$

where $\vec{F}$ is the **net force applied on the object** and $\vec{r}$ is **the position vector of the point of application of the force from a point on the axis of rotation** (Fig. 1.18a).

Angular momentum: The **angular momentum** $\vec{L}$ of a particle with respect to a chosen origin (Fig. 1.18b) is defined as the vector product of the position vector of the particle with respect to the origin and its linear momentum:

$$\vec{L} = \vec{r} \times \vec{p} \qquad (1.14b)$$

The **force acting on a point charge q moving with a velocity $\vec{v}$ in a magnetic field $\vec{B}$** is given by

$$\vec{F} = q\vec{v} \times \vec{B} \tag{1.14c}$$

The **force acting on an element $d\vec{l}$ of a current carrying conductor in a magnetic field B** is given by

$$\vec{F} = Id\vec{l} \times \vec{B} \tag{1.14d}$$

where I is the current through the conductor. We now state some properties of the vector product.

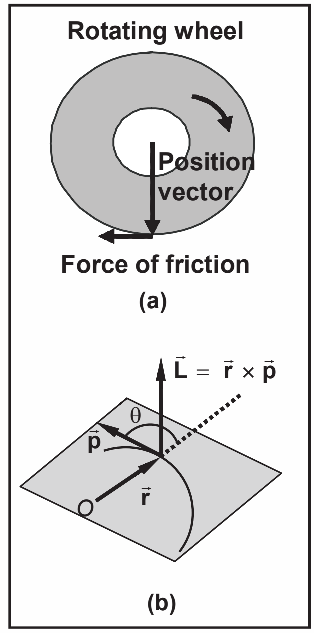


Fig. 1.18: Examples of vector products in physics. a) Torque on the rotating wheel points into the page perpendicular to the plane of the paper; b) angular momentum of a particle having position vector $\vec{r}$ and linear momentum $\vec{p}$.

PROPERTIES OF THE VECTOR PRODUCT

- The **vector product of two parallel vectors is a null vector** (Fig. 1.19a). Thus,

$$\vec{a} \times \vec{b} = \vec{0} \quad \text{since } \theta \text{ is zero and } \sin 0^\circ = 0 \tag{1.15a}$$

- The **vector product of a vector with itself** is a null vector:

$$\vec{a} \times \vec{a} = \vec{0} \tag{1.15b}$$

- From the definition of the vector product, it follows that

$$(k\vec{a}) \times \vec{b} = k(\vec{a} \times \vec{b}) = \vec{a} \times (k\vec{b}) \tag{1.15c}$$

- The **vector product of two vectors perpendicular to each other is maximum** (Fig. 1.19b). Thus,

$$\vec{a} \times \vec{b} = ab\,\hat{n} \quad \text{when } \theta \text{ is } 90^\circ \text{ since } \sin 90^\circ = 1 \tag{1.15d}$$

Here $\hat{n}$ is a unit vector perpendicular to the plane containing both $\vec{a}$ and $\vec{b}$ and its direction is given by the right-hand rule.

- **The vector product is anti-commutative:**

$$\vec{a} \times \vec{b} = -\,\vec{b} \times \vec{a} \tag{1.15e}$$

- **The vector product follows the distributive law, that is,**

$$\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c} \tag{1.15f}$$

- **The vector product is not associative, that is,**

$$\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c} \tag{1.15g}$$

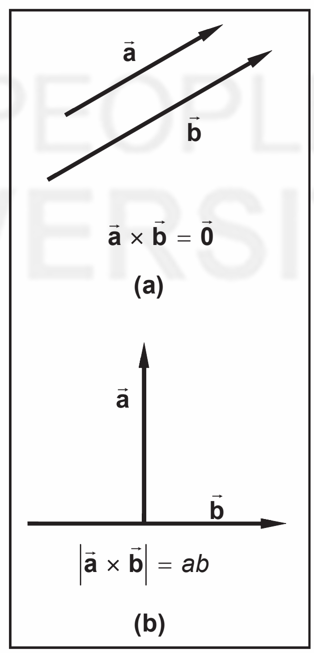


Fig. 1.19: Vector products of (a) parallel and (b) perpendicular vectors.

SAQ 5 – **Vector products**

a) Calculate the vector product $\vec{a} \times \vec{b}$ for the vectors $\vec{a}$ and $\vec{b}$ given in SAQ 4(a).

b) Show that torque at a point about the centre of force O due to a force $\vec{F} = F\,\hat{r}$, where r is the distance of the point from the centre of force, is zero.

The vector product also has a geometrical interpretation, which we now explain through an example.

Example 1.1: GEOMETRICAL INTERPRETATION OF THE VECTOR PRODUCT AS AREA VECTOR

Usually you think of area as a scalar quantity. However, in many applications in physics (e.g., while calculating flux), we also want to know the orientation of the surface of a given area. For example, suppose we want to calculate the rate at which water in a stream flows through a wire loop of a given area. The rates of flow of water will be different when we place the loop parallel to the stream and when we place it perpendicular to the stream. When the loop is parallel to the stream, no water flows through it and the rate of flow is zero. So let us now see how the vector product is used to specify the direction of an area.

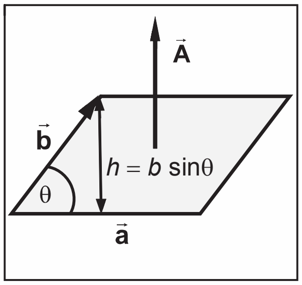


Fig. 1.20: **Geometrical interpretation of vector product as an area vector.**

Consider the area A of a parallelogram with the vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ as its two adjacent sides (Fig. 1.20). Note that the tails of the two vectors are at the same point and the angle between the vectors is θ. The vector product $(\vec{\mathbf{a}} \times \vec{\mathbf{b}})$ has a magnitude of $ab \sin\theta$. It is perpendicular to the plane containing the vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$. From Fig. 1.20, you can see that the area of the parallelogram is given by

$$A = a \times h = a(b \sin\theta) = \left|\vec{\mathbf{a}} \times \vec{\mathbf{b}}\right|$$

We can, therefore, define the vector representing the area of the parallelogram of adjacent sides $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ as the vector product of these vectors:

$$\vec{\mathbf{A}} = \vec{\mathbf{a}} \times \vec{\mathbf{b}}$$

By this definition, the direction of the area vector is perpendicular to the plane of the parallelogram and its sense is given by the right-hand rule. Thus, the area vector $\vec{\mathbf{A}}$ is normal to the surface.

REMEMBER, the sense of the area vector is arbitrary, but once we choose it, it is unique.

We now summarise the concepts you have studied in this unit.

1.5 SUMMARY

Concept	Description
Scalars	■ The physical quantities that are completely specified by a number followed by an appropriate unit of measurement are **scalars**.
Vectors	■ The physical quantities that are completely specified by a magnitude which is a non-negative scalar quantity and a direction in space are **vectors**. A vector is represented geometrically by an arrow (a directed line segment).
Equality of Vectors	■ Two free vectors are equal if they have the same magnitude and direction, regardless of the position of the tail of the vector. If a vector $\vec{\mathbf{b}}$ has the same magnitude but the opposite direction as any other vector $\vec{\mathbf{a}}$ then we can write $\vec{\mathbf{b}} = -\vec{\mathbf{a}}$

Unit Vector

- A vector of length or magnitude 1 is called a **unit vector**. By convention, unit vectors are taken to be dimensionless. A unit vector is used to denote a direction in space. Any vector $\vec{\mathbf{a}}$ can be represented as the product of its magnitude (a) and a unit vector along its direction denoted by $\hat{\mathbf{a}}$. Then we have:

$$\vec{\mathbf{a}} = a\,\hat{\mathbf{a}} \qquad \text{or} \qquad \hat{\mathbf{a}} = \frac{\vec{\mathbf{a}}}{a} = \frac{\vec{\mathbf{a}}}{|\vec{\mathbf{a}}|}$$

Addition and Subtraction of Vectors

- **Triangle Law of Vector Addition**: If two vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ to be added are represented in magnitude and direction by the two sides of a triangle taken in order (which means that the tail of $\vec{\mathbf{b}}$ is at the head of the vector $\vec{\mathbf{a}}$), then their sum or resultant is given in magnitude and direction by the third side of the triangle taken in the opposite order, that is from the tail of the first vector to the head of the second vector (Fig. 1.6).

- **Parallelogram Law of Vector Addition**: If the two vectors to be added are represented in magnitude and direction by the adjacent sides of a parallelogram, then their resultant is given in magnitude and direction by the diagonal of the parallelogram drawn through the point of intersection of the two given vectors.

- **Polygon Law of Vector Addition**: If a number of vectors are represented in magnitude and direction, by the sides of a polygon, taken in order, then the resultant vector is represented in magnitude and direction by the closing side of the polygon taken in the opposite order, that is from the tail of the first vector to the head of the last vector.

- **Vector addition is commutative and associative:**

$$\vec{\mathbf{a}} + \vec{\mathbf{b}} = \vec{\mathbf{b}} + \vec{\mathbf{a}} \quad \text{and} \quad (\vec{\mathbf{a}} + \vec{\mathbf{b}}) + \vec{\mathbf{c}} = \vec{\mathbf{a}} + (\vec{\mathbf{b}} + \vec{\mathbf{c}})$$

- Subtraction of a vector $\vec{\mathbf{b}}$ from a vector $\vec{\mathbf{a}}$ denoted by $\vec{\mathbf{a}} - \vec{\mathbf{b}}$ is just the sum of the vectors $\vec{\mathbf{a}}$ and $(-\vec{\mathbf{b}})$:

$$\vec{\mathbf{a}} - \vec{\mathbf{b}} = \vec{\mathbf{a}} + (-\vec{\mathbf{b}})$$

Multiplication of a vector by a scalar

- A vector $\vec{\mathbf{a}}$ when multiplied by a scalar quantity m, is equal to the vector $m\vec{\mathbf{a}}$, having the magnitude $|m||\vec{\mathbf{a}}|$. The following is true for the multiplication of a vector by a scalar:

$$m(n)\vec{\mathbf{a}} = (m)n\,\vec{\mathbf{a}} = mn\,\vec{\mathbf{a}} \qquad \textbf{Associative Law}$$

$$(m+n)\vec{\mathbf{a}} = m\vec{\mathbf{a}} + n\vec{\mathbf{a}} \quad \text{and} \quad m(\vec{\mathbf{a}} + \vec{\mathbf{b}}) = m\vec{\mathbf{a}} + m\vec{\mathbf{b}} \qquad \textbf{Distributive Laws}$$

If $m = 0$, then $m\vec{\mathbf{a}}$ is a **null** or **zero** vector, which has zero magnitude but no definite direction.

Scalar product

- The **scalar product** of two vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ called "a dot b" and denoted by $\vec{\mathbf{a}}.\vec{\mathbf{b}}$ is a scalar quantity defined as

$$\vec{\mathbf{a}}.\vec{\mathbf{b}} = |\vec{\mathbf{a}}|\,|\vec{\mathbf{b}}|\cos\theta = a\,b\cos\theta$$

Components of a vector in a given direction

- A vector can be resolved into its **component vectors** along any arbitrary direction. The components of a vector $\vec{\mathbf{a}}$ parallel and perpendicular to any other vector $\vec{\mathbf{b}}$ which makes an angle θ with the vector $\vec{\mathbf{a}}$ are given as:

The component of $\vec{\mathbf{a}}$ parallel to $\vec{\mathbf{b}} = \dfrac{\vec{\mathbf{a}}.\vec{\mathbf{b}}}{|\vec{\mathbf{b}}|} = |\vec{\mathbf{a}}|\cos\theta$

The component of $\vec{\mathbf{a}}$ perpendicular to $\vec{\mathbf{b}} = |\vec{\mathbf{a}}|\sin\theta$

Vector product

- The **vector product** of two vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ called "a cross b" and denoted by $\vec{\mathbf{a}} \times \vec{\mathbf{b}}$ is a vector quantity defined as

$$\vec{\mathbf{c}} = \vec{\mathbf{a}} \times \vec{\mathbf{b}} = ab\sin\theta\,\hat{\mathbf{c}} \text{ with magnitude } c = ab\sin\theta$$

The **direction of the vector product** is given by the unit vector $\hat{\mathbf{c}}$ which is a **unit vector perpendicular to the plane containing the vectors $\vec{\mathbf{a}}$** and $\vec{\mathbf{b}}$. We determine the **sense of $\hat{\mathbf{c}}$** from the **right-hand rule**.

1.6 TERMINAL QUESTIONS

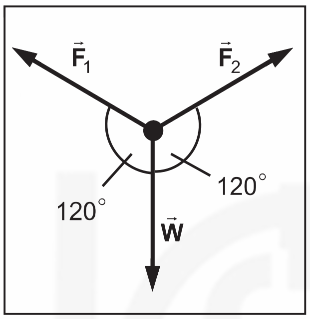


Fig. 1.21: **Forces on an object.**

1. A man walks 1.0 km east, and then walks 1.5 km in the direction 60° west of north. Determine the resultant displacement of the man using the graphical method.

2. An object is supported by two cables, which exert forces $\vec{\mathbf{F}}_1$ and $\vec{\mathbf{F}}_2$ as shown in Fig. 1.21. The weight of the object is $W = 400$ N. Determine F_1 and F_2 if the net force on the object is zero.

3. a) A bird flies directly opposite to the wind at a speed of 2.0 kmh^{-1} with respect to the wind. Wind flows from east to west at a speed of 1.0 kmh^{-1} with respect to the ground. What is the velocity of the bird with respect to the ground?

   b) A man rows a boat across the river at a speed of 2.0 ms^{-1}. The river is flowing at a speed 1.2 ms^{-1}. Determine the direction in which the man must row his boat if he wishes to land on the other bank at a point directly opposite to his starting point.

4. Show that for any two vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$, if the sum and difference of the vectors are perpendicular to each other, the vectors are equal in magnitude.

5. Determine the angle between any two vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ of non-zero magnitude given that $|\vec{\mathbf{a}} + \vec{\mathbf{b}}| = |\vec{\mathbf{a}} - \vec{\mathbf{b}}|$.

6. Show that for any two vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$,

$$|\vec{\mathbf{a}}.\vec{\mathbf{b}}|^2 + |\vec{\mathbf{a}}\times\vec{\mathbf{b}}|^2 = a^2b^2$$

7. A proton having a speed of 5.0×10^6 ms^{-1} moves vertically upward in a uniform magnetic field under a force of 8.0×10^{-14} N towards west. When there is no force on it, it moves horizontally in the northern direction. What is the magnitude and direction of the magnetic field in this region? Charge on the proton $= 1.6 \times 10^{-19}$ C.

8. Prove Eqs. (1.5a and b).

9. Determine $\vec{\mathbf{a}} \cdot (\vec{\mathbf{b}} + \vec{\mathbf{c}})$ given that $a = 2$ units, $b = 6$ units, $c = 1$ unit and the angles between $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$, and $\vec{\mathbf{a}}$ and $\vec{\mathbf{c}}$ are 0° and 90°, respectively.

10. The magnitudes of vectors $\vec{\mathbf{r}}$ and $\vec{\mathbf{s}}$ are 5 units and 6 units, respectively, and the value of $\vec{\mathbf{r}} \cdot \vec{\mathbf{s}}$ is 15. Calculate the angle between the two vectors.

11. Calculate the vector product of the vectors $\vec{\mathbf{r}}$ and $\vec{\mathbf{s}}$ given in TQ 10.

12. Determine the resultant of the forces $\vec{\mathbf{F}}_1$, $\vec{\mathbf{F}}_2$, $\vec{\mathbf{F}}_3$, $\vec{\mathbf{F}}_4$ and $\vec{\mathbf{F}}_5$, shown in Fig. 1.22. (*ABCDEF* is a regular hexagon.)

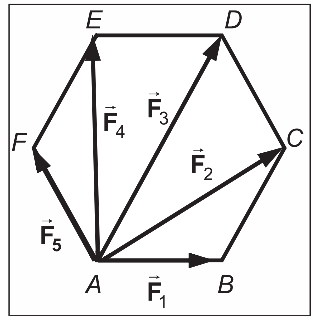


Fig. 1.22: **Diagram for TQ 12.**

13. The magnitude of a vector $\vec{\mathbf{a}}$ is 5 units and it is directed towards east. Vector $\vec{\mathbf{b}}$ has magnitude 4 units and is directed 45° west of north (Fig. 1.23). Determine the magnitude and direction of $\vec{\mathbf{a}} + \vec{\mathbf{b}}$ and $\vec{\mathbf{a}} - \vec{\mathbf{b}}$.

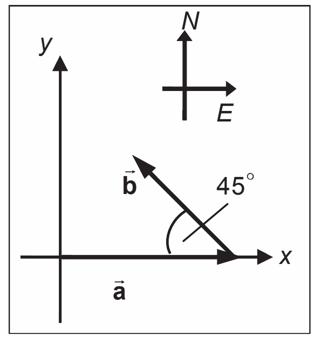


Fig. 1.23: Diagram for TQ 13.

14. A box is being pulled by a rope that makes an angle of 45° with the ground. The force on the box along the rope is 105 N (Fig. 1.24). Determine the horizontal and vertical components of the force. What is the work done by the force in moving the box 10 m along the ground?

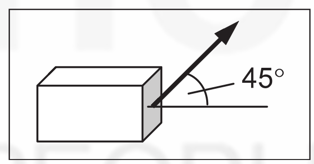


Fig. 1.24: Diagram for TQ 14.

15. Three vectors $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$ and $\vec{\mathbf{c}}$ satisfy the condition $\vec{\mathbf{a}} + \vec{\mathbf{b}} + \vec{\mathbf{c}} = \vec{\mathbf{0}}$. If $|\vec{\mathbf{a}}| = 2, |\vec{\mathbf{b}}| = 1$ and $|\vec{\mathbf{c}}| = 3$ determine the value of $\vec{\mathbf{a}}.\vec{\mathbf{b}} + \vec{\mathbf{b}}.\vec{\mathbf{c}} + \vec{\mathbf{c}}.\vec{\mathbf{a}}$.

1.7 SOLUTIONS AND ANSWERS

Pre-test

1. a) Temperature – scalar; b) acceleration – vector; c) speed – scalar;
   d) density – scalar; e) velocity – vector; f) wavelength – scalar;
   g) mass – scalar; h) displacement – vector; i) charge – scalar;
   j) melting point – scalar.

2. Force, momentum, impulse, electric field, velocity, displacement, magnetic field, weight. Draw arrows for each of these quantities and use a suitable notation as shown for two examples of force and electric field below:

Self-Assessment Questions

1. a) The vector $\vec{\mathbf{f}}$ is equal to the vector $\vec{\mathbf{a}}$. As you can see from Fig. 1.5, $\vec{\mathbf{f}}$ and $\vec{\mathbf{a}}$ are equal in both magnitude and direction.

   b) Refer to the unit vectors $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ in Fig. 1.25. The vector $\vec{\mathbf{a}} = 2.5\,\hat{\mathbf{i}}$ and the vector $\vec{\mathbf{b}} = 4.0\,\hat{\mathbf{j}}$.

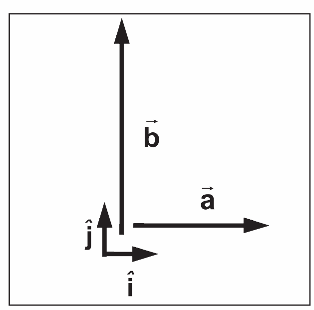


Fig. 1.25: **Diagram for SAQ 1(b)**.

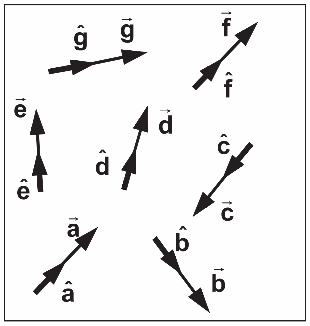


Fig. 1.26: **Diagram for SAQ 1(c). The unit vector along each vector is shown by the darker arrow.**

c) Refer to Fig. 1.26. The vectors in the unit vector notation are:

$$\vec{\mathbf{a}} = a\hat{\mathbf{a}} \qquad \vec{\mathbf{d}} = d\,\hat{\mathbf{d}} \qquad \vec{\mathbf{f}} = f\,\hat{\mathbf{f}}$$
$$\vec{\mathbf{b}} = b\hat{\mathbf{b}} \qquad \vec{\mathbf{e}} = e\hat{\mathbf{e}} \qquad \vec{\mathbf{g}} = g\,\hat{\mathbf{g}}$$
$$\vec{\mathbf{c}} = c\hat{\mathbf{c}}$$

d) The displacement is $\vec{\mathbf{0}}$, the null vector.

2. a) Refer to Fig. 1.27 where we have shown one way of adding up the vectors given in Fig. 1.10(a). Remember you would get the same resultant if you chose a different order of addition of vectors.

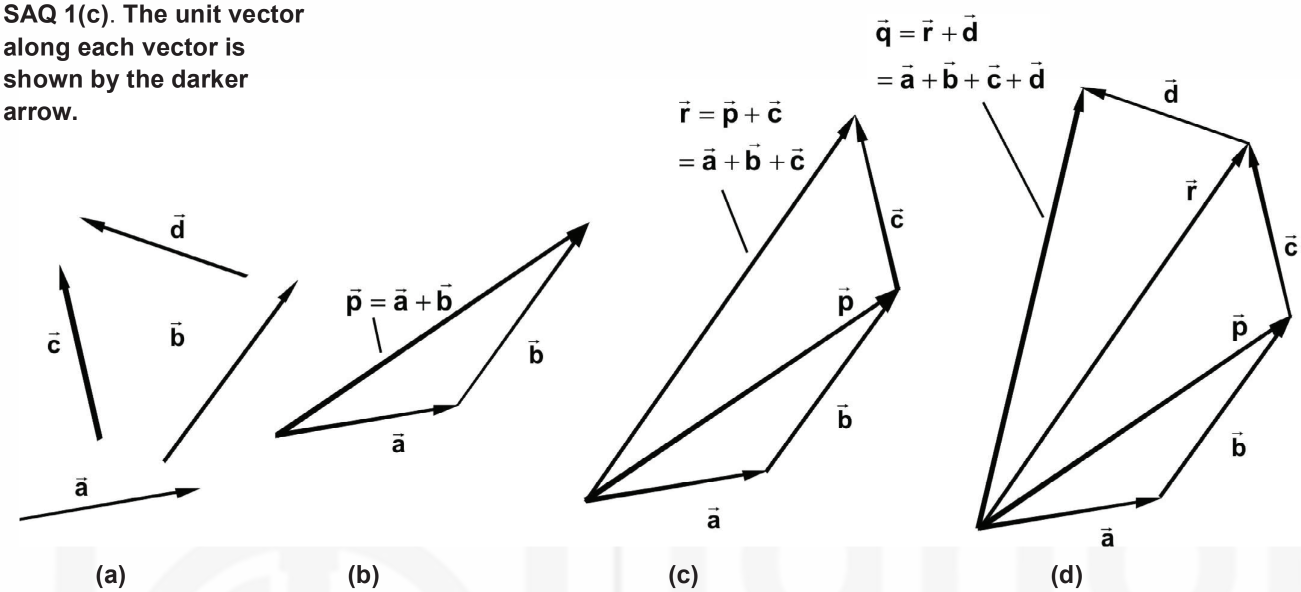


Fig. 1.27: **Addition of four vectors.**

We first add $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ to get $\vec{\mathbf{p}} = \vec{\mathbf{a}} + \vec{\mathbf{b}}$ (Fig. 1.27b) by the triangle law of addition. Next we add $\vec{\mathbf{p}}$ and $\vec{\mathbf{c}}$ to get $\vec{\mathbf{r}} = \vec{\mathbf{p}} + \vec{\mathbf{c}} = \vec{\mathbf{a}} + \vec{\mathbf{b}} + \vec{\mathbf{c}}$ (Fig. 1.27c) using the triangle law of addition for $\vec{\mathbf{p}}$ and $\vec{\mathbf{c}}$. Finally we add the vectors $\vec{\mathbf{r}}$ and $\vec{\mathbf{d}}$ to get the resultant vector $\vec{\mathbf{q}} = \vec{\mathbf{r}} + \vec{\mathbf{d}} = \vec{\mathbf{a}} + \vec{\mathbf{b}} + \vec{\mathbf{c}} + \vec{\mathbf{d}}$ (Fig. 1.27d), which is the sum of the four vectors.

b) The resultant force on the object is the vector sum of forces $\vec{\mathbf{F}}_1$, $\vec{\mathbf{F}}_2$ and $\vec{\mathbf{F}}_3$ (Fig. 1.28a). It is $\vec{\mathbf{F}}_4$ (Fig. 1.28b).

To ensure that the object does not move, a force equal and opposite to the net force $\vec{\mathbf{F}}_4$ must be exerted on the body. So the force to be applied (Fig. 1.28c) is $\vec{\mathbf{F}} = -\vec{\mathbf{F}}_4$.

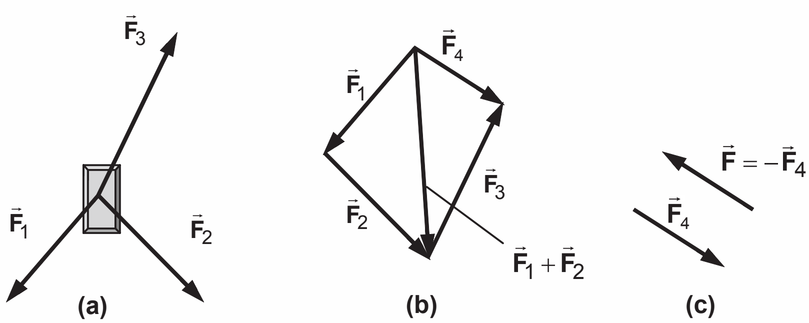


Fig. 1.28: **Diagram for SAQ 2b.**

3. The vectors $\vec{\mathbf{A}}$, $\vec{\mathbf{B}}$, $\vec{\mathbf{p}} = \vec{\mathbf{A}} + \vec{\mathbf{B}}$ and $\vec{\mathbf{q}} = \vec{\mathbf{A}} - \vec{\mathbf{B}}$ are shown in Figs. 1.29a and b.

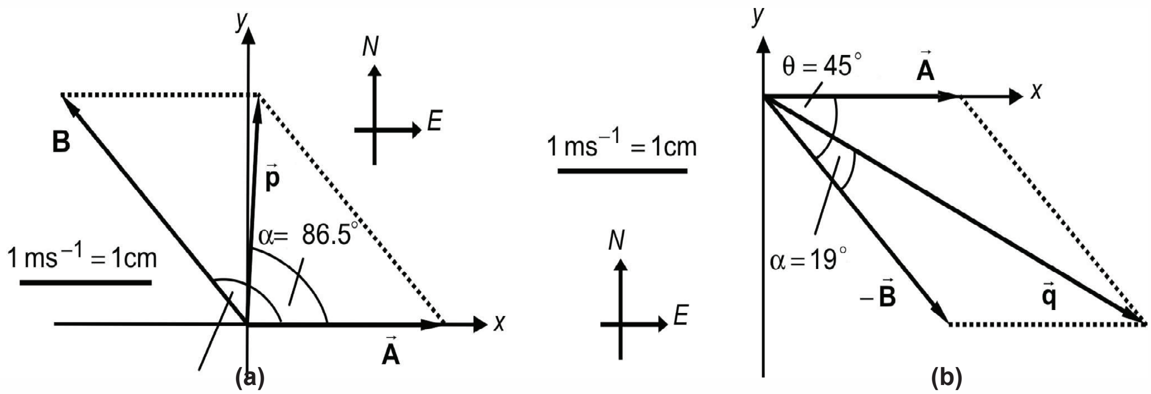


Fig. 1.29: **Diagram for SAQ 3.**

To determine $\vec{\mathbf{p}}$ (Fig. 1.29a), we use Eqs. (1.5a and b) with $\vec{\mathbf{b}} = \vec{\mathbf{A}}$ and $\vec{\mathbf{a}} = \vec{\mathbf{B}}$ and hence $b = 3.0\ \text{ms}^{-1}$, $a = 4.0\ \text{ms}^{-1}$ and $\theta = 135°$, respectively. The angle α is the angle the vector $\vec{\mathbf{p}}$ makes with the vector $\vec{\mathbf{A}}$. The magnitude and direction of $\vec{\mathbf{p}}$ are:

$$p = \sqrt{(3.0)^2 + (4.0)^2 + 2(3.0)(4.0)\cos 135^o} \approx 2.8\text{ms}^{-1},$$

$$\alpha = \tan^{-1}\left(\frac{(4.0)\sin 135^o}{(3.0) + (4.0)\cos 135^o}\right) = \tan^{-1}(16.5) = 86.5^o$$

To determine $\vec{\mathbf{q}}$ (Fig. 1.29b), we use Eqs. (1.5a and b) with $\vec{\mathbf{b}} = -\vec{\mathbf{B}}$ and $\vec{\mathbf{a}} = \vec{\mathbf{A}}$ and hence $b = 4.0\ \text{ms}^{-1}$, $a = 3.0\ \text{ms}^{-1}$ and $\theta = 45°$, respectively. The angle α is the angle the vector $\vec{\mathbf{q}}$ makes with the vector $-\vec{\mathbf{B}}$. The magnitude and direction of $\vec{\mathbf{q}}$ are:

$$q = \sqrt{(3.0)^2 + (4.0)^2 + 2(3.0)(4.0)\cos 45^o} \approx 6.5\text{ms}^{-1},$$

$$\alpha = \tan^{-1}\left(\frac{(3.0)\sin 45^\circ}{(4.0) + (3.0)\cos 45^\circ}\right) = \tan^{-1}(0.346) = 19^\circ$$

4. a) We use Eq. (1.10) to determine $\vec{\mathbf{a}}.\vec{\mathbf{b}}$:

i) For $a = 4$ units, $b = 5$ units, $\theta = 30°$

$\vec{\mathbf{a}}.\vec{\mathbf{b}} = 4 \times 5 \times \cos 30° = 10\sqrt{3}$ units

ii) For $a = 5$ units, $b = 5$ units, $\theta = 150°$

$\vec{\mathbf{a}}.\vec{\mathbf{b}} = 5 \times 5 \times \cos 150° = -\frac{25}{2}\sqrt{3}$ units

iii) For $a = 2$ units, $b = 3$ units, $\theta = 90°$

$\vec{\mathbf{a}}.\vec{\mathbf{b}} = 2 \times 3 \times \cos 90° = 0$ units

iv) For $a = 2$ units, $b = 3$ units, $\theta = 0°$, $\vec{\mathbf{a}}.\vec{\mathbf{b}} = 2 \times 3 \times \cos 0° = 6$ units

b) Let θ be the angle between the vectors. Since the magnitude of the vectors is not zero, from Eq. (1.10) for the scalar product, we can say that

$$\vec{\mathbf{a}}\,.\vec{\mathbf{b}} = ab\cos\theta = 0 \qquad \Rightarrow \qquad \cos\theta = 0 \quad \text{or} \quad \theta = 90^\circ$$

So the vectors are perpendicular to each other.

5. a) To find the vector product we use Eq. (1.12).

i) For $a = 4$ units, $b = 5$ units, $\theta = 30°$

$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = 4 \times 5 \times \sin 30°\ \hat{\mathbf{c}} = 10\ \hat{\mathbf{c}}\ \text{units}$$

ii) For $a = 5$ units, $b = 5$ units, $\theta = 150°$

$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = 5 \times 5 \times \sin 150°\,\hat{\mathbf{c}} = 12.5\,\hat{\mathbf{c}}\ \text{units}$$

iii) For $a = 2$ units, $b = 3$ units, $\theta = 90°$

$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = 2 \times 3 \times \sin 90°\,\hat{\mathbf{c}} = 6\,\hat{\mathbf{c}}\ \text{units}$$

iv) For $a = 2$ units, $b = 3$ units, $\theta = 0°$

$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = 2 \times 3 \times \sin 0°\hat{\mathbf{c}} = \vec{\mathbf{0}}$$

In each case, $\hat{\mathbf{c}}$ is the unit vector perpendicular to the plane containing vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$.

b) The torque $\vec{\boldsymbol{\tau}}$ is given by Eq. (1.14a): $\vec{\boldsymbol{\tau}} = \vec{\mathbf{r}} \times F\,\hat{\mathbf{r}}$

Writing $\vec{\mathbf{r}}$ as $r\,\hat{\mathbf{r}}$ we get, $\vec{\boldsymbol{\tau}} = r\,F\,(\hat{\mathbf{r}} \times \hat{\mathbf{r}}) = \vec{\mathbf{0}}$, because the cross product of a vector with itself is zero [see Eq. (1.15b)].

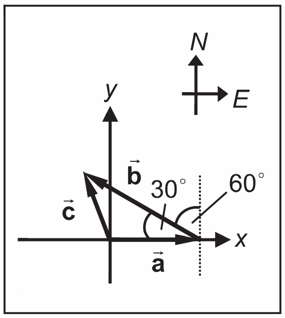


Fig. 1.30: **Diagram for Terminal Question 1**.

Terminal Questions

1. Refer to Fig. 1.30. The final displacement is $\vec{\mathbf{c}}$. The vector $\vec{\mathbf{a}}$ represents the displacement 1.0 km towards east on a chosen scale. Vector $\vec{\mathbf{b}}$ shows the displacement of 1.5 km along 60° west of north. The final displacement $\vec{\mathbf{c}}$ is obtained by joining the tail of $\vec{\mathbf{a}}$ to the head of $\vec{\mathbf{b}}$.

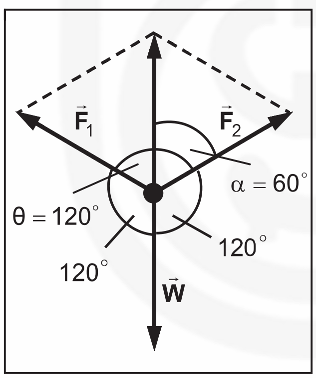


Fig. 1.31: **Diagram for Terminal Question 2**.

2. Here we reproduce Fig. 1.21 as Fig. 1.31. Since the net force on the object is zero, we have

$$\vec{\mathbf{F}}_1 + \vec{\mathbf{F}}_2 + \vec{\mathbf{W}} = \vec{\mathbf{0}} \Rightarrow \vec{\mathbf{F}}_1 + \vec{\mathbf{F}}_2 = -\vec{\mathbf{W}} \qquad \text{(i)}$$

So the vector sum $\vec{\mathbf{F}}$ of the forces $\vec{\mathbf{F}}_1$ and $\vec{\mathbf{F}}_2$ must be equal in magnitude and opposite in direction to $\vec{\mathbf{W}}$ as shown in Fig. 1.31. Let us now write the expression for the resultant $\vec{\mathbf{F}} = \vec{\mathbf{F}}_1 + \vec{\mathbf{F}}_2$ using Eqs. (1.5a and b) with $\vec{\mathbf{a}} = \vec{\mathbf{F}}_1$, $\vec{\mathbf{b}} = \vec{\mathbf{F}}_2$, $\theta = 120°$ and $\alpha = 60°$. Then we get:

$$F = \sqrt{F_1^2 + F_2^2 + 2F_1F_2\cos 120°} = \sqrt{F_1^2 + F_2^2 - F_1F_2} \qquad \text{(ii)}$$

$$\tan\alpha = \tan 60° = \sqrt{3} = \left(\frac{F_1 \sin 120°}{F_2 + F_1 \cos 120°}\right) \qquad \text{(iii)}$$

$F_1 \sin 120° =$

$\sqrt{3}\left(F_2 + F_1 \cos 120°\right)$

$F_1\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3}\left[F_2 - F_1\left(\frac{1}{2}\right)\right]$

$2F_1\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3}F_2$

or $F_1 = F_2$

Simplifying Eq. (iii) we get $F_1 = F_2$ (read the margin remark). This tells us that the magnitudes of $\vec{\mathbf{F}}_1$ and $\vec{\mathbf{F}}_2$ are equal. Substituting this result in Eq. (ii) and since $F = W$, we get:

$$F = \sqrt{F_1^2 + F_1^2 - F_1^2} = F_1 = 400\text{N} \ \text{and}\ F_2 = 400\text{N}$$

3. a) Let the velocity of the bird with respect to wind be $\vec{\mathbf{v}}_{BW}$ and the wind velocity with respect to ground be $\vec{\mathbf{v}}_{WG}$. The velocity of the bird with respect to ground ($\vec{\mathbf{v}}_{BG}$) is given by

$$\vec{\mathbf{v}}_{BG} = \vec{\mathbf{v}}_{BW} + \vec{\mathbf{v}}_{WG}$$

Since the bird is flying directly opposite to the wind, we can write

$$v_{BG} = v_{BW} - v_{WG}$$

$$\therefore \qquad v_{BG} = 2.0\ \text{kmh}^{-1} - 1.0\ \text{kmh}^{-1} = 1.0\ \text{kmh}^{-1}$$

Thus, the bird flies at a speed 1.0 kmh^{-1} with respect to the ground in the direction opposite to the wind, that is, from west to east.

b) Refer to Fig. 1.32. The velocity of the river is $\vec{\mathbf{v}}_R$. Let the velocity with which the man is rowing the boat be $\vec{\mathbf{v}}_B$. The resultant velocity $\vec{\mathbf{v}}_F$ of the boat should lie along AB as shown in Fig. 1.32. So the angle the resultant velocity $\vec{\mathbf{v}}_F$ makes with $\vec{\mathbf{v}}_R$ is $\alpha = 90°$. Let θ be the angle between $\vec{\mathbf{v}}_B$ and $\vec{\mathbf{v}}_R$. We now use Eq. (1.5b) to determine θ with $\vec{\mathbf{b}} = \vec{\mathbf{v}}_R$ and $\vec{\mathbf{a}} = \vec{\mathbf{v}}_B$. It is given that $v_R = 1.2\ \text{ms}^{-1}$ and $v_B = 2.0\ \text{ms}^{-1}$. Putting these values in Eq. (1.5b), we get:

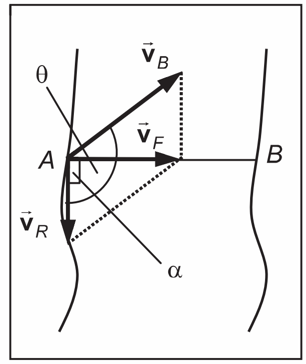


Fig. 1.32: **Diagram for Terminal Question 3**.

$$\tan\alpha = \tan 90^o = \frac{(2.0)\sin\theta}{(1.2) + (2.0)\cos\theta}$$

or $$\cot 90° = \frac{(1.2) + (2.0)\cos\theta}{(2.0)\sin\theta} = 0 \Rightarrow (1.2) + (2.0)\cos\theta = 0$$

or $$\cos\theta = -\frac{1.2}{2.0} \quad \Rightarrow \theta = \cos^{-1}(-0.6) \approx 127°$$

4. For any two vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$, the sum and difference of the two vectors are $\vec{\mathbf{a}} + \vec{\mathbf{b}}$ and $\vec{\mathbf{a}} - \vec{\mathbf{b}}$. Since $\vec{\mathbf{a}} + \vec{\mathbf{b}}$ and $\vec{\mathbf{a}} - \vec{\mathbf{b}}$ are perpendicular to each other, their scalar product is zero and we have:

$$\left(\vec{\mathbf{a}} + \vec{\mathbf{b}}\right) \cdot \left(\vec{\mathbf{a}} - \vec{\mathbf{b}}\right) = 0 \Rightarrow \vec{\mathbf{a}}.\vec{\mathbf{a}} + \vec{\mathbf{b}}.\vec{\mathbf{a}} - \vec{\mathbf{a}}.\vec{\mathbf{b}} - \vec{\mathbf{b}}.\vec{\mathbf{b}} = 0$$

or $$\vec{\mathbf{a}}.\vec{\mathbf{a}} - \vec{\mathbf{b}}.\vec{\mathbf{b}} = a^2 - b^2 = 0 \qquad \text{since } \vec{\mathbf{b}}.\vec{\mathbf{a}} = \vec{\mathbf{a}}.\vec{\mathbf{b}}$$

or $$a^2 = b^2 \ \Rightarrow a = b$$

Hence, the magnitudes of $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are equal.

5. Using Eq. (1.11c), we can write the magnitudes of $\vec{\mathbf{a}} + \vec{\mathbf{b}}$ and $\vec{\mathbf{a}} - \vec{\mathbf{b}}$ as

$$\left|\vec{\mathbf{a}} + \vec{\mathbf{b}}\right| = \sqrt{(\vec{\mathbf{a}} + \vec{\mathbf{b}}).(\vec{\mathbf{a}} + \vec{\mathbf{b}})} = \sqrt{a^2 + b^2 + 2ab\cos\theta}$$

$$\left|\vec{\mathbf{a}} - \vec{\mathbf{b}}\right| = \sqrt{(\vec{\mathbf{a}} - \vec{\mathbf{b}}).(\vec{\mathbf{a}} - \vec{\mathbf{b}})} = \sqrt{a^2 + b^2 - 2ab\cos\theta}$$

Since it is given that $|\vec{\mathbf{a}} + \vec{\mathbf{b}}| = |\vec{\mathbf{a}} - \vec{\mathbf{b}}|$, we can write

$$a^2 + b^2 + 2ab\cos\theta = a^2 + b^2 - 2ab\cos\theta$$

or $4ab\cos\theta = 0$ or $\cos\theta = 0$ since a and b are non-zero.

Therefore, $\theta = 90°$. So the angle between $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ is $90°$.

6. Let the angle between the vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ be θ. Then using Eqs. (1.10 and 1.12) for the scalar and vector products, respectively, we can write

$$\left|\vec{\mathbf{a}}.\vec{\mathbf{b}}\right|^2 + \left|\vec{\mathbf{a}} \times \vec{\mathbf{b}}\right|^2 = a^2b^2\cos^2\theta + a^2b^2\sin^2\theta$$
$$= a^2\,b^2\,(\cos^2\theta + \sin^2\theta) = a^2b^2$$

7. The force on the proton is given by $\vec{\mathbf{F}} = q\vec{\mathbf{v}} \times \vec{\mathbf{B}}$. When the proton's velocity is in the north direction in the horizontal plane, the force on it is zero. This implies that $\vec{\mathbf{v}}$ is parallel to $\vec{\mathbf{B}}$. The direction of $\vec{\mathbf{B}}$ is then towards north in the horizontal plane. Therefore, when the proton moves vertically upward, $\vec{\mathbf{v}}$ is perpendicular to $\vec{\mathbf{B}}$ and

$$F = qvB \sin 90° = qvb$$

or $$B = \frac{F}{qv} = \frac{8.0 \times 10^{-14}\,\text{N}}{1.6 \times 10^{-19}\,\text{C} \times 5.0 \times 10^{6}\,\text{ms}^{-1}} = 0.1 \text{ tesla}$$

8. In Fig. 1.7, redrawn here as Fig. 1.33, we can see that the sides of the parallelogram, *OA* and *OB*, represent the vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$, respectively, whereas the diagonal *OC* represents the resultant vector $\vec{\mathbf{c}} = \vec{\mathbf{a}} + \vec{\mathbf{b}}$. To determine the vector sum of $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ we need to determine the magnitude and direction of $\vec{\mathbf{c}}$. The magnitude of the vector $\vec{\mathbf{c}}$ is just the length of the diagonal *OC* of the parallelogram. To calculate *OC*, we extend the side *OB* of the parallelogram to *E* and drop a perpendicular *CD* on *OE* from the point *C*. In the right-angled triangle *COD*, we have

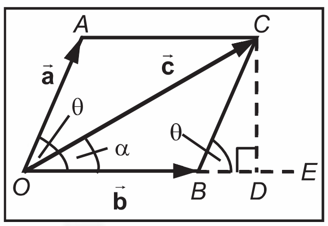


Fig. 1.33: **Parallelogram law of vector addition.**

$$OC = \sqrt{OD^2 + CD^2} = \sqrt{(OB + BD)^2 + CD^2} \qquad \text{(i)}$$

Now, we can see from Fig. 1.33 that in the right-angled triangle *CDB*:

$$BD = BC \cos\theta = OA \cos\theta = a \cos\theta \quad \text{and}$$

$$CD = BC \sin\theta = OA \sin\theta = a \sin\theta$$

Substituting these values in Eq. (i), we get *OC*, the magnitude of $\vec{\mathbf{c}}$:

$$c = \sqrt{OB^2 + BD^2 + 2(OB)(BD) + CD^2} = \sqrt{b^2 + a^2\cos^2\theta + 2ba\cos\theta + a^2\sin^2\theta}$$

$$= \sqrt{b^2 + a^2 + 2ba\cos\theta}$$

The direction of $\vec{\mathbf{c}}$ is defined by the angle α that the vector $\vec{\mathbf{c}}$ makes with the vector $\vec{\mathbf{b}}$. We have

$$\alpha = \tan^{-1}\left(\frac{CD}{OD}\right) = \tan^{-1}\left(\frac{CD}{OB + BD}\right) = \tan^{-1}\left(\frac{a\sin\theta}{b + a\cos\theta}\right)$$

9. Using the distributive property of the scalar product we can write:

$$\vec{\mathbf{a}} \,.\, (\vec{\mathbf{b}} + \vec{\mathbf{c}}) = \vec{\mathbf{a}}.\vec{\mathbf{b}} + \vec{\mathbf{a}}.\vec{\mathbf{c}} = ab \cos 0° + ac \cos 90° = ab = 12$$

10. Using Eq. (1.10), we can write $\cos\theta = \dfrac{\vec{\mathbf{r}} \,.\, \vec{\mathbf{s}}}{|\vec{\mathbf{r}}|\,|\vec{\mathbf{s}}|}$ or $\cos\theta = \dfrac{15}{5 \times 6} = \dfrac{1}{2}$

Thus, the angle between the vectors is $\theta = \cos^{-1}\left(\dfrac{1}{2}\right) = 60°$

11. The vector product of the vectors with magnitudes $r = 5$ units, $s = 5$ units is

$$\vec{\mathbf{r}} \times \vec{\mathbf{s}} = 5 \times 6 \times \sin 60° \,\hat{\mathbf{c}} = 15\sqrt{3}\ \hat{\mathbf{c}} \text{ units}$$

where the angle between them is $\theta = 60°$.

12. All sides of a regular hexagon are equal. From Fig. 1.22 (repeated here as Fig. 1.34), we can see that

$$\vec{\mathbf{F}}_2 + \vec{\mathbf{F}}_5 = \vec{\mathbf{F}}_3$$

($\because$ *AF* is parallel to *CD*, we can place $\vec{\mathbf{F}}_5$ along *CD* and use the triangle law of addition).

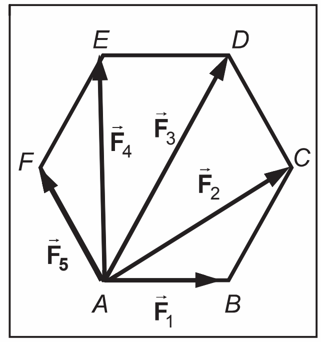


Fig. 1.34: **Diagram for TQ 12.**

Similarly:

$$\vec{\mathbf{F}}_1 + \vec{\mathbf{F}}_4 = \vec{\mathbf{F}}_3$$

$$\therefore\ \vec{\mathbf{F}}_1 + \vec{\mathbf{F}}_2 + \vec{\mathbf{F}}_3 + \vec{\mathbf{F}}_4 + \vec{\mathbf{F}}_5 = (\vec{\mathbf{F}}_2 + \vec{\mathbf{F}}_5) + \vec{\mathbf{F}}_3 + (\vec{\mathbf{F}}_1 + \vec{\mathbf{F}}_4) = 3\vec{\mathbf{F}}_3$$

13. To determine the magnitude of $\vec{\mathbf{c}} = \vec{\mathbf{a}} + \vec{\mathbf{b}}$ we use Eq. (1.5a) with $a = 4$, $b = 5$ and $\theta = (90° + 45°) = 135°$ (see Fig. 1.35a).

$$\therefore \quad c = \sqrt{(5)^2 + (2).(5)(4)\cos 135° + (4)^2} = 3.6 \text{ units}$$

$$\alpha = \tan^{-1}\frac{4\sin 135°}{5 + 4\cos 135°} = \tan^{-1}\left[\frac{4}{5\sqrt{2} - 4}\right] = \tan^{-1}[1.3] = 52.4°$$

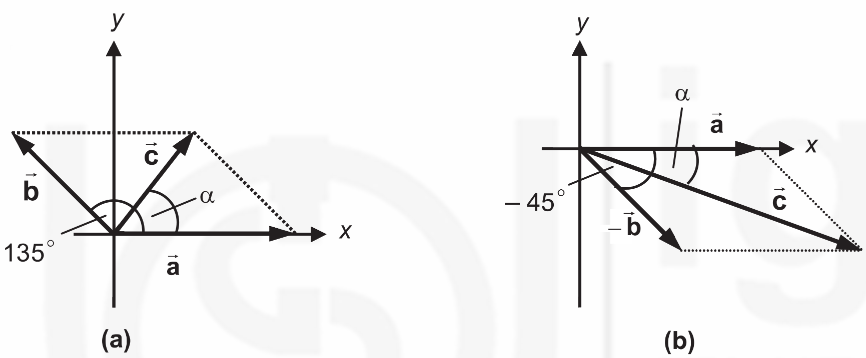


Fig. 1.35: **Diagram for Terminal Question 13.**

To calculate $\vec{\mathbf{a}} - \vec{\mathbf{b}}$ we see that the angle between $\vec{\mathbf{a}}$ and $-\vec{\mathbf{b}}$ is 315° (or – 45°) (Fig. 1.35b).

$$\therefore \quad |\vec{\mathbf{c}}| = \sqrt{(5)^2 + (4)^2 + (2)(5)(4)\cos 315°} = 8.3 \text{ units}$$

The angle that $\vec{\mathbf{c}}$ makes with $\vec{\mathbf{a}}$ is

$$\alpha = \tan^{-1}\left(\frac{4\sin 315°}{5 + 4\cos 315°}\right) = \tan^{-1}\left[\frac{-4}{5\sqrt{2} + 4}\right] = \tan^{-1}(-0.36) \approx 340°$$

14. The horizontal component of the force is

$$F_H = 105 \cos 45° = 74.3\text{N}$$

The vertical component of the force is

$$F_V = 105 \sin 45° = 74.3\text{N}$$

The work done by the force is

$$W = Fd \cos 45° = 105 \times 10 \times 45° = 743\text{J}$$

15. $\vec{\mathbf{a}}.\vec{\mathbf{a}} = |\vec{\mathbf{a}}|^2 = 4;$

$\vec{\mathbf{b}}.\vec{\mathbf{b}} = |\vec{\mathbf{b}}|^2 = 1;$

$\vec{\mathbf{c}}.\vec{\mathbf{c}} = |\vec{\mathbf{c}}|^2 = 9$

We can write $\vec{\mathbf{a}}.\,(\vec{\mathbf{a}} + \vec{\mathbf{b}} + \vec{\mathbf{c}}) = \vec{\mathbf{a}}.\vec{\mathbf{0}} = 0$

$$\Rightarrow \quad |\vec{\mathbf{a}}|^2 + \vec{\mathbf{a}}.\vec{\mathbf{b}} + \vec{\mathbf{b}}.\vec{\mathbf{c}} = 0 \Rightarrow \vec{\mathbf{a}}.\vec{\mathbf{b}} + \vec{\mathbf{b}}.\vec{\mathbf{c}} = -|\vec{\mathbf{a}}|^2 = -4 \qquad \text{(i)}$$

Similarly $\vec{\mathbf{b}}.(\vec{\mathbf{a}} + \vec{\mathbf{b}} + \vec{\mathbf{c}}) = 0 \Rightarrow \vec{\mathbf{b}}.\vec{\mathbf{a}} + \vec{\mathbf{b}}.\vec{\mathbf{c}} = -|\vec{\mathbf{b}}|^2 = -1 \qquad \text{(ii)}$

and $\vec{\mathbf{c}}\;.(\vec{\mathbf{a}} + \vec{\mathbf{b}} + \vec{\mathbf{c}}) = 0 \Rightarrow \vec{\mathbf{c}}.\vec{\mathbf{b}} + \vec{\mathbf{c}}.\vec{\mathbf{a}} = -|\vec{\mathbf{c}}|^2 = -9 \qquad \text{(iii)}$

Adding Eqs. (i, ii and iii), we get

$$2\,(\vec{\mathbf{a}}.\vec{\mathbf{b}} + \vec{\mathbf{b}}.\vec{\mathbf{c}} + \vec{\mathbf{c}}.\vec{\mathbf{a}}) = -4 - 1 - 9 = -14$$

$$\therefore \quad (\vec{\mathbf{a}}.\vec{\mathbf{b}} + \vec{\mathbf{b}}.\vec{\mathbf{c}} + \vec{\mathbf{c}}.\vec{\mathbf{a}}) = -14/2 = -7$$

The position of this train is a vector function of time. What is its velocity? This unit will help you answer such questions.

UNIT 2

VECTOR ALGEBRA-II

Structure

STUDY GUIDE

In Unit 1, you have revised elementary concepts of vector algebra using the **geometric/graphical** representation of **vectors as directed line segments**. In this unit, we discuss the **algebraic representation** of vectors in terms of their **components** relative to the Cartesian coordinate system. You may have studied about the components of vectors in your school mathematics or physics courses. If so, you could quickly revise Secs. 2.2 and 2.3 and solve the Examples and SAQs given in them. If not, you should study them thoroughly. The concepts discussed in Sec. 2.4 may be new for you. Recall that you have learnt how to differentiate functions of a single variable in your class 12 mathematics course. You may like to revise the methods. Then it would be easier for you to understand the section. Still, study it carefully. **There is a great deal of algebra in this unit. So, you should always keep a pen/pencil and paper with you while studying it.** Solve all steps in all Examples, SAQs and Terminal Questions. Of course, you should try to solve the problems on your own without first looking at the solutions and answers!

> ***"The book (of universe) is written in the mathematical language .. without whose help it is humanly impossible to comprehend a single word of it, and without which one wanders in vain through a dark labyrinth."***
>
> ***Galileo Galilee***

2.1 INTRODUCTION

In Unit 1, you have revised elementary concepts of vector algebra. You have used the graphical (or geometric) representation of vectors to add and subtract vectors without any reference to the system of coordinates. Defining **vectors and vector operations without any reference to a coordinate system** makes it possible to express the laws of physics using vectors. This is because these do not depend on the coordinate system. For example, in the vector notation, we write Newton's second law of motion as $\vec{\mathbf{F}} = m\vec{\mathbf{a}}$. This aspect of vectors makes them a very powerful tool in physics.

However, when vectors are represented by directed line segments, many a time, their addition/subtraction becomes difficult to visualize in three dimensions. Hence, the graphical/geometrical representation of vectors is of limited use in three dimensions. We find that it is far easier to represent vectors algebraically, in terms of their **components relative to a coordinate system**. In this unit, we use the algebraic representation of vectors for carrying out various vector operations such as addition/subtraction/scalar and vector products. You will see that vector algebra becomes quite simple when we represent vectors in terms of their components.

Cartesian coordinates are rectilinear, two-dimensional or three-dimensional coordinates, which are also called rectangular coordinates. By convention, the three axes of the three-dimensional Cartesian coordinates (denoted as the *x*, *y*, and *z*-axes) are chosen to be linear and mutually perpendicular. In three dimensions, each of the coordinates *x*, *y* and *z* may lie anywhere in the interval (– ∞, ∞).

In Sec. 2.2, we explain how to represent a vector algebraically in terms of its components in the 2D and 3D Cartesian coordinate systems. We also explain how to add and subtract vectors in component form. In Sec. 2.3, you will learn how to determine the scalar and vector products of vectors.

While several physical quantities are expressed as the sum and products of constant vectors, several physical quantities also change with time or vary from point to point in space. For example, the velocity of an object may change with time and the electric field of a point charge varies in space. Such physical quantities are represented by **vector functions** of position or time.

In Sec. 2.4, you will learn about vector functions or vector valued functions and their examples in physics. You also know from school physics that velocity and acceleration are the first and second time derivatives, respectively, of displacement. Displacement itself could be a function of time. In physics, you need to know how to determine the time derivative of vector functions. This is what you will learn in Sec. 2.4.2.

In the next unit, you will learn about first order ordinary differential equations.

NOTE

In your written work, always use an arrow above the letter you use to denote a vector, e.g., $\vec{r}$. Use a cap above the letter you use to denote a unit vector, e.g., $\hat{r}$.

Expected Learning Outcomes

After studying this unit, you should be able to:

- express a vector in terms of its components and the basis vectors in two and three-dimensional Cartesian coordinate systems;
- add and subtract vectors in their component form;
- calculate the scalar and vector products of vectors using their component form;
- differentiate vector functions; and
- solve physics problems based on the applications of vector algebra.

2.2 VECTOR COMPONENTS IN THE CARTESIAN COORDINATE SYSTEM

To define the components of a vector in a particular coordinate system, we need to define the unit vectors directed along the coordinate axes in the direction of increasing coordinates. To keep things simple, we first define the unit vectors along the coordinate axes in the two-dimensional Cartesian coordinate system.

2.2.1 Unit Vectors in the Cartesian Coordinate System

You are familiar with the two-dimensional (2D) **rectangular Cartesian** coordinate system from your school physics and mathematics courses. It consists of two mutually perpendicular axes denoted as the *x* and *y*-axes (Fig. 2.1). Note that the *x* and *y*-axes are **perpendicular to each other**. The point *O* at the intersection of the two axes is called the **origin** of the coordinate system. The positive values of the coordinates are measured from the origin along the axes in the direction specified by the arrows. We denote the unit vector along the +*x* direction by $\hat{\mathbf{i}}$ and the unit vector along the +*y* direction by $\hat{\mathbf{j}}$. These are constant vectors in the Cartesian coordinate system.

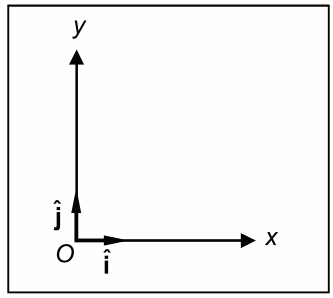


Fig. 2.1: **Unit vectors in the 2D Cartesian coordinate system.**

The three-dimensional (3D) Cartesian coordinate system has 3 axes: the *x*, *y* and *z*-axes. The unit vectors along the *x*, *y* and *z*-axes in this coordinate system are denoted by $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$, respectively. We could choose the *z*-axis in either of the two directions shown in Figs. 2.2a and b. This brings us to the notion of **right-handed** and **left-handed** Cartesian coordinate systems.

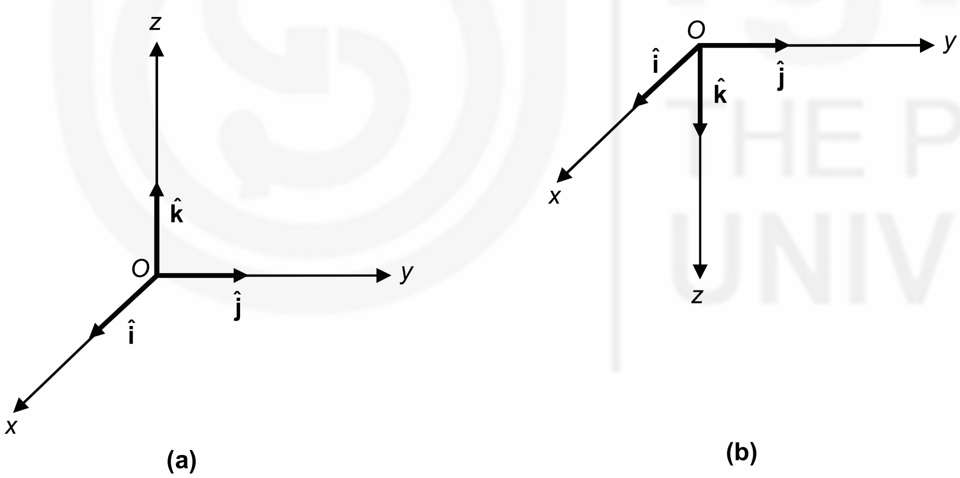


Fig. 2.2: Unit vectors in the a) right-handed and b) left-handed 3-dimensional (3D) rectangular Cartesian coordinate systems.

You may like to know: **What is meant by a right-handed and a left-handed Cartesian coordinate system**? By convention, the coordinate system shown in Fig. 2.2a is the right-handed Cartesian coordinate system. Note that for this system, the three coordinate axes form a right-handed triad. In this system, when you curl the fingers of your **right hand** around the *z*-axis so that your fingertips point in the direction of rotation of the positive *x*-axis towards the positive *y*-axis, your extended thumb points in the direction of the positive *z*-axis (Fig. 2.3). To understand this further, recall the definition of the vector product from Sec. 1.4.2 of Unit 1. For a right-handed system, the vector

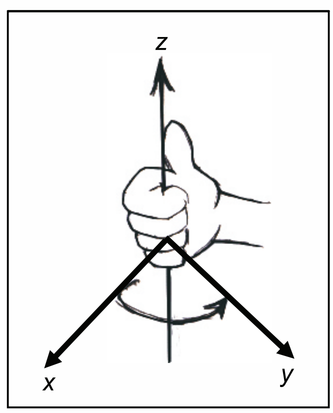


Fig. 2.3: **Right-hand rule for the right- handed coordinate system.**

products of the unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ along the coordinate axes follow the right-hand rule, that is, for the right-handed system: $\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}}$, $\hat{\mathbf{j}} \times \hat{\mathbf{k}} = \hat{\mathbf{i}}$ and $\hat{\mathbf{k}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}}$.

In the left-handed Cartesian coordinate system, shown in Fig. 2.2b, when you curl the fingers of your **left hand** around the *z*-axis so that your fingertips point in the direction of rotation of the positive *x*-axis towards the positive *y*-axis, the direction in which your extended thumb points is taken to be the direction of the positive *z*-axis. We must **always specify which of the two Cartesian coordinate systems** – **right-handed or left-handed** – **we are using**. In all our B. Sc. physics electives including this one, and in most other books in physics, the **right-handed Cartesian coordinate system is chosen for describing vectors**. Let us now see how a vector is represented in the Cartesian coordinate system in terms of its components.

2.2.2 Representing a Vector in terms of its Components

If **any one** of the unit vectors, $\hat{\mathbf{i}}, \hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ in the right-handed coordinate system is reversed, we would get a left-handed coordinate system.

We can express any vector in terms of the **unit vectors directed along the** *x*, *y* and *z*-axes, or in terms of the *x*, *y* and *z* **components** of the vector. The process of **determining the components of a vector along the coordinate axes** is also called **resolving the vector** or **resolution of the vector** along the coordinate axes. As before, we start by defining the components of a vector in the two-dimensional Cartesian coordinate system.

Let $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ be the unit vectors along the *x* and *y*-axes, respectively. Fig. 2.4 shows a vector $\vec{\mathbf{a}}$ in the *xy* plane represented geometrically by the directed line segment $\overrightarrow{\mathbf{OP}}$. The origin *O* is the tail of the vector and *P*, its head. Let the coordinates of the point *P* be (*x*, *y*). We draw the lines *PX* and *PY* perpendicular to the *x* and *y*-axes, respectively, from *P*.

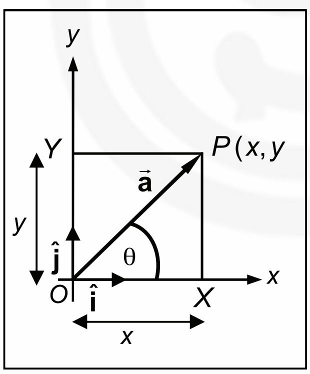


Fig. 2.4: **Vector components in the 2D Cartesian coordinate system.**

The **vector component** of the vector $\vec{\mathbf{a}}$, along the *x*-axis, is defined as the vector of magnitude *OX* along $\hat{\mathbf{i}}$. Recall from Sec. 1.4.1 that it is also the **projection** of $\vec{\mathbf{a}}$ along $\hat{\mathbf{i}}$ given by $\vec{\mathbf{a}} . \hat{\mathbf{i}}$. The **scalar component** of $\vec{\mathbf{a}}$ along the *x*-axis or $\hat{\mathbf{i}}$ (also called the *x* component of $\vec{\mathbf{a}}$) is denoted by a_x. It is a scalar quantity given in this case by $a_x = x$, the *x* coordinate of the point *P*.

Similarly, the **vector component** of $\vec{\mathbf{a}}$ along the *y*-axis is the vector of magnitude *OY* along $\hat{\mathbf{j}}$. The **scalar component** of $\vec{\mathbf{a}}$ along the *y*-axis or $\hat{\mathbf{j}}$ (also called the *y* component of $\vec{\mathbf{a}}$) is denoted by a_y. It is a scalar quantity given in this case by $a_y = y$, the *y* coordinate of the point *P*.

If θ is the angle the vector $\vec{\mathbf{a}}$ makes with the *x*-axis or with $\hat{\mathbf{i}}$, then from simple trigonometry (Fig. 2.4), we can write:

You can see from Fig. 2.4, that

$$\vec{\mathbf{a}} . \hat{\mathbf{j}} = a \cos(90^\circ - \theta) = a \sin\theta$$

$$a_x = OX = a\cos\theta = \vec{\mathbf{a}} . \hat{\mathbf{i}} \quad \text{and} \quad a_y = OY = a\sin\theta = \vec{\mathbf{a}} . \hat{\mathbf{j}} \tag{2.1}$$

From Fig. 2.4, you can also see that $\vec{\mathbf{a}}$ is the vector sum of its **vector components** $\overrightarrow{\mathbf{OX}} = \vec{\mathbf{a}}_x = a_x \hat{\mathbf{i}}$ and $\overrightarrow{\mathbf{OY}} = \vec{\mathbf{a}}_y = a_y \hat{\mathbf{j}}$ and, therefore,

$$\vec{\mathbf{a}} = \vec{\mathbf{a}}_x + \vec{\mathbf{a}}_y = a_x \hat{\mathbf{i}} + a_y \hat{\mathbf{j}} = a\cos\theta\, \hat{\mathbf{i}} + a\sin\theta\, \hat{\mathbf{j}} \tag{2.2a}$$

where we have used the results of Eq. (2.1) as well. To complete the description of the vector $\vec{\mathbf{a}}$ in terms of its components, we write down **the magnitude and direction of the vector in terms of its scalar components**. From Eq. (1.11c) of Unit 1 and Eq. (2.2a), the magnitude of $\vec{\mathbf{a}}$ is given as

$$a = \sqrt{\vec{\mathbf{a}}.\vec{\mathbf{a}}} = \sqrt{\left(a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}}\right).\left(a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}}\right)} = \sqrt{(a_x^2 + a_y^2)} \quad (2.2b)$$

The direction of the vector $\vec{\mathbf{a}}$ is specified by the angle θ, which is taken to be positive when measured anticlockwise from the *x*-axis. From Fig. 2.4, you can see that it is given by:

$$\theta = \tan^{-1}\frac{a_y}{a_x} \quad (2.2c)$$

Now $\vec{\mathbf{a}}$ could be any vector. But if it represents the **position vector** of the point *P*, we usually denote it by $\vec{\mathbf{r}}$ and write Eqs. (2.2a, b and c) as:

$$\vec{\mathbf{r}} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}, \qquad r = \sqrt{(x^2 + y^2)} \qquad \text{and} \qquad \theta = \tan^{-1}\frac{y}{x} \quad (2.3)$$

Further, Eqs. (2.2a, b and c) hold for any vector lying in the *xy* plane with its tail at the point $A(x_1, y_1)$ and head at the point $B(x_2, y_2)$ as shown in Fig. 2.5.

The vector and scalar components of the vector $\vec{\mathbf{a}}$ at an angle θ with the *x*-axis, are given as follows:

$$\vec{\mathbf{a}} = \vec{\mathbf{a}}_x + \vec{\mathbf{a}}_y = a_x\,\hat{\mathbf{i}} + a_y\,\hat{\mathbf{j}} \quad (2.4a)$$

$$a_x = x_2 - x_1 = \vec{\mathbf{a}}\,.\,\hat{\mathbf{i}} = a\cos\theta, \qquad a_y = y_2 - y_1 = \vec{\mathbf{a}}\,.\,\hat{\mathbf{j}} = a\sin\theta \quad (2.4b)$$

You may verify Eqs. (2.4a and b) from Fig. 2.5. Further,

$$a = \sqrt{(a_x^2 + a_y^2)} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2)} \quad (2.4c)$$

$$\text{and} \qquad \theta = \tan^{-1}\frac{a_y}{a_x} = \tan^{-1}\frac{(y_2 - y_1)}{(x_2 - x_1)} \quad (2.4d)$$

So far, you have learnt that any vector $\vec{\mathbf{a}}$ lying in the *xy* plane can be represented either **geometrically by the directed line segment $\overrightarrow{\mathbf{AB}}$** or by its **components** (a_x, a_y) in the two-dimensional Cartesian coordinate system.

You have also learnt that the magnitude and direction of the vector are completely specified in terms of its components relative to the two-dimensional Cartesian coordinate system. You may now like to determine the components of some vectors. Try the following SAQ.

Eq. (2.4a) is obtained when we apply the triangle law of vector addition to the vectors $a_x\hat{\mathbf{i}}$ and $a_y\hat{\mathbf{j}}$. You can also see from Fig. 2.5, that in the right-angled triangle *ACB*,

$$(AB)^2 = (AC)^2 + (CB)^2$$

$$\Rightarrow a^2 = a_x^2 + a_y^2$$

and

$$\tan\theta = \frac{BC}{AC} = \frac{a_y}{a_x}$$

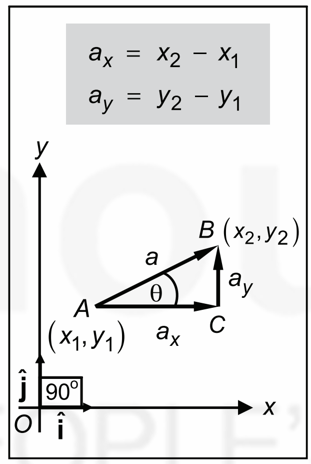


Fig. 2.5: **Resolution of a vector in a two-dimensional Cartesian coordinate system. If the coordinates of *A* and *B* are (x_1, y_1) and (x_2, y_2), respectively, then** $a_x = x_2 - x_1$ **and** $a_y = y_2 - y_1$.

SAQ 1 – Cartesian components of vectors

Determine the *x* and *y* components of vector $\vec{\mathbf{A}}$ of magnitude 3, vector $\vec{\mathbf{B}}$ of magnitude 4 and vector $\vec{\mathbf{C}}$ of magnitude 5 at the angles of 60°, 135° and 210° with the *x*-axis, respectively.

Before proceeding further, let us put all these results at one place.

Recap

2D CARTESIAN COMPONENTS OF A VECTOR

- Any vector $\vec{\mathbf{a}}$ in two-dimensional space with tail at the point (x_1, y_1) and head at the point (x_2, y_2) can be represented in terms of its *x* and *y* components a_x and a_y along the two-dimensional rectangular Cartesian coordinate axes (the *x* and *y*-axes) as

$$\vec{\mathbf{a}} = a_x\,\hat{\mathbf{i}} + a_y\,\hat{\mathbf{j}} \tag{2.4a}$$

- The scalar Cartesian components of $\vec{\mathbf{a}}$ (also called the *x* and *y* components) in 2D are, respectively,

$$a_x = x_2 - x_1 = a\cos\theta, \qquad a_y = y_2 - y_1 = a\sin\theta \tag{2.4b}$$

- The magnitude of $\vec{\mathbf{a}}$ is

$$a = \sqrt{(a_x^2 + a_y^2)} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2)} \tag{2.4c}$$

- The direction of the vector $\vec{\mathbf{a}}$ is given by the angle θ which the vector $\vec{\mathbf{a}}$ makes with the *x*-axis as

$$\theta = \tan^{-1}\frac{a_y}{a_x} \tag{2.4d}$$

You may know that a vector of the type $m\vec{\mathbf{a}} + n\vec{\mathbf{b}}$, where *m* and *n* are scalars is called a **linear combination** of $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$. So the vector $\vec{\mathbf{a}}$ in Eq. (2.4a) is a linear combination of the unit vectors $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$. The unit vectors $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ along the *x* and *y*-axes, constitute the **basis vectors** for the 2D Cartesian coordinate system. These vectors are sometimes called the **standard basis**.

The real world, however, is three-dimensional. So we now generalize what you have learnt about the vector components in a plane, to describe a vector in three-dimensional space. For this, we use the three-dimensional Cartesian coordinate system. We can represent **any vector** in three-dimensional space, in terms of its components along the right-handed 3D Cartesian coordinate axes.

Note that $\hat{\mathbf{i}}, \hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ are the unit vectors along the *x*, *y*, *z*-axes, respectively. Now, consider a vector $\vec{\mathbf{a}}$ having vector components $a_x\hat{\mathbf{i}}$, $a_y\hat{\mathbf{j}}$ and $a_z\hat{\mathbf{k}}$, respectively, with respect to the right-handed Cartesian coordinate system. For any vector $\vec{\mathbf{a}}$ in three dimensions, we can write

$$\vec{\mathbf{a}} = a_x\,\hat{\mathbf{i}} + a_y\,\hat{\mathbf{j}} + a_z\,\hat{\mathbf{k}} \tag{2.5}$$

Extending Eqs. (2.4a, b) to the 3D case, we can write the components of the vector $\vec{\mathbf{a}}$ along $\hat{\mathbf{i}}, \hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$, respectively, as (read the margin remark):

$$a_x = \vec{\mathbf{a}}.\hat{\mathbf{i}}, \qquad a_y = \vec{\mathbf{a}}.\hat{\mathbf{j}} \qquad \text{and} \qquad a_z = \vec{\mathbf{a}}.\hat{\mathbf{k}} \tag{2.6}$$

Since the three unit vectors $\hat{\mathbf{i}}, \hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$, are perpendicular to each other, we have

$\hat{\mathbf{i}}.\hat{\mathbf{i}} = 1.1\cos 0^\circ = 1,$

$\hat{\mathbf{j}}.\hat{\mathbf{i}} = 1.1\cos 90^\circ = 0$

$\hat{\mathbf{k}}.\hat{\mathbf{i}} = 1.1\cos 90^\circ = 0$

and

$\vec{\mathbf{a}}.\hat{\mathbf{i}} = (a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}} + a_z\hat{\mathbf{k}}).\hat{\mathbf{i}}$

$= a_x\hat{\mathbf{i}}.\hat{\mathbf{i}} + a_y\hat{\mathbf{j}}.\hat{\mathbf{i}} + a_z\hat{\mathbf{k}}.\hat{\mathbf{i}}$

$= a_x$

You can obtain a_y and a_z in the same way.

Let us now obtain the magnitude and direction of the vector $\vec{\mathbf{a}}$ in terms of its components (a_x, a_y, a_z). For this, you need to study Fig. 2.6. The vector $\vec{\mathbf{a}}$ shown in Fig. 2.6a has its tail at the origin and its head at the point $P(x_1, y_1, z_1)$. The magnitude of $\vec{\mathbf{a}}$ is given by:

$$a = \sqrt{a_x^2 + a_y^2 + a_z^2} \tag{2.7}$$

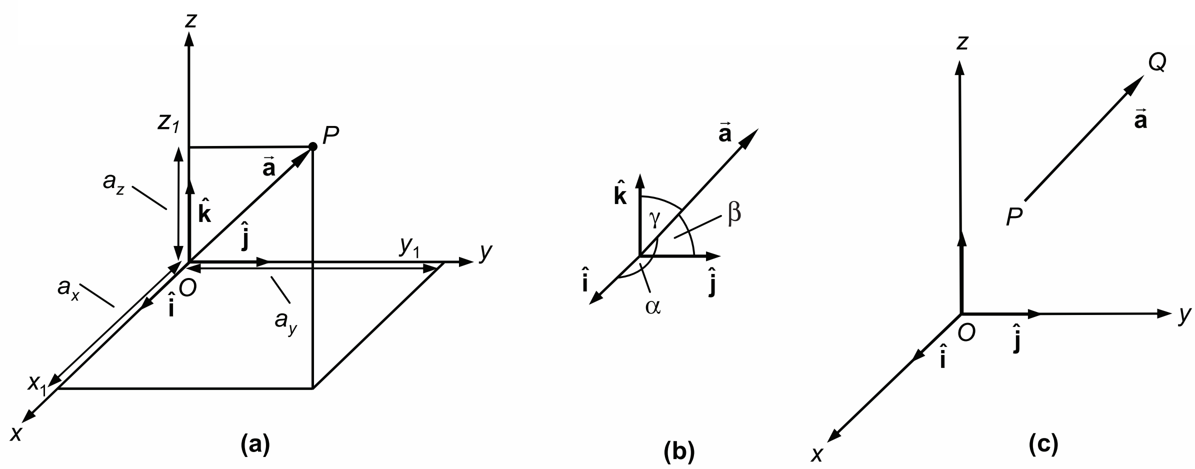


Fig. 2.6: a) Vector $\vec{a}$ in the 3D rectangular Cartesian coordinate system; b) angles between the vector and the *x*, *y* and *z*-axes and (c) vector with its tail at *P* and head at *Q* in the 3D rectangular Cartesian coordinate system.

The direction of the vector $\vec{a}$ is given by the cosines of the angles between the vector and the respective unit vectors in the *x*, *y* and *z* directions, which are cos α, cos β and cos γ. These are called the **direction cosines**. The angles α, β and γ are shown in Fig. 2.6b. Note that α is the angle between $\vec{a}$ and $\hat{i}$, denoted symbolically as $(\vec{a},\ \hat{i})$, β is $(\vec{a},\ \hat{j})$ and γ is $(\vec{a},\ \hat{k})$. So we can write

$$a_x = (\vec{a}\,.\,\hat{i}) = a\cos\alpha \tag{2.8a}$$

$$a_y = (\vec{a}\,.\,\hat{j}) = a\cos\beta \tag{2.8b}$$

$$a_z = (\vec{a}\,.\,\hat{k}) = a\cos\gamma \tag{2.8c}$$

Squaring Eqs. (2.8a, b and c), adding them, and comparing the result with Eq. (2.7), we get

$$a^2(\cos^2\alpha + \cos^2\beta + \cos^2\gamma) = a_x^2 + a_y^2 + a_z^2 = a^2$$

Thus, we have $$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \tag{2.9}$$

So **the sum of the squares of the direction cosines is equal to one**.

Now you have also studied in the two-dimensional case that **the tail of the vector need not necessarily lie at the origin**.

Suppose the vector $\vec{a}$ (as shown in Fig. 2.6c) has its tail at the point $P(x_1, y_1, z_1)$ and its head at the point $Q(x_2, y_2, z_2)$. In that case, as before, we can write the components of the vector as:

$$a_x = x_2 - x_1, \quad a_y = y_2 - y_1 \quad \text{and} \quad a_z = z_2 - z_1 \tag{2.10a}$$

The magnitude of vector $\vec{a}$ is the length of the line segment *PQ* (Fig. 2.6c). It is given by

$$a = \sqrt{a_x^2 + a_y^2 + a_z^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \tag{2.10b}$$

In general, you could choose any point to be the origin of the coordinate system. However, for simplicity, without any loss of generality you may also choose *P* to be the origin of the coordinate system. In that case the components of the vector would be the coordinates of the point *Q*.

Thus, any vector $\vec{\mathbf{a}}$, which is represented geometrically by a directed line segment $\overrightarrow{\mathbf{PQ}}$ in three-dimensional space may be represented algebraically by a set of components $a_x\ (= x_2 - x_1),\ a_y (= y_2 - y_1),\ a_z (= z_2 - z_1)$, where (x_1, y_1, z_1) and (x_2, y_2, z_2) are the coordinates of the points P and Q, respectively, relative to the origin in a right-handed Cartesian coordinate system. We now revisit the concepts of the null vector, equality of vectors and vector addition and subtraction in the algebraic representation.

The vector $\vec{\mathbf{a}}$ could represent a physical quantity like the velocity, acceleration, force or electric field, etc. Clearly, the magnitude of each of its components must have the unit appropriate to the physical quantity represented by the vector. For example you can have a force $\vec{\mathbf{F}} = 2\text{N}\,\hat{\mathbf{i}} + 3\text{N}\,\hat{\mathbf{j}} - 1\text{N}\,\hat{\mathbf{k}}$ which, by convention, we also write as $\vec{\mathbf{F}} = (2\text{N}, 3\text{N}, -1\text{N})$.

The magnitude of the force is given as $F = [2^2 + 3^2 + (-1)^2]^{1/2}\,\text{N}$. Similarly, you could have a displacement $\vec{\mathbf{d}} = (2\text{m})\,\hat{\mathbf{i}} + (1\text{m})\,\hat{\mathbf{j}}$ which has a magnitude of $\sqrt{5}\,\text{m}$. Note that vectors $\vec{\mathbf{d}}$ and $\vec{\mathbf{F}}$ are different because they represent different physical quantities.

<u>**Null vector**</u>: If $\vec{\mathbf{a}}$ is a null vector, which is a vector of zero magnitude, each of its components must be zero. In other words,

$$\vec{\mathbf{a}} = \vec{\mathbf{0}} \quad \Rightarrow \quad a_x = a_y = a_z = 0 \tag{2.11}$$

<u>**Equality of two vectors**</u>: If the vectors $\vec{\mathbf{a}} = a_x\,\hat{\mathbf{i}} + a_y\,\hat{\mathbf{j}} + a_z\,\hat{\mathbf{k}}$ and $\vec{\mathbf{b}} = b_x\,\hat{\mathbf{i}} + b_y\,\hat{\mathbf{j}} + b_z\,\hat{\mathbf{k}}$ are equal, their respective components along the x, y and z-axes must be equal:

$$a_x\,\hat{\mathbf{i}} + a_y\,\hat{\mathbf{j}} + a_z\,\hat{\mathbf{k}} = b_x\,\hat{\mathbf{i}} + b_y\,\hat{\mathbf{j}} + b_z\,\hat{\mathbf{k}} \quad \text{iff} \quad a_x = b_x,\ a_y = b_y,\ a_z = b_z \tag{2.12}$$

In Eq. (2.12), iff stands for the phrase 'if and only if'.

<u>**Addition and Subtraction of Vectors**</u>**:** For any two vectors

$$\vec{\mathbf{a}} = a_x\,\hat{\mathbf{i}} + a_y\,\hat{\mathbf{j}} + a_z\,\hat{\mathbf{k}} \quad \text{and} \quad \vec{\mathbf{b}} = b_x\,\hat{\mathbf{i}} + b_y\,\hat{\mathbf{j}} + b_z\,\hat{\mathbf{k}} \tag{2.13a}$$

the vectors $\vec{\mathbf{c}} = \vec{\mathbf{a}} + \vec{\mathbf{b}}$ and $\vec{\mathbf{d}} = \vec{\mathbf{a}} - \vec{\mathbf{b}}$ are, respectively, given by

$$\vec{\mathbf{c}} = \vec{\mathbf{a}} + \vec{\mathbf{b}} = (a_x + b_x)\,\hat{\mathbf{i}} + (a_y + b_y)\,\hat{\mathbf{j}} + (a_z + b_z)\,\hat{\mathbf{k}} \tag{2.13b}$$

and $$\vec{\mathbf{d}} = \vec{\mathbf{a}} - \vec{\mathbf{b}} = (a_x - b_x)\,\hat{\mathbf{i}} + (a_y - b_y)\,\hat{\mathbf{j}} + (a_z - b_z)\,\hat{\mathbf{k}} \tag{2.13c}$$

Let us take an example to apply these concepts in physics.

EXAMPLE 2.1: POSITION VECTOR AND DISPLACEMENT IN THE CARTESIAN COORDINATE SYSTEM

The **position vector** of a point in space gives its position relative to the origin of any given coordinate system. The position vector of the point P relative to the Cartesian coordinate system with its origin at O is shown in Fig. 2.7. The point P is completely specified by its Cartesian coordinates (x, y, z). The position vector of the point P is usually denoted by the vector $\vec{\mathbf{r}}$ directed along the line OP from O to P. Its magnitude is equal to the length of the line segment OP. In terms of its components along the x, y and z-axes, respectively, we can write,

$$\vec{\mathbf{r}} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}} \tag{2.14a}$$

The magnitude of $\vec{\mathbf{r}}$ is the length of the line segment OP given by

$$r = \sqrt{x^2 + y^2 + z^2} \tag{2.14b}$$

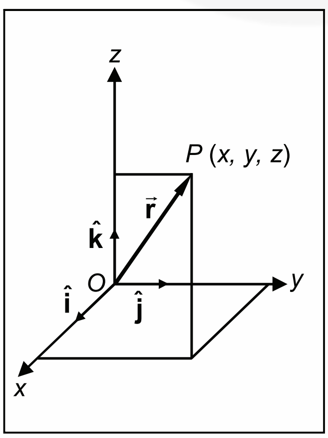


Fig. 2.7: The position vector $\vec{\mathbf{r}}$ of the point P in the Cartesian coordinate system.

Since the position vector is defined with respect to the origin of a coordinate system, it is not unique. Can you say why it is so?
It is because the origin of the coordinate system can be chosen to be anywhere and is not unique. For example, when you wish to locate the base of a tree, say, in a park, you could choose to do so using either the gate of the park or the middle of the park as the origin of the coordinate system. The position vectors of the base of the tree in these two cases would be different. Mathematically, these are two different ways of specifying the same physical point in space and these two descriptions must be related in some way. These relationships go by the name of **transformation equations** and are very important in physics.

Displacement of a particle: When a point particle moves from a point A, having position vector $\vec{\mathbf{r}}_1 = x_1\,\hat{\mathbf{i}} + y_1\,\hat{\mathbf{j}} + z_1\,\hat{\mathbf{k}}$, to a point B having position vector $\vec{\mathbf{r}}_2 = x_2\,\hat{\mathbf{i}} + y_2\,\hat{\mathbf{j}} + z_2\,\hat{\mathbf{k}}$, the displacement of the particle from A to B (Fig. 2.8) is defined as the vector

$$\Delta\vec{\mathbf{r}} = \vec{\mathbf{r}}_2 - \vec{\mathbf{r}}_1 \qquad (2.15a)$$

Geometrically, it is the vector drawn from the initial position A to the final position B of the particle. From Eq. (2.13c), the displacement vector, in component form, is given by

$$\Delta\vec{\mathbf{r}} = (x_2 - x_1)\hat{\mathbf{i}} + (y_2 - y_1)\hat{\mathbf{j}} + (z_2 - z_1)\hat{\mathbf{k}} \qquad (2.15b)$$

Note that unlike the position vector, the displacement vector does not depend on the choice of the origin or the coordinate system since it is the difference of two vectors.

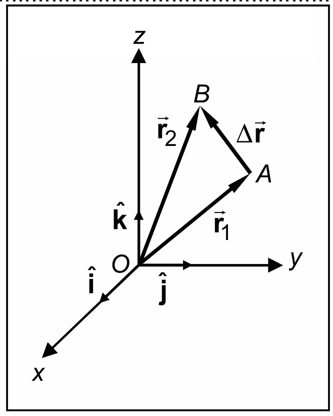


Fig. 2.8: **Displacement vector $\Delta\vec{\mathbf{r}}$ in the Cartesian coordinate system.**

So far, we have discussed the concept of the components of vectors in two and three-dimensional Cartesian coordinate systems. You may now like to work out an SAQ to check your understanding of these concepts.

SAQ 2 – Cartesian components of vectors

a) A ball is kicked 24 m at an angle 60° east of north. Write its displacement as a sum of its components along the x and y-axes.

b) Determine the magnitudes of the vectors $\vec{\mathbf{a}} = 3\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$, $\vec{\mathbf{b}} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\vec{\mathbf{c}} = 2\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$ and determine the unit vectors $\hat{\mathbf{a}}, \hat{\mathbf{b}}$ and $\hat{\mathbf{c}}$ along each vector.

Let us now summarise what you have learnt about the components of vectors in the 3D Cartesian coordinate system.

3D CARTESIAN COMPONENTS OF A VECTOR

Recap

- Any vector $\vec{\mathbf{a}}$ in three-dimensional space with tail at the point (x_1, y_1, z_1) and head at the point (x_2, y_2, z_2) can be represented in the three-dimensional Cartesian coordinate system as

$$\vec{\mathbf{a}} = a_x\,\hat{\mathbf{i}} + a_y\,\hat{\mathbf{j}} + a_z\,\hat{\mathbf{k}} \qquad (2.5)$$

NOTE

The vector $\vec{\mathbf{a}}$ in Eq. (2.5) is a linear combination of the unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ along the *x*, *y* and *z*-axes. These unit vectors constitute the **basis vectors** for the 3D Cartesian coordinate system. These vectors are also called the **standard basis vectors**. Note that these vectors are orthonormal, that is, mutually perpendicular and of unit magnitude.

where a_x, a_y and a_z are its *x*, *y* and *z* components along the coordinate axes.

- The scalar Cartesian components of $\vec{\mathbf{a}}$ in 3D (also called the *x*, *y* and *z* components) are, respectively,

$$a_x = x_2 - x_1, \quad a_y = y_2 - y_1, \quad a_z = z_2 - z_1 \tag{2.6}$$

- The magnitude of $\vec{\mathbf{a}}$ is

$$a = \sqrt{(a_x^2 + a_y^2 + a_z^2)} \tag{2.7}$$

- The direction of the vector $\vec{\mathbf{a}}$ is given by the direction cosines $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ where α, β and γ are the angles that the vector $\vec{\mathbf{a}}$ makes with the *x*, *y* and z-axes. Also

$$a_x = (\vec{\mathbf{a}}\,.\,\hat{\mathbf{i}}) = a\cos\alpha \tag{2.8a}$$

$$a_y = (\vec{\mathbf{a}}\,.\,\hat{\mathbf{j}}) = a\cos\beta \tag{2.8b}$$

$$a_z = (\vec{\mathbf{a}}\,.\,\hat{\mathbf{k}}) = a\cos\gamma \tag{2.8c}$$

We now express the scalar and vector products in component form.

2.3 SCALAR AND VECTOR PRODUCTS IN COMPONENT FORM

We first determine the scalar product of vectors.

2.3.1 Scalar Product in Component Form

Consider two vectors in 3D Cartesian coordinate system given by:

$$\vec{\mathbf{a}} = (a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}} + a_z\hat{\mathbf{k}}) \quad \text{and} \quad \vec{\mathbf{b}} = (b_x\hat{\mathbf{i}} + b_y\hat{\mathbf{j}} + b_z\hat{\mathbf{k}}) \tag{2.16a}$$

We can obtain their **scalar (or dot) product** in terms of their components as follows:

$$\begin{aligned}
\vec{\mathbf{a}}\,.\,\vec{\mathbf{b}} &= (a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}} + a_z\hat{\mathbf{k}})\,.\,(b_x\hat{\mathbf{i}} + b_y\hat{\mathbf{j}} + b_z\hat{\mathbf{k}}) \\
&= a_x\hat{\mathbf{i}}\,.\,(b_x\hat{\mathbf{i}} + b_y\hat{\mathbf{j}} + b_z\hat{\mathbf{k}}) + a_y\hat{\mathbf{j}}\,.\,(b_x\hat{\mathbf{i}} + b_y\hat{\mathbf{j}} + b_z\hat{\mathbf{k}}) + a_z\hat{\mathbf{k}}.\,(b_x\hat{\mathbf{i}} + b_y\hat{\mathbf{j}} + b_z\hat{\mathbf{k}}) \\
&= a_xb_x\,\hat{\mathbf{i}}\,.\,\hat{\mathbf{i}} + a_xb_y\hat{\mathbf{i}}\,.\,\hat{\mathbf{j}} + a_xb_z\hat{\mathbf{i}}\,.\,\hat{\mathbf{k}} \\
&\quad + a_yb_x\,\hat{\mathbf{j}}\,.\,\hat{\mathbf{i}} + a_yb_y\hat{\mathbf{j}}\,.\,\hat{\mathbf{j}} + a_yb_z\hat{\mathbf{j}}\,.\,\hat{\mathbf{k}} \\
&\quad + a_zb_x\,\hat{\mathbf{k}}\,.\,\hat{\mathbf{i}} + a_zb_y\hat{\mathbf{k}}\,.\,\hat{\mathbf{j}} + a_zb_z\hat{\mathbf{k}}\,.\,\hat{\mathbf{k}}
\end{aligned} \tag{2.16b}$$

From Eqs. (1.11c and b) of Unit 1 you know that the scalar product of a vector with itself is equal to the square of its magnitude and the scalar product of two vectors perpendicular to each other is zero. Therefore, we have

$$\hat{\mathbf{i}}\,.\,\hat{\mathbf{i}} = \hat{\mathbf{j}}\,.\,\hat{\mathbf{j}} = \hat{\mathbf{k}}\,.\,\hat{\mathbf{k}} = 1 \qquad [\because |\hat{\mathbf{i}}| = |\hat{\mathbf{j}}| = |\hat{\mathbf{k}}| = 1]$$

and $\hat{\mathbf{i}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{i}} = \hat{\mathbf{k}} \cdot \hat{\mathbf{i}} = \hat{\mathbf{i}} \cdot \hat{\mathbf{k}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{k}} = \hat{\mathbf{k}} \cdot \hat{\mathbf{j}} = 0$

[$\because$ $\hat{\mathbf{i}}, \hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ are perpendicular to each other] (2.16c)

Using the results of Eq. (2.16c) in Eq. (2.16b), we get

$$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = a_x b_x + a_y b_y + a_z b_z \quad (2.17a)$$

Recap

SCALAR PRODUCT IN COMPONENT FORM

The **scalar (or dot) product** of two vectors $\vec{\mathbf{a}} = a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}} + a_z\hat{\mathbf{k}}$ and $\vec{\mathbf{b}} = b_x\hat{\mathbf{i}} + b_y\hat{\mathbf{j}} + b_z\hat{\mathbf{k}}$ is the sum of the product of the corresponding components of the two vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$:

$$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = a_x b_x + a_y b_y + a_z b_z \quad (2.17a)$$

The scalar or dot product of two vectors $\vec{\mathbf{a}} = a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}}$ and $\vec{\mathbf{b}} = b_x\hat{\mathbf{i}} + b_y\hat{\mathbf{j}}$ is

$$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = a_x b_x + a_y b_y \quad (2.17b)$$

You have learnt in Unit 1 that the work done by a force can be expressed as the scalar product of force and displacement $W = \vec{\mathbf{F}} \cdot \vec{\mathbf{d}}$. Let us work out an example using Eq. (2.17a).

EXAMPLE 2.2: SCALAR PRODUCT OF VECTORS

Determine the work done in moving an object from the point (0, 1.0 m, – 1.0 m) to the point (3.0 m, 1.0 m, – 2.0 m) when a constant force $\vec{\mathbf{F}} = (2.0\text{N})\hat{\mathbf{i}} + (3.0\text{N})\hat{\mathbf{k}}$ is exerted on it.

SOLUTION ■ The displacement of the object is given by Eq. (2.15b) as

$$\Delta\vec{\mathbf{r}} = (3.0\,\text{m} - 0\,\text{m})\hat{\mathbf{i}} + (1.0\,\text{m} - 1.0\,\text{m})\hat{\mathbf{j}} + [-2.0\,\text{m} - (-1.0\,\text{m})]\hat{\mathbf{k}}$$

$$= (3.0\,\text{m})\hat{\mathbf{i}} + (-1.0\,\text{m})\hat{\mathbf{k}}$$

Therefore, the work done by the force is:

$$W = \vec{\mathbf{F}} \cdot \Delta\vec{\mathbf{r}} = [(2.0\text{N})\hat{\mathbf{i}} + (3.0\text{N})\hat{\mathbf{k}}].[(3.0\,\text{m})\hat{\mathbf{i}} + (-1.0\,\text{m})\hat{\mathbf{k}}]$$

Applying Eq. (2.17a), we get $W = (6.0\,\text{J}) - (3.0\,\text{J}) = 3.0\,\text{J}$

You may now like to apply Eq. (2.17a). Work out SAQ 3.

SAQ 3 – Scalar product in component form

a) Determine the scalar product of the vectors $\vec{\mathbf{a}} = 3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}}$ and $\vec{\mathbf{b}} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$. Also calculate the angle between the vectors.

b) Obtain the projection of the vector $\vec{\mathbf{a}} = 2\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ on the vector $\vec{\mathbf{b}} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 3\hat{\mathbf{k}}$.

Let us now determine the vector product in component form.

2.3.2 Vector Product in Component Form

Let us consider the vectors given by Eq. (2.16a). We can write the vector or cross product $\vec{\mathbf{a}} \times \vec{\mathbf{b}}$ of these two vectors $\vec{\mathbf{a}} = (a_x\,\hat{\mathbf{i}} + a_y\,\hat{\mathbf{j}} + a_z\,\hat{\mathbf{k}})$ and $\vec{\mathbf{b}} = (b_x\,\hat{\mathbf{i}} + b_y\,\hat{\mathbf{j}} + b_z\,\hat{\mathbf{k}})$ as

$$
\begin{aligned}
\vec{\mathbf{a}} \times \vec{\mathbf{b}} &= (a_x\,\hat{\mathbf{i}} + a_y\,\hat{\mathbf{j}} + a_z\,\hat{\mathbf{k}}) \times (b_x\,\hat{\mathbf{i}} + b_y\,\hat{\mathbf{j}} + b_z\,\hat{\mathbf{k}}) \\
&= a_x\,\hat{\mathbf{i}} \times (b_x\,\hat{\mathbf{i}} + b_y\,\hat{\mathbf{j}} + b_z\,\hat{\mathbf{k}}) + a_y\,\hat{\mathbf{j}} \times (b_x\,\hat{\mathbf{i}} + b_y\,\hat{\mathbf{j}} + b_z\hat{\mathbf{k}}) \\
&\quad + a_z\,\hat{\mathbf{k}} \times (b_x\,\hat{\mathbf{i}} + b_y\,\hat{\mathbf{j}} + b_z\,\hat{\mathbf{k}}) \\
&= a_x\,b_x\,(\hat{\mathbf{i}} \times \hat{\mathbf{i}}) + a_x\,b_y\,(\hat{\mathbf{i}} \times \hat{\mathbf{j}}) + a_x\,b_z\,(\hat{\mathbf{i}} \times \hat{\mathbf{k}}) \\
&\quad + a_y\,b_x\,(\hat{\mathbf{j}} \times \hat{\mathbf{i}}) + a_y b_y\,(\hat{\mathbf{j}} \times \hat{\mathbf{j}}) + a_y b_z\,(\hat{\mathbf{j}} \times \hat{\mathbf{k}}) \\
&\quad + a_z\,b_x\,(\hat{\mathbf{k}} \times \hat{\mathbf{i}}) + a_z\,b_y\,(\hat{\mathbf{k}} \times \hat{\mathbf{j}}) + a_z\,b_z\,(\hat{\mathbf{k}} \times \hat{\mathbf{k}})
\end{aligned}
\tag{2.18}
$$

Since the vector product of a vector with itself is the null vector $\vec{\mathbf{0}}$, we have

$$\hat{\mathbf{i}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}} \times \hat{\mathbf{k}} = \vec{\mathbf{0}} \tag{2.19a}$$

From the definition of vector product [Eq.(1.15d) of Unit 1], we have

$$\hat{\mathbf{i}} \times \hat{\mathbf{j}} = |\hat{\mathbf{i}}|\,|\hat{\mathbf{j}}| \sin 90^\circ \hat{\mathbf{n}} \tag{2.19b}$$

Here, from the right-hand rule, $\hat{\mathbf{n}}$ is a unit vector in a direction perpendicular to the plane containing the unit vectors $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$. But you know that the unit vector perpendicular to both $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ in the right-handed Cartesian coordinate system is $\hat{\mathbf{k}}$ (see Fig 2.2a). Thus, the vector $\hat{\mathbf{n}}$ is just the vector $\hat{\mathbf{k}}$. Also, since the vector product is not commutative, we have $\hat{\mathbf{j}} \times \hat{\mathbf{i}} = -\hat{\mathbf{i}} \times \hat{\mathbf{j}} = -\hat{\mathbf{k}}$. Thus,

$$\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}}; \qquad \hat{\mathbf{j}} \times \hat{\mathbf{i}} = -\hat{\mathbf{k}} \tag{2.19c}$$

$$\hat{\mathbf{j}} \times \hat{\mathbf{k}} = \hat{\mathbf{i}}; \qquad \hat{\mathbf{k}} \times \hat{\mathbf{j}} = -\hat{\mathbf{i}} \tag{2.19d}$$

and

$$\hat{\mathbf{k}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}}; \qquad \hat{\mathbf{i}} \times \hat{\mathbf{k}} = -\,\hat{\mathbf{j}} \tag{2.19e}$$

Substituting Eqs. (2.19a to e) in Eq. (2.18), we get

$$
\begin{aligned}
\vec{\mathbf{a}} \times \vec{\mathbf{b}} &= a_x\,b_x\,\vec{\mathbf{0}} + a_x\,b_y\,\hat{\mathbf{k}} + a_x\,b_z\,(-\hat{\mathbf{j}}) + a_y\,b_x\,(-\hat{\mathbf{k}}) + a_y b_y\,\vec{\mathbf{0}} + a_y\,b_z\,\hat{\mathbf{i}} \\
&\quad + a_z\,b_x\,\hat{\mathbf{j}} + a_z\,b_y\,(-\,\hat{\mathbf{i}}) + a_z\,b_z\,\vec{\mathbf{0}}
\end{aligned}
\tag{2.20}
$$

On collecting the terms corresponding to each unit vector, we get

$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = (a_y\,b_z - a_z\,b_y)\,\hat{\mathbf{i}} + (a_z\,b_x - a_x\,b_z)\,\hat{\mathbf{j}} + (a_x\,b_y - a_y\,b_x)\,\hat{\mathbf{k}} \tag{2.21a}$$

If you have studied determinants in school mathematics, you can recognise that the vector product given by Eq. (2.21a) is simply the expansion of the 3×3 determinant given below (read the margin remark):

$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} \tag{2.21b}$$

The determinant in Eq. (2.21b) can be expanded as

$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = \hat{\mathbf{i}} \begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} - \hat{\mathbf{j}} \begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} + \hat{\mathbf{k}} \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix}$$

Note that

$$\begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} = a_y b_z - a_z b_y$$

$$\begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} = a_x b_z - a_z b_x$$

$$\begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix} = a_x b_y - a_y b_x$$

We now summarise what you have learnt about the vector (or cross) product.

Recap

VECTOR PRODUCT IN COMPONENT FORM

The **vector (or cross) product** of two vectors $\vec{\mathbf{a}} = a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}} + a_z\hat{\mathbf{k}}$ and $\vec{\mathbf{b}} = b_x\hat{\mathbf{i}} + b_y\hat{\mathbf{j}} + b_z\hat{\mathbf{k}}$, is given by:

$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = (a_y b_z - a_z b_y)\hat{\mathbf{i}} + (a_z b_x - a_x b_z)\hat{\mathbf{j}} + (a_x b_y - a_y b_x)\hat{\mathbf{k}} \quad (2.21a)$$

It can also be written in the form of the following determinant:

$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} \quad (2.21b)$$

The cross product of two vectors $\vec{\mathbf{a}} = a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}}$ and $\vec{\mathbf{b}} = b_x\hat{\mathbf{i}} + b_y\hat{\mathbf{j}}$ is

$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = (a_x b_y - a_y b_x)\hat{\mathbf{k}} \quad (2.21c)$$

We now give an example from physics showing how to apply Eqs. (2.21a or b).

EXAMPLE 2.3: VECTOR PRODUCT IN COMPONENT FORM

Determine the torque about the origin due to the force $\vec{\mathbf{F}} = \hat{\mathbf{i}} - 4\hat{\mathbf{j}} + \hat{\mathbf{k}}$ acting at the point (3, 2, – 1). The unit of force is N and that of distance, m.

SOLUTION ■ The torque at a point due to a force $\vec{\mathbf{F}}$ about the origin is given by $\vec{\tau} = \vec{\mathbf{r}} \times \vec{\mathbf{F}}$, where $\vec{\mathbf{r}}$ is the position vector of the point.

The position vector of the point (3, 2, –1) is

$$\vec{\mathbf{r}} = (3\,\text{m})\hat{\mathbf{i}} + (2\,\text{m})\hat{\mathbf{j}} + (-1\text{m})\hat{\mathbf{k}}$$

Therefore, from Eq. (2.21b), the torque (in Nm) due to the force is

$$\vec{\tau} = \vec{\mathbf{r}} \times \vec{\mathbf{F}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 3 & 2 & -1 \\ 1 & -4 & 1 \end{vmatrix} = -2\hat{\mathbf{i}} - 4\hat{\mathbf{j}} - 14\hat{\mathbf{k}}$$

SAQ 4 – Vector product in component form

a) Obtain a unit vector perpendicular to the plane containing the vectors $\vec{\mathbf{a}} = 4\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\vec{\mathbf{b}} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$.

b) Calculate the area of the parallelogram whose adjacent sides are represented by the vectors $\vec{\mathbf{a}} = 2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\vec{\mathbf{b}} = 4\hat{\mathbf{i}} + 3\hat{\mathbf{k}}$.

So far in this unit, we have discussed how to resolve vectors into their components, and have revisited elementary concepts of vector algebra with vectors expressed in component form. For example, you have learnt the addition and subtraction of vectors and the scalar and vector products of vectors using the algebraic description. You will now learn about vector functions.

2.4 VECTOR FUNCTIONS

You must have studied scalar functions in your school mathematics courses.

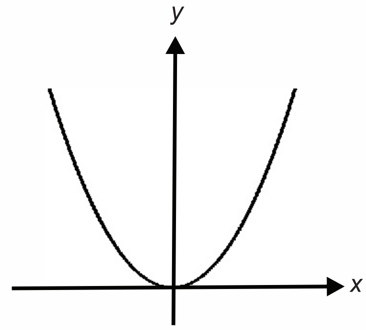
We begin this section by giving a brief introduction to scalar functions. In your school calculus course, you have studied about a **real valued function of a single real variable**, usually denoted by $f(x)$. Recall that

The **domain** of the function is the complete set of possible values of the independent variable, in this case, x. The **range** of the function is the complete set of all possible resulting values of the dependent variable, here y.

For every real value of x within some **domain**, the function $f(x)$ assigns a **unique real number**, which is the **value of the function** at x. So, when the input to the function is a real number, the output of the function is also a real number (see Fig. 2.9).

x	$f(x) = x^2$
– 3	9
– 2	4
– 1	1
0	0
1	1
2	4
3	9

(a)



(b)

Fig. 2.9: Scalar function $f(x) = x^2$ of one variable x. a) Table of values of $f(x)$ at some points; b) plot of the function $f(x)$. Note that it is a curve (a parabola) in the two-dimensional plane.

Let us now define a vector function.

2.4.1 Defining Vector Functions

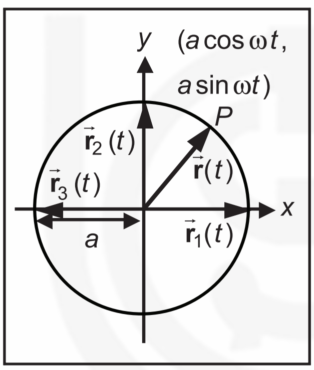


Fig. 2.10: The vector function $\vec{\mathbf{r}}(t)$ of Eq. (2.23). The head of the position vector traces out the path of the particle, as it changes. In this case, we can also say that the path of the particle is a circle of radius a units with its centre at the origin. How? Squaring the expressions for x and y in Eqs. (2.22a and b) we get:

$$x^2 + y^2 = a^2\cos^2(\omega t) + a^2\sin^2(\omega t)$$

or

$$x^2 + y^2 = a^2$$

which is the equation of a circle of radius 'a'.

Let us begin by asking: How do we describe the motion of a particle moving in a plane, say the xy plane? Let us understand this with the help of a specific example.

A particle moves in a circle of radius 'a' and the x and y coordinates of the particle change with time t as follows:

$$x = a\cos\omega t \qquad (2.22a)$$

$$y = a\sin\omega t \qquad (2.22b)$$

The position vector of the particle is the vector $\vec{\mathbf{r}}$, at any instant of time t (Fig. 2.10). Note that (x, y) are the coordinates of the point P and these change with t. So, the position vector also changes with time and we denote it as $\vec{\mathbf{r}}(t)$. It is, therefore, given as:

$$\vec{\mathbf{r}}(t) = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} = (a\cos\omega t)\,\hat{\mathbf{i}} + (a\sin\omega t)\,\hat{\mathbf{j}} \qquad (2.23)$$

Here $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ are the unit vectors along the x and y-axes, respectively. The vector $\vec{\mathbf{r}}(t)$ is an example of a **vector function**. In this case, $\vec{\mathbf{r}}(t)$ is a vector function of the variable t. This means that there exists a vector $\vec{\mathbf{r}}$ corresponding to each value of t. The set of all real numbers that correspond to the values of the independent variable t is called the **domain** of the vector function. **Any function whose range is a set of vectors is called a vector function.**

In Fig. 2.10, we have drawn the vector $\vec{\mathbf{r}}(t)$ described by Eqs. (2.22a, b) and (2.23) at three instants of time $t = 0$, $t = \dfrac{\pi}{2\omega}$ and $t = \dfrac{\pi}{\omega}$ and denoted them by $\vec{\mathbf{r}}_1 = \vec{\mathbf{r}}(t=0)$, $\vec{\mathbf{r}}_2 = \vec{\mathbf{r}}(t = \pi/2\omega)$ and $\vec{\mathbf{r}}_3 = \vec{\mathbf{r}}(t = \pi/\omega)$, respectively.

For different values of t, the position of the head of the vector $\vec{\mathbf{r}}(t)$ changes. As t changes continuously, the end point of the vector function traces out a continuous curve, which is a circle (Fig. 2.10).

Now suppose the particle moves on a right-circular cylinder of radius R (see Fig. 2.11). Its position vector also has a z-component, say $z = t$, and is given by

$$\vec{\mathbf{r}}(t) = R \cos \omega t\, \hat{\mathbf{i}} + R \sin \omega t\, \hat{\mathbf{j}} + t\, \hat{\mathbf{k}} \tag{2.24}$$

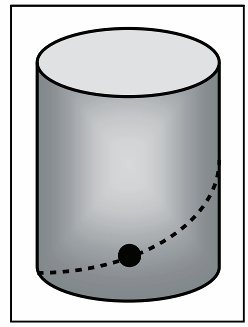


Fig. 2.11: **Particle moving on a right-circular cylinder.**

The **path** traced out by the particle is called a **right-circular helix**. For each value of t we have a vector $\vec{\mathbf{r}}(t)$.

In the two examples discussed so far, the vector function is the position vector. However, a vector function can represent any arbitrary vector quantity which depends on a scalar variable.

In general, we can express a vector function of a single variable t, in two-dimensional and three-dimensional Cartesian coordinate systems, respectively, as:

$$\vec{\mathbf{a}}(t) = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}} \tag{2.25a}$$

$$\vec{\mathbf{a}}(t) = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}} + h(t)\,\hat{\mathbf{k}} \tag{2.25b}$$

where $f(t), g(t)$ and $h(t)$ are **single-valued scalar functions** of t. Knowing the vector function $\vec{\mathbf{a}}(t)$ implies knowing $f(t)$, $g(t)$ and $h(t)$. The converse is also true, i.e., if you know $f(t)$, $g(t)$ and $h(t)$, you know the vector function $\vec{\mathbf{a}}(t)$. The name **vector function** arises because $\vec{\mathbf{a}}(t)$ is a vector, and it depends on the real independent variable t, such that at each t there is a unique vector $\vec{\mathbf{a}}(t)$ given by Eqs. (2.25a or b). The functions $f(t)$, $g(t)$ and $h(t)$ are called **component functions** of $\vec{\mathbf{a}}(t)$.

You can visualise a vector function if you know its geometrical meaning. Study Fig. 2.12a. We have used a 3-dimensional Cartesian coordinate system to show the vector function $\vec{\mathbf{a}}(t)$. Note that the initial point O of the vector function $\vec{\mathbf{a}}(t)$ is bound to the origin of the coordinates. For different values of t, its end point P changes because $\vec{\mathbf{a}}(t)$ depends on t. Fig. 2.12b shows three positions of P : P_1, P_2, P_3 for three values of t : t_1, t_2, t_3. Fig. 2.12c shows that as t varies continuously, the point P traces out a continuous curve in space. So you can visualise a vector function $\vec{\mathbf{a}}(t)$ in terms of a curve, as t varies continuously. A point P on the curve is described by the position vector $\vec{\mathbf{a}}(t)$.

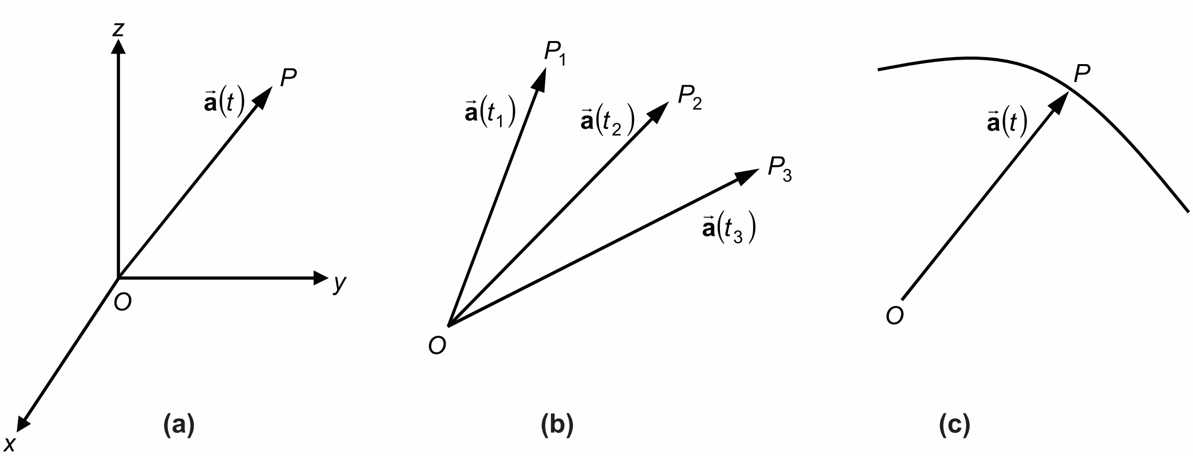


Fig. 2.12: Vector function of one variable interpreted as a curve in space.

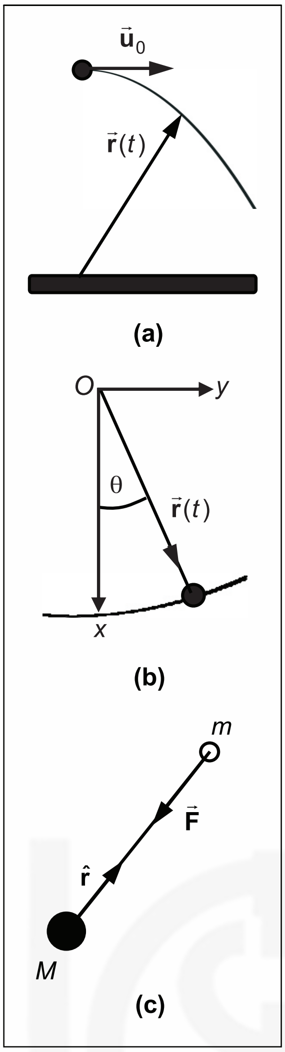


Fig. 2.13: **Some vector functions in physics.**

In the case of the **two-dimensional vector function**, the curve traced out by the tip of the vector $\vec{\mathbf{a}}(t)$ with changing t is called a **plane curve**. In **three dimensions**, the curve traced out by the tip of $\vec{\mathbf{a}}(t)$ is called a **space curve**.

The circle in Fig. 2.10 is an example of a plane curve and the helix in Fig. 2.11 is a space curve.

Alternatively, you can think of a three-dimensional vector function as three separate functions, $x = f(t)$, $y = g(t)$ and $z = h(t)$, that describe points (x, y, z) in space. Here the set of equations for the three component functions of the vector function tell us the position of the tip of the vector function $\vec{\mathbf{a}}(t)$ at each value of t. These equations are called the **parametric equations** describing the plane curve or the space curve. There are many examples of such vector functions in physics. For example, velocity, acceleration, force are all vector functions. Wouldn't you like to work out the following SAQ to identify some vector functions?

SAQ 5 – Vector functions

Which of the following functions are vector functions?

a) The position vector of a ball falling freely with an initial horizontal velocity $\vec{\mathbf{u}}_0$ (Fig. 2.13a), given as

$$\vec{\mathbf{r}}(t) = u_0 t\,\hat{\mathbf{j}} + \frac{1}{2} g t^2 \hat{\mathbf{k}}$$

b) The position vector of the bob of a simple pendulum of length L at any instant t, relative to O (Fig. 2.13b), given by

$$\vec{\mathbf{r}}(t) = L\cos\theta\,\hat{\mathbf{i}} + L\sin\theta\,\hat{\mathbf{j}} \quad \text{where } \theta = \theta(t)$$

c) A force $\vec{\mathbf{F}} = 3\hat{\mathbf{i}} - \hat{\mathbf{j}} + 4\hat{\mathbf{k}}$ N applied to a particle of mass 2 kg.

d) The gravitational force on a particle of mass m at a distance r from a particle of mass M (Fig. 2.13c), given by

$$\vec{\mathbf{F}} = -\frac{GMm}{r^2}\hat{\mathbf{r}}$$

Let us revise what you have learnt so far about the vector function.

Recap

VECTOR FUNCTION

A vector-valued function, or a **vector function**, is a function whose **domain** is a set of real numbers and whose **range** is a set of vectors. In two-dimensional and three-dimensional Cartesian coordinate systems, a vector function of a **single variable** t can be written as:

$$\vec{\mathbf{a}}(t) = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}} \quad \text{in two dimensions} \qquad (2.25a)$$

$$\vec{\mathbf{a}}(t) = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}} + h(t)\,\hat{\mathbf{k}} \quad \text{in three dimensions} \qquad (2.25b)$$

f(t), h(t) and *g(t)* are called the **component functions**.

You will now learn how to obtain the derivative of a vector function. In writing down the derivative of a vector, we must remember that a vector function has separate scalar functions as its components.

2.4.2 Derivative of a Vector Function

You know how to obtain the derivative of a scalar function $y = f(t)$. Recall from calculus that from first principles, the derivative of the function $f(t)$ with respect to t is defined as:

$$\frac{df(t)}{dt} = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t} \tag{2.26}$$

In the same way, we define the derivative of a vector function with respect to t as:

$$\frac{d\vec{\mathbf{a}}(t)}{dt} = \lim_{\Delta t \to 0} \frac{\vec{\mathbf{a}}(t+\Delta t) - \vec{\mathbf{a}}(t)}{\Delta t} \tag{2.27}$$

We can simplify the use of Eq. (2.27) if the vector function is given in its component form. We can express $\vec{\mathbf{a}}(t)$ in the three-dimensional Cartesian coordinate system in terms of its component functions:

$$\vec{\mathbf{a}}(t) = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}} + h(t)\hat{\mathbf{k}}$$

This is just Eq. (2.25b) written again. Since $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ are constant vectors, we can write

$$\frac{d\vec{\mathbf{a}}(t)}{dt} = \frac{df(t)}{dt}\hat{\mathbf{i}} + \frac{dg(t)}{dt}\hat{\mathbf{j}} + \frac{dh(t)}{dt}\hat{\mathbf{k}} \tag{2.28a}$$

So the components of $\frac{d\vec{\mathbf{a}}(t)}{dt}$ [the derivative of $\vec{\mathbf{a}}(t)$] are simply the derivatives of the components $f(t)$, $g(t)$ and $h(t)$ of the vector $\vec{\mathbf{a}}(t)$. For a two-dimensional vector $\vec{\mathbf{a}}(t) = f(t)\hat{\mathbf{i}} + g(t)\hat{\mathbf{j}}$, we can write the derivative as

$$\frac{d\vec{\mathbf{a}}(t)}{dt} = \frac{df(t)}{dt}\hat{\mathbf{i}} + \frac{dg(t)}{dt}\hat{\mathbf{j}} \tag{2.28b}$$

Let us now apply Eqs. (2.28a and b) to an example.

EXAMPLE 2.4: DIFFERENTIATING VECTOR FUNCTIONS

The position vector of a particle as a function of time is given by:

$$\vec{\mathbf{r}}(t) = 5\cos(2t)\hat{\mathbf{i}} + 5\sin(2t)\hat{\mathbf{j}} + t\hat{\mathbf{k}}$$

Determine its velocity and acceleration. Show that both its speed and the magnitude of its acceleration are constant.

SOLUTION ■ The velocity $\vec{\mathbf{v}}(t)$ of the particle is the first derivative of the position vector:

$$\vec{\mathbf{v}}(t) = \frac{d}{dt}[\vec{\mathbf{r}}(t)] = \frac{d}{dt}[5\cos(2t)\hat{\mathbf{i}} + 5\sin(2t)\hat{\mathbf{j}} + t\hat{\mathbf{k}}]$$

$$= -10\sin(2t)\hat{\mathbf{i}} + 10\cos(2t)\hat{\mathbf{j}} + \hat{\mathbf{k}}$$

To determine the acceleration $\vec{\mathbf{a}}(t)$ of the particle we differentiate its velocity with respect to time:

$$\vec{\mathbf{a}}(t) = \frac{d}{dt}[\vec{\mathbf{v}}(t)] = \frac{d}{dt}[-10\sin(2t)\hat{\mathbf{i}} + 10\cos(2t)\hat{\mathbf{j}} + \hat{\mathbf{k}}]$$

$$= -20\cos(2t)\hat{\mathbf{i}} - 20\sin(2t)\hat{\mathbf{j}}$$

The speed of the particle is the magnitude of its velocity. It is given by

$$|\vec{\mathbf{v}}(t)| = \left[(-10\sin 2t)^2 + (10\cos 2t)^2 + 1^2\right]^{1/2}$$

$$= \left[100\sin^2 2t + 100\cos^2 2t + 1\right]^{1/2} = \sqrt{101}$$

Note that the particle's speed is constant: it does not change with time, though the velocity does. The magnitude of the particle's acceleration is

$$|\vec{\mathbf{a}}(t)| = \left[(-20\cos 2t)^2 + (-20\sin 2t)^2\right]^{1/2} = 20$$

It is also a constant.

Let us now briefly summarise the rules of vector differentiation of a vector function of a single variable in two and three dimensions in terms of the component functions.

Recap

DERIVATIVE OF A VECTOR FUNCTION

The derivative of a vector function of a **single variable** in three dimensions:

$$\vec{\mathbf{a}}(t) = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}} + h(t)\,\hat{\mathbf{k}}$$

with respect to the variable t is given by:

$$\frac{d\vec{\mathbf{a}}(t)}{dt} = \frac{df(t)}{dt}\hat{\mathbf{i}} + \frac{dg(t)}{dt}\hat{\mathbf{j}} + \frac{dh(t)}{dt}\hat{\mathbf{k}} \qquad (2.28a)$$

The derivative of a vector function of a **single variable** in two dimensions:

$$\vec{\mathbf{a}}(t) = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}}$$

can be written as:

$$\frac{d\vec{\mathbf{a}}(t)}{dt} = \frac{df(t)}{dt}\hat{\mathbf{i}} + \frac{dg(t)}{dt}\hat{\mathbf{j}} \qquad (2.28b)$$

You may like to work out a simple problem on what you have learnt so far.

SAQ 6 – Differentiating vector functions

The position vector of a particle is given by

$$\vec{\mathbf{r}}(t) = (6t - t^2)\,\hat{\mathbf{i}} + (2t)\,\hat{\mathbf{j}} + t\,\hat{\mathbf{k}}$$

Determine the velocity and acceleration of the particle.

When you differentiate a vector function, three types of situations may arise:

There is a change

i) only in its ***magnitude*** or

ii) only in its ***direction*** or

iii) in both ***direction and magnitude*** (see Fig. 2.14).

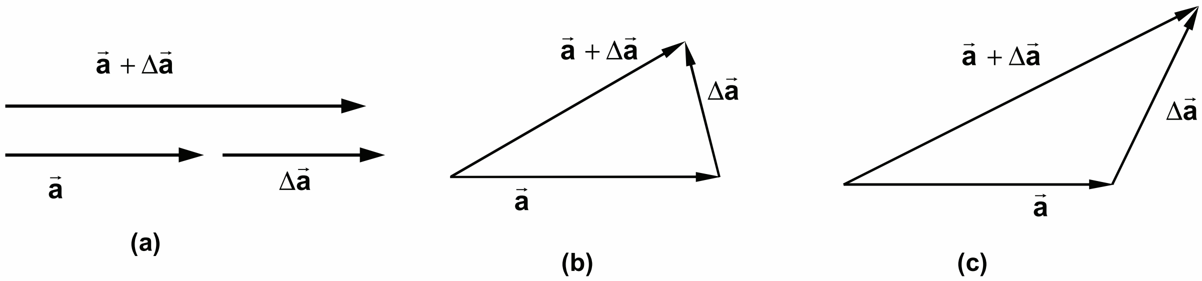


Fig. 2.14: **Derivative of a vector function when there is a change a) only in magnitude; b) only in direction; c) both in magnitude and direction.**

In physics, you will need to determine derivatives of product of a scalar with a vector function, and scalar and vector products of vector functions. Note that we can write the scalar and vector products in terms of their components in a Cartesian coordinate system. Then the problem is reduced to determining the derivatives of scalar functions. So you can use Eqs. (2.28a or b) to determine the derivatives of scalar and vector products of vector functions. We just give the results here:

$$\frac{d}{dt}\left(s\vec{\mathbf{f}}\right) = \frac{ds}{dt}\vec{\mathbf{f}} + s\frac{d\vec{\mathbf{f}}}{dt} \tag{2.29}$$

$$\frac{d}{dt}\left(\vec{\mathbf{a}}.\vec{\mathbf{b}}\right) = \frac{d\vec{\mathbf{a}}}{dt}.\vec{\mathbf{b}} + \vec{\mathbf{a}}.\frac{d\vec{\mathbf{b}}}{dt} \tag{2.30a}$$

and

$$\frac{d}{dt}\left(\vec{\mathbf{a}}\times\vec{\mathbf{b}}\right) = \frac{d\vec{\mathbf{a}}}{dt}\times\vec{\mathbf{b}} + \vec{\mathbf{a}}\times\frac{d\vec{\mathbf{b}}}{dt} \tag{2.30b}$$

Before we end this discussion, we would like to explain the **meaning of the derivative of a vector function**. Let us go back to Eq. (2.27). We explain briefly: **What is the meaning of the derivative of a vector function defined by Eq. (2.27)**? You will not be examined on the matter given in the box.

DERIVATIVE OF A VECTOR FUNCTION

We can interpret the derivative of a vector function $\vec{\mathbf{a}}(t)$ defined by Eq. (2.27) geometrically as the tangent vector to the curve described by it. It also gives the **rate of change of the vector** with the variable *t*. To understand this point, let us consider a scalar variable *t* and the corresponding vector function $\vec{\mathbf{a}}(t)$. Study Fig. 2.15. It shows the vector function $\vec{\mathbf{a}}(t)$ in the form of a curve in space (as in Fig. 2.12c). Now study Fig. 2.16a. Let the vector $\overrightarrow{\mathbf{OP}}(t)$ in Fig. 2.16a denote $\vec{\mathbf{a}}(t)$ at any value of *t*. We now increase the scalar *t* by an amount Δt. The vector corresponding to the scalar $t+\Delta t$ is $\vec{\mathbf{a}}(t+\Delta t)$. It is denoted by $\overrightarrow{\mathbf{OQ}}$.

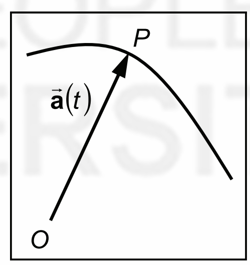


Fig. 2.15: **Vector function of one variable interpreted as a curve in space.**

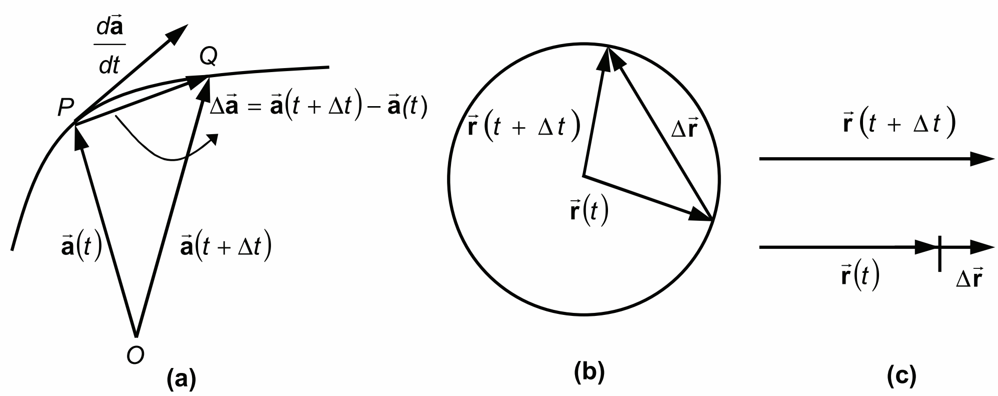


Fig. 2.16: **a) The derivative of a vector interpreted as its rate of change; b) $\Delta\vec{\mathbf{r}}$ is not in the same direction as $\vec{\mathbf{r}}$ for circular motion; c) an exception in which $\Delta\vec{\mathbf{r}}$ is along $\vec{\mathbf{r}}$.**

The change in $\vec{\mathbf{a}}(t)$ corresponding to the change Δt in t is

$$\Delta\vec{\mathbf{a}} = \vec{\mathbf{a}}(t+\Delta t) - \vec{\mathbf{a}}(t)$$

From the triangle law of addition you can see that $\Delta\vec{\mathbf{a}}$ is the vector $\overrightarrow{\mathbf{PQ}}$. Note that $\Delta\vec{\mathbf{a}}$ is not in the direction of $\vec{\mathbf{a}}$. This is true generally. For example, when a particle moves along a parabolic path, in a circle or in an ellipse, the change $\Delta\vec{\mathbf{r}}$ in its position vector is not in the direction of $\vec{\mathbf{r}}$ (Fig. 2.16b). Can you think of an exception? Obviously, $\Delta\vec{\mathbf{r}}$ is in the same direction as $\vec{\mathbf{r}}$ for straight line motion, when we choose the origin at the starting point (Fig. 2.16c). Let us now go back to Fig. 2.16a. Since Δt is a scalar, $\frac{\Delta\vec{\mathbf{a}}}{\Delta t}$ is a vector in the same direction as $\overrightarrow{\mathbf{PQ}}$.

As $\Delta t \to 0$, Q approaches P and $\lim\limits_{\Delta t\to 0}\frac{\Delta\vec{\mathbf{a}}}{\Delta t}$ becomes the vector **tangent to the curve** at P, whenever it exists, and is not zero.

It is directed in the sense in which point P would move if the value of t is increased. Thus, the derivative $\frac{d\vec{\mathbf{a}}(t)}{dt} = \vec{\mathbf{a}}'(t)$ is the **tangent to the space curve** $\vec{\mathbf{a}} = \vec{\mathbf{a}}(t)$, whenever $\vec{\mathbf{a}}'(t)$ exists and $\vec{\mathbf{a}}'(t) \neq 0$. Now $\frac{\Delta\vec{\mathbf{a}}}{\Delta t}$ also gives the change in $\vec{\mathbf{a}}$ per unit value of the variable t during the interval Δt. So the derivative $\vec{\mathbf{a}}'(t) = \lim\limits_{\Delta t\to 0}\frac{\Delta\vec{\mathbf{a}}}{\Delta t}$ is the **rate of change** of $\vec{\mathbf{a}}$ with respect to t.

You can see that the geometrical meaning of $\frac{d\vec{\mathbf{a}}(t)}{dt}$ is analogous to the one obtained for the derivative of a scalar function, say $\frac{du(t)}{dt}$.

However, there is one important difference between the derivatives of a vector function and a scalar function. **Since $\vec{\mathbf{a}}$ is a vector, it can change both in magnitude and direction, whereas a scalar function changes only in magnitude**.

Let us consider the example of uniform circular motion to apply these concepts.

EXAMPLE 2.5: UNIFORM CIRCULAR MOTION

Apply Eq. (2.30a) to analyse the uniform circular motion of a particle.

SOLUTION ■ For a particle in uniform circular motion, the magnitudes of the position and velocity vectors are constant but their directions are changing as the particle moves in the circle. Let the speed of the particle be v, and let it move in a circle of radius r. Then we can write

$$\vec{\mathbf{r}}\,.\,\vec{\mathbf{r}} = r^2 = \text{constant and } \vec{\mathbf{v}}\,.\,\vec{\mathbf{v}} = v^2 = \text{constant} \qquad \text{(i)}$$

where $\vec{\mathbf{r}}$ and $\vec{\mathbf{v}}$ are the position vector and velocity of the particle, respectively.

Using Eq. (2.30a), we can then write:

$$\frac{d}{dt}\left(\vec{\mathbf{r}}\,.\,\vec{\mathbf{r}}\right) = \vec{\mathbf{r}}\,.\frac{d\vec{\mathbf{r}}}{dt} + \frac{d\vec{\mathbf{r}}}{dt}.\vec{\mathbf{r}} = \frac{d}{dt}\left(r^2\right) = 0 \quad \text{since } r \text{ is constant}$$

or
$$\vec{\mathbf{r}}.\frac{d\vec{\mathbf{r}}}{dt} = \vec{\mathbf{r}}\,.\,\vec{\mathbf{v}} = 0 \quad \left(\text{since } \frac{d\vec{\mathbf{r}}}{dt} = \vec{\mathbf{v}}\right) \qquad \text{(ii)}$$

Eq. (ii) tells us that for a particle undergoing uniform circular motion, **the velocity is perpendicular to the position vector at each instant of time** (Fig. 2.17). Let us now apply the results above to the velocity of the particle. Since it is constant, we can write

$$\frac{d}{dt}(\vec{\mathbf{v}}\,.\,\vec{\mathbf{v}}) = 2\,\vec{\mathbf{v}}\,.\frac{d\vec{\mathbf{v}}}{dt} = \frac{d}{dt}\left(v^2\right) = 0$$

or $$\vec{\mathbf{v}}\,.\,\frac{d\vec{\mathbf{v}}}{dt} = \vec{\mathbf{v}}\,.\,\vec{\mathbf{a}} = 0 \qquad \left(\text{since } \frac{d\vec{\mathbf{v}}}{dt} = \vec{\mathbf{a}}\right) \qquad \text{(iii)}$$

where $\vec{\mathbf{a}}$ is the acceleration of the particle. Eq. (iii) tells us that for uniform circular motion, **the acceleration of the particle is perpendicular to its velocity**. What then can we say about the direction of the acceleration?

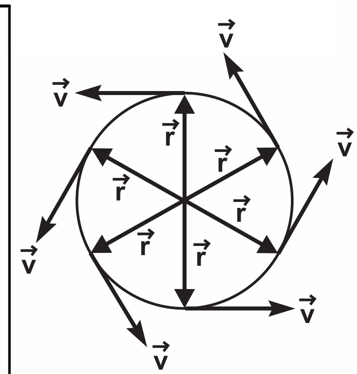


Fig. 2.17: In uniform circular motion, the velocity of the particle is perpendicular to its position vector at all instants of time.

We already know that $\vec{\mathbf{r}}$ is perpendicular to the velocity $\vec{\mathbf{v}}$ and also that $\vec{\mathbf{v}}$ is perpendicular to the acceleration $\vec{\mathbf{a}}$. Since circular motion is confined to a plane, this suggests that $\vec{\mathbf{r}}$ and $\vec{\mathbf{a}}$ **must be parallel or anti parallel to each other**. Let us find out which of these two is actually the case. We first determine the first derivative of $(\vec{\mathbf{r}}\,.\,\vec{\mathbf{v}})$. Since $\vec{\mathbf{r}}\,.\,\vec{\mathbf{v}} = 0$, we can write:

$$\frac{d}{dt}[\vec{\mathbf{r}}\,.\,\vec{\mathbf{v}}] = \frac{d\vec{\mathbf{r}}}{dt}\,.\,\vec{\mathbf{v}} + \vec{\mathbf{r}}\,.\,\frac{d\vec{\mathbf{v}}}{dt} = \vec{\mathbf{v}}\,.\,\vec{\mathbf{v}} + \vec{\mathbf{r}}\,.\,\vec{\mathbf{a}} = 0 \qquad \text{(iv)}$$

or $$\vec{\mathbf{r}}\,.\,\vec{\mathbf{a}} + v^2 = 0 \;\Rightarrow\; \vec{\mathbf{r}}\,.\,\vec{\mathbf{a}} = -v^2 \qquad \text{(v)}$$

Since the right hand side of Eq. (v) is negative, the angle between them has to be $180°$ so that $\cos\theta = -1$ [recall the definition of scalar product in Eq. (1.10)]. Thus, from Eq. (v), we get

$$-\,ar = -\,v^2$$

or $$a = \frac{v^2}{r} \qquad \text{(vi)}$$

Thus, the acceleration $\vec{\mathbf{a}}$ of a particle in uniform circular motion is anti- parallel or opposite in direction to the position vector. It has magnitude v^2/r. It is called the **centripetal acceleration** because it points towards the centre of the circle at all instants of time (Fig. 2.18).

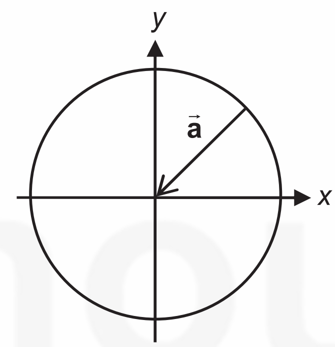


Fig. 2.18: In uniform circular motion, the acceleration of the particle points towards the centre at all instants of time.

Let us now summarise what you have studied in this unit.

2.5 SUMMARY

Concept	Description
2D *Cartesian components of a vector*	■ Any vector $\vec{\mathbf{a}}$ in two-dimensional space with tail at the point (x_1, y_1) and head at the point (x_2, y_2) can be represented in terms of its x and y components as $\vec{\mathbf{a}} = a_x\,\hat{\mathbf{i}} + a_y\,\hat{\mathbf{j}}$ where $a_x = x_2 - x_1 = a\cos\theta$, $a_y = y_2 - y_1 = a\sin\theta$ The **magnitude** of the vector is given by $\lvert\vec{\mathbf{a}}\rvert = a = \sqrt{a_x^2 + a_y^2}$ and its **direction** is given by the angle θ that the vector makes with the positive x-axis: $\theta = \tan^{-1}\left(\frac{a_y}{a_x}\right)$

3D Cartesian components of a vector

- Any vector $\vec{\mathbf{a}}$ in three-dimensional space with tail at the point (x_1, y_1, z_1) and head at the point (x_2, y_2, z_2) can be represented in terms of its *x*, *y* and *z* components as

$$\vec{\mathbf{a}} = a_x\,\hat{\mathbf{i}} + a_y\,\hat{\mathbf{j}} + a_z\,\hat{\mathbf{k}}$$

where $a_x = x_2 - x_1, \quad a_y = y_2 - y_1, \quad a_z = z_2 - z_1$

The **magnitude** of $\vec{\mathbf{a}}$ is given by $a = \sqrt{(a_x^2 + a_y^2 + a_z^2)}$

The **direction** of the vector $\vec{\mathbf{a}}$ is given by the direction cosines $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ where α, β and γ are the angles that the vector $\vec{\mathbf{a}}$ makes with the *x*, *y* and z-axes. Thus

$$a_x = (\vec{\mathbf{a}}\,.\,\hat{\mathbf{i}}) = a\cos\alpha$$

$$a_y = (\vec{\mathbf{a}}\,.\,\hat{\mathbf{j}}) = a\cos\beta$$

$$a_z = (\vec{\mathbf{a}}\,.\,\hat{\mathbf{k}}) = a\cos\gamma$$

Scalar product in the component form

- The **scalar** (**or dot**) **product** of the vectors $\vec{\mathbf{a}} = a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}} + a_z\hat{\mathbf{k}}$ and $\vec{\mathbf{b}} = b_x\hat{\mathbf{i}} + b_y\hat{\mathbf{j}} + b_z\hat{\mathbf{k}}$, is given by

$$\vec{\mathbf{a}}\,.\,\vec{\mathbf{b}} = a_x b_x + a_y b_y + a_z b_z$$

The scalar or dot product of two vectors $\vec{\mathbf{a}} = a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}}$ and $\vec{\mathbf{b}} = b_x\hat{\mathbf{i}} + b_y.$ is given by

$$\vec{\mathbf{a}}\,.\,\vec{\mathbf{b}} = a_x b_x + a_y b_y$$

Vector product in the component form

- The **vector** (**or cross**) **product** of two vectors $\vec{\mathbf{a}} = a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}} + a_z\hat{\mathbf{k}}$ and $\vec{\mathbf{b}} = b_x\hat{\mathbf{i}} + b_y\hat{\mathbf{j}} + b_z\hat{\mathbf{k}}$ is given by:

$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = (a_y\, b_z - a_z\, b_y)\,\hat{\mathbf{i}} + (a_z\, b_x - a_x\, b_z)\,\hat{\mathbf{j}} + (a_x\, b_y - a_y\, b_x)\,\hat{\mathbf{k}}$$

It can also be written in the form of the determinant as

$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

The vector product of the vectors $\vec{\mathbf{a}} = a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}}$ and $\vec{\mathbf{b}} = b_x\hat{\mathbf{i}} + b_y\hat{\mathbf{j}}$ is given by

$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = (a_x b_y - b_x a_y)\,\hat{\mathbf{k}}$$

Vector function of a scalar variable

- A vector may be a function of one or more independent scalar variables. In the Cartesian coordinate system, a **vector function** $\vec{\mathbf{a}}(t)$ of a **single variable** can be expressed in its component form as

$$\vec{\mathbf{a}}(t) = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}} \qquad \text{in two dimensions}$$

$$\vec{\mathbf{a}}(t) = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}} + h(t)\hat{\mathbf{k}} \qquad \text{in three dimensions}$$

where $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ are the unit vectors along the *x*, *y* and *z*-axes, and *f*(*t*), *h*(*t*) and *g*(*t*) are called the **component functions**.

Derivative of the vector function of a single variable

- The **derivative** of $\vec{\mathbf{a}}(t)$ with respect to t, found by differentiating each component of $\vec{\mathbf{a}}(t)$ separately w.r.t. t, is given by

$$\frac{d\vec{\mathbf{a}}(t)}{dt} = \frac{df(t)}{dt}\hat{\mathbf{i}} + \frac{dg(t)}{dt}\hat{\mathbf{j}} + \frac{dh(t)}{dt}\hat{\mathbf{k}}$$

and

$$\frac{d\vec{\mathbf{a}}(t)}{dt} = \frac{df(t)}{dt}\hat{\mathbf{i}} + \frac{dg(t)}{dt}\hat{\mathbf{j}}$$

Derivative of the products of vector functions

- The **derivatives** of the product of a scalar s and vector function $\vec{\mathbf{f}}(t)$, and the scalar and vector products of vector functions $\vec{\mathbf{a}}(t)$ and $\vec{\mathbf{b}}(t)$ are given as:

$$\frac{d}{dt}\left(s\vec{\mathbf{f}}\right) = \frac{ds}{dt}\vec{\mathbf{f}} + s\frac{d\vec{\mathbf{f}}}{dt}$$

$$\frac{d}{dt}\left(\vec{\mathbf{a}}.\vec{\mathbf{b}}\right) = \frac{d\vec{\mathbf{a}}}{dt}.\vec{\mathbf{b}} + \vec{\mathbf{a}}.\frac{d\vec{\mathbf{b}}}{dt}$$

and $$\frac{d}{dt}\left(\vec{\mathbf{a}}\times\vec{\mathbf{b}}\right) = \frac{d\vec{\mathbf{a}}}{dt}\times\vec{\mathbf{b}} + \vec{\mathbf{a}}\times\frac{d\vec{\mathbf{b}}}{dt}$$

2.6 TERMINAL QUESTIONS

1. Two velocity vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$, when added together have a resultant of $\vec{\mathbf{V}} = 3.0\ \text{ms}^{-1}\ \hat{\mathbf{i}} + 1.0\ \text{ms}^{-1}\ \hat{\mathbf{j}}$. If $\vec{\mathbf{u}} = 1.5\ \text{ms}^{-1}\ \hat{\mathbf{i}} + 2.0\ \text{ms}^{-1}\ \hat{\mathbf{j}}$, determine $\vec{\mathbf{v}}$.

2. Calculate the area of the ΔABC if the coordinates of the vertices *A, B* and *C* are (2, – 1, 0), (3, 2, 1) and (1, 2, – 2), respectively.

3. Determine the projection of $\vec{\mathbf{a}} + 2\vec{\mathbf{b}}$ on $\vec{\mathbf{a}}$ where $\vec{\mathbf{a}} = 3\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\vec{\mathbf{b}} = -2\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$.

4. Determine a vector of magnitude two units, perpendicular to each of the vectors $\vec{\mathbf{a}} = 2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ and $\vec{\mathbf{b}} = \hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$.

5. For two vectors $\vec{\mathbf{a}} = 2\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\vec{\mathbf{b}} = 3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}}$, obtain $(\vec{\mathbf{a}} + \vec{\mathbf{b}})\times(\vec{\mathbf{a}} - \vec{\mathbf{b}})$.

6. Find two unit vectors perpendicular to both $\vec{\mathbf{A}} = \hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ and $\vec{\mathbf{B}} = -2\hat{\mathbf{i}} + 4\hat{\mathbf{j}}$.

7. Determine the torque about the point (1, 0, – 1) due to a force $\vec{\mathbf{F}} = 3\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}$ being exerted at the point (2, – 1, – 4).

8. Determine *x* and *y* if both vectors $\vec{\mathbf{A}} = x\hat{\mathbf{i}} + 3\hat{\mathbf{j}}$ and $\vec{\mathbf{B}} = 2\hat{\mathbf{i}} + y\hat{\mathbf{j}}$ are parallel to the vector $\vec{\mathbf{C}} = 4\hat{\mathbf{i}} + 9\hat{\mathbf{j}}$.

9. Is it possible to have a vector $\vec{\mathbf{A}} = x\hat{\mathbf{i}} + 3\hat{\mathbf{j}}$ such that the relation: $\left(2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}}\right)\times\vec{\mathbf{A}} = 4\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 4\hat{\mathbf{k}}$ is true? Explain.

10. The components of a force, on a particle, in the *x* and *y* directions are 5N and 8N, respectively. If the particle moves from point (2, 6) to point (7, 9), calculate the work done by the force on it. Take the unit of displacement to be in m.

11. If $\vec{\mathbf{r}}(t) = 3t\hat{\mathbf{i}} + 2t^2\hat{\mathbf{j}} + t^3\hat{\mathbf{k}}$, calculate (i) $\dfrac{d\vec{\mathbf{r}}}{dt}$ and (ii) $\dfrac{d^2\vec{\mathbf{r}}}{dt^2}$.

12. Determine the velocity of a particle having the following position vector:

$$\vec{\mathbf{r}}(t) = \cos^2 t\hat{\mathbf{i}} + \sin^2 t\,\hat{\mathbf{j}} + \cos 2t\,\hat{\mathbf{k}}$$

13. The position vector of an object of mass m moving along a curve is given by

$$\vec{\mathbf{r}}(t) = at^2\,\hat{\mathbf{i}} + \sin bt\,\hat{\mathbf{j}} + \cos bt\,\hat{\mathbf{k}},\ 0 \le t \le 1,$$

where a and b are constants. Calculate the force being exerted on the object.

14. A curve is described by the following parametric equations

$$x = 2, \qquad y = 3t^2 + 2, \qquad z = 4 - t$$

Determine the unit tangent vector to the curve at the point $t = 1$.

15. Given two vector functions $\vec{\mathbf{a}}(t) = (2t+3)\,\hat{\mathbf{i}} + (t^2 - 2)\,\hat{\mathbf{j}} + t\,\hat{\mathbf{k}}$ and $\vec{\mathbf{b}}(t) = (4-t)\,\hat{\mathbf{i}} + (3t^2+1)\,\hat{\mathbf{j}} + (2t-1)\hat{\mathbf{k}}$, determine the derivative of $\vec{\mathbf{a}}(t)\,.\,\vec{\mathbf{b}}(t)$ at $t = 1$.

2.7 SOLUTIONS AND ANSWERS

Self-Assessment Questions

1. We use Eq. (2.1). For $\vec{\mathbf{a}} = \vec{\mathbf{A}}$, $a = A = 3$, $\theta = 60^\circ$ and we get

$$A_x = A\cos\theta = 3\cos 60^\circ = \frac{3}{2} \text{ and } A_y = A\sin\theta = 3\sin 60^\circ = \frac{3\sqrt{3}}{2}$$

For $\vec{\mathbf{a}} = \vec{\mathbf{B}}$, $a = B = 4$, $\theta = 135^\circ$ and we get

$$B_x = B\cos 135^\circ = \frac{-4}{\sqrt{2}} = -2\sqrt{2} \text{ and } B_y = B\sin 135^\circ = \frac{4}{\sqrt{2}} = 2\sqrt{2}$$

For $\vec{\mathbf{a}} = \vec{\mathbf{C}}$, $a = C = 5$, $\theta = 210^\circ$ and we get

$$C_x = C\cos 210^\circ = \frac{-5\sqrt{3}}{2} \text{ and } C_y = C\sin 210^\circ = -\frac{5}{2}$$

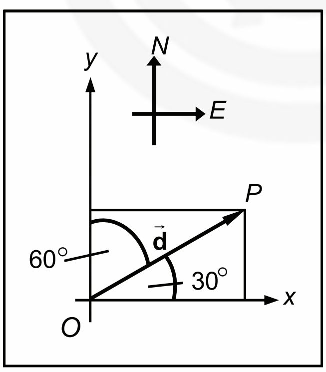


Fig. 2.19: **Diagram for SAQ 2a.**

2. a) Refer to Fig. 2.19; $\overrightarrow{\mathbf{OP}}$ represents the displacement $\vec{\mathbf{d}}$. The angle the displacement makes with the x-axis is $\theta = 90^\circ - 60^\circ = 30^\circ$. Using Eq. (2.1) with $\vec{\mathbf{a}} = \vec{\mathbf{d}}$, $a = d = 24\,\text{m}$ and $\theta = 30^\circ$, the x and y components of $\vec{\mathbf{d}}$ are,

$$d_x = d\cos\theta = (24\cos 30^\circ) = 20.8\,\text{m} \approx 21\text{ m}$$

$$d_y = d\sin\theta = (24\sin 30^\circ) = 12\text{ m}$$

$$\therefore \qquad \vec{\mathbf{d}} = 21\,\text{m}\,\hat{\mathbf{i}} + 12\,\text{m}\,\hat{\mathbf{j}}$$

b) We can use Eq. (2.7) to calculate the magnitudes of the vectors.

$$a = \sqrt{3^2 + (-2)^2 + 1^2} = \sqrt{14}$$

$$b = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

$$c = \sqrt{2^2 + (-1)^2 + 1^2} = \sqrt{6}$$

The unit vectors along $\vec{\mathbf{a}}, \vec{\mathbf{b}}$ and $\vec{\mathbf{c}}$ which are $\hat{\mathbf{a}}, \hat{\mathbf{b}}, \hat{\mathbf{c}}$, respectively, are found using Eq. (1.3b).

$$\hat{\mathbf{a}} = \frac{\vec{\mathbf{a}}}{a} = \frac{1}{\sqrt{14}}(3\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}});\ \hat{\mathbf{b}} = \frac{\vec{\mathbf{b}}}{b} = \frac{1}{\sqrt{3}}(\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}});\ \hat{\mathbf{c}} = \frac{\vec{\mathbf{c}}}{c} = \frac{1}{\sqrt{6}}(2\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}})$$

3. a) We use Eq. (2.17a) to determine the scalar product of the two vectors and Eq. (1.11d) to find the angle between the vectors.

   We are given

   $a_x = 3,\ a_y = 2,\ a_z = -1$ and $b_x = 1,\ b_y = 1,\ b_z = 2$

   $$\therefore \quad \vec{\mathbf{a}}\,.\,\vec{\mathbf{b}} = 3\times1 + 2\times1 + (-1)\times2 = 3$$

   The magnitudes of the vectors are:

   $$a = \sqrt{3^2 + 2^2 + (-1)^2} = \sqrt{14} \text{ and } b = \sqrt{1^2 + 1^2 + 2^2} = \sqrt{6}$$

   The angle θ between $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ is

   $$\theta = \cos^{-1}\left(\frac{\vec{\mathbf{a}}\,.\,\vec{\mathbf{b}}}{ab}\right) = \cos^{-1}\left(\frac{3}{\sqrt{14}\,.\,\sqrt{6}}\right) = \cos^{-1}\left(\frac{3}{\sqrt{84}}\right)$$

   b) We use Eq. (1.11g) to write the projection of $\vec{\mathbf{a}}$ on $\vec{\mathbf{b}}$ as,

   $$\frac{\vec{\mathbf{a}}\,.\,\vec{\mathbf{b}}}{|\,\vec{\mathbf{b}}\,|} = \frac{(2\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 3\hat{\mathbf{k}})\,.\,(\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 3\hat{\mathbf{k}})}{\sqrt{1^2 + 2^2 + 3^2}} = -\frac{15}{\sqrt{14}}$$

4. a) The vector product of two vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ is a vector perpendicular to the plane containing the vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$. Therefore, the unit vector $\hat{\mathbf{n}}$ perpendicular to the plane containing any two vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ is

   $$\hat{\mathbf{n}} = \frac{\vec{\mathbf{a}}\times\vec{\mathbf{b}}}{|\vec{\mathbf{a}}\times\vec{\mathbf{b}}|} \qquad \text{(i)}$$

   where $|\vec{\mathbf{a}}\times\vec{\mathbf{b}}|$ is the magnitude of the vector product $(\vec{\mathbf{a}}\times\vec{\mathbf{b}})$. Using Eq. (2.21b) with $\vec{\mathbf{a}} = 4\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\vec{\mathbf{b}} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$, we get:

   $$\vec{\mathbf{a}}\times\vec{\mathbf{b}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 4 & -2 & 1 \\ 1 & 2 & 3 \end{vmatrix} = -8\hat{\mathbf{i}} - 11\hat{\mathbf{j}} + 10\hat{\mathbf{k}}$$

   and $\quad |\vec{\mathbf{a}}\times\vec{\mathbf{b}}| = \sqrt{(-8)^2 + (-11)^2 + (10)^2} = \sqrt{285}$

   $$\therefore \quad \hat{\mathbf{n}} = \frac{1}{\sqrt{285}}(-8\hat{\mathbf{i}} - 11\hat{\mathbf{j}} + 10\hat{\mathbf{k}})$$

   b) From Example 1.1 we know that the area of the parallelogram whose adjacent sides are given by the vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ is $|\vec{\mathbf{a}}\times\vec{\mathbf{b}}|$ and here

   $$\vec{\mathbf{a}}\times\vec{\mathbf{b}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & -3 & 1 \\ 4 & 0 & 3 \end{vmatrix} = -9\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 12\hat{\mathbf{k}}$$

   $$\therefore \quad \text{Area} = \left|\vec{\mathbf{a}}\times\vec{\mathbf{b}}\right| = \sqrt{(-9)^2 + (2)^2 + (12)^2} = \sqrt{229} \text{ units}$$

5. The vectors given in (a),(b) and (d) are vector functions. The function in (a) is a function of time *t*. The function in (b) is a function of angle θ. The function in (d) is a function of the space coordinates (*x, y, z*) through the variable $r = (x^2 + y^2 + z^2)^{1/2}$. But the force in (c) is not a vector function as it is constant.

6. The velocity of the particle is

$$\vec{\mathbf{v}}(t) = \frac{d}{dt}\vec{\mathbf{r}}(t) = \frac{d}{dt}[(6t - t^2)\,\hat{\mathbf{i}} + 2t\,\hat{\mathbf{j}} + t\,\hat{\mathbf{k}}] = (6 - 2t)\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} + \hat{\mathbf{k}}$$

The acceleration of the particle: $\vec{\mathbf{a}}(t) = \frac{d}{dt}\vec{\mathbf{v}}(t) = \frac{d}{dt}\left[(6 - 2t)\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} + \hat{\mathbf{k}}\right] = -2\,\hat{\mathbf{i}}$

Terminal Questions

You can check from Fig. 2.20 **that the vector from** A **to** B**, that is,** $\vec{\mathbf{a}}$**, is given by**

$\vec{\mathbf{a}} = \vec{\mathbf{r}}_B - \vec{\mathbf{r}}_A$

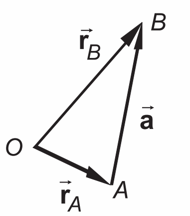


Fig. 2.20

In the same way, the vector from B to C, that is, $\vec{\mathbf{b}}$, is given by

$\vec{\mathbf{b}} = \vec{\mathbf{r}}_C - \vec{\mathbf{r}}_B$

1. Since $\vec{\mathbf{u}} + \vec{\mathbf{v}} = \vec{\mathbf{V}},\ \vec{\mathbf{v}} = \vec{\mathbf{V}} - \vec{\mathbf{u}}$

Given $\vec{\mathbf{u}} = 1.5\ \text{ms}^{-1}\,\hat{\mathbf{i}} + 2.0\,\text{ms}^{-1}\,\hat{\mathbf{j}}$ and $\vec{\mathbf{V}} = 3.0\ \text{ms}^{-1}\,\hat{\mathbf{i}} + 1.0\,\text{ms}^{-1}\,\hat{\mathbf{j}}$, we get

$$\vec{\mathbf{v}} = (3.0\,\text{ms}^{-1}\,\hat{\mathbf{i}} + 1.0\,\text{ms}^{-1}\,\hat{\mathbf{j}}) - (1.5\,\text{ms}^{-1}\,\hat{\mathbf{i}} + 2.0\,\text{ms}^{-1}\,\hat{\mathbf{j}})$$

$$= 1.5\,\text{ms}^{-1}\,\hat{\mathbf{i}} - 1.0\,\text{ms}^{-1}\,\hat{\mathbf{j}}$$

2. We can write the position vectors of the three points A, B and C as

$$\vec{\mathbf{r}}_A = 2\hat{\mathbf{i}} - \hat{\mathbf{j}}, \qquad \vec{\mathbf{r}}_B = 3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}} \quad \text{and} \quad \vec{\mathbf{r}}_C = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$$

Let the side AB be represented by the vector $\vec{\mathbf{a}}$ and BC by the vector $\vec{\mathbf{b}}$. Then you can see that $\vec{\mathbf{a}} = \vec{\mathbf{r}}_B - \vec{\mathbf{r}}_A$ (see Fig. 2.20) and $\vec{\mathbf{b}} = \vec{\mathbf{r}}_C - \vec{\mathbf{r}}_B$ (read the margin remark). Then

Area of the $\Delta ABC = \frac{1}{2}|\vec{\mathbf{a}} \times \vec{\mathbf{b}}|$ where

$$\vec{\mathbf{a}} = \vec{\mathbf{r}}_B - \vec{\mathbf{r}}_A = (3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}}) - (2\hat{\mathbf{i}} - \hat{\mathbf{j}}) = \hat{\mathbf{i}} + 3\hat{\mathbf{j}} + \hat{\mathbf{k}}$$

and $$\vec{\mathbf{b}} = \vec{\mathbf{r}}_C - \vec{\mathbf{r}}_B = (\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 2\hat{\mathbf{k}}) - (3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}}) = -2\hat{\mathbf{i}} - 3\hat{\mathbf{k}}$$

Area of the ΔABC

$$= \frac{1}{2}|\vec{\mathbf{a}} \times \vec{\mathbf{b}}| = \frac{1}{2}|(\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + \hat{\mathbf{k}})| \times (-2\hat{\mathbf{i}} - 3\hat{\mathbf{k}}) = \frac{1}{2}|(-9\hat{\mathbf{i}} + \hat{\mathbf{j}} + 6\hat{\mathbf{k}})|$$

$$\therefore \quad \text{Area} = \frac{1}{2}\sqrt{(-9)^2 + (1)^2 + 6^2} = \frac{1}{2}\sqrt{118}\ \text{units}$$

3. Let $\vec{\mathbf{d}} = \vec{\mathbf{a}} + 2\vec{\mathbf{b}}$. Substituting for $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ we can write

$$\vec{\mathbf{d}} = 3\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}} + 2(-2\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 4\hat{\mathbf{k}}) = -\hat{\mathbf{i}} - 6\hat{\mathbf{j}} + 9\hat{\mathbf{k}}$$

From Eq. (1.11g), we can write the projection of $\vec{\mathbf{d}}$ on $\vec{\mathbf{a}}$ as

$$\frac{\vec{\mathbf{d}}\ .\ \vec{\mathbf{a}}}{a} = \frac{(-\hat{\mathbf{i}} - 6\hat{\mathbf{j}} + 9\hat{\mathbf{k}})\ .\ (3\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}})}{\sqrt{(3)^2 + (-2)^2 + 1^2}} = \frac{18}{\sqrt{14}}$$

4. Using the method outlined in the solution for SAQ 4a, we can write down the unit vector perpendicular to both $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ as

$$\hat{\mathbf{n}} = \frac{\vec{\mathbf{a}} \times \vec{\mathbf{b}}}{|\vec{\mathbf{a}} \times \vec{\mathbf{b}}|} = \frac{(2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}}) \times (\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}})}{|(2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}}) \times (\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}})|} = \frac{2\hat{\mathbf{i}} - 2\hat{\mathbf{j}} - 3\hat{\mathbf{k}}}{|2\hat{\mathbf{i}} - 2\hat{\mathbf{j}} - 3\hat{\mathbf{k}}|} = \frac{2\hat{\mathbf{i}} - 2\hat{\mathbf{j}} - 3\hat{\mathbf{k}}}{\sqrt{17}}$$

Now a vector, $\vec{\mathbf{D}}$, of magnitude 2 units, which is, perpendicular to both $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ is given by

$$\vec{\mathbf{D}} = 2\hat{\mathbf{n}} = \frac{2}{\sqrt{17}}(2\hat{\mathbf{i}} - 2\hat{\mathbf{j}} - 3\hat{\mathbf{k}})$$

5. With $\vec{\mathbf{a}} = 2\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\vec{\mathbf{b}} = 3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}}$

$\vec{\mathbf{a}} + \vec{\mathbf{b}} = 5\hat{\mathbf{i}} + 3\hat{\mathbf{j}}$ and $\vec{\mathbf{a}} - \vec{\mathbf{b}} = -\hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$

From Eq. (2.21b), the cross product is

$$(\vec{\mathbf{a}} + \vec{\mathbf{b}}) \times (\vec{\mathbf{a}} - \vec{\mathbf{b}}) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 5 & 3 & 0 \\ -1 & -1 & 2 \end{vmatrix} = 6\hat{\mathbf{i}} - 10\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$$

In TQ 5, you may also write

$$(\vec{\mathbf{a}} + \vec{\mathbf{b}}) \times (\vec{\mathbf{a}} - \vec{\mathbf{b}}) = \vec{\mathbf{a}} \times \vec{\mathbf{a}} + \vec{\mathbf{b}} \times \vec{\mathbf{a}} - \vec{\mathbf{a}} \times \vec{\mathbf{b}} - \vec{\mathbf{b}} \times \vec{\mathbf{b}} = 2(\vec{\mathbf{b}} \times \vec{\mathbf{a}})$$

since

$$-\vec{\mathbf{a}} \times \vec{\mathbf{b}} = \vec{\mathbf{b}} \times \vec{\mathbf{a}}$$

So

$$(\vec{\mathbf{a}} + \vec{\mathbf{b}}) \times (\vec{\mathbf{a}} - \vec{\mathbf{b}}) = 2\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 3 & 2 & -1 \\ 2 & 1 & 1 \end{vmatrix} = 6\hat{\mathbf{i}} - 10\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$$

6. We first determine $\vec{\mathbf{R}} = (\vec{\mathbf{A}} \times \vec{\mathbf{B}})$ because $\vec{\mathbf{R}}$ is perpendicular to both $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$:

$$\vec{\mathbf{R}} = (\vec{\mathbf{A}} \times \vec{\mathbf{B}}) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & -2 & 3 \\ -2 & 4 & 0 \end{vmatrix} = \hat{\mathbf{i}}(-12) + \hat{\mathbf{j}}(-6) + \hat{\mathbf{k}}(0)$$

$$= -12\hat{\mathbf{i}} - 6\hat{\mathbf{j}}$$

The unit vector along $\vec{\mathbf{R}}$ is

$$\hat{\mathbf{R}} = \frac{\vec{\mathbf{R}}}{|\vec{\mathbf{R}}|} = \frac{-12\hat{\mathbf{i}} - 6\hat{\mathbf{j}}}{\sqrt{144 + 36}} = \frac{-12\hat{\mathbf{i}} - 6\hat{\mathbf{j}}}{\sqrt{180}} = \frac{(-2\hat{\mathbf{i}} - \hat{\mathbf{j}})}{\sqrt{5}}$$

Another unit vector perpendicular to $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ would be

$$\hat{\mathbf{R}}' = -\hat{\mathbf{R}} = \frac{(2\hat{\mathbf{i}} + \hat{\mathbf{j}})}{\sqrt{5}}$$

7. The torque is $\vec{\boldsymbol{\tau}} = \vec{\mathbf{r}} \times \vec{\mathbf{F}}$ and $\vec{\mathbf{r}}$ is the position vector of the point (2, – 1, – 4) relative to point (1, 0, – 1) given by

$$\vec{\mathbf{r}} = (2\hat{\mathbf{i}} - \hat{\mathbf{j}} - 4\hat{\mathbf{k}}) - (\hat{\mathbf{i}} - \hat{\mathbf{k}}) = \hat{\mathbf{i}} - \hat{\mathbf{j}} - 3\hat{\mathbf{k}}$$

$$\therefore \quad \vec{\boldsymbol{\tau}} = \vec{\mathbf{r}} \times \vec{\mathbf{F}} = (\hat{\mathbf{i}} - \hat{\mathbf{j}} - 3\hat{\mathbf{k}}) \times (3\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}) = 4\hat{\mathbf{i}} - 8\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$$

8. If $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ are parallel to $\vec{\mathbf{C}}$, then $\vec{\mathbf{A}} \times \vec{\mathbf{C}} = \vec{\mathbf{B}} \times \vec{\mathbf{C}} = \vec{\mathbf{0}}$ from Eq. (1.15a) of Unit 1, and we get

$$\vec{\mathbf{A}} \times \vec{\mathbf{C}} = (9x - 12)\hat{\mathbf{k}} = \vec{\mathbf{0}} \Rightarrow x = \frac{4}{3}$$

$$\vec{\mathbf{B}} \times \vec{\mathbf{C}} = (18 - 4y)\hat{\mathbf{k}} = \vec{\mathbf{0}} \Rightarrow y = 9/2$$

9. If the given relation is true, then the vector $(4\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 4\hat{\mathbf{k}})$ is perpendicular to both $(2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}})$ and $\vec{\mathbf{A}}$ (property of the vector product). If $(4\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 4\hat{\mathbf{k}})$ is perpendicular to $(2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}})$, their scalar product should be zero. But $(2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}}).(4\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 4\hat{\mathbf{k}}) = -17 \neq 0$. Therefore, It is not possible to have a vector $\vec{\mathbf{A}}$ if the given relation is true.

10. The force is $\vec{\mathbf{F}} = (5\text{N})\hat{\mathbf{i}} + (8\text{N})\hat{\mathbf{j}}$. The displacement of the particle in m is

$$\vec{\mathbf{r}} = (x_2 - x_1)\hat{\mathbf{i}} + (y_2 - y_1)\hat{\mathbf{j}} = (7 - 2)\hat{\mathbf{i}} + (9 - 6)\hat{\mathbf{j}} = 5\text{m}\,\hat{\mathbf{i}} + 3\text{m}\,\hat{\mathbf{j}}$$

Work done = $\vec{\mathbf{F}}.\vec{\mathbf{r}} = (5\text{N}\,\hat{\mathbf{i}} + 8\text{N}\,\hat{\mathbf{j}}).(5\text{m}\,\hat{\mathbf{i}} + 3\text{m}\,\hat{\mathbf{j}}) = (25 + 24)\text{J} = 49\,\text{J}$

11. Using Eq. (2.28a), we get

$$\frac{d\vec{\mathbf{r}}}{dt} = \frac{d}{dt}(3t)\hat{\mathbf{i}} + \frac{d}{dt}(2t^2)\hat{\mathbf{j}} + \frac{d}{dt}(t^3)\hat{\mathbf{k}} = 3\hat{\mathbf{i}} + 4t\,\hat{\mathbf{j}} + 3t^2\,\hat{\mathbf{k}}$$

and $$\frac{d^2\vec{\mathbf{r}}}{dt^2} = \frac{d}{dt}\left(\frac{d\vec{\mathbf{r}}}{dt}\right) = \frac{d}{dt}(3)\hat{\mathbf{i}} + \frac{d}{dt}(4t)\hat{\mathbf{j}} + \frac{d}{dt}(3t^2)\hat{\mathbf{k}} = 4\hat{\mathbf{j}} + 6t\,\hat{\mathbf{k}}$$

12. The velocity of the particle is

$$\frac{d\vec{\mathbf{r}}}{dt} = \frac{d}{dt}(\cos^2 t)\,\hat{\mathbf{i}} + \frac{d}{dt}(\sin^2 t)\,\hat{\mathbf{j}} + \frac{d}{dt}(\cos 2t)\,\hat{\mathbf{k}}$$

$$= -2\cos t \sin t\,\hat{\mathbf{i}} + 2\sin t \cos t\,\hat{\mathbf{j}} - 2\sin 2t\,\hat{\mathbf{k}}$$

$$= -\sin 2t\,\hat{\mathbf{i}} + \sin 2t\,\hat{\mathbf{j}} - 2\sin 2t\,\hat{\mathbf{k}} = -\sin 2t\left[-\hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\hat{\mathbf{k}}\right]$$

13. The force on the object is: $\vec{\mathbf{F}} = m\dfrac{d^2\vec{\mathbf{r}}}{dt^2}$

Using Eq. (2.28a) with $\vec{\mathbf{r}} = at^2\,\hat{\mathbf{i}} + \sin bt\,\hat{\mathbf{j}} + \cos bt\,\hat{\mathbf{k}}$, we get

$$\frac{d\vec{\mathbf{r}}}{dt} = 2at\,\hat{\mathbf{i}} + b\cos bt\,\hat{\mathbf{j}} - b\sin bt\,\hat{\mathbf{k}}$$

and $$\frac{d^2\vec{\mathbf{r}}}{dt^2} = \frac{d}{dt}\left[\frac{d\vec{\mathbf{r}}}{dt}\right] = 2a\,\hat{\mathbf{i}} - b^2\sin bt\,\hat{\mathbf{j}} - b^2\cos bt\,\hat{\mathbf{k}}$$

$\therefore$ $$\vec{\mathbf{F}} = 2ma\,\hat{\mathbf{i}} - mb^2\sin bt\,\hat{\mathbf{j}} - mb^2\cos bt\,\hat{\mathbf{k}}$$

14. The position vector function representing the curve corresponding to the given parametric equation is

$$\vec{\mathbf{r}}(t) = 2\,\hat{\mathbf{i}} + (3t^2 + 2)\,\hat{\mathbf{j}} + (4 - t)\,\hat{\mathbf{k}}$$

The derivative $\dfrac{d\vec{\mathbf{r}}}{dt}$ at any value t defines the tangent to the curve at that value of t. So we can write the tangent vector as

$$\vec{\mathbf{T}} = \frac{d\vec{\mathbf{r}}}{dt} = \frac{d}{dt}[2\,\hat{\mathbf{i}} + (3t^2 + 2)\,\hat{\mathbf{j}} + (4 - t)\,\hat{\mathbf{k}}] = 6t\,\hat{\mathbf{j}} - \hat{\mathbf{k}}$$

At $t = 1$, the tangent vector is $\vec{\mathbf{T}} = 6\,\hat{\mathbf{j}} - \hat{\mathbf{k}}$

The unit tangent vector is $\hat{\mathbf{T}} = \dfrac{\vec{\mathbf{T}}}{|\vec{\mathbf{T}}|} = \dfrac{6\,\hat{\mathbf{j}} - \hat{\mathbf{k}}}{|6^2 + 1^2|} = \dfrac{1}{\sqrt{37}}\,[6\,\hat{\mathbf{j}} - \hat{\mathbf{k}}]$

15. We use Eq. (2.30a) to write

$$\frac{d}{dt}(\vec{\mathbf{a}}.\vec{\mathbf{b}}) = \left[(2t+3)\,\hat{\mathbf{i}} + (t^2 - 2)\,\hat{\mathbf{j}} + t\,\hat{\mathbf{k}}\right].\left[\frac{d}{dt}[(4-t)\,\hat{\mathbf{i}} + (3t^2+1)\,\hat{\mathbf{j}} + (2t-1)\,\hat{\mathbf{k}}]\right]$$

$$+\left(\frac{d}{dt}[(2t+3)\,\hat{\mathbf{i}} + (t^2-2)\,\hat{\mathbf{j}} + t\,\hat{\mathbf{k}}]\right).\left[(4-t)\,\hat{\mathbf{i}} + (3t^2+1)\,\hat{\mathbf{j}} + (2t-1)\,\hat{\mathbf{k}}\right]$$

$$= [(2t+3)\,\hat{\mathbf{i}} + (t^2-2)\,\hat{\mathbf{j}} + t\,\hat{\mathbf{k}}].[-\hat{\mathbf{i}} + 6t\,\hat{\mathbf{j}} + 2\hat{\mathbf{k}}]$$

$$+\ [2\,\hat{\mathbf{i}} + 2t\,\hat{\mathbf{j}} + \hat{\mathbf{k}}].[(4-t)\,\hat{\mathbf{i}} + (3t^2+1)\,\hat{\mathbf{j}} + (2t-1)\,\hat{\mathbf{k}}]$$

We can now substitute $t = 1$. Then

$$\frac{d}{dt}(\vec{\mathbf{a}}.\vec{\mathbf{b}}) = (5\,\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}).(-\hat{\mathbf{i}} + 6\,\hat{\mathbf{j}} + 2\hat{\mathbf{k}})$$

$$+(2\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} + \hat{\mathbf{k}}).(3\hat{\mathbf{i}} + 4\,\hat{\mathbf{j}} + \hat{\mathbf{k}}) = -5 - 6 + 2 + 6 + 8 + 1 = 6$$

What should the velocity of a spacecraft be if it is to escape from the Earth? Attempt Terminal Question 2 to find the answer!

UNIT 3

FIRST ORDER ORDINARY DIFFERENTIAL EQUATIONS

Structure

STUDY GUIDE

We hope that you have studied calculus at the senior secondary (+ 2) level. We shall take it for granted that you know how to determine the first order derivatives, partial derivatives and integrals of various functions. However, we have briefly explained how to calculate partial derivatives in an **Appendix** to this unit. You may like to go through it to revise them. You have to make sure that you can solve integrals of various functions very well and then study the unit. You may like to **revise the methods of integration** that you have studied in + 2 mathematics course. You should study the unit thoroughly and make sure that you can solve the problems given in this unit. You may take about 5 to 10 minutes to solve the SAQs given in this unit. Some of the Terminal Questions are more challenging. If you are able to solve the SAQs and Terminal Questions, then you have grasped the concepts of the unit very well. Of course, you should try to solve the problems on your own without first looking at the solutions and answers!

> ***"It is impossible to explain honestly the beauties of the laws of nature in a way that people can feel, without their having some deep understanding of mathematics. I am sorry, but this seems to be the case."***
>
> ***Richard P. Feynman***

3.1 INTRODUCTION

In your school physics, you have studied Newton's laws of motion and applied them to simple systems such as the motion of a particle falling under the force of gravity, projectile motion and rocket motion. The laws of mechanics help us study the motion of objects using **an equation containing an unknown variable and its first and/or second order ordinary derivatives**. Such equations are called **ordinary differential equations**. Henceforth, in this block, we shall refer to them as ODEs.

ODEs have applications in all such problems of physics in which we usually wish to know how a physical variable is changing with respect to another variable, e.g., rate of change of velocity with time in mechanics. Other examples are radioactive decay, rate of fluid flow, growth/decay of current in an electric circuit and even the rate of change of oil slicks formed in seas/oceans in environmental sciences.

NOTE

The **order** of an ODE is the **order of the highest derivative appearing in it**.

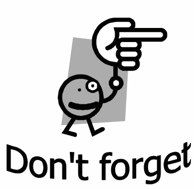
We shall begin the unit by briefly explaining the classification of ODEs and the concept of general and particular solutions (Sec. 3.2). In the remaining sections, we shall discuss how to solve first order differential equations with special focus on applications in physics. In Sec. 3.3, you will learn how to solve first order ODEs in which the variables can be separated. Such equations are very common in mechanics and may be solved using the **method of separation of variables** or the **method of substitution**. You will also learn how to solve **homogeneous** first order ODEs.

In Sec. 3.4, you will learn how to solve **exact equations**, which appear frequently in thermal physics. Finally in Sec. 3.5, we explain the method of solving first order non-homogeneous ODEs using the method of integrating factors. This method is used in solving the ODEs for currents in *LR* and *RC* circuits. In each section, you will solve problems applying the technique presented there. In Unit 4, you will learn how to solve second order ODEs with constant coefficients.

Expected Learning Outcomes

After studying this unit, you should be able to:

- state the degree of first order ordinary differential equations and classify them as linear/non-linear, homogeneous/non-homogeneous;
- solve separable and exact first order ordinary differential equations; and
- solve first order non-homogeneous ordinary differential equations using the method of integrating factors.



You must be able to determine the first order derivatives and integrals of various functions. Revise the integrals given in the table at the end of this block.

3.2 CLASSIFICATION AND SOLUTION OF ODEs

Let us begin by considering some examples of ODEs from physics. Do you recall Newton's laws of mechanics from your school physics course? Let us apply Newton's second law of motion ($\vec{\mathbf{F}} = m\vec{\mathbf{a}}$) to a particle (say, a ball or a parachute) of mass *m* falling near the Earth's surface. Let the force of gravity be the only force on the particle (Fig. 3.1). So, its equation of motion can be written as

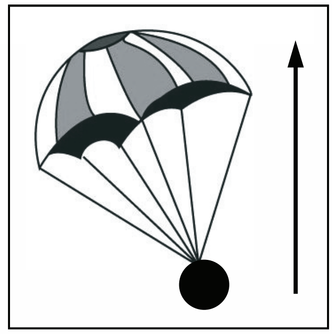


Fig. 3.1: **A particle falling under the force of gravity.**

$$ma = m\frac{dv}{dt} = -mg \quad \text{or} \quad \frac{dv}{dt} = -g \tag{3.1a}$$

where *v* is the particle's speed and *g*, the acceleration due to gravity. In writing Eq. (3.1a), we have chosen the vertical coordinate axis in the upward direction. Using Eq. (3.1a), we can obtain a relation between the independent variable *t* and the dependent variable *v*. It is simply $v = u - gt$ (read the margin remark), an equation you know very well from your school physics. Let us consider another example from school physics – the law of radioactive decay. It tells us that the rate at which the atoms of a radioactive substance disintegrate is proportional to the number of atoms (*N*) present in it. You know from calculus that we can represent the rate of disintegration of atoms as $\left(-\frac{dN}{dt}\right)$, where *t* represents time. The negative sign appears because *N* decreases with time. So we can write

You can solve Eq. (3.1a) by integrating both sides with respect to time.
This gives
$v(t) = -gt + c$
Then applying the initial condition that at
$t = 0$, $v(0) = u$,
we get
$v = u - gt$

$$\frac{dN}{dt} = -\lambda N \quad \text{or} \quad \frac{dN}{dt} + \lambda N = 0 \tag{3.1b}$$

where λ is a constant. You may also have studied the simple harmonic oscillations in your school physics course. The equation for one-dimensional simple harmonic motion is given as

An equation similar to Eq. (3.1b) is obtained when we consider collisions of gas molecules in a container.

$$m\frac{d^2x}{dt^2} + kx = 0 \tag{3.1c}$$

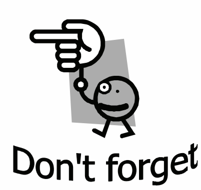
where *m* is the mass of the oscillator and *k*, the force constant. Eq. (3.1c) contains a term involving *x* and another term that is its second derivative with respect to time. Note that Eqs. (3.1a, b and c) contain **ordinary** derivatives of only the **dependent** variable. Let us consider yet another example from physics. Suppose a current *i* flows through an electric circuit for an infinitesimal duration of time, *dt*. Then the charge that flows during this time is given by

$$dq = i\,dt \tag{3.1d}$$

The abbreviation w.r.t. stands for 'with respect to'.

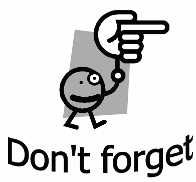
Eq. (3.1d) involves the differentials *dq* and *dt*. Equations like (3.1a to d) are called **ordinary differential equations**.

> An equation that contains differentials or only ordinary derivatives of one or more dependent variable w.r.t. ***a single independent variable***, is said to be an **ordinary differential equation**.



You will learn about similar classification of second order ODEs in Unit 4.

Now before you learn how to solve equations such as (3.1a to d), you should know **how to classify ODEs**. The most basic classification of ODEs is on the basis of their **order** and **degree**. You must always remember the following definitions:



Order and degree of ODEs

The **order** of an ODE is the **order of the highest derivative** appearing in it.

The **degree** of an ODE is the **power of the highest order derivative** appearing in it, after it has been recast in a form that has no negative or fractional powers of derivatives.

Can you use these definitions to tell the order and degree of Eqs. (3.1a to d)? Answer the following SAQ.

SAQ 1 – **Order and degree of ODEs**

State the order and degree of Eqs. (3.1a to d).

NOTE

Henceforth, you must make it a habit to classify an ODE the moment you see it. This means that you must be able to state its order and degree, and whether it is linear or non-linear. You should also be able to classify it as homogeneous or non-homogeneous after you have leamt these concepts in Secs. 3.3 and 3.5 of this unit, and Unit 4.

We now focus on the **first order ODEs of degree one**. Note that first order ODEs of degree one contain terms involving *the variables* and only their **first order ordinary derivatives** with respect to any other variable. Also the highest power of the first order derivative is one. In order to solve the common first order ODEs in physics, we need to classify them further as **linear/non-linear** and **homogeneous/non-homogeneous** (read the margin remark).

3.2.1 Further Classification of First Order ODEs

In this section, you will learn how to classify a first order ODE as **linear/non-linear** and **homogeneous/non-homogeneous**. You should be able to tell whether a first order, first degree ODE is linear or non-linear, homogeneous or non-homogeneous by just looking at it. Then you will be able to decide on which method to use for solving it. We first define linear/non-linear first order ODEs.

LINEAR AND NON-LINEAR FIRST ORDER ODEs

A first order ODE is **linear** if

i) The unknown function and its first order derivatives occur only to the first degree;

ii) There are no products involving the unknown function and its first order derivatives or products of the first order derivatives; and

iii) There are no transcendental functions involving the unknown function or any of its first order derivatives.

A first order ODE that does not satisfy any one or more of the above conditions (i) to (iii) is said to be **non-linear**.

You can verify that the first order ODEs given by Eqs. (3.1a, b and d) are **linear**. We give below some more examples of first order **linear** ODEs commonly used in physics:

1. $L\dfrac{di}{dt} + Ri = E$ for current i in an LR circuit with constant E (3.2a)

2. $\dfrac{dq}{dt} + \dfrac{q}{RC} = \dfrac{E}{R}$ for charge q in an RC circuit with constant E (3.2b)

3. $v\dfrac{dv}{dr} = -\dfrac{GM}{r^2}$ for escape velocity of a particle from an object of mass M (3.2c)

You may like to further practice classifying linear and non-linear ODEs. Go through the definition given in the box again and then solve SAQ 2.

SAQ 2 – **Linear and non-linear ODEs**

Classify the following first order ODEs as linear or non-linear:

a) $m\dfrac{dv}{dt} = mg - kv(t)$

b) $L\dfrac{di}{dt} + Ri = E\sin\omega t$

c) $y' + (y')^2 = 0$ where y' stands for the derivative $\dfrac{dy}{dx}$.

A first order **linear** ODE is also classified as **homogeneous** or **non-homogeneous**. You will learn the definitions of the first order homogeneous and non-homogeneous ODEs in Secs. 3.3.3 and 3.5 of this unit, when you learn how to solve them. Before you learn the methods of solving first order ODEs, you need to know the concepts of general and particular solutions.

Generally, in text books, you will find that the dependent variable is denoted by y and the independent variable by x and we write $y = y(x)$ or $y = f(x)$. However, we can also denote the dependent variable by x and the independent variable by y and then we write

$x = x(y)$ or $x = f(y)$

3.2.2 General Solution and Particular Solution

A differential equation may have more than one solution. Generally, in physics, we first determine all solutions of a given differential equation. Then we retain the solutions that are relevant for the physical problem. You have seen for Eq. (3.1a) that depending on the value of the constant c (in the margin remark), there can be many solutions of the equation. Let us take another example. Consider the differential equation

$$y' = \sin x \tag{3.3}$$

You can easily verify that each of the functions $y = -\cos x$, $y = -\cos x + 5$, $y = -\cos x - 9$, $y = -\cos x + \dfrac{5}{8}$ is a solution of Eq. (3.3). You can express them generally as

$$y = -\cos x + C \tag{3.4}$$

In Eq. (3.3),

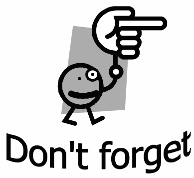
$$y' = \frac{dy}{dx} = \sin x$$

Integrating this, we get

$$\int dy = \int \sin x\, dx$$

or $y = -\cos x + C$

where C is an *arbitrary constant*. Eq. (3.4) is called a **general solution** of Eq. (3.3) as it can yield any number of solutions.



A solution involving arbitrary constant(s) is known as the **general solution.**

Note that Eq. (3.3) is a first order differential equation and its general solution given by Eq. (3.4) has **one** arbitrary constant. But the general solution of a second order differential equation has two arbitrary constants (as explained in Unit 4). So, we can say that **the number of arbitrary constants appearing in the solution of a differential equation is equal to its order.** Let us now impose the following condition on Eq. (3.4): $y = 0$ when $x = 0$. Then we get

$$0 = 0 + C \quad \text{or} \quad C = 0 \quad \text{so that} \quad y = -\cos x \tag{3.5}$$

So by imposing a condition on Eq. (3.4), we can assign a specific value to the arbitrary constant C. The solution so obtained is called a **particular solution**. For example, $y = -\cos x + 2$ is a particular solution of Eq. (3.3). We have shown one particular solution of Eq. (3.4) for $C = 0$ in Fig. 3.2.

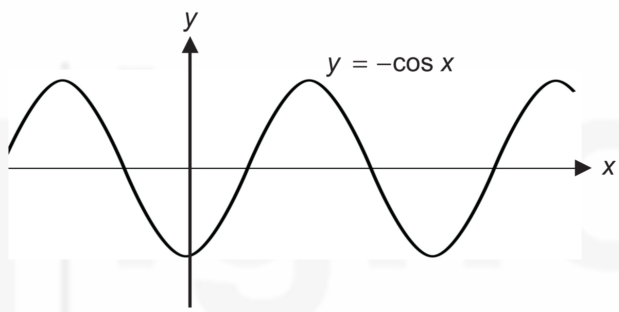
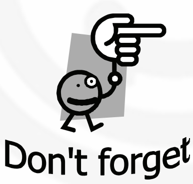


Fig. 3.2: **A particular solution of the differential equation:** $y' = \sin x$.



If a definite value can be assigned to each arbitrary constant appearing in a general solution, then we get a **particular solution.**

Conditions of the type $y(x_0) = C_1,\ y'(x_0) = C_2$ are called **initial conditions** and together with the ODE constitute the **initial value problem**. Conditions of the type $y(x_0) = C_1,\ y(x_1) = C_2$ are called **boundary conditions** and together with the ODE constitute the **boundary value problem**.

You will learn about these concepts with much more rigour in your Mathematics course on differential equations, if you have opted for the PCM combination in B. Sc. In general, if the conditions specified for an ODE are only for **one value** of the independent variable, we get an **initial value problem**. For 2^{nd} order and higher order ODEs, if the conditions specified for an ODE are for **two or more values** of the independent variable, we get a **boundary value problem** (read the margin remark). Now that you know what an ODE is and can classify the first order ODEs as linear/non-linear, you are ready to learn the methods for solving them. We begin with the simplest case of first order linear ODEs in which the variables can be separated.

3.3 SEPARABLE FIRST ORDER ODEs

In several first order ODEs, the dependent and independent variables and their functions can be separated. Such ODEs can be solved by integrating the separated parts. Many other first order ODEs can be reduced to a separable form by making a suitable substitution (change of variable). In this section, we explain the methods of solving separable equations.

3.3.1 Method of Separation of Variables

Let us consider a general first order ordinary differential equation of the form

$$y' = f(x, y) \tag{3.6}$$

Suppose we can write $f(x, y)$ as

$$f(x, y) = M(x)N(y) \tag{3.7}$$

Note that $M(x)$ is a function of only x and $N(y)$ is a function of only y. Using Eq. (3.7), we can recast Eq. (3.6) as follows:

$$\frac{dy}{N(y)} = M(x)dx \tag{3.8}$$

First order ODEs of the form $y' = M(x)N(y)$ are said to be separable. Note that in Eq. (3.8), the variables x and y and their functions are separated. On integrating Eq. (3.8), we get the general solution of Eq. (3.6) as

$$\int M(x)dx - \int \frac{dy}{N(y)} = C_1 \tag{3.9}$$

where C_1 is a constant of integration. Eq. (3.9) is the required general solution of the ODE (3.6), provided we can solve the integrals. Note that the function $f(x, y)$ is given by Eq. (3.7). Let us now consider a few simple examples to illustrate this method.

The differential of $y(x)$ is defined as

$$dy = \frac{dy}{dx}dx \qquad \text{(i)}$$

We substitute $y' = f(x, y)$ in equation (i) and rewrite it as

$$dy = f(x, y)dx \qquad \text{(ii)}$$

Now if

$$f(x, y) = M(x)N(y)$$

then we can write (ii) as

$$dy = M(x)N(y)dx$$

Thus, we get Eq. (3.8):

$$\frac{dy}{N(y)} = M(x)dx$$

EXAMPLE 3.1: METHOD OF SEPARATION OF VARIABLES

Solve the first order ODE $(y+1)y' + x = 0$ given that $y = 2$ at $x = 0$.

SOLUTION ■ We can write this equation in the form of Eq. (3.6) as

$$y' = -\frac{x}{y+1} \equiv f(x, y) \qquad \text{(i)}$$

Comparing $f(x, y)$ of Eq. (i) with the form given in Eq. (3.7), we have $M(x) = -x$ and $N(y) = \frac{1}{y+1}$. From Eq. (3.9), we get the solution as

$$\int x\,dx + \int (y+1)dy = C \quad \text{or} \quad \frac{x^2}{2} + \frac{y^2}{2} + y = C \qquad \text{(ii)}$$

or $$x^2 + y^2 + 2y = 2C \qquad \text{(iii)}$$

Do you recognise what Eq. (iii) represents? Recall the coordinate geometry you have studied in school. You can verify that Eq. (iii) represents a family of concentric circles with their centre at $(0, -1)$ and of radii $\sqrt{2C+1}$ (see Fig. 3.3). We obtain the particular solution by substituting $y = 2$ at $x = 0$ in (iii) so that $C = 4$. Thus, the particular solution is the equation of a circle of radius 3, centred at $(0, -1)$:

$$x^2 + y^2 + 2y = 8 \qquad \text{(iv)}$$

In Example 3.1, we could also take

$$M(x) = x, \; N(y) = -\frac{1}{y+1}$$

You can check that the result would be the same.

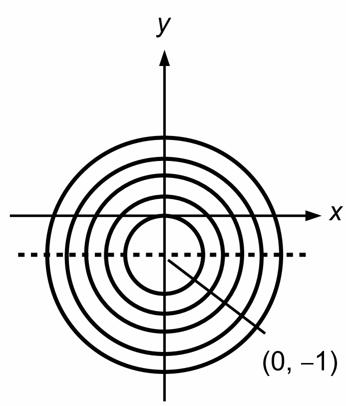


Fig. 3.3: A family of concentric circles with centre at (0, –1)**.**

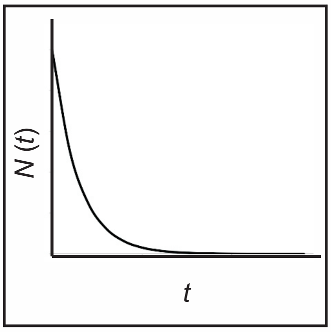


Fig. 3.4: **Exponential decay.**

Example 3.2: RADIOACTIVE DECAY/EXPONENTIAL DECAY

Solve the first order ODE modelling radioactive decay: $\dfrac{dN(t)}{dt} = -\lambda N(t)$.

SOLUTION ■ Comparing this ODE with Eq. (3.7), we note that $M = -1$ and $N = N(t)$. From Eq. (3.9), the solution is obtained as follows:

$$\int \frac{dN}{N} = -\lambda \int dt + c \quad \text{or} \quad \ln|N| = -\lambda t + \ln C \qquad \text{(i)}$$

Since $N > 0$, $|N| = N$ and the solution is $N = C\exp(-\lambda t)$ (ii)

Can you interpret Eq. (ii)? It represents **exponential decay** of atoms in a radioactive sample (see Fig. 3.4). If the initial number of atoms at time $t = 0$ is N_0, then from (ii), we get: $C = N_0$ and the particular solution is

$$N = N_0 \exp(-\lambda t)$$

A particle falling only under the force of gravity is called a freely falling particle.

Let us consider an example from mechanics of a particle falling under the force of gravity. Suppose we wish to know the effect of air resistance on the motion of the freely falling particle modelled by Eq. (3.1a). Such equations are useful for studying the motion of parachutes or skydiving, etc. (Fig. 3.5). Usually, air resistance is taken to be proportional to the velocity of the particle. From Newton's second law, the equation of motion is given by

$$m\vec{\mathbf{a}} = m\vec{\mathbf{g}} + k\,\vec{\mathbf{v}}(t)$$

For the particle falling vertically downwards, it takes the form (see Fig. 3.5):

$$-ma = -m\frac{dv}{dt} = -mg + kv(t) \quad \text{or} \quad ma = m\frac{dv}{dt} = mg - kv(t)$$

Let us solve this equation using the method of separation of variables.

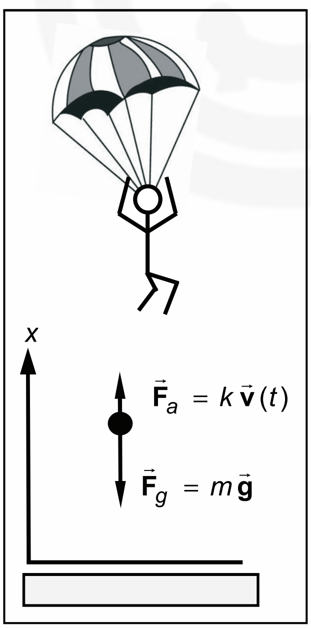


Fig. 3.5: **A particle falling under the force of gravity $\vec{\mathbf{F}}_g = m\vec{\mathbf{g}}$ and air resistance $\vec{\mathbf{F}}_a = k\,\vec{\mathbf{v}}(t)$.**

Example 3.3: EFFECT OF AIR RESISTANCE ON MOTION

Obtain the general solution of the following ODE to determine the effect of air resistance on a particle falling under the force of gravity:

$$m\frac{dv}{dt} = mg - kv(t) \qquad \text{(i)}$$

SOLUTION ■ We can rewrite this equation in the form of Eq. (3.6) as

$$\frac{dv}{dt} = g - \frac{k}{m}v \qquad \text{(ii)}$$

Comparing Eq. (ii) with the form given in Eq. (3.7), we can write $M(t) = 1$ and $N(v) = g - \dfrac{k}{m}v$. Using Eq. (3.9), we can write the solution as

$$\int \frac{dv}{\left(g - \frac{k}{m}v\right)} = \int dt + C_1 \qquad \text{(iii)}$$

On integration, we get the general solution (read the margin remark):

$$v(t) = \frac{mg}{k} - \frac{m}{k} C_1 \exp\left(-\frac{k}{m}t\right) \qquad \text{(iv)}$$

Note that for large values of t, the second term in Eq. (iv) becomes small and approaches zero as t tends to ∞. Thus, v becomes constant and is given by:

$$v = \frac{mg}{k} \qquad \text{(v)}$$

This constant velocity is known as **terminal velocity**. So the effect of air resistance (proportional to the velocity of the particle) on a freely falling particle is that after a large interval of time, it attains a constant velocity.

In Eq. (iii), we put

$$y = g - \frac{k}{m}v \text{, so that}$$

$$dy = -\frac{k}{m}dv$$

So Eq. (iii) can be rewritten as

$$\int \frac{dy}{y} = -\frac{k}{m}\int dt + C'$$

On evaluating this integral, we get

$$\ln|y| = -\frac{kt}{m} + \ln C_1$$

On taking antilog, we can write the solution as

$$|y| = C_1 \exp\left(-\frac{k}{m}t\right)$$

and

$$v = \frac{mg}{k} - \frac{m}{k}|y|$$

Did you note that in Example 3.3 we changed the variable to simplify the integrals? This brings us to the method of substitution. But before studying further, you may like to solve an SAQ based on this section.

***SAQ 3* – Method of separation of variables**

a) Obtain the general solution of the first order ODE $yy' = -x$.

b) Determine the particular solution of the first order ODE $y' = -2xy$ for $y(0) = 3$.

Before studying the method of substitution, you may like to quickly revise the method of separation of variables.

METHOD OF SEPARATION OF VARIABLES

Recap

Step 1: Write the first order ODE in the form $y' = M(x)N(y)$

Then $$\frac{dy}{N(y)} = M(x)dx$$

Step 2: Integrate to obtain the solution.

Always check the solution. You should **always** substitute the solution back into the ODE and check whether you get an identity. Sometimes, you get the ODE by simply differentiating the solution.

3.3.2 Method of Substitution

Some first order linear ODEs may look non-separable at first glance. But we can make them separable by making some substitution. In some cases we can know what substitution to make by just inspecting the equation. Let us take up an example to illustrate this technique of substitution.

EXAMPLE 3.4: THE METHOD OF SUBSTITUTION

Solve the first order ODE $\frac{dy}{dx} = (x+y)$.

SOLUTION ■ The ODE seems non-separable because of the factor $(x+y)$. But it is not so. We put $u = x + y$ and get,

$$\frac{du}{dx} = 1 + \frac{dy}{dx} \quad \text{or} \quad \frac{dy}{dx} = \frac{du}{dx} - 1 \tag{i}$$

Putting $\frac{dy}{dx} = (x+y)$ in Eq. (i), we get

$$\frac{du}{dx} = 1 + u$$

This is separated in *u* and *x*:

$$\frac{du}{u+1} = dx \tag{ii}$$

Integrating Eq. (ii), we get

$$\ln|u+1| = x + c \tag{iii}$$

or $$|u+1| = C\,e^{x} \;\Rightarrow\; |y + x + 1| - Ce^{x} = 0$$

This is the required general solution. You may like to check it.

Checking the solution

Differentiating the solution w.r.t. *x*, we have

$$\frac{dy}{dx} + 1 = Ce^{x}$$

or $$\frac{dy}{dx} + 1 = y + x + 1$$

or $$\frac{dy}{dx} = (x+y)$$

which is the ODE being solved. Therefore, the solution is correct.

Now how about trying an SAQ similar to Example 3.4 before studying further?

SAQ 4 – Method of substitution

Obtain the general solution of the first order ODE: $y' = (x + y)^2$.

We now discuss a very typical method of substitution suitable for ODEs of the form $y' = f(y/x)$, where *f* is a function of y/x, e.g., $(y/x)^3$, $\sin(y/x)$, etc. We can solve such first order ODEs by substituting $y = vx$. This method is suitable for a special category of first order ODEs: the **first order homogeneous ODEs**. We now define such ODEs and then explain the method of solving them.

NOTE

Note that **second order homogeneous ODEs are defined in a different manner**. You will learn about this in Unit 4.

3.3.3 First Order Homogeneous ODEs

Let us begin by asking: What is a first order **homogeneous** ODE? A first order ODE of the form

$$M(x,y)\,dx + N(x,y)\,dy = 0 \tag{3.10}$$

is called **homogeneous if *M* and *N* are homogeneous functions of the same degree**.

Now, you may ask: **What is a homogeneous function**? A **function** $f(x,y)$ is said to be **homogeneous** of degree n in x and y, if, for every k, we can write

$$f(kx,ky) = k^n f(x,y),$$

where k is a real parameter. For example, $f(x,y) = x^2 + xy + y^2$ is a homogeneous function of degree 2 since

$$f(kx,ky) = (kx)^2 + kxky + (ky)^2 = k^2(x^2 + xy + y^2) = k^2 f(x,y)$$

Some other examples of homogeneous functions are given in the margin. What can you say about the function $f(x,y) = x^2 + y^2 + 2$? Is it homogeneous? Let us check:

$$f(kx,ky) = (kx)^2 + (ky)^2 + 2$$

So we cannot express it as the product of some power of k and the function itself. Therefore, $f(x,y)$ is **not homogeneous**.

You may like to practice identifying homogeneous first order ODEs before learning how to solve them. Try the following SAQ.

The function $f(x,y) = \sqrt{x-y}$ is a homogeneous function of degree $\frac{1}{2}$ since

$$f(kx,ky) = \sqrt{kx-ky} = \sqrt{k}\sqrt{x-y} = k^{1/2} f(x,y)$$

The function

$$f(x,y) = y + \sqrt{x^2 - y^2}$$

is a homogeneous function of degree 1 as

$$f(kx,ky) = ky + \sqrt{(kx)^2 - (ky)^2} = ky + k\sqrt{x^2 - y^2} = k\,f(x,y)$$

SAQ 5 – First order homogeneous ODEs

Identify the first order homogeneous ODEs from among the following:

i) $(x+y)dx + (x-y)dy = 0$, ii) $(x^2+1)dx + xy\,dy = 0$

iii) $(x+y)dx + (x^2+y)dy = 0$, iv) $(x^2+y^2)dx - xy\,dy = 0$

An alternative way of identifying a first order homogeneous ODE is to check whether it can be cast in the form in which y' is a function of $\frac{y}{x}$:

$$y' = g\left(\frac{y}{x}\right) \tag{3.11}$$

For example, we can rewrite the first order ODE $(2x-y)y' = (x-3y)$

in the form $$\frac{dy}{dx} = \frac{1-3\left(\frac{y}{x}\right)}{2-\left(\frac{y}{x}\right)}$$

So it is homogeneous. You can check that the first order ODE $xy\,y' + 4x^2 + 3y^2 = 0$ can be written as

$$\left(\frac{y}{x}\right)y' + 4 + 3\left(\frac{y}{x}\right)^2 = 0 \quad \text{or} \quad y' = -\frac{4+3\left(\frac{y}{x}\right)^2}{\left(\frac{y}{x}\right)}$$

Since $M(x, y)$ and $N(x, y)$ are homogeneous, we can write

$$\frac{M(kx, ky)}{N(kx, ky)} = \frac{k^n M(x, y)}{k^n N(x, y)} = \frac{M(x, y)}{N(x, y)}$$

By now, you should be able to identify first order homogeneous ODEs using either of the two ways explained above.

You may now like to learn how to solve such ODEs. Let us go back to Eqs. (3.10) and (3.11). Note that, in Eq. (3.10), $M(x, y)$ and $N(x, y)$ are homogeneous functions of the same degree, say n. You can verify that $\frac{M(x, y)}{N(x, y)}$ is a **homogeneous function of degree zero** (read the margin remark).

From Eq. (3.10), we can write

$$\frac{dy}{dx} = -\frac{M(x, y)}{N(x, y)}$$

Thus, we can say that $\frac{dy}{dx}$ is a homogeneous function of degree zero.

Equations of the form (3.10) and (3.11) are first order homogeneous ODEs and we can solve them by making the substitution $y = vx$.

We now take up an example to illustrate this method of solving first order homogeneous ODEs.

Example 3.5: FIRST ORDER HOMOGENEOUS ODEs

Solve the first order ODE $(x^2 + y^2)dx - xydy = 0$.

SOLUTION ■ While solving SAQ 5, you have verified that this ODE is homogeneous and of first order. Let us now rearrange the equation as:

$$\frac{dy}{dx} = \frac{x^2 + y^2}{xy} = \frac{x}{y} + \frac{y}{x} \tag{i}$$

If we put $y = vx$, we can write (i) as $\frac{dy}{dx} = \frac{1}{v} + v$ (ii)

Now differentiating $y = vx$ w.r.t. x, we get $\frac{dy}{dx} = v + x\frac{dv}{dx}$ (iii)

Substituting $\frac{dy}{dx}$ from Eq. (ii) in Eq. (iii), we can write:

$$\frac{1}{v} + v = v + x\frac{dv}{dx} \quad \text{or} \quad x\frac{dv}{dx} = \frac{1}{v} \tag{iv}$$

Note that in Eq. (iv), the variables v and x are separated. Now you can solve it.

SAQ 6 – First order homogeneous ODEs

a) Solve Eq. (iv) in Example 3.5.

b) Solve the first order ODE $(x + y)dx - x\,dy = 0$.

Before discussing the next method, let us summarise the method of substitution and the method of solving homogeneous first order ODEs.

Recap

THE METHOD OF SUBSTITUTION

Substitution by inspection

Step 1: Inspect the first order ODE and make an appropriate substitution so that it can be recast in the separable form given by Eq. (3.8).

Step 2: Integrate the resulting expression on both sides to obtain the desired solution.

The method of solving a first order homogeneous ODE

Step 1: Write the ODE in the form $M(x,y)dx + N(x,y)dy = 0$

Step 2: Determine whether $M(x,y)$ and $N(x,y)$ are **homogeneous functions** of the same degree or check if it can be recast in the form of Eq. (3.11).

Step 3: If yes, separate variables by making the **substitution** $y = vx$ and solve the ODE.

So far you have learnt how to solve first order ODEs using the methods of separation of variables and also by substitution. You have also learnt that the first order homogeneous ODEs can be solved by making the substitution $y = vx$.

In some cases, the first order ODEs are neither separable nor can be made separable by the method of substitution. In such cases we use other methods for solving them. In the next section, we shall first define an **exact equation** and then use it to solve first order non-homogeneous ODEs that have many applications in physics (Sec. 3.5).

3.4 FIRST ORDER EXACT ODEs

ODEs of the form $M(x,y)dx + N(x,y)dy = 0$ given by Eq. (3.10) can sometimes be recast in the form

$$df(x,y) = 0 \tag{3.12a}$$

Then the solution is simply

$$f(x,y) = C, \text{ a constant} \tag{3.12b}$$

The first order ODEs that can be recast as Eq. (3.12a) are said to be **exact equations**. We now explain how to identify exact equations and solve them. Let us consider Eq. (3.10) once again: $M(x,y)dx + N(x,y)dy = 0$

We need to determine the conditions on $M(x,y)$ and $N(x,y)$ that should be satisfied so that we may recast Eq. (3.10) in the form (3.12a). For this, we use a result from calculus for partial differentiation. Suppose a function depends on two or more variables. Then we can express an infinitesimal change in the function in terms of the total differential of the function. The total differential of the function involves its partial derivatives w.r.t. the independent variables. You should read the note in the margin and the Appendix to this unit to understand how to calculate the partial derivatives of a function of two variables.

NOTE

Note that $\left(\dfrac{\partial f}{\partial x}\right)_y$ denotes a partial derivative of $f(x, y)$ w.r.t. x. To calculate it, differentiate the function $f(x, y)$ w.r.t. x treating the variable y as constant. Similarly, $\left(\dfrac{\partial f}{\partial y}\right)_x$ denotes a partial derivative of $f(x, y)$ w.r.t y. To calculate it, differentiate the function $f(x, y)$ w.r.t y treating the variable x as constant. We have explained how to calculate partial derivatives of a function in an **Appendix** to the unit. You may like to study it before studying Sec. 3.4 further.

The expression for the total differential df of a function $f(x, y)$ is given as

$$df = \left(\frac{\partial f}{\partial x}\right)_y dx + \left(\frac{\partial f}{\partial y}\right)_x dy \tag{3.12c}$$

Using Eq. (3.12c), we can rewrite Eq. (3.12a) as

$$df = \left(\frac{\partial f}{\partial x}\right)_y dx + \left(\frac{\partial f}{\partial y}\right)_x dy = 0 \tag{3.12d}$$

Now we compare Eqs. (3.10) and (3.12d). Suppose there exists a function $f(x, y)$ such that

$$M(x, y) = \left(\frac{\partial f(x, y)}{\partial x}\right) \quad \text{and} \quad N(x, y) = \left(\frac{\partial f(x, y)}{\partial y}\right) \tag{3.12e}$$

where M and N are continuous functions and have continuous partial derivatives. Then we can express Eq. (3.10) in the form (3.12e) and its solution is given by Eq. (3.12b). This gives us the definition of a first order **exact** ODE.

DEFINITION OF A FIRST ORDER EXACT ODE

The equation $M(x, y)dx + N(x, y)dy = 0$ is said to be a **first order exact ODE** if there exists a function f of two variables x and y, which has continuous partial derivatives such that

$$M(x, y) = \left(\frac{\partial f(x, y)}{\partial x}\right) \quad \text{and} \quad N(x, y) = \left(\frac{\partial f(x, y)}{\partial y}\right) \tag{3.13a}$$

so that $df(x, y) = 0$. The general solution of the equation is

$$f(x, y) = C \tag{3.13b}$$

Now, you may ask: **How do we check whether or not an ODE of the form of Eq. (3.10) is exact**? We use the following result of calculus to arrive at a test for exactness:

If $f(x, y)$ is continuous and its first order derivatives are also continuous, then

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} \tag{3.14a}$$

Let us now take the partial derivatives of M with respect to y and N with respect to x. Then using the result at Eq. (3.14a), we can write (read the margin remark):

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

Now

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$$

Thus, we get Eq. (3.14b).

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial N}{\partial x} \qquad (3.14b)$$

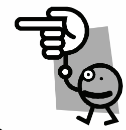
Thus, we arrive at the following test for exactness

Test for Exactness

A first order ODE of the form $M\,dx + N\,dy = 0$ is **exact** iff (if and only if)

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \qquad (3.15)$$

provided *M* and *N* are continuous and have continuous partial derivatives.



Don't forget

If Eq. (3.15) holds, Eq. (3.10) can be written as $df(x, y) = 0$ and the solution is simply $f(x, y) = C$ [Eq. (3.12b or 3.13b)].

Now you may like to know: **How is a first order exact ODE solved**? Let us illustrate this with an example.

EXAMPLE 3.6: FIRST ORDER EXACT ODE

Solve the first order ODE $xy' + ax + y = 0$ where *a* is a constant.

SOLUTION ■ We can write this equation in the form of Eq. (3.10) as

$$(ax + y)dx + xdy = 0 \qquad \text{(i)}$$

where $M(x) = ax + y$ and $N(y) = x$.

You can verify that $\dfrac{\partial M}{\partial y} = 1$ and $\dfrac{\partial N}{\partial x} = 1.$ So Eq. (3.15) is satisfied and the given ODE is exact. Now we integrate the first equation in Eq. (3.13a) w.r.t. *x*, keeping *y* constant, and obtain

$$f(x, y) = \int M(x, y)dx + z(y) = \int (ax + y)dx + z(y) \qquad \text{(ii)}$$

or $$f(x, y) = \frac{ax^2}{2} + yx + z(y) \qquad \text{(iii)}$$

We now need to determine $z(y)$ in (iii). For this, we differentiate (iii) w.r.t. *y* keeping *x* constant. We then use the second equation in Eq. (3.13a) and equate the result to $N(y)$ in (i). So we have

$$\left(\frac{\partial f(x, y)}{\partial y}\right) = x + \frac{\partial z}{\partial y} = x \quad \text{or} \quad \frac{\partial z}{\partial y} = 0 \Rightarrow z(y) = C$$

where *C* is a constant. Therefore, using Eq. (iii), we can write the general solution of the given exact first order ODE as

$$f(x, y) = \frac{ax^2}{2} + yx + C$$

Let us summarise the method of solving an exact equation.

Recap

SOLVING FIRST ORDER EXACT ODEs

Step 1: Write the differential equation in the form of Eq. (3.10) as

$$M(x,y)dx + N(x,y)dy = 0$$

Check to make sure that Eq. (3.15) is satisfied: $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Step 2: Using $\frac{\partial f}{\partial x} = M(x,y)$ and keeping y constant, evaluate $f(x,y) = \int M(x,y)dx + z(y)$.

Step 3: Next use $\frac{\partial f}{\partial y} = N(x,y)$ to evaluate $z(y)$. For this differentiate $f(x,y)$ obtained in Step 2 w.r.t. y keeping x constant.

Step 4: The general solution is $f(x,y) = C$.

Alternatively, if it is easier to work with $N(x,y)$, follow the steps given below:

Step 5: Using $\frac{\partial f}{\partial y} = N(x,y)$ and keeping x constant, evaluate $f(x,y) = \int N(x,y)dy + g(x)$.

Step 6: Use $\frac{\partial f}{\partial x} = M(x,y)$ to evaluate $g(x)$. For this differentiate $f(x,y)$ obtained in Step 5 w.r.t. x keeping y constant.

Step 7: The general solution is $f(x,y) = C$

NOTE

Use either of the set of steps (1 to 4) or (1 and 5 to 7) depending upon which function, M or N, is simpler to integrate.

You should now solve a first order exact ODE to quickly practice this method.

SAQ 7 – First order exact ODE

Show that the ODE $(3x^2y - 6x)dx + (x^3 + 2y)dy = 0$ is exact and hence solve it.

So far we have explained how to solve first order homogeneous and exact ODEs. But in physics you will come across many situations which can be modelled using non-homogeneous ODEs. For example, growth and/or decay of current in an *LR* circuit having an *AC* or *DC* voltage source is the most common example.

Therefore, we now explain how to solve first order linear and non-linear non-homogeneous ODEs.

3.5 FIRST ORDER NON-HOMOGENEOUS ODEs

An ODE of the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = f(x) \tag{3.16}$$

defined on an interval in x is called a **first order non-homogeneous ODE**. It is linear if it satisfies the conditions of linearity explained in Sec. 3.2 1. Otherwise, it is non-linear. Note that in Eq. (3.16), $a_1(x) \neq 0$. You can see that Eq. (3.16) is of the first order as it contains only the first derivative of y w.r.t. x. It is **non-homogeneous** as it contains a function of only x that is not zero: $f(x) \neq 0$. It will be linear if $f(x)$ is not a transcendental function. On dividing both sides of Eq. (3.16) by $a_1(x)$, we can write

$$\frac{dy}{dx} + p(x)y = q(x) \tag{3.17a}$$

where $p(x) = \dfrac{a_0(x)}{a_1(x)}$ and $q(x) = \dfrac{f(x)}{a_1(x)}$ (3.17b)

This is the **standard form** of a **first order <u>linear</u> non-homogeneous differential equation** provided $q(x)$ is not a transcendental function. It could also be a non-linear equation depending on the form of $q(x)$. Equations of the form of Eq. (3.17a) may or may not be exact. If not exact, these can be made **exact** by obtaining an **<u>integrating factor</u>** $v(x)$ which is **only** a function of x.

Let us write Eq. (3.17a) in the form

$$[p(x)y - q(x)]dx + dy = 0$$

If such a function $v(x)$ exists then the equation

$$v(x)[py - q]dx + v(x)dy = 0 \tag{3.18}$$

must be exact. For Eq. (3.18) to be exact, it should satisfy Eq. (3.13a).

$$\therefore \quad \frac{\partial v}{\partial x} = \frac{\partial}{\partial y}[v(py - q)] \tag{3.19a}$$

or

$$\frac{dv}{dx} = vp \tag{3.19b}$$

Using the method of separation of variables, we get

$$\frac{dv}{v} = p\,dx \tag{3.20a}$$

Integrating both sides, we get

$$\ln|v| = \int p(x)dx \tag{3.20b}$$

Thus, the integrating factor is

$$v(x) = \exp[h(x)] \qquad \text{where} \qquad h(x) = \int p(x)dx \tag{3.21}$$

We now multiply Eq. (3.17a) by $v(x)$ given by Eq. (3.21) and get

$$e^h (y' + py) = e^h q \tag{3.22a}$$

Since from Eq. (3.21), $h' = p,$ we can write Eq. (3.22a) as

$$\frac{d}{dx}(ye^h) = e^h q \tag{3.22b}$$

You may like to verify Eq. (3.22b) by differentiating (ye^h) w.r.t. x before studying further. Now integrating Eq. (3.22b) on both sides, we get

$$y\,e^h = \int e^h q\, dx + C, \quad \text{where} \quad h = \int p(x)\,dx \tag{3.22c}$$

Dividing both sides of Eq. (3.22c) by e^h we get the **general solution of a first order linear non-homogeneous ODE of the form of Eq. (3.17a)**:

$$y = e^{-h}\left[\int e^h q\, dx + C\right], \qquad h = \int p(x)\,dx \tag{3.22d}$$

Let us take up a simple example of this method.

EXAMPLE 3.7: LINEAR NON-HOMOGENEOUS ODE

Solve the first order ODE $\dfrac{dy}{dx} + axy = bx.$

SOLUTION ■ Note that this ODE is linear non-homogeneous of the form of Eq.(3.17a) where $p(x) = ax.$ From Eq. (3.21), the integrating factor is

$$v(x) = \exp\left[\int ax dx\right] = e^{ax^2/2} \tag{i}$$

From Eq. (3.22d), the general solution of the given ODE is

$$y(x) = e^{-ax^2/2}\left[b\int xe^{ax^2/2} dx + C_1\right] \tag{ii}$$

where C_1 is an arbitrary constant. To evaluate the integral on the RHS of Eq. (ii), we put $t = \dfrac{ax^2}{2}.$ Then $dt = ax\,dx$ and we can write the integral $\int xe^{ax^2/2} dx$ in Eq. (ii) as $\dfrac{1}{a}\int e^t dt = \dfrac{e^t}{a} + C_2.$ Hence, we get the general solution as

$$y(x) = \frac{b}{a} + Ce^{-ax^2/2} \quad \text{where } C \text{ is a constant.}$$

Let us now apply this method to an example from physics. You may have studied about series RC circuits in your school physics. Recall that in a series RC circuit, a capacitor and a resistor are connected in series to an alternating voltage source (see Fig. 3.6a). We can write the equation governing the change in charge in an RC circuit with time as

$$R\frac{dq}{dt} + \frac{q}{C} = E(t) \tag{3.23a}$$

where $E(t)$ denotes the alternating voltage. Usually, a sinusoidal voltage source or a battery is connected in the circuit. A similar equation for the change in current with time is obtained for a series LR circuit. In such a circuit, a resistor and an inductor are connected in series to an alternating voltage source (Fig. 3.6b) or a battery:

$$L\frac{di}{dt} + Ri = E(t) \qquad (3.23b)$$

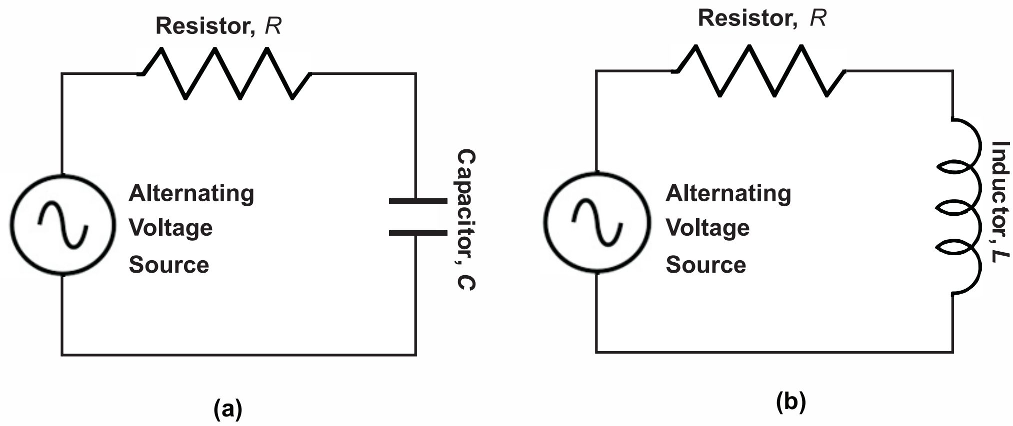


Fig. 3.6: ***AC*** **circuits in which a) a capacitor and a resistor; b) an inductor and a resistor are connected in series to an alternating voltage source. These are called series** ***RC*** **and series** ***LR*** **circuits, respectively.**

We now solve Eq. (3.23b) for a series LR circuit.

EXAMPLE 3.8: NON-LINEAR NON-HOMOGENEOUS ODE

The current i in a series LR circuit having an alternating voltage source $E\sin\omega t$ satisfies the equation $L\frac{di}{dt} + Ri = E\sin\omega t$ where R, L and E are constant. Solve this first order ODE.

SOLUTION ■ We can write this equation in the form of Eq. (3.17a) as

$$\frac{di}{dt} + \frac{R}{L}i = \frac{E}{L}\sin\omega t \qquad \text{(i)}$$

Note that here $p = \frac{R}{L}$ is constant. Therefore, from Eq. (3.21), the integrating factor is $v(t) = \exp\left[\int \frac{R}{L}dt\right] = e^{Rt/L}$

Using Eq. (3.22d), we can write the general solution as

$$i(t) = \frac{E}{L}e^{-Rt/L}\left[\int e^{Rt/L}\sin\omega t\, dt + C\right] \qquad \text{(ii)}$$

where C is an arbitrary constant. On integrating Eq. (ii) by parts, we get (see the margin remark for complete solution)

$$i = \frac{E\sin(\omega t - \theta)}{\sqrt{R^2 + \omega^2 L^2}}$$

Integrating $J = \int e^{Rt/L}\sin\omega t\, dt$ by parts, we get

$$J = e^{Rt/L}\left(-\frac{\cos\omega t}{\omega}\right) + \frac{R}{L\omega}\int e^{Rt/L}\cos\omega t\, dt$$

$$= -e^{Rt/L}\frac{\cos\omega t}{\omega} + \frac{R}{L\omega}\left[e^{Rt/L}\left(\frac{\sin\omega t}{\omega}\right) - \frac{R}{L\omega}\int e^{Rt/L}\sin\omega t\, dt\right]$$

or $$J = -e^{Rt/L}\frac{\cos\omega t}{\omega} + \frac{R}{L\omega}e^{Rt/L}\left(\frac{\sin\omega t}{\omega}\right) - \left(\frac{R}{L\omega}\right)^2 J$$

Simplifying for J, we get

$$J = \frac{Le^{Rt/L}}{(R^2 + \omega^2L^2)}[R\sin\omega t - \omega L\cos\omega t]$$

Let $\cos\theta = \frac{R}{\sqrt{R^2 + \omega^2L^2}}$

and $\sin\theta = \frac{\omega L}{\sqrt{R^2 + \omega^2L^2}}$

$$J = \frac{Le^{Rt/L}\sin(\omega t - \theta)}{(R^2 + \omega^2L^2)^{1/2}}$$

where $\theta = \tan^{-1}(\omega L/R)$.

You may now like to solve a first order non-homogeneous ODE. Try the following SAQ.

SAQ 8 – First order non-homogeneous ODE

Solve $xy' + 2y = x^3$

We now summarise the procedure for solving a first order non-homogeneous ODE using the integrating factor.

Recap

SOLVING FIRST ORDER NON-HOMOGENEOUS ODEs

Step 1: Put the equation into the standard form $y' + p(x)y = q(x)$.

(**<u>Note</u>:** The coefficient of y' must be 1).

Step 2: Identify $p(x)$ and compute $v(x) = \exp[\int p(x)dx]$.

Step 3: Multiply the standard form of the equation by $v(x)$.

Step 4: Integrate both sides of the modified equation and solve for y.

Let us now summarise what you have learnt in this unit.

3.6 SUMMARY

Concept	Description
Ordinary differential equation (ODE)	■ An equation that contains derivatives or differentials of one or more dependent variables with respect to <u>**one**</u> independent variable is called an **ordinary differential equation (ODE).**
Classifying first order ODEs	■ An ODE in which the highest derivative is of order 1 is called the first order ODE. It is further classified by its **degree** and as **linear/non-linear, homogeneous/non-homogeneous**.
General and particular solutions of an ODE	■ A function $y = \phi(x)$ is a **solution** of an ODE on some interval if $\phi(x)$ is defined and differentiable throughout that interval and is such that the ODE becomes an identity when y is replaced by $\phi(x)$ in it. A solution involving arbitrary constant(s) is called a **general solution**. If definite value(s) can be assigned to the arbitrary constant(s) in a general solution by specifying certain conditions then it becomes a **particular solution**. Depending on the way the conditions are specified we get an **initial value problem** or a **boundary value problem.**
Methods of solving first order ODEs	■ The methods of solving a **first order ODE** depend on its classification. You have learnt the following methods. ➢ **Separable Equations**: An equation is **separable** if it can be put in the form $$\frac{dy}{N(y)} = M(x)dx$$

The solution is obtained by integrating both sides of the equation. An ODE may be reduced to a separable form by an appropriate **substitution** or **change of variables.**

- **Homogeneous Equations**: The first order ordinary differential equation $M(x,y)dx + N(x,y)dy = 0$ is said to be **homogeneous** of first order if $M(x,y)$ and $N(x,y)$ are **homogeneous functions** of the same degree or if it can be put in the form $y' = f\left(\frac{y}{x}\right)$, that is, the derivative of y is a function of $\left(\frac{y}{x}\right)$. Then it can be made separable by making the substitution $y = vx$.

- **Exact Equations**: The ordinary differential equation $M(x,y)dx + N(x,y)dy = 0$ is said to be **exact** if $M(x,y)dx + N(x,y)dy$ is an exact differential $[df(x,y)]$. When M and N are continuous and have continuous partial derivatives, then

$$\partial M / \partial y = \partial N / \partial x$$

is a necessary and sufficient condition for $M\,dx + N\,dy = 0$ to be exact. Then there exists some function f for which $M(x,y) = \partial f / \partial x$ and $N(x,y) = \partial f / \partial y$. The method of solving an exact ODE starts by integrating either of these expressions.

- **Non-homogeneous Equations**: If a first order ODE can be put in the form

$$\frac{dy}{dx} + p(x)y = q(x), \text{ with the coefficient of } \frac{dy}{dx} \text{ being 1,}$$

it can be reduced to the exact form by multiplying it by an **integrating factor**

$$\exp\left[\int p(x)dx\right]$$

We can solve this equation by integrating both sides of the equation

$$\frac{d}{dx}\left[\left\{\exp(\int p(x)dx\right\}y\right] = \left\{\exp(\int p(x)dx\right\}q(x)$$

Thus, the general solution of the equation is

$$y = e^{-h}\int e^{h} q(x)dx + C \quad \text{where} \quad h = \int p(x)dx$$

3.7 TERMINAL QUESTIONS

1. Obtain the general solutions of the following first order ODEs:

a) $\frac{dN(t)}{dt} = Ct$, where C is a constant.

b) $\frac{df(t)}{dt} = C\sqrt{f(t)}$, where C is a constant and $f(0) = C_0$

c) $\frac{dw}{dx} + 2w = 0$

d) $yw\frac{dw}{dy} + w^2 = 0$

e) $(x+1)y + y' = 0$

2. Solve Eq. (3.2c) and obtain the expression for escape velocity of a particle having the initial speed $v = v_0$ at $r = R$, where R is the Earth's radius.

3. Obtain the general solution of the following ODEs:

   a) $(1 + \cos\theta)dr = r\sin\theta\, d\theta$

   b) $(x - 2y - 1) = (x - 2y + 7)y'$

   c) $\frac{dy}{dx} = \frac{(x-y)}{(x+y)}$

   d) $xy' + 2y = x^5$

   e) $y' - 2y = 8e^x$

4. Solve the following ODEs:

   a) $(e^x + y - 1)dx + (3e^y + x - 7)dy = 0$

   b) $x\,dy - (y - \sqrt{x^2 + y^2})dx = 0$ when $y = 4$ for $x = 3$

5. a) The one-dimensional equation of motion of a simple linear harmonic oscillator can be reduced to a first order ODE given by

$$v\frac{dv}{dx} + \omega^2 x = 0,$$

where v is its linear velocity, x, its distance from its mean position and ω, its angular frequency. Solve the first order ODE and further, use the equation $v = \frac{dx}{dt}$ to obtain the relation between x and t given that $v = 0$ when $x = a$.

   b) Solve Eq. (3.23a) for a series RC circuit for constant E given that the initial charge in the circuit is q_0. What happens when $E = 0$?

3.8 SOLUTIONS AND ANSWERS

Self-Assessment Questions

1. Eq. (3.1a) is first order, first degree ODE; Eq. (3.1b) is first order, first degree ODE; Eq. (3.1c) is second order, first degree ODE and Eq. (3.1d) is first order, first degree ODE.

2. a) This is a linear ODE; b) this is a non-linear ODE as it contains the transcendental function $\sin \omega t$; c) this is also a non-linear ODE as it contains the product of the first order derivative of y with itself (y'^2).

3. a) We can write the ODE $yy' = -x$ as $y\,dy = -x\,dx$

   Integrating both sides yields $\frac{y^2}{2} = -\frac{x^2}{2} + C_1$ where C_1 is the constant of integration. Hence, the general solution of the given ODE is $x^2 + y^2 = C$ where $C = 2C_1$.

   b) The initial value problem $y' = -2xy, \quad y(0) = 3$ is also a separable first order ODE and we solve it as follows:

$$\int \frac{dy}{y} = -2\int x\,dx + C \quad \text{or} \quad \ln|y| = -x^2 + C$$

$$\Rightarrow \quad y = C_1 e^{-x^2} \quad \text{where} \quad C_1 = \ln|C|$$

   From the initial condition, $C_1 = 3$ and the particular solution is $y = 3e^{-x^2}$

4. We follow Example 3.4 and put $x + y = u$ in the ODE $\frac{dy}{dx} = (x+y)^2$.

   Then using Eq. (i) of Example 3.4, we can write the ODE as

$$\frac{du}{dx} = 1 + u^2 \qquad \text{(i)}$$

   This is separated in u and x: $\frac{du}{u^2 + 1} = dx$ (ii)

   Integrating Eq. (ii), we get $\tan^{-1} u = x + c$ or $\tan^{-1}(x + y) = x + c$

   This is the required general solution.

5. Eq. (i) is homogeneous because the functions $(x + y)$ and $(x - y)$ in Eq. (i) are homogeneous:

$$f(kx, ky) = (kx + ky) = k(x + y) = k\,f(x, y) \quad \text{and}$$
$$f(kx, ky) = (kx - ky) = k(x - y) = k\,f(x, y)$$

   Eqs. (ii) and (iii) are not homogeneous. Eq. (iv) is homogeneous because the functions $(x^2 + y^2)$ and xy in Eq. (iv) are homogeneous:

$$f(kx, ky) = (kx)^2 + (ky)^2 = k^2(x^2 + y^2) = k^2 f(x, y)$$

   and $f(kx, ky) = (kxky) = k^2(x\,y) = k^2 f(x, y)$

6. a) We can write Eq. (iv) of Example 3.5 as $v\,dv = \frac{dx}{x}$

   Its solution is given as $\frac{v^2}{2} = \ln|x| + c$ or $\frac{y^2}{2x^2} = \ln|x| + c$

   b) You can check that the ODE is homogeneous. So following Example 3.5, we rewrite it as $\frac{dy}{dx} = \left(1 + \frac{y}{x}\right)$ and substitute $y = vx$. We substitute $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$ from Eq. (iii) of Example 3.5 in the above ODE and write

$$v + x\frac{dv}{dx} = 1 + v \text{ or } x\frac{dv}{dx} = 1 \text{ which is a separable equation.}$$

It can be solved as follows:

$$\int dv = \int \frac{dx}{x} \quad \Rightarrow \quad v = \ln|x| + C \quad \text{or} \quad y = x\ln|x| + Cx$$

7. Here $\quad M = 3x^2y - 6x, \qquad N = x^3 + 2y$ (i)

$$\therefore \quad \frac{\partial M}{\partial y} = 3x^2, \quad \frac{\partial N}{\partial x} = 3x^2. \text{ Thus } \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \qquad \text{(ii)}$$

So the equation is exact.

Now, we have to solve the given ODE, which is exact. We can therefore say that there exists a function $f(x, y)$ for which Eq. (3.13a) is satisfied.

We can now use either Steps 2 and 3 or Steps 5 and 6 of the method of solving first order exact ODEs. From Step 2 we get

$$f(x, y) = \int M(x, y)dx + z(y) = \int (3x^2y - 6x)dx + z(y) = x^3y - 3x^2 + z(y) \qquad \text{(iii)}$$

Since $\dfrac{\partial f}{\partial y} = N(x, y)$, from Eqs. (i) and (iii), we obtain

$$\frac{\partial f}{\partial y} = x^3 + \frac{dz}{dy} = x^3 + 2y$$

$$\therefore \quad \frac{dz}{dy} = 2y \quad \text{or} \quad z(y) = y^2 + k, \qquad \text{(iv)}$$

where k is an arbitrary constant. Thus, $\quad f(x, y) = x^3y - 3x^2 + y^2 + k$

So the required solution is $f(x, y) =$ constant

or $\quad x^3y - 3x^2 + y^2 = C, \quad$ where C is a constant.

8. We can express the given ODE as

$$y' + \frac{2}{x}y = x^2$$

The integrating factor $= \exp\left(\int \frac{2}{x}dx\right) = \exp[2\ln|x|] = \exp\left[\ln\left|x^2\right|\right] = x^2$

So, we have

$$\frac{d}{dx}(x^2y) = x^4$$

or $\quad x^2y = \int x^4dx + C \quad$ or $\quad x^2y - \dfrac{x^5}{5} = C$, which is the required solution.

Terminal Questions

1. a) This is a first order, first degree ODE that is separable and we can solve it by integrating it as follows:

$$dN(t) = Ct\,dt \quad \text{or} \quad \int dN = C\int t dt$$

or $N = C\dfrac{t^2}{2} + C_1$ where C_1 is the constant of integration. This is the general solution of the ODE.

b) This is a first order, first degree ODE that is separable. We can solve it by integrating it as follows:

$$\frac{df(t)}{\sqrt{f(t)}} = C\,dt \quad \text{or} \quad \int \frac{df}{\sqrt{f}} = C\int dt \quad \text{or} \quad 2\sqrt{f} = Ct + C_1$$

Applying the initial condition $f(0) = C_0$, we get $C_1 = 2\sqrt{C_0}$

Therefore, the particular solution is $2\sqrt{f} = Ct + 2\sqrt{C_0}$

c) This is a first order, first degree ODE that is separable:

$$\frac{dw}{2w} + dx = 0$$

We can solve it by integrating it as follows:

$\dfrac{1}{2}\int\dfrac{dw}{w} + \int dx = C,$ where C is the constant of integration.

or $\dfrac{1}{2}\ln|w| + x = C$

$\therefore$ $|w| = Ae^{-2x},$ where $A = e^{2c}$

d) This is a first order, first degree ODE that is separable. We can write it as

$$\frac{dw}{w} + \frac{dy}{y} = 0$$

Integrating, we get $\ln|w| + \ln|y| = C$ or $\ln|w|\,|y| = C$

In most physical situations, the parameters w and y are positive quantities. Therefore, in solving physics problems, you will find that this solution would be written as $\ln w + \ln y = C$ and so the general solution is

$$wy = e^C$$

e) This is a first order, first degree ODE that is separable. We can write it as $\dfrac{dy}{y} = -(x+1)dx$. We take y to be positive and integrate the expression to get the general solution as $\ln(y) = -(\frac{x^2}{2} + x) + C_1$

or $y = C\exp\left[-(\frac{x^2}{2} + x)\right]$, where C is a constant.

2. Eq. (3.2c) is separable and we can write it as $vdv = -\frac{GM}{r^2}dr$

On integration, we get $\int v\,dv = -GM\int\frac{dr}{r^2} + C$

Thus, the general solution is $v^2 = \frac{2GM}{r} + C$

Applying the initial condition that $v = v_0$ at $r = R$, we get

$$C = v_0^2 - \frac{2GM}{R}$$

The particular solution is given by $v^2 = \frac{2GM}{r} + v_0^2 - \frac{2GM}{R}$

For the particle to escape, $v \geq 0$ for all values of r. Now $v \geq 0$, iff

$$v_0^2 - \frac{2GM}{R} \geq 0 \quad \Rightarrow \quad v_0 \geq \sqrt{\frac{2GM}{R}}$$

If $v_0^2 - \frac{2GM}{R} < 0$, then a value of r will exist for which $v = 0$. The particle will stop, v will become negative and the particle will return.

The minimum value of v_0 is called the escape velocity:

$$v_e = \sqrt{\frac{2GM}{R}}$$

3. a) $(1+\cos\theta)dr = r\sin\theta\,d\theta$

$$\therefore \quad \frac{dr}{r} - \frac{\sin\theta\,d\theta}{1+\cos\theta} = 0 \quad \text{or} \quad \int\frac{dr}{r} + \int\frac{(-\sin\theta)d\theta}{1+\cos\theta} = 0$$

$$\text{or} \quad \ln|r| + \ln|1+\cos\theta| = \ln|C| \qquad \left[\because \frac{d}{d\theta}(1+\cos\theta) = -\sin\theta\right]$$

Thus, the general solution is $r(1+\cos\theta) = C$

b) By inspection, we can suggest the substitution $x - 2y = v$

Differentiating w.r.t x we get $1 - 2\frac{dy}{dx} = \frac{dv}{dx}$ or $y' = \frac{1}{2}(1 - v')$

Substituting in the original ODE we get $(v-1) = \frac{(v+7)}{2}\left(1 - \frac{dv}{dx}\right)$

$$\text{or} \qquad \frac{dv}{dx} = 1 - \frac{2v-2}{v+7} = \frac{-v+9}{v+7}$$

Using the method of separation of variables we have

$$\int\frac{v+7}{v-9}dv = -\int dx + C \quad \text{or} \quad \int\left(1 + \frac{16}{v-9}\right)dv = -\int dx + C$$

$$\text{or} \quad v + 16\ln|v-9| = -x + C$$

Since $v = x - 2y$, we get the general solution in the form

$$x - 2y + 16\ln\left|x - 2y - 9\right| = -x + C$$

or $$2x - 2y + 16\ln\left|x - 2y - 9\right| = C$$

c) This is a first order linear homogeneous ODE and we can solve it following Example 3.5 by substituting $y = vx$.

$$\frac{dy}{dx} = \frac{(x - y)}{(x + y)} = \frac{1 - \frac{y}{x}}{1 + \frac{y}{x}}$$

We put $y = vx$ so that $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$$\therefore \quad v + x\frac{dv}{dx} = \frac{1-v}{1+v}$$

or $$x\frac{dv}{dx} = \frac{1-v}{1+v} - v = \frac{1-2v-v^2}{1+v} \quad \text{or} \quad \frac{dx}{x} = \frac{(1+v)dv}{1-2v-v^2}$$

or $$\int\frac{dx}{x} + \int\frac{(1+v)dv}{v^2 + 2v - 1} = 0$$

or $\ln|x| + \frac{1}{2}\ln|u| = \ln|C|$, $u = v^2 + 2v - 1$ or $xu^{1/2} = C_1$

$$\therefore \quad x(v^2 + 2v - 1)^{1/2} = C_1 \quad \Rightarrow \quad (y^2 + 2yx - x^2)^{1/2} = C_1$$

or $$y^2 + 2yx - x^2 = C_1^2$$

Let $I = \int\frac{(1+v)dv}{v^2 + 2v - 1}$

We put

$u = v^2 + 2v - 1$

$\therefore du = (2v + 2)dv$

$= 2(v + 1)dv$

or $I = \frac{1}{2}\int\frac{du}{u}$

$= \frac{1}{2}\ln|u|$

d) The given ODE may be expressed as $y' + \frac{2}{x}y = x^4$

Integrating factor $= \exp\left(\int\frac{2}{x}dx\right) = \exp[2\ln|x|] = \exp\left[\ln\left|x^2\right|\right] = x^2$

So, we have $\frac{d}{dx}(x^2 y) = x^6$ or $x^2 y = \int x^6 dx + C$

Thus, $x^2 y - \frac{x^7}{7} = C$ is the required solution.

e) You can see that the given ODE $y' - 2y = 8e^x$ is a first order linear non-homogeneous ODE. We note that $p(x) = -2$.

So the integrating factor is $v(x) = \exp\left[-\int 2\,dx\right] = \exp(-2x)$

Multiplying the given ODE by e^{-2x}, we get

$$e^{-2x}y' - 2ye^{-2x} = 8e^{-x} \quad \text{or} \quad \frac{d}{dx}(ye^{-2x}) = 8e^{-x}$$

Integrating both sides yields $ye^{-2x} = -8e^{-x} + C$

So the general solution is $y = -8e^x + Ce^{2x}$

4. a) To solve the ODE $(e^x + y - 1)dx + (3e^y + x - 7)dy = 0$, we check whether it is exact. Here $M = e^x + y - 1$, $N = 3e^y + x - 7$

$$\therefore \frac{\partial M}{\partial y} = 1, \frac{\partial N}{\partial x} = 1 \text{ and } \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$ Thus, the equation is exact.

$$\therefore \frac{\partial f}{\partial x} = e^x + y - 1 \text{ and we get } f = e^x + xy - x + z(y)$$

Hence, we have $\frac{\partial f}{\partial y} = 3e^y + x - 7 = x + \frac{dz}{dy}$

$$\therefore \quad \frac{dz}{dy} = 3e^y - 7 \quad \text{or} \quad z(y) = 3e^y - 7y + C_1$$

The solution is $f(x, y) = C'$ or $e^x + xy + 3e^y - x - 7y + C = 0.$

Checking the Solution

We can rewrite the general solution

$$y + \sqrt{y^2 + x^2} = C \qquad \text{(i)}$$

as

$$x^2 + y^2 = (C - y)^2$$

or $\quad x^2 = C^2 - 2Cy \qquad$ (ii)

This gives us

$$2x\,dx = -2C\,dy$$

or $\quad dy = -\frac{x}{C}dx \qquad$ (iii)

Substituting $\sqrt{x^2 + y^2} = C - y$ from (i), and dy from (iii) in the ODE, we get

$$x\left(-\frac{x}{C}dx\right) - dx(y - C + y) = 0$$

or

$$dx\left[-\frac{x^2}{C} - 2y + C\right] = 0 \qquad \text{(iv)}$$

Substituting (ii) in (iv) gives us an identity. Thus, (i) is a solution of the given ODE.

b) We can write the given ODE $x\,dy - (y - \sqrt{x^2 + y^2})dx = 0$ as

$$\frac{dy}{dx} = \frac{y - \sqrt{x^2 + y^2}}{x}$$

You can check that it is a first order homogeneous ODE. We put $y = vx$ so that $\frac{dy}{dx} = v + x\frac{dv}{dx}$.

The right hand side $= \frac{vx - \sqrt{1 + v^2}}{x} = v - \sqrt{1 + v^2}$

So we get $\quad v + x\frac{dv}{dx} = v - \sqrt{1 + v^2}$

$$\text{or} \quad \frac{dv}{\sqrt{1 + v^2}} = -\frac{dx}{x} \quad \Rightarrow \quad \int \frac{dv}{\sqrt{1 + v^2}} + \int \frac{dx}{x} = 0$$

$$\text{or } \ln\left|v + \sqrt{v^2 + 1}\right| + \ln|x| = \ln|C| \quad \text{or} \quad \ln\left|x\left(v + \sqrt{v^2 + 1}\right)\right| = \ln|C|$$

$$\therefore \qquad x\left[v + \sqrt{v^2 + 1}\right] = C$$

i.e., $\quad y + \sqrt{y^2 + x^2} = C$ is the required general solution.

Let us now apply the initial condition: $y = 4$ for $x = 3$

$$4 + \sqrt{4^2 + 3^2} = C \quad \text{or} \quad C = 9$$

Hence, the particular solution is $\quad y + \sqrt{y^2 + x^2} = 9$

5. a) We can write the given ODE $v\frac{dv}{dx} + \omega^2 x = 0$ as $v\,dv + \omega^2 x\,dx = 0$

On integrating, we get $\frac{v^2}{2} + \frac{\omega^2 x^2}{2} = C$, where C is an arbitrary constant, i.e., $v^2 + \omega^2 x^2 = C'$, where $C' = 2C$

But $\frac{dx}{dt} = v = 0$, when $x = a \Rightarrow C' = \omega^2 a^2$ and

$$v^2 = \omega^2 (a^2 - x^2) \quad \text{or} \quad v = \frac{dx}{dt} = \pm\omega\sqrt{a^2 - x^2}$$

or $\frac{dx}{\pm\sqrt{a^2 - x^2}} = \omega\, dt$

$$\therefore \quad \int \frac{dx}{\pm\sqrt{a^2 - x^2}} = \omega t + \delta$$ where δ is an arbitrary constant.

or $$\left.\begin{matrix} \sin^{-1}\frac{x}{a} \\ \\ \cos^{-1}\frac{x}{a} \end{matrix}\right| = \omega t + \delta$$

Thus, $$\frac{x}{a} = \begin{matrix} \sin(\omega t + \delta) \\ \text{or} \\ \cos(\omega t + \delta) \end{matrix}$$

Thus, the required solutions are: $x = a\sin(\omega t + \delta)$ and $x = a\cos(\omega t + \delta)$

b) We follow Example 3.8 to solve Eq. (3.23a) for an RC circuit given by $\frac{dq}{dt} + \frac{q}{RC} = \frac{E}{R}$ where E is constant. The initial condition is that the initial charge or charge at time $t = 0$ in the circuit is q_0. This equation is in the form of Eq. (3.17a) and note that here too $p = \frac{1}{RC}$ is constant. Therefore, from Eq. (3.21), the integrating factor is

$$v(t) = \exp\left[\int \frac{dt}{RC}\right] = e^{t/RC} \qquad \text{(i)}$$

Using Eq. (3.22d), we can write the general solution as

$$q(t) = e^{-t/RC}\left[\frac{E}{R}\int e^{t/RC}\, dt + C'\right] \qquad \text{(ii)}$$

where C' is an arbitrary constant. On integrating (ii), we get

$$q(t) = EC + C' e^{-t/RC}$$

Applying the initial condition $q = q_0$ at $t = 0$, we get $C' = q_0 - EC$

Hence, the particular solution is given by

$$q(t) = EC\,(1 - e^{-t/RC}) + q_0 e^{-t/RC}$$

When $E = 0$, the solution becomes $q(t) = q_0 e^{-t/RC}$

Here RC is called the time constant of the circuit. At $t = RC$, the charge on the capacitor is $\frac{q_0}{e}$, that is, $t = RC$ is the time at which the charge on the capacitor is $\frac{q_0}{e}$ or $\frac{1}{e}$ of its initial value.

APPENDIX : PARTIAL DERIVATIVES

By definition, the partial derivative of a function $f(x, y, z)$ with respect to x is

$$\frac{\partial f(x, y, z)}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y, z) - f(x, y, z)}{\Delta x} \qquad (1)$$

The function $\partial f/\partial x$ is obtained by differentiating the function $f(x, y, z)$ with respect to x as in ordinary calculus, treating **other variables** y, z **as constants**. You can similarly determine $\partial f/\partial y$ and $\partial f/\partial z$. The partial derivatives $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ of a function $f(x, y, z)$ give us, respectively, the rate of change of f in the directions of x, y or z-axes. Thus, $\frac{\partial f}{\partial x}$ gives the rate of change of f with respect to x at a given point in space.

Let us explain how to calculate the partial derivatives of a function $f(x, y, z)$ with respect to x, y and z **holding other variables to be constant**.

For example, let $f(x,y,z) = 2x^2 y z^3$. Then

$$\frac{\partial f}{\partial x} = \left[\frac{\partial}{\partial x}(x^2)\right](2yz^3) = 4xyz^3 \text{ since } y \text{ and } z \text{ are treated as constants.}$$

Similarly, for the partial derivative with respect to any other variable, we keep the remaining variables as constant. Thus,

$$\frac{\partial f}{\partial y} = \left[\frac{\partial}{\partial y}(y)\right](2x^2 z^3) = 2x^2 z^3 \quad \text{and} \quad \frac{\partial f}{\partial z} = \left[\frac{\partial}{\partial z}(z^3)\right](2x^2 y) = 6x^2 yz^2$$

You may quickly work out a couple of exercises to learn how to calculate partial derivatives of a function.

---

a) Compute $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for $f(x, y) = x^2 y^3 + \exp(x^2 y)$.

b) For the function $u(x, y, z) = 2x + yz - xy$, evaluate $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial z}$.

The solutions are as follows:

a) $$\frac{\partial f}{\partial x} = \left[\frac{\partial}{\partial x}(x^2 y^3) + \frac{\partial}{\partial x}\exp(x^2 y)\right] = \left[\frac{\partial}{\partial x}(x^2)\right]y^3 + 2x\exp(x^2 y)$$

$$= 2xy^3 + 2x\exp(x^2 y)$$

$$\frac{\partial f}{\partial y} = \left[\frac{\partial}{\partial y}(x^2 y^3) + \frac{\partial}{\partial y}\exp(x^2 y)\right]$$

$$= \left[\frac{\partial}{\partial y}(y^3)\right]x^2 + \exp(x^2 y) = 3y^2 x^2 + \exp(x^2 y)$$

b) $$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(2x + yz - xy) = 2 - y \text{ since } y \text{ and } z \text{ are treated as constants.}$$

Similarly, $\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(2x + yz - xy) = z - x$ and $\frac{\partial u}{\partial z} = \frac{\partial}{\partial z}(2x + yz - xy) = y$

How long will a swing take to stop if it is not pushed continuously? You could find the answer after studying this unit!

UNIT 4

SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

Structure

STUDY GUIDE

In this unit, we will use some concepts of physics and mathematics that you have studied in your senior secondary (+ 2) classes. We shall take it for granted that you know how to solve a system of linear equations, and evaluate determinants, partial derivatives and integrals of various functions. You should also know how to solve quadratic equations and determine their roots. You may like to **revise** these concepts from the senior secondary (+ 2) mathematics course. You also have to revise the concepts of Unit 3 and make sure that you can solve first order ODEs and integrals of various functions very well. Only then you will enjoy studying this unit.

In order to meet the expected learning outcomes, try to solve all steps explained in the text and examples. Keep a paper and pen/pencil at hand or use the margins of the unit. Make sure that you can solve the problems given in this unit before studying the remaining course.

"The real goal of physics is to come up with an equation that could explain the universe but still be small enough to fit on a T-shirt."

Leon Lederman

4.1 INTRODUCTION

In Unit 3, you have learnt several methods for solving first order differential equations. You have learnt how to apply these methods to physical problems such as the motion of a body falling freely and under air resistance, radioactive decay, change in current and charge in electrical circuits.

However, for many physical and biological problems, we need to solve second order homogeneous ODEs. For example, such ODEs are used in the study of wave (sound and light) propagation, mechanical and electrical vibrations while designing bridges over rivers or highways. These are also used in studying the transmission of radio/TV signals. In some cases, these equations can be reduced to first order ODEs. In this unit, we discuss techniques of solving **second order homogeneous ODEs with constant coefficients**.

In Sec. 4.2, you will learn the basic terminology that will help you to identify such ODEs.

You will apply such ODEs on different systems in the remaining blocks of this course. In the laboratory, you will get an opportunity to perform experiments to study the motion of a flywheel or determine the depression in horizontal beams. These are governed by homogeneous second order ODEs with constant coefficients. You also need to solve such ODEs when you study the growth/decay of current in an *LCR* circuit or planetary motion in the solar system. You will learn the basic techniques to solve such equations in Sec. 4.3.

In the next block, you will study the concepts of mechanics and use the mathematical techniques you have learnt in this block.

Expected Learning Outcomes

After studying this unit, you should be able to:

- compute the Wronskian of a given ODE;
- obtain the auxiliary/characteristic equation for second order linear, homogeneous ODEs with constant coefficients; and
- determine the general and particular solutions of second order linear, homogeneous ODEs with constant coefficients using the method of exponential functions.

4.2 SOME TERMINOLOGY

While studying first order ordinary differential equations in Unit 3, you have learnt terms such as the **order** and **degree** of an ODE. You know the conditions for an ODE to be **linear** or **non-linear**. You have also learnt about first order **homogeneous** and **non-homogeneous** ODEs.

Let us repeat the terminology for second order ODEs.

CLASSIFYING SECOND ORDER ODEs

- If the highest derivative appearing in an ODE is of **order** 2, it is called a second order ODE. The **degree** of a second order ODE is the power of the highest order derivative appearing in it.
- An ODE is termed as **linear** if
  i) the unknown function and its derivatives occur only to the first degree in it;
  ii) there are no products involving the unknown function and its derivatives or products of two or more derivatives in it; and
  iii) there are no transcendental functions involving the unknown function or any of its derivatives in it.

An ODE that does not satisfy any one or more of the above conditions will **not** be linear. It is called **non-linear**.

In general, an nth order ODE has n linearly independent solutions. Furthermore, a linear combination of linearly independent solutions is also a solution.

We now explain some more basic terms for second order ODEs as you should know them too. A second order **linear** ordinary differential equation can be written as

$$\frac{d^2y}{dx^2} + p_1(x)\frac{dy}{dx} + p_2(x)y = g(x) \tag{4.1}$$

The function $g(x)$ is termed as the **forcing function**, and $p_1(x)$ and $p_2(x)$ are called **coefficient functions**. These are continuous over the interval where the solution exists.

A second order ODE is termed as **homogeneous** if $g(x) = 0$.

It is termed as **non-homogeneous** if $g(x) \neq 0$.

Note that

$$y' = \frac{dy}{dx}$$

and $y'' = \dfrac{d^2y}{dx^2}$

Now consider a **linear, homogeneous second order ODE** of the form

$$y'' + p_1(x)y' + p_2(x)y = 0 \tag{4.2}$$

Suppose that y_1 and y_2 are linearly independent (read the margin remark) solutions of Eq. (4.2). Then their linear combination

$$y = C_1y_1 + C_2y_2 \tag{4.3}$$

is a general solution of Eq. (4.2). Here C_1 and C_2 are arbitrary non-zero constants. This is the **principle of superposition**. Let us explain this concept with the help of an example. Consider the ODE:

Two functions $y_1(x)$ and $y_2(x)$ are said to be linearly independent on an interval I, where both functions are defined, if and only if we can determine constants C_1 and C_2 such that the relation

$$C_1y_1 + C_2y_2 = 0$$

is satisfied only for $C_1 = C_2 = 0$.

$$\frac{d^2y}{dt^2} + \omega^2 y = 0 \tag{4.4a}$$

Fig. 4.1: A spring-mass system is an example of a harmonic oscillator.

This ODE is the equation of motion of an undamped harmonic oscillator (Fig. 4.1). You can check that $y_1 = \sin \omega t$ and $y_2 = \cos \omega t$ are solutions of this equation (read the margin remark). So the general solution of this equation is

$$y(t) = C_1 \sin \omega t + C_2 \cos \omega t \tag{4.4b}$$

You may now ask: What do we mean by linearly independent solutions? How do we test their linear independence? Will a linear combination of linearly independent solutions necessarily lead to a different solution? When does a set of solutions constitute the general solution of a linear differential equation? Let us now seek answers to these questions for ODEs of second order.

4.2.1 Linearly Independent Solutions and the Wronskian

$$y_1' = \omega \cos \omega t$$

and

$$y_1'' = -\omega^2 \sin \omega t$$

$$\Rightarrow -\omega^2 \sin \omega t + \omega^2 \sin \omega t = 0$$

Similarly,

$$y_2' = -\omega \sin \omega t$$

$$y_2'' = -\omega^2 \cos \omega t$$

Therefore,

$$-\omega^2 \cos \omega t + \omega^2 \cos \omega t = 0$$

Let us first answer the question: **What do we mean by linearly independent solutions**? We say that two solutions y_1 and y_2 are linearly independent on an interval if the identity given in Eq. (4.3) is satisfied only when $C_1 = C_2 = 0$.

To understand this point, let us suppose that the constants C_1 and C_2 are non-zero. Then from Eq. (4.3), we would get

$$\frac{y_2}{y_1} = K, \quad \text{a constant} \tag{4.5}$$

Thus, y_1 and y_2 would be proportional on the given interval. Then, by definition, y_1 and y_2 would be linearly dependent functions on that interval.

Let us now answer the next question: **How do we test the linear independence of solutions**?

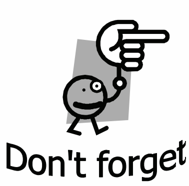
You have learnt that linear dependence of y_1 and y_2 means that the ratio y_2 / y_1 is constant. This implies that the solutions y_1 and y_2 would be linearly independent if the ratio y_2 / y_1 is not a constant. This also means that the differential of this ratio is not identically equal to zero:

$$\frac{y_2' y_1 - y_1' y_2}{y_1^2} \neq 0 \quad \text{or} \quad y_2' y_1 - y_1' y_2 \neq 0 \tag{4.6}$$

We can express the LHS of Eq. (4.6) in the form of a determinant and write the condition of linear independence of two solutions y_1 and y_2 as follows:

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0 \tag{4.7}$$

The determinant $W(y_1, y_2)$ is called the **Wronski determinant** or the **Wronskian** of the given differential equation. We may, therefore, conclude that



> Two solutions y_1 and y_2 of a second order homogeneous second order ODE with constant coefficients are linearly independent on an interval [*a*, *b*], **if and only if their Wronskian is non-zero** for $a \le x \le b$.

Let us check this for a harmonic oscillator. Substituting $y_1 = \sin \omega t$ and $y_2 = \cos \omega t$ in Eq. (4.6), we get

$$W(x) = \begin{vmatrix} \sin \omega t & \cos \omega t \\ \omega \cos \omega t & -\omega \sin \omega t \end{vmatrix} = -\omega \neq 0 \qquad (4.8)$$

Thus, the Wronskian of $y_1 = \sin \omega t$ and $y_2 = \cos \omega t$ is non-zero and these are linearly independent solutions.

We can also say that y_1 and y_2 are **linearly dependent** solutions on an interval [*a*, *b*], if and only if their Wronskian is zero for some $x = x_0$ in the given interval. To grasp this concept, you may like to answer an SAQ.

SAQ 1 – Testing linear independence

The solutions of the equation $y'' + 4y = 0$ are given by $y_1 = \sin 2x$ and $y_2 = \cos 2x$. Are these solutions linearly independent?

Before proceeding further, let us revise what you have learnt in this section.

Recap

LINEAR INDEPENDENCE OF SOLUTIONS OF 2ND ORDER HOMOGENEOUS ODEs

- A second order homogeneous ODE has two solutions, say y_1 and y_2.
- The solutions y_1 and y_2 are said to be **linearly independent** on an interval [*a*, *b*], if and only if, their Wronskian is **non-zero** for $a \le x \le b$:

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0$$

We hope that you are now equipped with all the necessary basic terminology. We now explain how to solve **second order, linear, homogeneous ODEs with constant coefficients**. Many techniques of solving such equations have been developed. Here we will discuss the method of exponential functions.

4.3 METHOD OF EXPONENTIAL FUNCTIONS

We know that a second order linear, homogeneous ODE with constant coefficients is of the form

$$ay'' + by' + cy = 0 \qquad (4.9)$$

where a, b and c are real numbers. From the method of separation of variables (Sec. 3.3.1 of Unit 3), you can verify that the solution of the first order linear homogeneous ODE $y' + y = 0$ is an exponential function of the form $y = A \exp(-kx)$.

Here also, we seek a solution of Eq. (4.9) of the form

$$y = A\exp(m\,x) \tag{4.10}$$

where the dimensions of m are inverse of those of x. This ensures that the power of the exponential function is dimensionless.

On differentiating Eq. (4.10) with respect to x twice, we get

$$y' = A\,m\exp(m\,x) \tag{4.11a}$$

and $$y'' = A\,m^2\exp(m\,x) \tag{4.11b}$$

The roots of a quadratic equation

$$ax^2 + bx + c = 0$$

are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

On substituting these results in Eq. (4.9), we get

$$(am^2 + bm + c)A\exp(m\,x) = 0$$

Since $A\exp(m\,x)$ is finite, this equation will be satisfied only if

$$am^2 + bm + c = 0 \tag{4.12}$$

This is a quadratic equation in m and is called the **characteristic equation** (or **auxiliary equation**). Its roots are given by

$$m_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad m_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \tag{4.13}$$

So we can write the solutions of Eq. (4.9) as

$$y_1(x) = A\exp(m_1 x) \tag{4.14a}$$

and $$y_2(x) = A\exp(m_2 x) \tag{4.14b}$$

Before we write the general solution, let us check the linear independence of these solutions. For this, we calculate their Wronskian:

$$W(x) = \begin{vmatrix} A\exp(m_1 x) & A\exp(m_2 x) \\ m_1 A\exp(m_1 x) & m_2 A\exp(m_2 x) \end{vmatrix}$$

$$= (m_2 - m_1)B\exp[(m_1 + m_2)\,x] \tag{4.15}$$

Here $B = A^2$ is another constant. This shows that if $m_1 \neq m_2$, $W(x) \neq 0$ and the solutions will be linearly independent. That is, **if the roots of the auxiliary equation are different, the solutions of the given second order ODE with constant coefficients will be linearly independent**.

Using Eq. (4.3), i.e., the principle of superposition, the general solution of Eq. (4.9) can be expressed as a linear combination of the two linearly independent solutions. We can write it as

$$y(x) = C_1\exp(m_1 x) + C_2\exp(m_2 x) \tag{4.16}$$

The constants C_1 and C_2 are determined using initial or boundary conditions.

So solving a second order linear, homogeneous ODE with constant coefficients with an exponential function as a solution has now become as simple as obtaining the roots of a quadratic equation.

Let us consider an example to illustrate this method.

Example 4.1: SECOND ORDER ODE WITH CONSTANT COEFFICIENTS

Solve the ODE $y'' + 6y' - 7y = 0$

SOLUTION ■ The given ODE is a second order linear homogeneous ODE with constant coefficients. Let its solution be given by $y = A\exp(mx)$. You can verify that the auxiliary equation for this ODE is

$$m^2 + 6m - 7 = 0 \qquad \text{(i)}$$

The roots of Eq. (i) are $m = 1, -7$.

So we get two solutions of the given ODE as

$$y_1(x) = A\exp(x) \qquad \text{(ii)}$$

and $$y_2(x) = A\exp(-7x) \qquad \text{(iii)}$$

The general solution is $y(x) = C_1 \exp(x) + C_2 \exp(-7x)$ (iv)

For Eq. (i) in Example 4.1,

$$m = \frac{-6 \pm \sqrt{36 - 4\times 1\times(-7)}}{2}$$
$$= 1, -7$$

Note that roots of the auxiliary equation [Eq. (4.12)] can be

1. **real** and **distinct** if $b^2 - 4ac > 0$ or $b^2 > 4ac$
2. **real** and **equal** if $b^2 - 4ac = 0$ or $b^2 = 4ac$ and
3. **complex conjugate** if $b^2 - 4ac < 0$ or $b^2 < 4ac$

Let us obtain the solutions corresponding to these three cases.

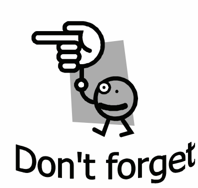
4.3.1 Real and Distinct Roots

For distinct real roots, $\exp(m_1 x)$ and $\exp(m_2 x)$ will be linearly independent and the general solution can be written as follows:

General solution for real and distinct roots of auxiliary equation

$$y = C_1 \exp(m_1 x) + C_2 \exp(m_2 x)$$
$$= \exp\left[-\left(\frac{bx}{2a}\right)\right][C_1 \exp(\alpha x) + C_2 \exp(-\alpha x)] \qquad (4.17)$$

where $$\alpha = \frac{\sqrt{b^2 - 4ac}}{2a}$$



Note that the constants C_1 and C_2 are determined using given initial and boundary conditions. We now illustrate this method with an example.

EXAMPLE 4.2: CASE OF DISTINCT REAL ROOTS

Solve the ODE $y'' + 3y' + 2y = 0$ subject to the initial conditions $y(0) = 1$ and $y'(0) = 2$.

SOLUTION ■ You can verify that the auxiliary equation for this ODE is

$$m^2 + 3m + 2 = 0 \qquad \text{(i)}$$

and its roots are $m = -1$ and $m = -2$. So the general solution of the ODE is

$$y = C_1 e^{-x} + C_2 e^{-2x} \qquad \text{(ii)}$$

To determine C_1 and C_2, we first use the condition that at $x = 0, y = 1$. This gives

$$1 = C_1 + C_2 \Rightarrow C_1 = 1 - C_2 \qquad \text{(iii)}$$

Further, since $y' = -C_1 e^{-x} - 2C_2 e^{-2x}$, we can write

$$y'(0) = 2 = -C_1 - 2C_2 \qquad \text{(iv)}$$

On solving Eqs. (iii) and (iv) for C_1 and C_2, you will get $C_1 = 4$ and $C_2 = -3$. Hence, the desired particular solution of the given ODE can be expressed as

$$y(x) = 4\exp(-x) - 3\exp(-2x)$$

You may like to solve another SAQ on this method for practice.

SAQ 2 – Case of distinct and real roots

Solve the ODE $y'' - 5y' + 6y = 0$ given that $y(0) = 2$ and $y'(0) = 2$.

4.3.2 Real and Equal Roots

When two roots are real and equal, that is, $m_1 = m_2$, the Wronskian is zero:

$$W(x) = 0$$

This means that $e^{m_1 x}$ and $e^{m_2 x}$ will be linearly dependent. What does this imply? It implies that Eq. (4.16) does not hold. It also means that our starting assumption is not true. You may now ask: **How can we obtain two linearly independent functions when two equal roots of the auxiliary equation are equal**? (We also call this as the case of **repeated real roots**.) In such a situation, we use the **method of reduction of order** to construct a second linearly independent solution.

Repeated Real Roots

When the auxiliary equation of a second order ordinary differential equation has two equal real roots, we obtain the correct form of the second solution by assuming that

$$y_2 = u(x)\exp(mx) \tag{4.18}$$

where m is a root of the auxiliary equation (4.12). Let us obtain the expression for $u(x)$. For this, we first differentiate Eq. (4.18) twice with respect to x and get

$$y_2' = u' \exp(mx) + mu \exp(mx) \tag{4.19a}$$

and $$y_2'' = u'' e^{mx} + 2mu' e^{mx} + m^2 u\, e^{mx} \tag{4.19b}$$

Substituting these results in Eq. (4.9), we get

$$(am^2 + bm + c)u(x)e^{mx} + (2ma + b)e^{mx}u' + a\,e^{mx}u'' = 0 \tag{4.19c}$$

The first term in Eq. (4.19c) vanishes in view of Eq. (4.12) and the coefficient of u' in the second term is zero since in this case $m = -b/2a$ [see Eq. (4.13)]. Hence, Eq. (4.19c) simplifies to

$$\exp(mx)u'' = 0 \tag{4.19d}$$

On multiplying Eq. (4.19d) by $\exp(-mx)$ and integrating the resultant expression, we get

$$u' = K \tag{4.20a}$$

where K is an arbitrary constant of integration. Integrating Eq. (4.20a) again, we get

$$u = Kx + C \tag{4.20b}$$

Since we are seeking only a second linearly independent solution, we now take $C = 0$ and $K = 1$. Then we can write the desired solution as

$$y_2 = x\,e^{mx} = x\,e^{-bx/2a} \tag{4.21}$$

Hence, the general solution of a homogeneous second order differential equation with constant coefficients, **when the auxiliary equation has repeated real roots**, is

$$y(x) = C_1 e^{mx} + C_2 x\,e^{mx} \quad \text{where } m = -\frac{b}{2a} \tag{4.22a}$$

$$= (C_1 + C_2 x)\exp(mx) \tag{4.22b}$$

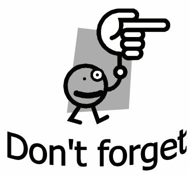
where C_1 and C_2 are arbitrary constants.

To test that the solutions $e^{-bx/2a}$ and $x\,e^{-bx/2a}$ are linearly independent, we calculate their Wronskian:

$$W(x) = \begin{vmatrix} \exp\left(-\dfrac{bx}{2a}\right) & x\exp\left(-\dfrac{bx}{2a}\right) \\ -\dfrac{b}{2a}\exp\left(-\dfrac{bx}{2a}\right) & -\dfrac{b}{2a}x\exp\left(-\dfrac{bx}{2a}\right) + \exp\left(-\dfrac{bx}{2a}\right) \end{vmatrix}$$

$$= e^{-(bx/a)} > 0 \quad \text{for} \quad a \le x \le b \tag{4.23}$$

It implies that $e^{-(bx/a)}$ and $xe^{-bx/2a}$ are acceptable solutions. The arbitrary constants C_1 and C_2 occurring in Eq. (4.22a or b) can be determined using specified initial or boundary conditions. We may, therefore, conclude as follows:



General solution for repeated roots of auxiliary equation

When the auxiliary equation for a second order ODE with constant coefficients has repeated real roots $(m_1 = m_2 = m)$, the general solution is given by

$$y(x) = (C_1 + C_2 x)\exp(mx) \tag{4.22b}$$

where C_1 and C_2 are arbitrary constants.

You may now like to solve an SAQ to be sure that you have grasped this method.

SAQ 3 **– Case of repeated roots**

Solve the initial value problem $y'' + 6y' + 9y = 0, \quad y(0) = 2$ and $y'(0) = 1.$

4.3.3 Complex Roots

Let us now consider the case for which the auxiliary equation has **complex roots** of the form $m = \alpha + i\beta$, where α and β are real. From your school mathematics, you may recall that complex roots of a real polynomial equation always occur in conjugate pairs. That is, if

$$m_1 = \alpha + i\beta \tag{4.24a}$$

is one of the roots, then

$$m_2 = \alpha - i\beta \tag{4.24b}$$

is also a root. As before, we can obtain the following general solution as a linear combination of two linearly independent solutions:

$$\begin{aligned} y &= Ay_1 + By_2 = A\exp(m_1 x) + B\exp(m_2 x) \\ &= Ae^{(\alpha+i\beta)x} + Be^{(\alpha-i\beta)x} = e^{\alpha x}(Ae^{i\beta x} + Be^{-i\beta x}) \end{aligned} \tag{4.25}$$

You will note that this solution is complex. Can you express it as a real solution? To do so, we use **Euler's formula**, which is given below:

$$e^{\pm i\theta} = \cos\theta \pm i\sin\theta \tag{4.26}$$

Now we write

$$y_1 = e^{m_1 x} = e^{\alpha x} e^{i\beta x} = e^{\alpha x}(\cos\beta x + i\sin\beta x) \tag{4.27a}$$

and $$y_2 = e^{m_2 x} = e^{\alpha x} e^{-i\beta x} = e^{\alpha x}(\cos\beta x - i\sin\beta x) \tag{4.27b}$$

Then we can write the real general solution as follows:

$$y = C_1 Y_1 + C_2 Y_2 \tag{4.27c}$$

where $Y_1 = \frac{1}{2}(y_1 + y_2) = e^{\alpha x}\cos\beta x$ (4.27d)

and $Y_2 = \frac{1}{2i}(y_1 - y_2) = e^{\alpha x}\sin\beta x$ (4.27e)

Putting $C_1 = C\cos\phi$ and $C_2 = C\sin\phi$, we can rewrite Eq. (4.27c) as

$$y = Ce^{\alpha x}\cos(\beta x - \phi) \tag{4.28}$$

where C and ϕ are arbitrary constants. These are related to C_1 and C_2 by

$$C = \sqrt{C_1^2 + C_2^2} \quad \text{and} \quad \tan\phi = \frac{C_2}{C_1} \tag{4.29}$$

Thus, the general solution of a second order ODE with constant coefficients can be written as the product of an exponential and a trigonometric function when the roots of the auxiliary equation are complex. Therefore, we conclude as follows:

General solution for complex roots of auxiliary equation

If the auxiliary equation of a second order ODE with constant coefficients has complex roots of the form $\alpha \pm i\beta$, the general solution is of the form $y = Ce^{\alpha x}\cos(\beta x - \phi)$ (4.28)

Let us now apply this method to a physical system: the undamped spring-mass system.

EXAMPLE 4.3: UNDAMPED SPRING-MASS SYSTEM

An undamped spring-mass system executes simple harmonic motion along the x-axis (see Fig. 4.1). Solve the differential equation governing its motion:

$$\frac{d^2x}{dt^2} + \omega_0^2 x = 0, \text{ where } \omega_0^2 = \frac{k}{m}$$

SOLUTION ■ Note that the auxiliary equation is: $m^2 + \omega_0^2 = 0$

which has complex roots $m_1 = i\,\omega_0$ and $m_2 = -i\,\omega_0$.

Hence, the general solution is given by Eq. (4.28) with $\alpha = 0$:

$$x(t) = C\cos(\omega_0 t - \phi)$$

Theoretically, the oscillatory motion of the spring-mass system should continue indefinitely. In practice, however, several factors lead to loss of energy of the system. As a result, the amplitude of oscillations tends to decrease. In the next example, we illustrate the method discussed in this section to study the motion of a **damped oscillator**. But before that, let us revise what you have learnt in this section so far.

Recap

METHOD OF EXPONENTIAL FUNCTIONS TO SOLVE 2nd ORDER ODEs WITH CONSTANT COEFFICIENTS

- Obtain the auxiliary/characteristic equation and determine its roots, say, m_1 and m_2.
- When the roots are **distinct** and **real**, there exist two linearly independent solutions of the form $\exp(m_1 x)$ and $\exp(m_2 x)$ and the general solution is given by

$$y(x) = C_1 \exp(m_1 x) + C_2 \exp(m_2 x)$$

- When the roots are **real** and **equal**, the two linearly independent functions are of the form $\exp(mx)$ and $x\exp(mx)$, and the general solution is given by

$$y(x) = (C_1 + C_2 x)\exp(mx)$$

- When the roots are **complex**, the two linearly independent solutions are of the form $\exp(\alpha x)\sin\beta x$ and $\exp(\alpha x)\cos\beta x$, and the general solution can be written as

$$y(x) = \exp(\alpha x)(C_1 \cos\beta x + C_2 \sin\beta x) = C\exp(\alpha x)\cos(\beta x - \phi)$$

EXAMPLE 4.4: DAMPED SPRING-MASS SYSTEM

Consider a spring-mass system which is damped by air resistance (Fig. 4.2). Let the air resistance be linearly proportional to velocity. The differential equation that describes its motion is given by

$$M\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + kx = 0$$

where M is mass, k is spring constant and γ is characteristic of damping. Obtain the roots of corresponding auxiliary equation.

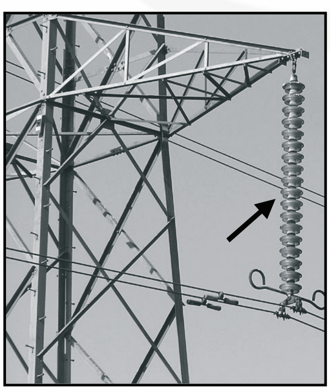


Fig. 4.2: **Example of a damped spring-mass system in a transmission line.**

SOLUTION ■ We can rewrite the given equation of motion as

$$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + \omega_0^2 x = 0 \qquad \text{(i)}$$

where we have put $2b = \gamma / M$ and $\omega_0^2 = k / M$.

On comparing this equation with Eq. (4.9), we can write the auxiliary equation as $m^2 + 2bm + \omega_0^2 = 0$, which has roots

$$m_1 = -b + \sqrt{b^2 - \omega_0^2}$$

and $m_2 = -b - \sqrt{b^2 - \omega_0^2}$

These roots depend on damping and determine the motion of the oscillator. Depending on the value of $(b^2 - \omega_0^2)^{1/2}$, we have three possibilities.

Case 1: If $b > \omega_0$, $\sqrt{b^2 - \omega_0^2}$ is positive and there are two **real**, **distinct** roots.

Case 2: If $b = \omega_0$, $b^2 - \omega_0^2 = 0$ the roots are **real** and **repeated**.

Case 3: If $b < \omega_0$, $b^2 - \omega_0^2$ is negative and $\sqrt{b^2 - \omega_0^2}$ is imaginary, i.e., there is a **complex conjugate pair** of roots.

So the equation of motion of a damped spring-mass system can be written as:

$$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + \omega_0^2 x = 0 \qquad (4.30)$$

Let us now obtain its acceptable solutions. It is important to mention here that solutions corresponding to three roots discussed in Example 4.4 will be used in Unit 18 in Block 4 of this course. Therefore, you should study them carefully.

You will learn about damped oscillations in detail in Unit 18.

Case 1: Distinct and Real Roots

When the roots are **real** and **distinct**, the general solution of Eq. (4.30) is given by

$$x(t) = \exp(-bt)[C_1 \exp(\beta t) + C_2 \exp(-\beta t)] \qquad (4.31)$$

where $\beta = \sqrt{b^2 - \omega_0^2}$.

This represents non-oscillatory behaviour. The system is said to be **heavily damped** and such a motion is called **dead beat**.

The actual displacement of any such system is determined by the initial conditions. To know how this is done, we would like you to solve the following SAQ.

SAQ 4 – Damped oscillator

A heavily damped oscillator in its equilibrium position is suddenly kicked so that at $t = 0$, $x = 0$ and $\frac{d}{dt}x(0) = v_0$. Obtain the expression for resultant displacement and interpret the result.

On solving SAQ 4, you would have noted that the displacement of a heavily damped oscillator is determined by an increasing hyperbolic function and a decaying exponential. As a result, the displacement increases initially, attains a maximum and thereafter decays exponentially (Fig. 4.3).

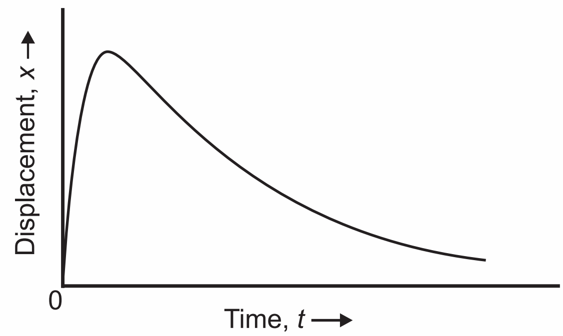


Fig. 4.3: Displacement-time graph for a heavily damped spring-mass system.

Case 2: Repeated Real Roots

When the roots of the auxiliary equation are repeated and real, the general solution of Eq. (4.30) is given by

$$x(t) = (C_1 + C_2 t)\exp(-bt) \tag{4.32}$$

Note that here C_1 has dimension of length and the dimensions of C_2 are those of velocity. As before, these constants can be determined by specifying initial conditions. You can verify that for initial conditions given in SAQ 4, $C_1 = 0$ and $C_2 = v_0$ so that the complete solution is

$$x(t) = v_0 t \exp(-bt) \tag{4.33}$$

Such a system is said to be **critically damped**. The typical graph of a critically damped system is shown in Fig. 4.4.

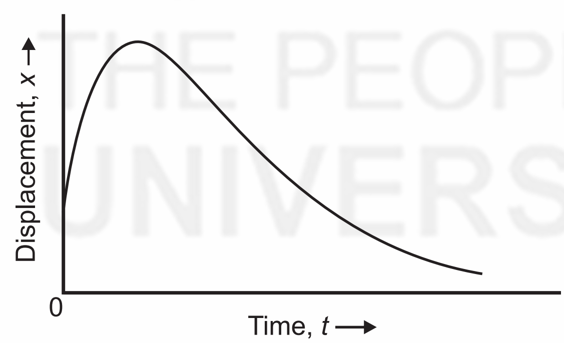


Fig. 4.4: Displacement-time graph for a critically damped spring-mass system.

Case 3: Complex Roots

When the roots are complex, let us write

$$\sqrt{b^2 - \omega_0^2} = \sqrt{-1}(\omega_0^2 - b^2)^{1/2} = i\,\omega_d$$

where $i = \sqrt{-1}$ and $\omega_d = \sqrt{\omega_0^2 - b^2}$ is a real positive quantity.

Following Eq. (4.28), the displacement for this case is given by

$$x(t) = C\exp(-bt)\cos(\omega_d t - \phi) \tag{4.34}$$

where C and ϕ are arbitrary constants.

Note that Eq. (4.34) represents oscillatory motion whose amplitude decreases exponentially at a rate governed by b. Such a system is said to be **weakly damped**. The displacement of a weakly damped system is depicted in Fig. 4.5. Physically, this case is of maximum interest.

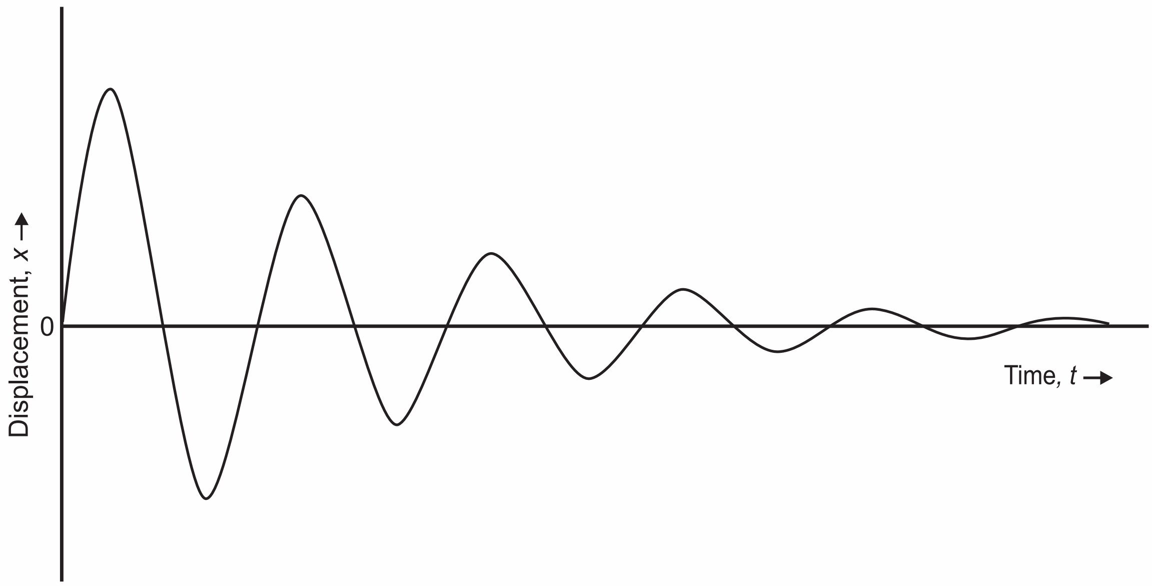


Fig. 4.5: Oscillations of a weakly damped spring-mass system.

With this, we come to an end of this unit and summarise its contents.

4.4 SUMMARY

Concept	Description
Linear independence of solutions, Wronskian	■ If y_1 and y_2 are solutions of a second order ODE, then these are linearly independent, if and only if, their Wronskian is non-zero. Mathematically, we can write $$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0$$
Solution of a second order homogeneous ODE with constant coefficients	■ A second order homogeneous ODE with constant coefficients can be solved either by using the method of exponential functions or by reduction of its order.
Method of exponential functions	■ In the method of exponential functions, the form of the solution depends on the roots of the auxiliary equation. ➢ For **distinct real roots**, there exist two linearly independent functions of the form $\exp(m_1 x)$ and $\exp(m_2 x)$.

Method of exponential function

The general solution for **distinct real roots** is given by

$$y(x) = C_1 \exp(m_1 x) + C_2 \exp(m_2 x)$$

- When **roots are repeated and real** $(m_1 = m_2 = m)$**,** the general solution is given by

$$y(x) = (C_1 + C_2 x)\exp(mx)$$

- For a **complex conjugate pair of roots**, the two linearly independent solutions are of the form $\exp(\alpha x)\sin\beta x$ and $\exp(\alpha x)\cos\beta x$, and the general solution can be written as

$$y(x) = \exp(\alpha x)(C_1 \cos\beta x + C_2 \sin\beta x)$$

$$= C \exp(\alpha x)\cos(\beta x - \phi)$$

4.5 TERMINAL QUESTIONS

1. Determine the general solutions for the following second order ODEs:

   i) $y'' + y' - 12y = 0$

   ii) $2y'' + 3y' - y = 0$

   iii) $4y'' + 12y' + 9y = 0$

   iv) $y'' - 2y' + y = 0$

   v) $y'' + 10y' + 26y = 0$

   vi) $y'' + 2y' + y = 0$

2. Solve the following initial value problems:

   i) $y'' + 16y = 0; \quad y(\pi/4) = -1; \; y'(\pi/4) = 4$

   ii) $4y'' - 4y' + y = 0; \quad y(0) = 2; \; y'(0) = -1$

3. Solve the following boundary value problems:

   i) $4y'' + y = 0; \quad y(0) = 3; \; y(\pi) = -2$

   ii) $y'' - 6y' + 9y = 0; \quad y(0) = 2; \; y(1) = 0$

4. In an *LCR* circuit, an inductance *L*, a resistance *R* and a capacitance *C* are connected in series. The variation of charge *q* flowing through with time *t* in the circuit is given by the differential equation:

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = 0$$

   Solve this equation to determine *q* as a function of *t*.

5. The undamped back and forth (torsional) vibrations of a wheel attached to a thin elastic wire are given by the following ODE:

$$I\frac{d^2\theta}{dt^2} + k\theta = 0$$

where I is the moment of inertia of the wheel about an axis passing through its centre and k is the torsional constant of the wire. The angle θ is the angular displacement from the equilibrium position (Fig. 4.6). Solve the given equation to determine θ as a function of t for $k/I = 16\ \text{s}^{-1}$, $\theta(t=0) = 30°$, $\dot{\theta}(t=0) = 10\ \text{rad s}^{-1}$.

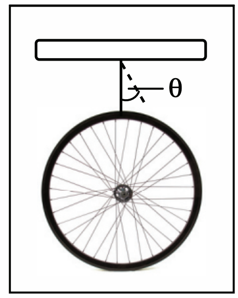


Fig. 4.6: **Torsional vibrations of a wheel.**

4.6 SOLUTIONS AND ANSWERS

Self-Assessment Questions

1. The solutions of the given equation are $\sin 2x$ and $\cos 2x$. To verify linearly independence of these functions, we have to calculate their Wronskian:

$$W(x) = \begin{vmatrix} \sin 2x & \cos 2x \\ 2\cos 2x & -2\sin 2x \end{vmatrix}$$

$$= -2\sin^2 2x - 2\cos^2 2x$$

$$= -2$$

Since $W(x) \neq 0$ for all x, the functions $\sin 2x$ and $\cos 2x$ are linearly independent.

2. The auxiliary equation of the given ODE is

$$m^2 - 5m + 6y = 0 \qquad \text{(i)}$$

The roots of Eq. (i) are 3 and 2. Hence, the general solution is

$$y(x) = C_1 \exp(3x) + C_2 \exp(2x) \qquad \text{(ii)}$$

To determine C_1 and C_2, we first use the condition $y(0) = 2$. Hence,

$$C_1 + C_2 = 2 \Rightarrow C_1 = 2 - C_2 \qquad \text{(iii)}$$

To use the second condition $y'(0) = 2$, we differentiate $y(x)$ of Eq. (ii) with respect to x. Then, we get

$$y'(x) = 3C_1 \exp(3x) + 2C_2 \exp(2x)$$

$$\therefore \quad y'(0) = 2 = 3C_1 + 2C_2 \qquad \text{(iv)}$$

On substituting the value of C_1 from Eq. (iii) in Eq. (iv), we note that

$$2 = 3(2 - C_2) + 2C_2$$

$$= 6 - C_2$$

so that $C_2 = 4$

and $C_1 = -2$

Hence, the desired solution of the given equation can be expressed as

$$y(x) = -2\exp(3x) + 4\exp(2x)$$

3. The auxiliary equation corresponding to the ODE is

$$m^2 + 6m + 9 = 0$$

The repeated root of this equation is $m = -3$. Hence, the desired acceptable solution is given by

$$y(x) = (C_1 + C_2 x)\exp(-3x) \quad \text{(i)}$$

We can obtain the value of C_1 using the condition $y(0) = 2$:

$$C_1 = 2 \quad \text{(ii)}$$

On differentiating Eq. (i) with respect to *x*, we get

$$\frac{dy}{dx} = C_2 \exp(-3x) - 3(C_1 + C_2 x)\exp(-3x)$$

Using the condition $y'(0) = 1$, we get the relation

$$C_2 - 3C_1 = 1 \Rightarrow C_2 = 7 \quad \text{(iii)}$$

Hence the expression for the particular solution is

$$y(x) = (2 + 7x)\exp(-3x)$$

4. From Eq. (4.31) we recall that for a heavily damped oscillator

$$x(t) = \exp(-bt)[C_1 e^{\beta t} + C_2 e^{-\beta t}] \quad \text{(i)}$$

where $\beta = \sqrt{b^2 - \omega_0^2}$. It is given that at $t = 0,\ x = 0$.

$$\therefore \qquad 0 = C_1 + C_2 \Rightarrow C_1 = -C_2$$

On differentiating (i), we get

$$\frac{dx}{dt} = \left[C_1 e^{\beta t - bt}\ (\beta - b) - C_2 \exp^{-\beta t - bt}\ (\beta + b)\right]$$

$$\therefore \qquad \left.\frac{dx}{dt}\right|_{t=0} = v_0 = (\beta - b)C_1 - C_2(\beta + b)$$

$$= (\beta - b)C_1 + C_1(\beta + b)$$

$$= 2\beta C_1 \Rightarrow C_1 = \frac{v_0}{2\beta} \quad \text{and} \quad C_2 = -\frac{v_0}{2\beta}$$

Hence, $\quad x(t) = \exp(-bt)\,\dfrac{v_0}{2\beta}[\exp(\beta t) - \exp(-\beta t)]$

$$= \frac{v_0}{\beta}\sinh \beta t\, e^{-bt}$$

Terminal Questions

1. i) The auxiliary equation for the given ODE is $m^2 + m - 12 = 0$

We rewrite it as $(m+4)(m-3) = 0$

So the roots of the auxiliary equation are – 4 and 3 and the general solution is:

$$y = C_1 e^{-4x} + C_2 e^{3x}$$

ii) The auxiliary equation is $2m^2 + 3m - 1 = 0$

and roots are $m_1 = \dfrac{-3+\sqrt{17}}{4}$, $m_2 = \dfrac{-3-\sqrt{17}}{4}$

So the solution is $y = e^{-3x/4}\left[C_1 e^{\sqrt{17}x/4} + C_2 e^{-\sqrt{17}x/4}\right]$

iii) The auxiliary equation is $4m^2 + 12m + 9 = 0$

and the repeated root is $m_1 = -3/2 = m_2$

Since the roots are equal, the solution of the given ODE is

$$y = (C_1 + C_2 x)e^{-(3/2)x}$$

iv) The auxiliary equation is $m^2 - 2m + 1 = 0$

and the roots are repeated: $m_1 = 1 = m_2$

The solution of the given ODE is $y = (C_1 + C_2 x)e^x$

v) The auxiliary equation is $m^2 + 10m + 26 = 0$

The roots of the auxiliary equation are: $m_1 = -5 + i$, $m_2 = -5 - i$

From Eq. (4.28), the solution of the given ODE is $y = Ce^{-5x}\cos(x - \phi)$

vi) The auxiliary equation is $m^2 + 2m + 1 \Rightarrow (m+1)^2 = 0$

The repeated root of this equation is $m = -1$

Hence the general solution is $y(x) = (C_1 + C_2 x)\exp(-x)$

2. i) The auxiliary equation is $m^2 + 16 = 0$

and the roots of the equation are: $m_1 = 4i$, $m_2 = -4i$

Hence, from the Eq. (4.27c), the general solution of the given equation is

$$y = C_1 \cos 4x + C_2 \sin 4x \qquad \text{(i)}$$

where C_1 and C_2 are constants.

To determine these, we use the initial conditions. So,

$$y(\pi/4) = -1 \Rightarrow C_1 \cos\pi + C_2 \sin\pi = -1 \Rightarrow C_1 = 1 \qquad \text{(ii)}$$

because $\sin\pi = 0$ and $\cos\pi = -1$. Further

$$y'(\pi/4) = 4 \Rightarrow -4C_1 \sin\pi + 4C_2 \cos\pi = 4 \qquad \text{(iii)}$$

$$\Rightarrow \quad -4C_2 = 4 \Rightarrow C_2 = -1 \qquad \text{(iv)}$$

$\therefore$ The solution is: $y = \cos 4x - \sin 4x$

ii) The auxiliary equation is: $4m^2 - 4m + 1 = 0$

The roots are $m_1 = m_2 = \dfrac{1}{2}$

So the solution of the given ODE is: $y = (C_1 + C_2 x)e^{x/2}$ (i)

Using the initial condition $y(0) = 2$, we get $C_1 = 2$ (ii)

Next, we differentiate Eq. (i) to obtain

$$y'(x) = \frac{1}{2}(C_1 + C_2 x)e^{x/2} + C_2\, e^{x/2} \tag{iii}$$

Using the initial condition $y'(0) = -1$ we get

$$y'(0) = -1 \Rightarrow \frac{C_1}{2} + C_2 = -1 \Rightarrow C_2 = -2 \tag{iv}$$

So the solution of given ODE is $y = 2(1-x)e^{x/2}$

3. i) The auxiliary equation corresponding to the given ODE is $4m^2 + 1 = 0$

The roots of the equation are $m_1 = \dfrac{1}{2}i$, and $m_2 = -\dfrac{1}{2}i$

And from Eq. (4.27c), the general solution is $y(x) = C_1 \cos\dfrac{x}{2} + C_2 \sin\dfrac{x}{2}$

At $x = 0$, $y(x) = 3$ $\Rightarrow$ $3 = C_1 \cos 0 + C_2 \sin 0$ $\Rightarrow$ $C_1 = 3$

since $\sin 0 = 0$ and $\cos 0 = 1$.

At $x = \pi$, $y(x) = -2$ $\Rightarrow$ $-2 = C_1 \cos(\pi/2) + C_2 \sin(\pi/2) \Rightarrow C_2 = -2$

since $\sin \pi/2 = 1$ and $\cos \pi/2 = 0$.

The particular solution of the given equation is $y(x) = 3\cos\dfrac{x}{2} - 2\sin\dfrac{x}{2}$

ii) The auxiliary equation in this case is $m^2 - 6m + 9 = 0$, for which the roots are real and repeated:

$$m_1 = m_2 = 3$$

So the general solution of the ODE is

$$y(x) = (C_1 + C_2 x)e^{3x}$$

At $x = 0$, $y(x) = 2$

$\Rightarrow$ $C_1 = 2$

At $x = 1$, $y(1) = 0$

$\Rightarrow$ $0 = (2 + C_2)e^3 \Rightarrow C_2 = -2$

The particular solution of the given equation is $y(x) = (2 - 2x)e^{3x}$

4. The ODE can be rewritten as

$$\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{q}{LC} = 0$$

for which the auxiliary equation is

$$m^2 + \frac{R}{L}m + \frac{1}{LC} = 0$$

The roots of this equation are

$$m_{1,2} = \frac{-\frac{R}{L} \pm \sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}}{2}$$

$$= -\frac{R}{2L} \pm \frac{1}{2L}\left[R^2 - \frac{4L}{C}\right]^{1/2}$$

So from Eq. (4.17), the solution of the ODE is

$$q(t) = e^{-\frac{R}{2L}t}[C_1 e^{\alpha t} + C_2 e^{-\alpha t}]$$

where $\alpha = \frac{1}{2L}\sqrt{R^2 - \frac{4L}{C}}$.

5. We can rewrite the ODE as

$$\ddot{\theta} + \frac{k}{I}\theta = 0, \quad \frac{k}{I} = 16\,\text{s}^{-1}$$

The auxiliary equation is $m^2 + 16 = 0 \Rightarrow m_1 = 4i, \quad m_2 = -4i$

The general solution of the ODE from Eq. (4.27c) is

$$\theta(t) = C_1 \cos 4t + C_2 \sin 4t \tag{i}$$

Using the condition that $\theta = 30° = \frac{\pi}{6}$ rad at $t = 0$ in Eq. (i) we get

$$C_1 = \pi/6$$

Differentiating Eq. (i) with respect to t, we get

$$\dot{\theta}(t) = -4C_1 \sin 4t + 4C_2 \cos 4t \tag{ii}$$

Using $\dot{\theta}(t) = 10\,\text{rad s}^{-1}$ at $t = 0$ in Eq. (ii), we get

$$C_2 = \frac{5}{2}$$

On substituting for C_1 and C_2 in Eq. (i), we can write

$$\theta(t) = \frac{\pi}{6}\cos 4t + \frac{5}{2}\sin 4t$$

APPENDIX

BASIC CONCEPTS OF CALCULUS

The study of mechanics involves calculus which is not new for you but we would like to revise the basic concepts of the derivative and the integral in calculus. We hope that you have studied mathematics at the + 2 level. If you have taken the subjects Physics, Chemistry and Mathematics in B. Sc., you would be studying the course on Calculus along with this course. So you may not need to study this appendix.

In this appendix, we briefly revise the basic concepts of the **derivative** and the **integral** of a function being used in this course. Do try to master these concepts. Then, you will be able to understand the physics given in this course better.

Mathematically, a **function** is a rule or correspondence which associates to each number x in a set A, a **unique** number $y = f(x)$ in a set B.

You may understand a **function** as an equation in y and x, such that for any value of x the equation will yield exactly one value, i.e., a unique value of y.

A1.1 THE CONCEPT OF DERIVATIVE

The concept of the derivative is very useful in physics. For example, it helps us in describing motion in a simple way as you have learnt in school physics courses. In mechanics, we need to describe how the distance travelled by a body or its displacement **changes** with time. We also wish to describe how the speed/velocity of a body **changes** with time.

Suppose a quantity y changes with respect to x. We say that y is a **function** of x. Here y is termed the dependent variable and x the independent variable. For example, the distance d travelled changes with respect to time t and we say that d is a function of t. (Here we shall not go into the precise mathematical definition of the function, which is stated in the margin only for reference.)

Now suppose we want to know the rate at which a given function y changes with respect to the change in x. For example, suppose the distance travelled by a body changes with time. How do we find the speed of the body? We can find the rate of change of y with respect to x by calculating the **derivative** of y with respect to x. **How do we find the derivative of a function**?

We can approach the calculation of the derivative in two different ways: The **geometrical** way (as a **slope** of a curve) and the **physical** way (as a **rate of change**). Here we shall discuss both the ways. Our emphasis will be on the use of the derivative as a tool in physics rather than on mathematical rigour. Let us first consider the geometrical concept of the derivative.

The Geometrical Concept of the Derivative

Suppose a quantity y depends on x, and x and y are real numbers.
To begin with, let us consider a linear function $y = x$. You may like to draw its graph. It is a straight line passing through the origin (OP in Fig. A1.1). We have drawn it in a plane using a two-dimensional Cartesian coordinate system.

Let us now ask: What is the slope of this straight line?

To calculate the slope of a straight line, we mark any two points on the line and divide the difference in their *y*-values by the difference in their corresponding *x*-values.

If we plot the <u>graph</u> of *y* against *x*, **the derivative measures the <u>slope</u> of this graph at each point**. The functional relationship between y and *x* is often denoted by $y = f(x)$ where *f* denotes the function.

Do this exercise for different sets of points on the straight line in Fig. A1.1. What is the value of the slope of the line $y = x$? You will find that it is 1. We now draw another straight line *AB* on Fig. A1.1. What is its slope?

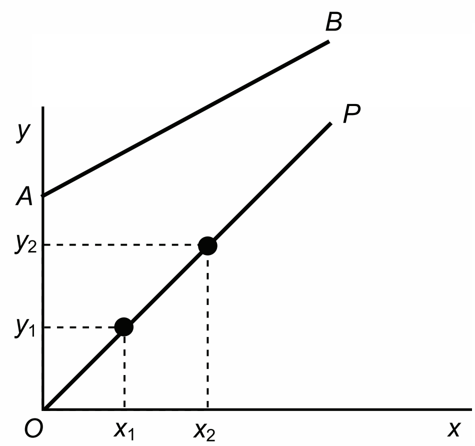


Fig. A1.1: **Slope of a straight line**.

You can calculate it by taking any 2 points (x_1, y_1) and (x_2, y_2) on the line and determining the following ratio:

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}$$

Here Δy denotes the difference $(y_2 - y_1)$ and is pronounced as "delta y". Similarly, Δx denotes the difference $(x_2 - x_1)$ and is pronounced as "delta x".

The ratio $\frac{\Delta y}{\Delta x}$ gives the slope of the straight line.

You have seen that 'slope' is a concept that can easily be applied to linear functions. It is the change in *y* divided by the change in *x*. Any two points can be used in determining the slope of a straight line, **because its slope is constant throughout**.

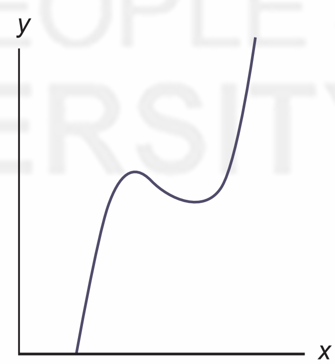


Fig. A1.2: **What is the slope of a curve at a given point**?

Let us now try to find the slope of the curve in Fig. A1.2, which is the graph of the function: $f(x) = x^3 + 4x^2 - 3$. You can see that there is no single slope for this figure. Instead, the curve has a different slope at each separate point. Therefore, for non-linear functions (functions that are not represented by straight lines), it makes sense only to talk about the slope of the curve at a particular point.

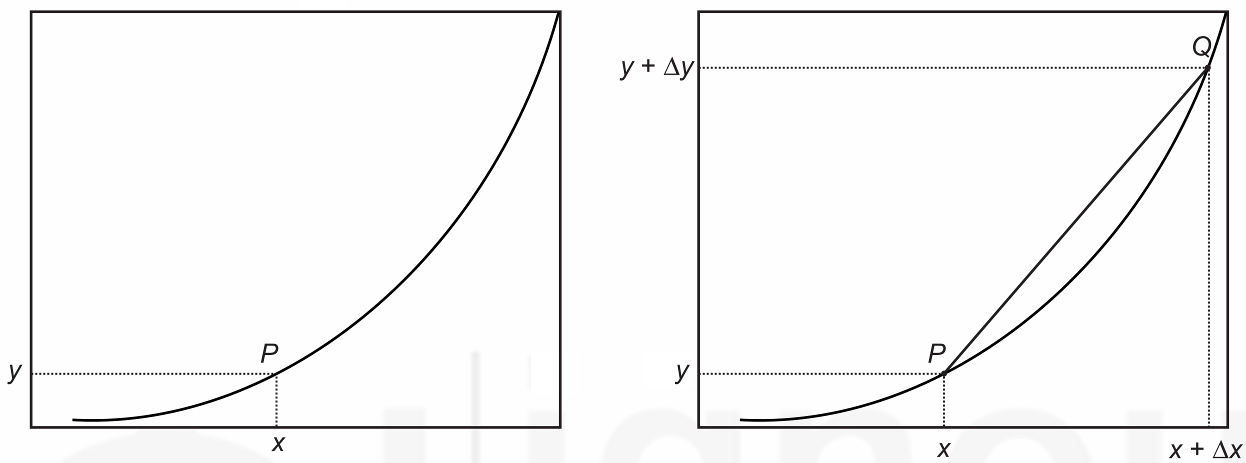
How do we find it?

Slope of a Curve at a Point

To visualize what needs to be done, let us consider some function *y* of *x* and choose an arbitrary point *P* on it (Fig. A1.3a).

We have to find the slope of the curve at this arbitrary point P. Let its coordinates be (x, y). You have learnt that to find the slope, you need to take two points on the curve and calculate $\dfrac{y_2 - y_1}{x_2 - x_1}$. Since P is the point on the graph where we want to find the slope, one of the points we should use is the point (x, y). But what should we choose as the other point? You may say that no other point would yield the right answer, since we are interested in the slope at the single point (x, y) only.

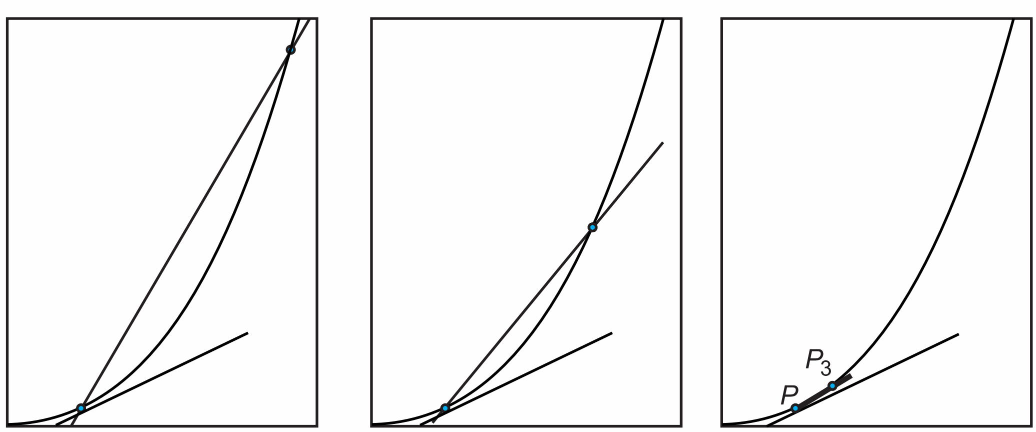
Still, let us pick an arbitrary point Q, Δx units away from point P on the x-axis. Thus, its x coordinate is $x + \Delta x$. Let its y coordinate be $y + \Delta y$ (Fig. A1.3b).



(a) (b)

Fig. A1.3: **(a) An arbitrary point** $P(x, y)$ **on a curve; (b) slope of the curve between points** $P(x, y)$ **and** $Q(x + \Delta x, y + \Delta y)$ **of the curve.**

Let us now calculate the **slope of the straight line** that goes through the points P at (x, y) and Q at $(x + \Delta x, y + \Delta y)$. It is $\Delta y/\Delta x$. You can see that it does not represent the slope of the curve at the point P. Let us now bring the point Q closer and closer to the point P. In other words, let us make Δx smaller and smaller. What happens when we do that? Let us consider the function shown in Fig. A1.4. Study Figs. A1.4a, b and c. As Δx gets smaller and smaller, note that the corresponding region of the curve becomes very nearly a straight line.



(a) (b) (c)

Fig. A1.4: **As the distance separating two points on the x-axis is made smaller, the corresponding region of the curve becomes very nearly a straight line.**

You can look at any region of the graph where Δx is very small. Since this graph is a smooth curve, it is nearly straight for any sufficiently small Δx. For example, in Fig. A1.4c, the part of the curve between the points P_3 and P is very nearly straight.

In fact, as Δx gets smaller and smaller, the slope of the straight line joining the two points (x, y) and $(x + \Delta x, y + \Delta y)$ approaches a constant value. Moreover, the line itself looks more and more like the tangent to the curve at the point P. This also means that the **slope of the line** is getting closer and closer to the **slope of the tangent**.

Thus, **if we could make Δx arbitrarily small, the slope of the line given by $\Delta y / \Delta x$ would get arbitrarily close to the slope of the tangent and approach a constant value**.

The definition of the derivative follows from the above process. The **geometrical concept of the derivative of a function** at a given point is that it is

- **the slope of the curve representing the function at the point, and**
- **the slope of the tangent line to the curve at the point.**

If the interval Δx is chosen sufficiently small so that it approaches zero, the ratio $\Delta y/\Delta x$ approaches a **constant limiting value**, which is called the **derivative** of y with respect to x. Mathematically, we express this result as

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{(\Delta y)}{(\Delta x)} \tag{A1.1}$$

We say that the derivative of y with respect to x is equal to the ratio $\frac{(\Delta y)}{(\Delta x)}$ in the limit as Δx tends to zero. The derivative itself is also a function, because for every x value that it is given, it gives a value that is equal to the slope of the tangent to the function at x.

The Physical Concept of the Derivative

This approach was used by Newton in the development of his Classical Mechanics. You have come across two kinds of quantities in mechanics: constants having fixed values (e.g., position of a fixed object) and variables (e.g., position or displacement of a moving object that changes with time). Let us consider the one-dimensional motion of a particle shown in Fig. A1.5. It shows us how the distance travelled by the particle depends on time.

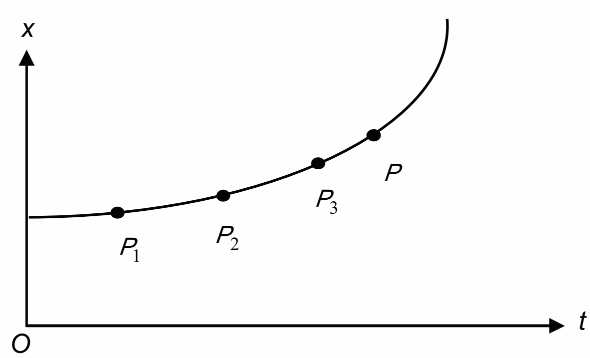


Fig. A1.5: **The path of a moving particle showing its positions at different instants of time.**

If we want to describe in greater detail *how* a particle moves, we need to specify its positions at successive instants of time at regular intervals. For example, we may choose to specify its position with respect to a coordinate system at regular time intervals of 1s. We can show the sequence of positions at these time intervals in a diagram by points. Each successive point indicates the positions of the particle at these successive instants (Fig. A1.5). Let P be the position of the particle at the instant t. Let its positions at the instants t_1, t_2 and t_3 be P_1, P_2 and P_3, respectively.

We now ask the question: How fast does the particle move? For this, we would need to determine the speed (average or instantaneous) of the particle. How will we do it? Recall the definition of average speed of a particle from your school physics

$$\textbf{\textit{AVERAGE SPEED}} = \frac{\textbf{\textit{TOTAL DISTANCE TRAVELLED}}}{\textbf{\textit{TOTAL TIME TAKEN}}}$$

What is the average speed of the particle as it moves from P_1 to P between the instants t_1 and t? From Fig. A1.6a, you can see that it is

$$v_{av} = \frac{(x - x_1)}{(t - t_1)} = \frac{\Delta x}{\Delta t} \tag{A1.2a}$$

Here Δt is pronounced as "delta t" and Δx is pronounced as "delta x". Now let us reduce the time interval and ask: What is the average speed of the particle as it moves from P_2 to P between the instants t_2 and t? From Fig. A1.6b, you can see that it is

$$v_{av} = \frac{(x - x_2)}{(t - t_2)} \tag{A1.2b}$$

As we make the time interval Δt smaller and smaller, we notice that the corresponding region of the curve is very nearly a straight line. For example, in Fig. A1.6b, the part of the curve between the points P_3 and P is very nearly straight. Correspondingly, the average speed (which is the ratio $\Delta x / \Delta t$) is constant for this straight portion of the curve.

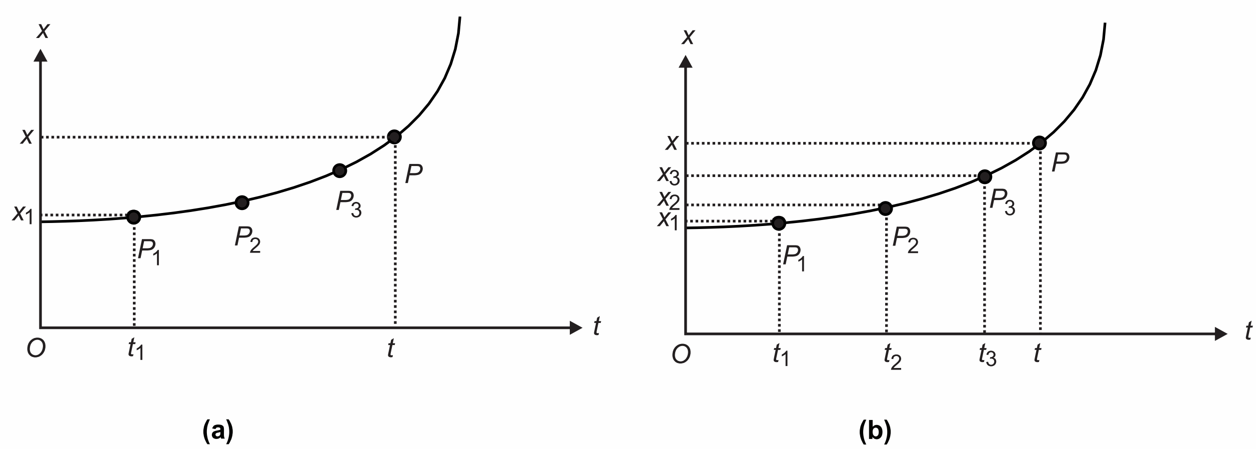


Fig. A1.6: **The path of a moving particle showing its positions at different instants of time.**

If we choose the time interval Δt to be sufficiently small so that it approaches zero, the average speed approaches a **constant limiting value**, which is called the **instantaneous speed** at time t. Mathematically, we express this as

$$v_{inst} = \lim_{\Delta t \to 0} \frac{(\Delta x)}{(\Delta t)} = \frac{dx}{dt} \tag{A1.3}$$

We say that the instantaneous speed v_{inst} is equal to the ratio $\frac{(\Delta x)}{(\Delta t)}$ in the limit as Δt tends to zero. It is also called the derivative of distance with respect to time. It gives us the rate of change of distance with time. You have to understand this limiting process very well. It is important for you to understand **the geometrical and physical meaning of the derivative**.

MEANING OF THE DERIVATIVE

Physically, the derivative gives the rate of change of one quantity (the dependent variable, e.g., distance) in terms of the other (independent variable, e.g., time). Geometrically, it is the slope of the curve (say $x(t)$), at that point (at t). It is the slope of the tangent to the curve at the point.

In the expression $\frac{dx}{dt}$, dx and dt are not products of d and x or of d and t, as in algebra. The combined expression $\frac{dx}{dt}$ is called a *derivative*. It expresses the rate of change of x with respect to t and is equal to the slope of the curve of $x\ (t)$ at the point t.

You should also understand the term **'differentials': dx and dt are called the differentials of x and t, respectively.** These expressions represent **incremental changes; dx represents an incremental change in distance x, and dt represents an incremental change in time t.** The ideas of **differentials** and **derivatives** are frequently used when we study the physics of systems that change. These are called **dynamic systems.** You may like to remember the differences in the meaning of various terms that we use in physics. These are given in the box below:

TERMINOLOGY FOR REPRESENTING CHANGE IN A DYNAMIC SYSTEM

- **When expressions are written using deltas, they represent changes, e.g., Δx is the *change* in distance.**
- **dx represents the differential of x, i.e., an *infinitesimal* (a very small) *change* in distance.**
- **$\frac{dx}{x}$ is the *fractional change* in distance, and**
- **$\frac{dx}{dt}$ is the *rate of change* of distance with respect to time.**

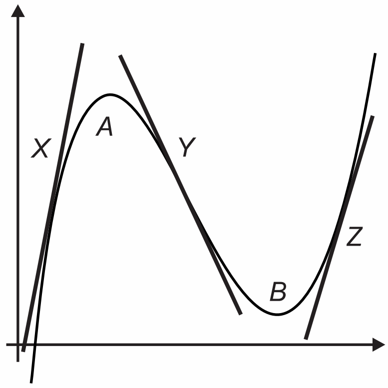


Fig. A1.7: **The derivative is positive to the left of point *A* and to the right of point *B*. It is negative between points *A* and *B* of the function and zero at the points *A* and *B*. The function shown in the graph has a local maximum at *A* and a local minimum at *B*.**

You can also visualise the following statements from what you have studied so far (see Fig. A1.7):

- A **positive** derivative (a positive slope) at a point means that the function is increasing at that point.
- A **negative** derivative (a negative slope) at a point means that the function is decreasing at that point.

Sometimes the derivative is **zero**. This means that the function has some special behaviour at the given point. It may have a local maximum or a local minimum. We now discuss the concept of integral of a function.

A1.2 THE CONCEPT OF AN INTEGRAL

You can understand the need for integration through the following example: Suppose we want to fill a tank with water. There can be two ways of doing this. We can pour buckets of water one after the other or open a tap into the tank. The second method ensures a continuous supply of water until the tank is full. If the rate of flow of water filling the tank is constant and equal to v_1 ccs^{-1} and it takes t_1 s to fill the tank, the volume *V* of the tank is given by

$$V = (v_1 t_1)\text{cc} = \text{Flow rate} \times \text{time taken} \qquad \text{(A1.4)}$$

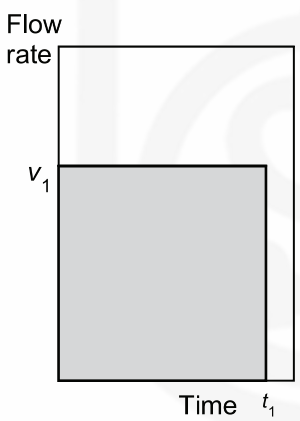


Fig. A1.8: **Integral as area under a curve.**

Suppose the flow of water is constant. Let us plot the graph of flow rate against the time taken (Fig. A1.8). Then *V* can be represented as the area of a rectangle of breadth v_1 and length t_1. This is actually equal to the area under a graph of flow rate versus time. However, if the flow is non-uniform then a simple product like this cannot be used. What do we do in that case?

We divide the time of flow into a very large number of very short intervals of duration Δt each (during which the rate of flow is nearly the same). Then, we find the volume of water flowing in each interval (flow rate $v(t) \times \Delta t$). Next, we sum up the volumes for all the small intervals (Fig. A1.9).

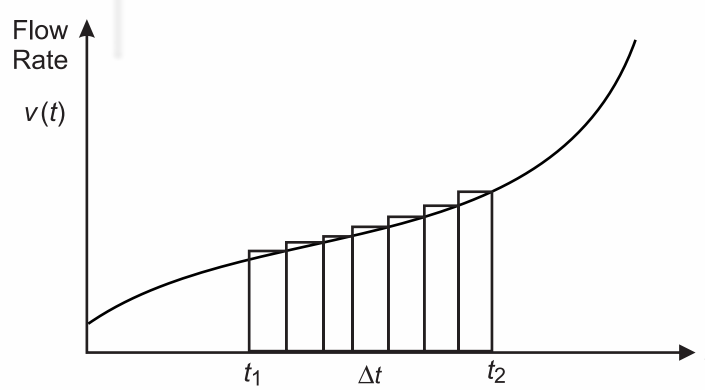


Fig. A1.9: **Integral as area under the curve. The volume of water flowing in the area under the curve is given by** $\lim\limits_{\Delta t \to 0} \sum v(t)\Delta t$ **and represented by** $\int v(t)\,dt$.

Let us now make the time interval Δt smaller and smaller so that it approaches a very small value close to zero. The curve for each small interval Δt

resembles a straight line. Then the area under the curve for each Δt is given by the area of the corresponding rectangle.

We can then write the **area under the curve** as the sum of the areas of all such rectangles for which Δt is very small or in the limit as Δt goes to zero:

$$\lim_{\Delta t \to 0} \sum v(t)\Delta t$$

We represent this sum as an integral of the function $v(t)$ and denote it as follows:

$$\lim_{\Delta t \to 0} \sum v(t)\Delta t = \int v(t)\,dt \qquad \text{(A1.5)}$$

You may find it interesting to note that the symbol $\int$ is an elongated *S* denoting the sum.

So, you have learnt that **integration is also a limiting process**: **We sum up the areas of a very large number of rectangles with one side (given by the independent variable) of these rectangles being extremely small and tending to zero.**

Shri Rajsekhar Basu, a famous literary figure from West Bengal was also a mathematician. He used to call the symbol of integration $\int$, the trunk of an elephant.

Geometrically, the integral represents the area under the graph of the function. You can appreciate this point better if you do an actual calculation. For some function of a variable *x* or *t*, you should calculate the sum for several sets of values of Δx or Δt, making the interval smaller and smaller for each set. You will find that the sum converges to some value, which is the area under the curve. The simplest calculation you could do is for a straight line passing through the origin $(y = x)$. From Fig. A1.10, you can see that the area is $\frac{x^2}{2}$.

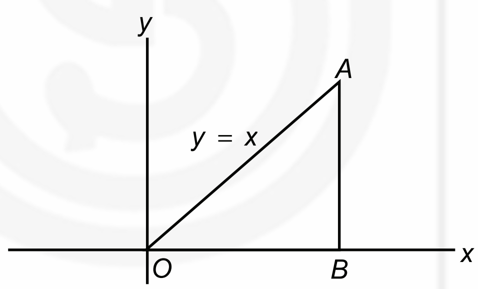


Fig. A1.10: **Integral as area under the curve for a straight line** $y = x$:

$$\int_0^x x\,dx = \frac{1}{2}x^2.$$

We can also understand the concept of an **integral as an anti-derivative**:

If $F(x) = f'(x)$ is the derivative of a function $f(x)$ then $f(x)$ is termed the anti-derivative or the integral of the function $F(x)$.

Thus, if $F(x) = \frac{d}{dx}[f(x)]$ then $f(x)$ is called the integral of $F(x)$ with respect to *x*. It is denoted by

$$f(x) = \int F(x)\,dx + c \qquad \text{(A1.6)}$$

where c is a constant. You should understand that if we add c in this equation, we would still have the same derivative $F(x)$ since the derivative of a constant is zero. Its value is determined from certain given conditions. You will understand these concepts further when you study their applications in mechanics. You will need to determine the derivatives and many indefinite and definite integrals of various functions while solving problems in physics. We have given the values of the derivatives and integrals of some common functions in Tables A1.1 and A1.2.

Table A1.1: Derivatives of simple functions

S. No.	df/dx	S. No.	df/dx
1.	$\frac{d}{dx}(c) = 0$, c constant	10.	$\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2},\ x \neq 0$
2.	$\frac{d}{dx}(x) = 1$	11.	$\frac{d}{dx}(x^{-n}) = -\frac{n}{x^{n+1}}$
3.	$\frac{d}{dx}(x^n) = nx^{n-1}$	12.	$\frac{d}{dx}[g(x) + h(x)] = \left[\frac{d}{dx}g(x)\right] + \left[\frac{d}{dx}h(x)\right]$
4.	$\frac{d}{dx}(\sin x) = \cos x$	13.	$\frac{d}{dx}[f(x)\,g(x)] = \left[\frac{df}{dx}\right]g + f\left[\frac{dg}{dx}\right]$
5.	$\frac{d}{dx}(\cos x) = -\sin x$	14.	$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g\frac{df}{dx} - f\frac{dg}{dx}}{g^2},\ g \neq 0$
6.	$\frac{d}{dx}(\tan x) = \sec^2 x$	15.	$\frac{d}{dx}(e^x) = e^x$
7.	$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}}$	16.	$\frac{d}{dx}\ln x = \frac{1}{x}$
8.	$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1 - x^2}}$	17.	$\frac{d}{dx}(c^x) = c^x \ln c,\ c > 0$
9.	$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1 + x^2}$	18.	$\frac{d}{dx}\log_c x = \frac{1}{x \ln c}\quad c \neq 1,\ c > 0$

Table A1.2: Integrals of simple functions

S. No.	Integral	S. No.	Integral
1.	$\int a\,dx = ax + c$, a and c constant	5.	$\int \sin x\,dx = -\cos x + c$, c constant
2.	$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$, c constant	6.	$\int \cos x\,dx = \sin x + c$, c constant
3.	$\int \frac{1}{x}\,dx = \ln\lvert x\rvert + c$, c constant	7.	$\int \tan x\,dx = \ln\lvert\sec x\rvert + c$, c constant
4.	$\int e^x\,dx = e^x + c$, c constant	8.	$\int e^{ax}\,dx = \frac{e^{ax}}{a} + c$, a and c constants

TABLE OF PHYSICAL CONSTANTS

Symbol	Quantity	Value
c	Speed of light in vacuum	$3.00 \times 10^{8}\ \text{ms}^{-1}$
μ_0	Permeability of free space	$1.26 \times 10^{-6}\ \text{NA}^{-2}$
ε_0	Permittivity of free space	$8.85 \times 10^{-12}\ \text{C}^2\,\text{N}^{-1}\,\text{m}^{-2}$
$1/4\pi\varepsilon_0$		$8.99 \times 10^{9}\ \text{Nm}^2\,\text{C}^{-2}$
e	Charge of the proton	$1.60 \times 10^{-19}\ \text{C}$
$-e$	Charge of the electron	$-1.60 \times 10^{-19}\ \text{C}$
h	Planck's constant	$6.63 \times 10^{-34}\ \text{Js}$
$\hbar$	$h / 2\pi$	$1.05 \times 10^{-34}\ \text{Js}$
m_e	Electron rest mass	$9.11 \times 10^{-31}\ \text{kg}$
$-e/m_e$	Electron charge to mass ratio	$-1.76 \times 10^{11}\ \text{Ckg}^{-1}$
m_p	Proton rest mass	$1.67 \times 10^{-27}\ \text{kg}$ (1 amu)
m_n	Neutron rest mass	$1.68 \times 10^{-27}\ \text{kg}$
a_0	Bohr radius	$5.29 \times 10^{-11}\ \text{m}$
N_A	Avogadro constant	$6.02 \times 10^{23}\ \text{mol}^{-1}$
R	Universal gas constant	$8.31\ \text{Jmol}^{-1}\text{K}^{-1}$
k_B	Boltzmann constant	$1.38 \times 10^{-23}\ \text{J K}^{-1}$
G	Universal gravitational constant	$6.67 \times 10^{-11}\ \text{Nm}^2\,\text{kg}^{-2}$

Astrophysical Data

Celestial Body	Mass (kg)	Mean radius (m)	Mean distance from the centre of Earth (m)
Sun	1.99×10^{30}	6.96×10^{8}	1.50×10^{11}
Moon	7.35×10^{22}	1.74×10^{6}	3.84×10^{8}
Earth	5.97×10^{24}	6.37×10^{6}	0

LIST OF BLOCKS AND UNITS: BPHCT-131

SYLLABUS: MECHANICS (BPHCT-131) 4 Credits

Vector Algebra: Geometrical and algebraic representation of vectors, Vector algebra; Scalar and vector products; Derivatives of a vector with respect to a scalar.

First Order Ordinary Differential Equations: First order homogeneous differential equations (separable and linear first order differential equations).

Second Order Ordinary Differential Equations: 2^{nd} order homogeneous differential equations with constant coefficients.

Laws of Motion: Frames of reference; Newton's Laws of motion; Straight line motion; Motion in a plane; Uniform circular motion; 3-d motion.

Applications of Newton's Law of Motion: Friction; Tension; Gravitation; Spring-mass system – Hooke's law; Satellite in circular orbit and applications; Geosynchronous orbits; Basic idea of global positioning system (GPS); Weight and Weightlessness.

Linear Momentum and Impulse: Conservation of momentum; Impulse; impulse-momentum Theorem; Motion of rockets.

Work and Energy: Work and energy; Conservation of energy; Head-on and 2-d collisions.

Kinematics of Angular Motion: Kinematics of angular motion: Angular displacement, velocity and angular acceleration; General angular motion.

Dynamics of Rotational Motion: Torque; Rotational inertia; Kinetic energy of rotation; Angular momentum; Conservation of angular momentum and its applications.

Motion under Central Force Field: Motion of a particle in a central force field (motion in a plane, conservation of angular momentum constancy of areal velocity); Kepler's Laws (statement only).

Dynamics of Many Particle Systems: Dynamics of a system of particles. Centre of Mass, determination of the centre of mass of discrete mass distributions, centre of mass of a rigid body (qualitative).

Conservation Laws: Linear momentum, angular momentum and energy conservation for many-particle systems.

Simple Harmonic Motion: Simple Harmonic Motion; Differential equation of SHM and its solutions; Kinetic Energy, Potential Energy, and Total Energy of SHM and their time averages.

Superposition of Harmonic Oscillations: Linearity and Superposition Principle; Superposition of Collinear Oscillations having equal frequencies and having different frequencies (beats); Superposition of Orthogonal Oscillations with equal and unequal frequency; Lissajous Figures and their uses.

Damped Oscillations: Equation of Motion of Damped Oscillations and its solution (without derivation); Qualitative description of the solution for Heavy, Critical and Weak Damping; Characterising Damped Oscillations – Logarithmic Decrement, Relaxation Time and Quality Factor.

Wave Motion: Qualitative Description (Wave formation and Propagation; Describing Wave Motion, Frequency, Wavelength and Velocity of Wave; Mathematical Description of Wave Motion).

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