Indira Gandhi National
Open University
School of Sciences

BPHCT-131
MECHANICS

Block

3

ROTATIONAL MOTION AND MANY-PARTICLE SYSTEMS

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CONTENTS

BLOCK 3: ROTATIONAL MOTION AND MANY-PARTICLE SYSTEMS

In Block 2 of this course, you have studied the basic concepts of **kinematics** and **dynamics** that include **Newton's laws of motion** and the concepts **of force**, **linear momentum**, **impulse**, **gravitation, work** and **energy**. You have studied the laws of conservation of linear momentum and energy, which makes it easier for us to study complex mechanical phenomena. The discussion in Block 2 is about the translational motion of objects, in which the velocity of all points on an object remains the same. In this block, you will study **angular/rotational motion** of objects in which the motion is about some axis. There are many examples of such motion around us. The motion of wheels, ceiling fans and clock hands, as also the motion of planets around the Sun are all examples of angular/ rotational motion. In this block, you will study the related concepts of **torque** and **angular momentum**, the law of **conservation of angular momentum** and **applications of this law**.

In your studies so far, you have represented each object, be it a ball, a car or even the Moon as a single particle. However, there are many situations in which we study the motion of systems of many particles. For example, the solar system comprising the Sun, the planets and their satellites, comets and asteroids is an example of many-particle system. So is a cylinder containing gas or a rigid body. In this block, we will discuss how the concepts of motion that you have studied so far and the laws of **conservation of linear momentum**, **energy** and **angular momentum** can be applied to many simple and complex physical situations involving the motion of two-particle and many-particle systems. An important application of these laws in two particle systems is in collisions, which scientists study to understand the properties of atomic and sub-atomic particles. This block has 5 units.

Unit 11 deals with the kinematics of **angular motion** with a special focus on circular motion. You will learn about angular displacement, angular velocity and angular acceleration and how they relate to the corresponding variables in translational motion. Angular kinematic variables are convenient for the description of different types of angular motion, for example, the motion of planets around the Sun which you will study in Unit 13. In **Unit 12**, we discuss the concepts related to the dynamics of rotational motion. We focus on the dynamics of uniform and non-uniform circular motion. You will study the concept of **torque**, which explains why a particle moves the way it does in **angular/rotational motion**. We also discuss **angular momentum** and the law of **conservation of angular momentum.** This law can be used to explain, for example, why ice skaters can increase their speed while skating by drawing in their arms or why a collapsing star acquires a very high rotational speed.

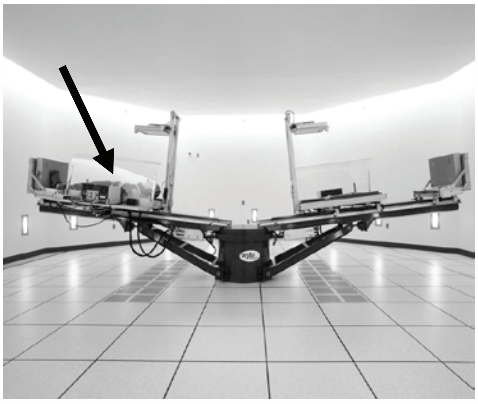
In **Unit 13**, you will study about the motion of objects under special kind of forces called **central forces**, in particular **central conservative forces** with a special focus on the inverse square force of gravitation. Motion under central conservative forces has some special properties, which will help you to understand qualitatively the motion of planets, their satellites and comets in the solar system. You will also see how Kepler's laws of planetary motion, which you have also studied in school physics, are a simple consequence of Newton's laws of motion and gravitation.

In **Unit 14**, we consider the **dynamics of two-particle** and **many-particle systems**. When we apply Newton's second law of motion to each particle in this system, it becomes a complex mathematical problem. However, the dynamics of these systems are simplified by introducing the concept of **centre of mass**. In this unit, you will learn how to describe the motion of two-particle systems in terms of the centre of mass and relative coordinates and extend the concept to many-particle systems.

In **Unit 15**, you will learn how to express the linear momentum, angular momentum and energy of two-particle and many-particle systems. Then we present the conservation laws of energy, linear momentum and angular momentum for these systems. In particular, you will apply the laws of conservation of linear momentum and energy to study collisions of two particles.

The concepts presented in this block may be new to you and the discussion at some places may be rather mathematical in nature. You may need to study them more than once to master them. Once again, please do work through all the examples and problems given in each unit on your own.

We hope you enjoy studying the block and once again wish you success.



What is the acceleration of astronauts in the centrifuge used for training them? You will find the answer to this question in this unit!

UNIT 11

KINEMATICS OF ANGULAR MOTION

Structure

STUDY GUIDE

In this unit, you will study about the **kinematics of angular motion** and we mainly discuss uniform and non-uniform circular motion. You will revise the concepts of **angular position, angular displacement**, **angular velocity** and **angular acceleration**, which you have learnt in school physics. You will notice that these concepts are similar to the kinematical concepts used for describing translational motion. That is why we shall be using the terms "**analogy**" or "**analogous to**" when we discuss them in the unit.

You may wonder: Why do we need to study these concepts? This is because just as every particle on an object in pure translation has the same velocity, **every particle** (with the exception of the particles on the axis of rotation) **on an object in pure rotation has the same angular velocity**. Therefore, describing rotational motion in terms of these angular variables is far easier. You will understand this point when you relate the two sets of kinematical variables. Secs. 11.3 and 11.4 may be new for you and you may need more time to study them. We shall be using the concepts related to **vectors** and suggest that you revise them before studying this unit.

"It is wrong to think that the task of physics is to find out how Nature is. Physics concerns what we say about nature."	***Neils Bohr***

11.1 INTRODUCTION

So far in this course, we have discussed **translational** motion of objects, that is, motion in which the **velocity of all points on the object remains the same**. We now consider **angular or rotational motion of an object or a particle**, in which it moves **about some axis**. The motion of the Moon in its orbit around the Earth, the motion of planets orbiting the Sun, the motion of the Earth about its axis are examples of angular/rotational motion. Circular motion is the simplest example of angular motion and is very common in the world around us. Any particle on an object rotating about its axis executes circular motion. Fig. 11.1 shows many objects, which rotate (turn) about some axis as they move. We come across such rotating objects everywhere around us.

Fig. 11.1: **Objects around us that rotate about an axis. Any particle situated on them executes circular motion as they rotate about the axis.**

You can see that some of the commonest rotating objects around us include wheels (of bicycles, buses, cars and trains, and potter's wheel), ceiling fans, and clock hands. You are opening and shutting doors and books all the time and this involves rotating the doors or books about their axes. And of course we live on the Earth, which rotates around its axis and orbits the Sun. Rotation of objects is involved in many games and fun activities. For example, you may have taken joyrides on giant wheels, merry-go-rounds or the see-saw, ridden bicycles, or played with marbles and spinning tops. It is also involved in many other daily activities such as opening nuts, bolts or juice cartons by pulling the tabs. Why don't you list some more such rotating objects around you?

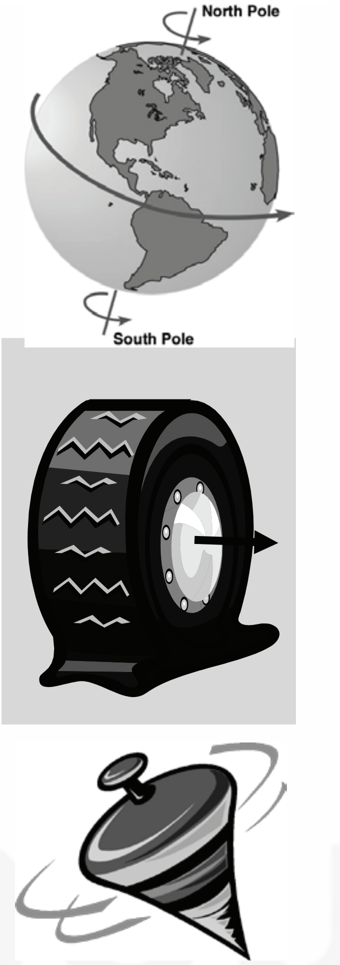


Fig. 11.2: **When an object rotates about an axis passing through it, all points on the object move in a circle. The centres of the circles lie on a straight line that is called the axis of rotation.**

If you consider the motion of just one particle on any of these rotating objects, for example, on the tread of a tyre, the spinning top or the rotating Earth, what do you find? You can see that it follows a circular path (see Fig. 11.2). You have studied about uniform circular motion in Unit 6 and 7. In this unit, we further explore the **kinematics** of angular motion in Sec. 11.2. You will revisit the concepts of **angular position**, **angular displacement**, **angular speed** and **angular acceleration**, the kinematical variables used to describe angular motion. In Sec. 11.3, we briefly explain the vector nature of the angular quantities. Then in Sec. 11.4, we relate the kinematical variables for translational motion, namely, displacement, velocity and acceleration, to those for angular motion, viz. angular displacement, angular velocity and angular acceleration. We mainly apply these concepts to describe **uniform and non-uniform circular motion**, the simplest examples of angular motion, in terms of angular kinematical variables.

In the next unit, we discuss the **dynamics of angular/rotational motion** and introduce the concepts of torque, rotational inertia and angular momentum, which play the same role in angular motion as force, mass and linear momentum in translational motion.

Expected Learning Outcomes

After studying this unit, you should be able to:

- ❖ determine the angular position, angular velocity and angular acceleration of a particle executing uniform and non-uniform circular motion;
- ❖ relate the angular displacement, angular velocity and angular acceleration of a particle in angular motion to its displacement, velocity and acceleration; and
- ❖ solve problems related to the kinematics of uniform and non-uniform circular motion.

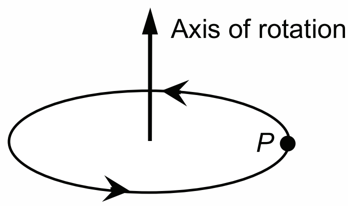


Fig. 11.3: **A particle** ***P*** **moving in a circle in the counter-clockwise direction about a fixed axis. This axis is known as the axis of rotation.**

11.2 KINEMATICAL VARIABLES

As is usual in physics, we begin by considering the simplest form of angular motion, namely, **circular motion**. Suppose a particle moves in a circle of radius r about a fixed axis passing through the centre of the circle and perpendicular to the plane of the circle (Fig. 11.3).

The fixed axis about which the particle moves in the circle is called the **axis of rotation**. You have learnt in Unit 6 that we need only a two-dimensional coordinate system to describe the motion of such a particle.

Let us begin by defining the **kinematical variables** for describing angular motion, namely, **angular position**, **angular displacement**, **angular speed** and **angular acceleration** just as we define displacement, speed and acceleration for translational motion.

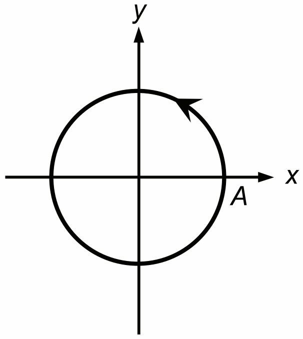


Fig. 11.4: **Particle in circular motion as seen from a point above the plane of the circle.**

11.2.1 Angular Position

Let us choose the *xy* coordinate system to describe the motion of the particle of Fig. 11.3 and attach it to the plane of the circle in which the particle moves. Let the particle move **counter-clockwise** in the circle. Fig. 11.4 shows the motion of the particle when we observe it from some point above the plane of the circle. Let the particle start moving counter-clockwise at the instant $t = 0$ from a point *A* on the positive *x*-axis, which we take as the **reference axis**. Suppose it is at point *P* at any instant *t* (Fig. 11.5). Let $\vec{\mathbf{r}}$ be the **position vector** of the particle with respect to the centre of the circle *O*.

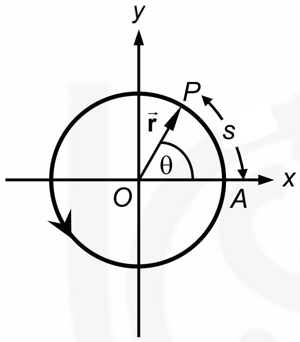


Fig. 11.5: **The angular position θ at an instant *t* of the particle of Fig. 11.4.**

Let us now define its angular position (Fig. 11.5). We measure the angular position of the particle with respect to the reference *x*-axis. We take the position of the particle at point *A* on the reference axis to be the **zero angular position**. Then the **angle** θ defines the **angular position** of the particle at any instant *t* with respect to the reference axis (the positive direction of the *x*-axis). Note that θ is the angle between the position vector of the particle and the reference axis. In the time interval *t*, the particle travels along the circular arc *AP*. The arc extends from the zero angular position to the angular position of the particle at the instant *t*. Let us denote the arc's length *AP* by *s*.

Then we define the **angular position** for circular motion as follows:

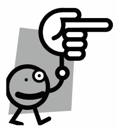
ANGULAR POSITION FOR CIRCULAR MOTION

The **angular position** of a particle in circular motion at any instant of time *t* with respect to the positive *x*-axis passing through the centre of the circle (Fig. 11.5) is given by the **angle** θ. It is the angle that its position vector (with respect to the centre of the circle) makes at that instant with the axis. It is given by

Angular position

$$\theta = \frac{s}{r} \qquad \text{(θ is measured in radians)} \qquad (11.1)$$

where *s* is the length of the circular arc travelled by the particle in time *t* and *r,* the radius of the circle.



Don't forget

The angle θ defined by Eq. (11.1) is measured only in radians and not in degrees or revolutions. Radian is written in short as rad. Always convert θ into radians.

The radian is a ratio of two lengths and hence, it is a pure number (that is, it is dimensionless). Since the circumference of a circle of radius r is $2\pi r$, for one revolution in the circle, Eq. (11.1) gives us:

$$1 \text{ revolution} = \frac{2\pi r}{r} = 2\pi \text{ rad} = 360^\circ \tag{11.2a}$$

So, **there are 2π radians in one complete revolution of a circle**.

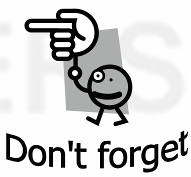
$$\therefore \; 1 \text{ rad} = \frac{360^\circ}{2\pi} = \frac{1}{2\pi} \text{ revolution} \tag{11.2b}$$

Relation between degrees and radians

Moreover, if the particle moves more than once in the circle, **we do not reset the value** of θ **to zero** every time it crosses the reference axis. It takes the increased value every time.

For example, if the particle completes 2 revolutions from the zero reference position, its angular position is given by $(2 \times 2\pi)$ rad $= 4\pi$ rad. If it completes 3 revolutions from the zero reference position, its angular position is given by $(3 \times 2\pi)$ rad $= 6\pi$ rad, and so on. If it is at an angle θ with the reference line after completing 2 revolutions, its angular position is given by $(2 \times 2\pi + \theta)$ rad $= (4\pi + \theta)$ rad. What is its angular position if it is at an angle ϕ after completing 4 revolutions around the circle? It is $(4 \times 2\pi + \phi)$ rad $= (8\pi + \phi)$ rad. Thus, the angular position of a particle at an angle θ after completing n revolutions around the circle is given by $(2\pi n + \theta)$ rad.

Just as the position x of a particle can take positive and negative values, its angular position θ can also be positive or negative (Fig. 11.6). Since the particle can move in a circle in only two ways: **clockwise** or **counter-clockwise**, we adopt the following sign convention for θ:



By convention, θ is positive for counter-clockwise circular motion and negative for clockwise motion.

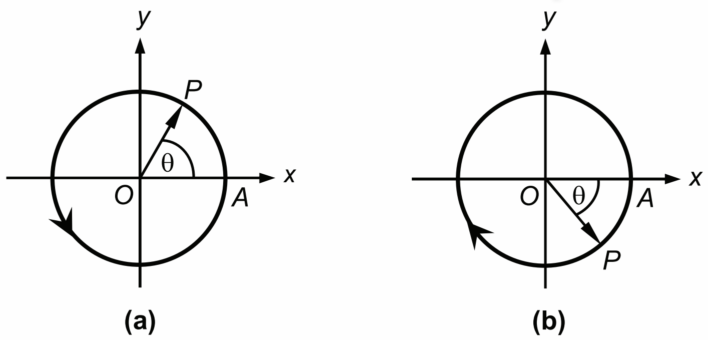


Fig. 11.6: **The angular position θ is a) positive for the particle moving counter-clockwise in the circle; b) negative for the particle moving clockwise in the circle.**

You can memorise the phrase "**clocks are negative**" to remember this sign convention.

Recall from school physics that if you know $x(t)$, that is, the particle's position as a function of time, you can determine its velocity, acceleration and know all there is to know about its motion.

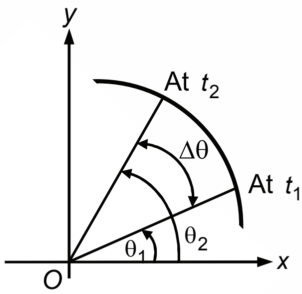


Fig. 11.7: The angular displacement of a particle moving counter-clockwise in a circle. Here we show only the arc of the circle.

In the same way, if we know $\theta(t)$, that is, the particle's angular position as a function of time, we can know all there is to know about its angular motion.

We can determine its **angular displacement**, and hence **angular velocity** and **angular acceleration**.

11.2.2 Angular Displacement

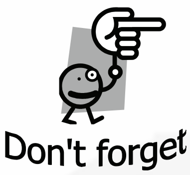
Consider the particle shown in Fig. 11.5 or Fig. 11.6a. Note that it is moving counter-clockwise in the circle. Its angular positions at the instants t_1 and t_2 are given by θ_1 and θ_2 as shown in Fig. 11.7.

Then its **angular displacement** is given by

Angular displacement

$$\Delta\theta = \theta_2 - \theta_1 \qquad (\Delta\theta \text{ is measured in radians}) \tag{11.3}$$

The sign convention for angular displacement is the same as that for angular position.



Angular displacement is positive for counter-clockwise circular motion, and negative for clockwise circular motion. It is measured in radians.

Let us now consider an example for determining angular displacement.

EXAMPLE 11.1: ANGULAR DISPLACEMENT

A point at the edge of a rotating compact disc (CD) of radius 6.05 cm moves from position *A* to position *B* in the time interval Δt (Fig. 11.8a). What is its angular displacement in this time interval? It is given that the arc lengths of these two positions with respect to the horizontal reference line are 2.00 cm and 6.00 cm, respectively.

SOLUTION ■ The **KEY IDEA** here is to use Eq. (11.1) to determine the angular positions in radians and then use Eq. (11.3) to determine the angular displacement.

Let us look at the rotating CD from above and draw the coordinate axes as shown in Fig. 11.8b. From Eq. (11.1), we get

$$\theta_1 = \frac{s}{r} = \frac{AP}{OP} = \frac{2.00\text{ cm}}{6.05\text{ cm}} = 0.331\text{ rad},$$

$$\theta_2 = \frac{BP}{OP} = \frac{6.00\text{ cm}}{6.05\text{ cm}} = 0.992\text{ rad}$$

Hence, $\Delta\theta = \theta_2 - \theta_1 = (0.992 - 0.331)\text{ rad} = 0.661\text{ rad}$. It is positive because the point is moving in the counter-clockwise direction.

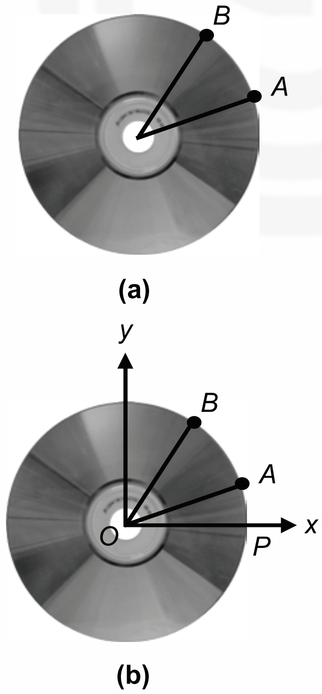


Fig. 11.8: **Angular displacement of a point on a rotating CD.**

SAQ 1 – Angular displacement

Consider a point on any of the rotating objects in Fig. 11.1. Determine the angular displacement for the following angular positions of the point:

θ_1 rad	+ 5	+ 7	– 5	– 3	– 5
θ_2 rad	+ 12	– 2	+ 2	– 4	– 1

Let us now define the angular velocity and angular acceleration of a particle in circular motion.

11.2.3 Angular Velocity and Angular Acceleration

Consider the particle motion shown in Fig. 11.7 repeated here as Fig. 11.9 for ready reference. Its angular positions at the instants t_1 and t_2 are given by θ_1 and θ_2, respectively. We first define the **average angular speed** ω_{avg} of the particle in the time interval Δt from t_1 to t_2 as follows:

ω is a Greek symbol, which is pronounced as omega.

$$\text{Average angular speed } \omega_{avg} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t} \tag{11.4}$$

Average angular speed

The SI unit of angular speed is radian per second and is written as rad s^{-1}. The other units used for angular speed are revolutions per second (rev s^{-1}) and revolutions per minute (rev/min or rpm). You may have seen the angular speed of old musical records given in units of rpm, for example, 45 rpm, 78 rpm, etc. We now define the **instantaneous angular speed**, which is termed angular speed. We denote it by the Greek symbol ω. Note that it is the **magnitude of the instantaneous angular velocity**. We shall define **the direction of the angular velocity** in the next section.

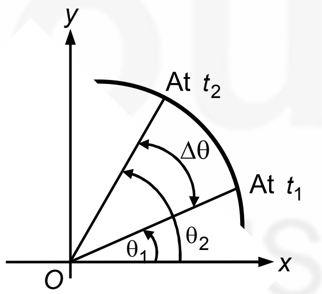


Fig. 11.9: The angular displacement of a particle moving counter-clockwise in a circle. Here we show only the arc of the circle.

ANGULAR SPEED

The **magnitude of the instantaneous angular velocity** ω (**called the angular speed**) is the limit of the ratio in Eq. (11.4) as Δt tends to zero:

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt} \tag{11.5}$$

It is the rate of change of θ with time.

Angular speed

NOTE

$\because$ 1 revolution = 2π rad,

$\therefore$ 1 rpm $= \dfrac{(2\pi)\text{rad}}{60\,\text{s}}$

$1\,\text{rad s}^{-1} = \dfrac{60}{2\pi}\text{rpm}$

Thus, if we know $\theta(t)$, that is the angular position θ as a function of time, we can obtain the angular speed by differentiating it with respect to t. Henceforth, we shall refer to the instantaneous angular velocity as angular velocity. The **angular velocity of an object can be positive or negative**:

- **Particle moves counter-clockwise:** **Angular velocity is positive.**
- **Particle moves clockwise:** **Angular velocity is negative.**

If the angular velocity of a particle changes with time, then the particle has a non-zero **angular acceleration** (just like an object that has a changing velocity has non-zero acceleration). Suppose the angular speeds of the particle in Fig. 11.9 at the instants t_1 and t_2 are given by ω_1 and ω_2, respectively. We define the **magnitude of the average angular acceleration** of the particle in the time interval Δt from t_1 to t_2 as follows:

α is a Greek symbol, which is pronounced as alpha.

$$\text{Average angular acceleration } \alpha_{avg} = \frac{\omega_2 - \omega_1}{t_2 - t_1} = \frac{\Delta\omega}{\Delta t} \qquad (11.6)$$

The SI unit of angular acceleration is rad s^{-2}.

ANGULAR ACCELERATION

Angular acceleration

The **magnitude** of **instantaneous angular acceleration** α **(called the angular acceleration)** is the limit of the ratio in Eq. (11.6) as Δt tends to zero:

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt} \qquad (11.7)$$

It is the rate of change of ω with time.

NOTE

Notice from Eqs. (11.8a to c) and the corresponding equations for straight line motion that the angular position θ is *analogous* to position *x*, the angular velocity ω is *analogous* to velocity *v* and the angular acceleration α is *analogous* to acceleration *a*. Thus, angular motion with constant angular acceleration is completely *analogous* to straight-line motion with constant acceleration.

Thus, if we know $\omega(t)$, that is, ω as a function of time, we can obtain the angular acceleration by differentiating it with respect to *t*. Recall the equations of kinematics for particles moving with constant acceleration from school physics. In the same way, we can write the equations of kinematics for angular motion with constant angular acceleration as follows:

EQUATIONS OF KINEMATICS

Angular motion $(\alpha = \text{constant})$		**Linear motion** $(a = \text{constant})$
$\omega = \omega_0 + \alpha t$	(11.8a)	$v = v_0 + at$
$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$	(11.8b)	$x = x_0 + v_0 t + \frac{1}{2}at^2$
$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$	(11.8c)	$v^2 = v_0^2 + 2a(x - x_0)$

Let us now apply Eqs. (11.8a to c) to determine the angular position, angular speed and angular acceleration of particles.

EXAMPLE 11.2: AVERAGE ANGULAR SPEED

a) A grinding stone completes 50 clockwise revolutions in 10 s. Determine the average angular speed of a particle on it.

SOLUTION ■ The **KEY IDEA** here is to use Eq. (11.4) to determine the angular speed along with Eq. (11.2a) since $\Delta\theta$ is to be in radians.

For a particle on the grinding stone

$$\Delta\theta = (-50 \text{ revolutions})\, 2\pi \text{ rad} = -314 \text{ rad} \approx -3.1\times 10^{2} \text{ rad}$$

The minus sign here indicates that the grindstone and the particle on it is rotating clockwise. The average angular speed of the particle is:

$$\omega_{avg} = \frac{\Delta\theta}{\Delta t} = \frac{-314\,\text{rad}}{10\,\text{s}} = -31.4 \text{ rads}^{-1} \approx -31\,\text{rads}^{-1}$$

b) The blades of a helicopter rotate clockwise with an average angular speed of -50 rads^{-1} (Fig. 11.10). Consider a particle on the tip of a blade. What is its angular position after 10 s if its initial angular position is -12 rad?

SOLUTION ■ The **KEY IDEA** here is to use Eq. (11.8b) to determine the angular displacement. $\alpha = 0$, since ω_0 is constant and $t = 10$ s:

$$\theta = \theta_0 + \omega_0 t = -12\text{rad} - (50 \text{ rads}^{-1} \times 10\,\text{s})$$
$$= -512 \text{ rad} \approx -5.1\times 10^{2}\,\text{rad}$$

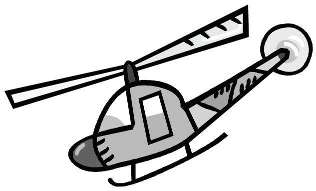


Fig. 11.10: **Average angular speed of a particle on the tip of a rotating helicopter blade.**

EXAMPLE 11.3: CONSTANT ANGULAR ACCELERATION

A bicycle wheel is rotating counter-clockwise with a constant angular acceleration of 2.0 rads^{-2}. The angular speed of a particle on the wheel at the instant $t = 0$ is 5.0 rads^{-1}.

a) What will its angular speed be at $t = 5.0$ s?

b) Suppose a spoke of the wheel is horizontal at the instant $t = 0$. What is the angular position of a particle on the spoke at $t = 5.0$ s? How many revolutions has it completed during this time interval?

c) What is the angle that the spoke makes with the horizontal at $t = 5.0$ s?

SOLUTION ■ The **KEY IDEA** here is that angular acceleration is constant and we can use the equations of kinematics (Eqs. 11.8 a to c) for angular motion. We also make use of the fact that **all particles on the rotating wheel move with the same angular speed**.

a) The angular speed of the particle on the wheel at $t = 5.0$ s is given by Eq. (11.8a): $\omega = \omega_0 + \alpha t$

$$\therefore\ \omega = 5.0\text{rads}^{-1} + (2.0 \text{ rads}^{-2})\times(5.0\,\text{s}) = 15 \text{ rads}^{-1}$$

b) The angular position of the particle on the spoke of the wheel at $t = 5.0\,\text{s}$ is given by Eq. (11.8b). Here $\theta_0 = 0$.

$$\theta = (5.0\text{rads}^{-1})\times(5.0\,\text{s}) + \frac{1}{2}(2.0 \text{ rads}^{-2})\times(5.0\,\text{s})^2 = 50 \text{ rad}$$

The number of revolutions can be determined from Eq. (11.2b):

$$\theta = 50 \times \frac{1}{2\pi} \text{ rev} = 7.95\text{rev} \approx 8.0 \text{ rev}$$

NOTE

Note that the **angular speed will be the same for all particles** situated on a rotating object (except for those on the axis of rotation). This is the advantage of using angular variables to describe angular motion.

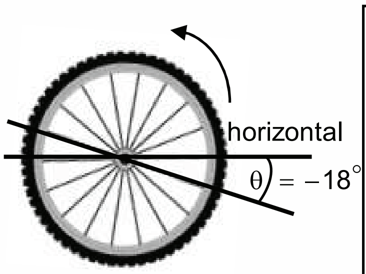


Fig. 11.11: **Angle made by the spoke of the wheel with the horizontal is** $342°$ **or** $-18°$.

c) The particle as well as the spoke turns through 7 complete revolutions in 5.0s plus an additional 0.95 rev, which in radians is equal to

$$0.95\,\text{rev} = (0.95\text{ rev}) \times (2\pi\,\text{rad}.\text{rev}^{-1}) = 5.97\,\text{rad}$$

and in degrees is equal to

$$0.95\,\text{rev} = (0.95\text{ rev}) \times (360^\circ) = 342^\circ$$

The spoke turns by an angle of 342° in the last revolution. Hence, the angle that the spoke makes with the horizontal is $342°$ or also $-18°$. You can understand this point from Fig. 11.11.

You may like to practice using the equations of kinematics for angular motion before studying further. Try the following SAQ.

SAQ 2 – Angular speed and angular acceleration

a) A disc of radius 1.0 m rotates by 50° about an axis passing through its centre and perpendicular to its plane. What is the distance travelled by its centre?

b) The average angular acceleration of a stone lodged in a rotating car wheel is $150\ \text{rad}\,\text{s}^{-2}$. What will the stone's final angular speed be after 2.0 s, if it starts from rest?

In our discussion so far, you must have noted that we have referred to angular **displacement**, angular **velocity** and angular **acceleration**. You would also have noted that we took angular displacement as positive if the particle moved counter-clockwise in circular motion and negative if it moved clockwise. In this sense, we have defined the direction of angular displacement.

You may like to know: **Can we also define a direction for angular velocity and angular acceleration**? This is what you will learn in the next section.

11.3 DIRECTION OF ANGULAR VELOCITY AND ANGULAR ACCELERATION

NOTE

In this course, we shall mainly consider **angular or rotational motion about a fixed axis. For such cases, we do not need to take into account the vector nature of angular quantities.**

You must have noted that we have associated **angular quantities** with particles or objects **rotating** about some axis. Therefore, we define their directions in a different way from those of displacement, velocity and acceleration.

You have seen that **rotations about a fixed axis** seen along the axis are either **clockwise** or **counter-clockwise**.

We follow the **right-hand rule** to determine which of these is associated with positive direction of angular velocity and which one with the negative direction.

Let us define the direction of angular velocity using the **right-hand rule**.

DIRECTION OF ANGULAR VELOCITY

The **right-hand rule** for determining the direction of angular velocity: **Curl the fingers of your right-hand around the axis of rotation** so that the fingers point in the direction of rotation. Your **extended thumb points** along the axis **in the direction of the angular velocity vector** (Fig. 11.12). Angular velocity is **positive** for **counter-clockwise** angular motion and **negative** for **clockwise** angular motion.

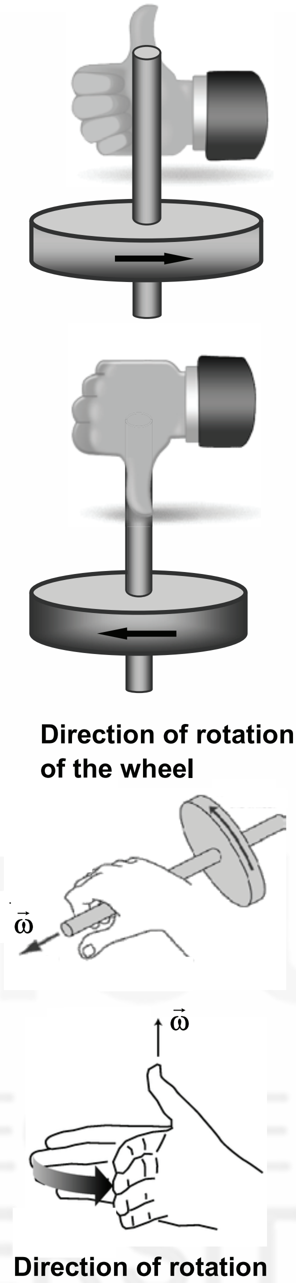


Fig. 11.12: **The direction of the angular velocity vector $\vec{\omega}$ of a rotating object is given by the right-hand rule**.

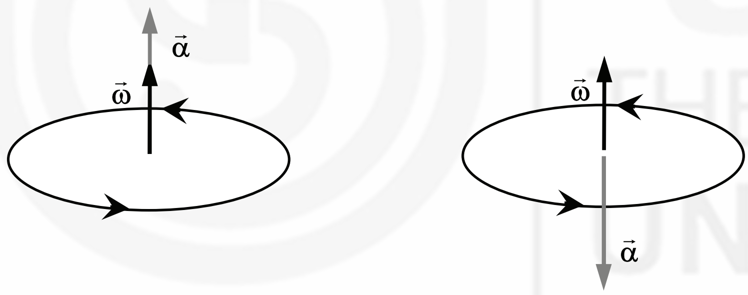
How do we determine the direction of angular acceleration? Since angular acceleration is directed along the change in angular velocity, this is how we do it.

DIRECTION OF ANGULAR ACCELERATION

For rotation about a fixed axis, the direction of **angular acceleration** is along the axis of rotation and the **same as that of the <u>change</u> in the angular velocity** (Fig. 11.13).

Thus, if the magnitude of the angular velocity is increasing, the angular acceleration vector points along the angular velocity vector.

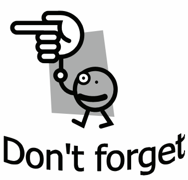
If the magnitude of the angular velocity is decreasing, the angular acceleration vector points opposite to the direction of the angular velocity vector.



(a) (b)

Fig. 11.13: The direction of the angular acceleration vector $\vec{\alpha}$ of a rotating object is a) along the angular velocity vector if the angular speed increases; b) opposite to the angular velocity vector if angular speed decreases.

No part of the rotating object moves in the direction of the angular velocity vector. In translational motion, this is not the case as we expect motion to be <u>along</u> the direction of a vector.



In Unit 6, you have studied uniform circular motion and learnt about the centripetal acceleration that a particle experiences when it travels with a constant speed v in a circle of radius r. In this unit, you have learnt about angular variables for a particle moving in a circle of radius r. You may like to know: **What is the relation between the variables** v, a **for translational motion and angular variables** ω, α?

11.4 RELATING VARIABLES FOR ANGULAR AND TRANSLATIONAL MOTION

You may get an intuitive feel for the relation by trying the following activity or have someone do it for you:

Activity

VARIABLES IN ANGULAR AND TRANSLATIONAL MOTION

Spread out your arms and spin around like a dancer about a vertical axis passing through your body. Or else, you could put a bicycle on its stand as shown in Fig. 11.14 and rotate the bicycle wheel. Can you find out which point on your body or the bicycle wheel moves the fastest? Is it a point nearer the axis of rotation (for example, a point on your shoulder or a point near the wheel's axis)? Or is it a point farther away from the axis of rotation (for example, a point on your fingertip or a point on the wheel's rim)?

Fig. 11.14: **In a rotating body, the velocity of points farther away from the axis of rotation is greater than the points nearer the axis.**

You can get the answer by determining v for the two particles as follows:

Let us consider two particles P_1 and P_2 on the rotating wheel at distances r_1 and r_2, respectively such that P_1 is closer to the axis of rotation ($r_1 < r_2$) as shown in Fig. 11.15. Note that the farther away a point is from the axis of rotation, the larger is the radius of the circle in which it is moving and hence the larger is the circumference $(2\pi r)$ of the circle. Since both particles (P_1 and P_2) are moving in the circle with the **same angular speed**, the **time taken** by them **to complete one revolution is the same**. This means that both particles travel different distances ($2\pi r_1$ and $2\pi r_2$) in the same time. Now you can answer the question: Which particle (P_1 or P_2) will have the greater speed? The answer is that particle P_2 will have greater speed v_2. This is because it travels greater distance than P_1 in the same amount of time T (the time taken to complete one revolution around the circle). Hence,

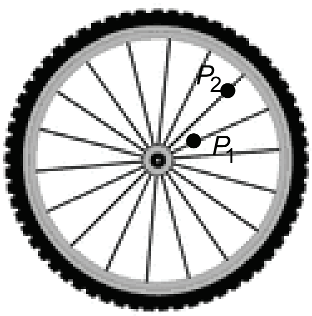


Fig. 11.15: **The velocity of point P_2 at a distance r_2 from the axis of rotation is greater than that at the point P_1 at a distance r_1 since $r_2 > r_1$.**

$$v_2 = \frac{2\pi r_2}{T} \quad > \quad v_1 = \frac{2\pi r_1}{T} \qquad (\text{for } r_2 > r_1)$$

Thus, the farther a particle is from the axis of rotation, the greater is its speed/velocity although its angular speed or angular velocity is equal to that of other particles.

How do we express this result in general terms?

11.4.1 Relating ω and v

Consider a particle moving in a circle of radius r. Recall Eq. (11.1), which relates the angle of rotation (θ) of a particle with respect to a reference axis with the length of the corresponding circular arc (s) and the radius r. Using Eq. (11.1), we can write:

$$s = r\theta \qquad (11.9a)$$

Since s is the **distance travelled by the particle** along the arc, Eq. (11.9a) gives the relation between angular displacement and distance travelled. **Remember** that the angle θ *is measured in radians*. If we differentiate Eq. (11.9a) with respect to time and note that r is constant for a circle, we get

$$v = \frac{ds}{dt} = r\frac{d\theta}{dt} = r\omega \tag{11.9b}$$

Here ω is measured in $\text{rad}\,\text{s}^{-1}$. **For motion in a circle**, the velocity of the particle is directed along the **tangent to the circle**. Hence, we also refer to v for circular motion as **tangential speed** and denote it by v_t. At this stage, we would like you to recall Eq. (6.11) from Unit 6, which relates the speed v of the particle with its time period T. We repeat it here.

$$v = \frac{2\pi r}{T} \quad \text{or} \quad T = \frac{2\pi r}{v} \tag{11.9c}$$

Substituting v from Eq. (11.9b) in Eq. (11.9c), we get an important result relating the time period and angular speed:

$$\boxed{T = \frac{2\pi}{\omega}} \tag{11.10}$$

Time period

Now that we have related θ and s, and ω and v, you may like to know: **What is the relation between α and a for non-uniform circular motion**?

11.4.2 Relating α and a for Non-uniform Circular Motion

You might say that we can use Eq. (6.9c) for $\vec{\mathbf{a}}$ and substitute for v from Eq. (11.9b) in it to obtain the relation that we want. What do we get? We get

$$\vec{\mathbf{a}}_r = -\frac{v^2}{r}\,\hat{\mathbf{r}} = -\omega^2 r\,\hat{\mathbf{r}} \quad \text{and} \quad a_r = \frac{v^2}{r} = \omega^2 r \tag{11.11}$$

Centripetal acceleration

Note that the direction of $\vec{\mathbf{a}}_r$ is **towards the centre of the circle** at all instants of time, which is why it is called the **centripetal acceleration**.

But as you can see, this equation does not contain α in it. We, therefore, go back to the first principles and see what result we get. There is a finer point in relating the two accelerations, which you will understand by the end of this section.

Note that we are considering **<u>non-uniform circular motion</u>** (that is, the **angular speed of the particle is changing with time** and it has a finite angular acceleration).

Since ω changes with time, the speed v given by Eq. (11.9b) also changes with time. Let the particle's speed be v_1 at the instant of time t_1 and v_2 at a later instant of time t_2.

Then the average acceleration of the particle in the time interval $\Delta t = t_2 - t_1$ is given by

$$a_{av} = \frac{v_2 - v_1}{\Delta t} = \frac{r\omega_2 - r\omega_1}{\Delta t} = \frac{r\Delta\omega}{\Delta t} \quad (11.12)$$

Note that both v_2 and v_1 are **speeds**, that is, the **magnitudes** of the velocity and a_{av} above represents the average acceleration or average change in speeds at the respective instants of time. In the limit as $\Delta t \to 0$, Eq. (11.12) gives the **instantaneous acceleration** directed along the tangent. Denoting its magnitude by a_t, we can write

$$a_t = \lim_{\Delta t \to 0} \frac{v_2 - v_1}{\Delta t} = \lim_{\Delta t \to 0} \frac{r\Delta\omega}{\Delta t}$$

Tangential acceleration

or

$$a_t = \frac{dv}{dt} = r\frac{d\omega}{dt} = \alpha r \quad (11.13)$$

Note that the acceleration with magnitude a_t given by Eq. (11.13) is different from the centripetal acceleration given by Eq. (11.11), because it is **directed along the tangent to the circle towards the change in the magnitude of velocity** (v).

We call this as <u>**tangential acceleration**</u> or the ***tangential component of acceleration***. **This arises due to <u>change in the magnitude</u> of the velocity** and it is therefore **zero for uniform circular motion**. We denote it by $\vec{\mathbf{a}}_t$ and the centripetal acceleration by $\vec{\mathbf{a}}_r$. We can summarise this result as follows:

Recap

ACCELERATION IN NON-UNIFORM CIRCULAR MOTION

The acceleration of a particle in **non-uniform circular motion** has two components: **radial** and **tangential** (see Fig. 11.16).

The radial component arises due to the change in the <u>direction</u> of velocity. It is **directed along the radius towards the centre of the circle** and its **magnitude** is given by

$$a_r = \frac{v^2}{r} = \omega^2 r \quad (11.14a)$$

The tangential component arises due to the change in the <u>magnitude</u> of velocity. It is directed **along the tangent to the circle** and points towards the **change** (increasing or decreasing) **in speed**. Its magnitude is given by

$$a_t = \alpha r \quad (11.14b)$$

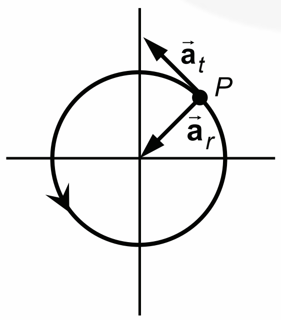


Fig. 11.16: **Acceleration of a particle** P **in non-uniform circular motion has two components**: **radial component** $\vec{a}_r$ **and tangential component** $\vec{a}_t$.

Let us put all the results relating the angular and translational variables together for **non-uniform circular motion**.

NON-UNIFORM CIRCULAR MOTION

In this case *r* **is constant but** ω **is changing**, **that is**, α **is non-zero**:

$$s = r\theta \tag{11.15a}$$

$$v = r\omega \tag{11.15b}$$

The velocity vector is along the tangent to the circle pointing towards the direction in which the particle moves in the circle (Fig. 11.17a). The acceleration of the particle has two components (Fig. 11.17b):

i) The **radial or centripetal acceleration component directed along the radius towards the centre of the circle** with **magnitude** given by

$$a_r = \frac{v^2}{r} = \omega^2 r \tag{11.15c}$$

ii) The **tangential acceleration component directed along the tangent to the circle** pointing towards the **change** (increasing or decreasing) **in speed** with magnitude given by

$$a_t = \alpha r \tag{11.15d}$$

The **resultant acceleration** of a particle in circular motion will be the **vector sum** of the radial and tangential accelerations and we express the **acceleration for non-uniform circular motion** as follows:

$$\vec{\mathbf{a}} = \vec{\mathbf{a}}_r + \vec{\mathbf{a}}_t = -\omega^2 r\,\hat{\mathbf{r}} + \alpha r\,\hat{\boldsymbol{\theta}} \tag{11.16a}$$

where $$\vec{\mathbf{a}}_r = -\omega^2 r\,\hat{\mathbf{r}} \tag{11.16b}$$

and $$\vec{\mathbf{a}}_t = \alpha r\,\hat{\boldsymbol{\theta}} \tag{11.16c}$$

Here

$\hat{\mathbf{r}}$: the **unit vector in the radial direction pointing away** from the centre of the circle (see Fig. 11.18a).

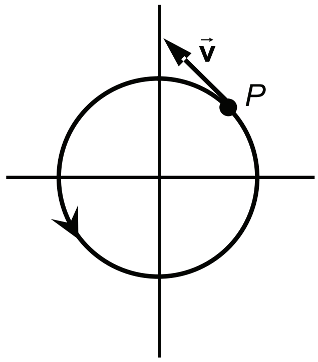
$\hat{\boldsymbol{\theta}}$: the **unit vector perpendicular to** $\hat{\mathbf{r}}$ **along the tangent in the direction** shown in Fig. 11.18a in the direction of increasing θ.

Also $$a = \sqrt{a_r^2 + a_t^2} = \sqrt{(\omega^2 r)^2 + (\alpha r)^2} \tag{11.16d}$$

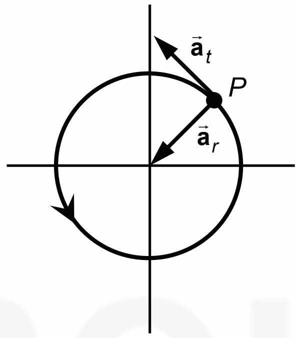
$$\tan\beta = \frac{a_t}{a_r} = -\frac{\alpha}{\omega^2} \tag{11.16e}$$

Here β is the angle that the acceleration vector $\vec{\mathbf{a}}$ makes with the radial direction.

Recap



(a)



(b)

Fig. 11.17: **a) The direction of the velocity vector; b) the radial and tangential components of the acceleration in non-uniform circular motion**.

NOTE

The **negative signs** in Eqs. (11.16a and 11.16b) indicate that the **radial** (or **centripetal**) component of the **acceleration is directed opposite to the unit vector** $\hat{\mathbf{r}}$. This is because of the way we have defined the direction of this unit vector.

The tangential component is also called the **transverse component**.

NOTE

For general angular motion, r is not a constant in $\vec{\mathbf{r}} = r\hat{\mathbf{r}}$.

Using the results (see margin remark on page 77, Unit 13):

$$\frac{d\hat{\mathbf{r}}}{dt} = \dot{\theta}(-\sin\theta\ \hat{\mathbf{i}} + \cos\theta\ \hat{\mathbf{j}})$$

and

$$\frac{d\hat{\boldsymbol{\theta}}}{dt} = -\dot{\theta}(\cos\ \theta\ \hat{\mathbf{i}} + \sin\theta\ \hat{\mathbf{j}})$$

we can write the expression for the velocity $\vec{\mathbf{v}}$ and acceleration $\vec{\mathbf{a}}$ as,

$$\vec{\mathbf{v}} = \dot{r}\,\hat{\mathbf{r}} + r\dot{\theta}\,\hat{\boldsymbol{\theta}} = \vec{\mathbf{v}}_r + \vec{\mathbf{v}}_t$$

$$\vec{\mathbf{a}} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}} = \vec{\mathbf{a}}_r + \vec{\mathbf{a}}_t$$

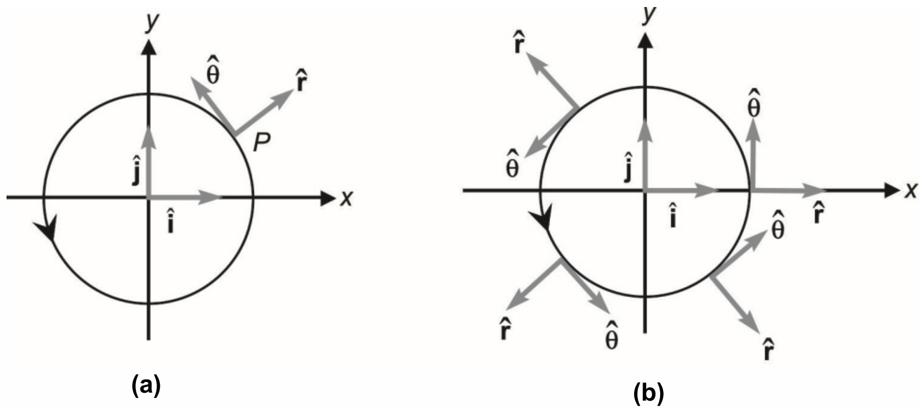


Fig. 11.18: **a) The unit vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ at a point *P*; b) Note that unlike the unit vectors $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ along the *x* and *y*-axes, the directions of unit vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ change continuously for angular motion.**

Note that the vector $\hat{\mathbf{r}}$ is a unit vector in the radial direction, that is, it is directed along the line joining the particle to the centre of the circle at all instants of time. Moreover, it points away from the centre of the circle.

Note also that the unit vector $\hat{\boldsymbol{\theta}}$ is perpendicular to $\hat{\mathbf{r}}$ at all instants of time pointing in the direction shown in Fig. 11.18a and b. As the particle moves in the circle, the radial direction changes and so do the directions of these unit vectors as shown in Fig. 11.18b. In this sense, **$\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ are not constant vectors like the unit vectors $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$** directed along the positive *x* and *y*-axes, respectively.

We also give these results for uniform circular motion for ready reference.

Recap

UNIFORM CIRCULAR MOTION

In this case r and ω are **constant**:

$$s = r\theta \tag{11.17a}$$

$$v = r\omega \tag{11.17b}$$

and

$$a_r = \frac{v^2}{r} = \omega^2 r \tag{11.17c}$$

The velocity vector is along the tangent to the circle pointing towards the direction in which the particle moves in the circle (see Fig. 11.19). The acceleration vector is **directed along the radius towards the centre of the circle**.

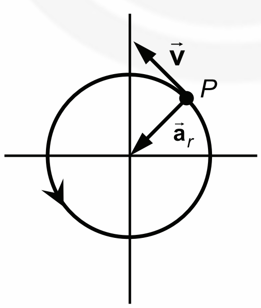


Fig. 11.19: **The velocity and centripetal acceleration vectors of a particle *P* in uniform circular motion.**

Let us now solve an example to show you how to use these relations for circular motion.

EXAMPLE 11.4: RELATING VARIABLES

Astronauts are put in a rotating circular chamber to be trained in how to withstand high accelerations. In one such chamber, they travel in a circle of radius 10 m (Fig. 11.20).

a) What is the acceleration experienced by an astronaut in the chamber in terms of g if it rotates with a constant angular speed of 14 rpm?

b) Suppose the chamber starts from rest and rotates with constant angular acceleration to reach the angular speed of 14 rpm in t = 2 minutes. What is the acceleration experienced by the astronaut in terms of g in this time interval? Take $g = 9.8\text{ms}^{-2}$.

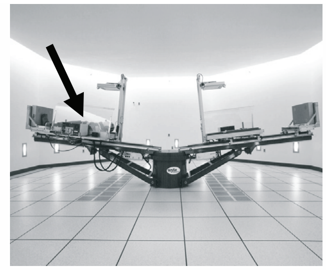


Fig. 11.20: **Acceleration in circular motion. The centrifuge used by NASA for astronaut training (the black arrow points to an astronaut strapped to the seat).**

SOLUTION ■ The **KEY IDEA** in part (a) is that the angular speed is constant and hence, the angular acceleration is zero. Therefore, the acceleration has only the radial component. In part (b), the angular speed is changing and hence, the angular acceleration is non-zero. The acceleration has both radial and tangential components.

a) Using 1 rev $= 2\pi$ rad and Eq. (11.15c), we get

$$a_r = \omega^2 r = (1.46\text{ rad s}^{-1})^2 \times 10\text{m} = 21.3\text{ ms}^{-2} = 2.2g$$

b) From Eq. (11.8a), the constant angular acceleration at $t = 120\text{s}$ is

$$\alpha = \frac{(\omega - \omega_0)\text{ rad s}^{-1}}{t\text{ s}} = \frac{(1.46 - 0)\text{ rad s}^{-1}}{120\text{s}} = 0.01\text{ rad s}^{-2}$$

At $t = 120\text{s}$, $\quad a_t = \alpha r = (0.01 \times 10)\text{ms}^{-2} = 0.10\text{ms}^{-2} = 0.01g$

And since $\omega = 14\text{rpm} = 1.46\text{ rad s}^{-1}$, $\quad a_r = 21.3\text{ms}^{-2} = 2.2g$

From Eq. (11.16d), the net acceleration at $t = 120\text{s}$ is

$$a = \sqrt{(2.2)^2 + (0.01)^2}\, g = 2.2g$$

From Eq. (11.16e), $\quad \tan\beta = \dfrac{a_t}{a_r} = \dfrac{0.01}{2.2} = 0.0045 \quad \Rightarrow \quad \beta \approx 0.26^\circ$

Therefore, the net acceleration on the astronauts points almost along the radial direction towards the centre of the chamber and has magnitude $2.2g$.

SAQ 3 – Variables in angular and translational motion

The average angular acceleration of a stone lodged in a rotating car wheel is 150 rad s^{-2}. What will the stone's speed be after 2.0 s, if it starts from rest? What is its acceleration at this instant of time? It is given that the distance of the stone from the axis of rotation is 0.10 m.

We now summarise what you have learnt in this unit.

11.5 SUMMARY

Concept	Description
Angular position	■ The **angular position** of a particle in circular motion about a fixed axis at any instant of time t is given by the **angle** θ that its position vector (with respect to the centre of the circle) makes at that instant with a fixed reference axis passing through the centre of the circle: $$\theta = \frac{s}{r} \qquad (\theta \text{ is measured in radians})$$ Here s is the length of the arc travelled by the particle and r is the radius of the circle. By convention, θ is taken to be positive for counter-clockwise motion and negative for clockwise motion in a circle. (θ is measured only in radians, not in degrees or revolutions.) The radian measure of the angle is related to the angle measure in revolutions and degrees by $$1 \text{ revolution} = 2\pi \text{ rad} = 360^\circ$$ $$\text{and} \quad 1 \text{ rad} = \left(\frac{360}{2\pi}\right)^\circ = \left(\frac{1}{2\pi}\right) \text{revolution}$$
Angular displacement	■ The **angular displacement** of a particle which undergoes change in its angular position from θ_1 to θ_2 is given by: $$\Delta\theta = \theta_2 - \theta_1 \qquad (\Delta\theta \text{ is measured in radians})$$ where by convention, $\Delta\theta$ is taken to be positive for counter-clockwise motion and negative for clockwise motion in a circle.
Average angular velocity and angular speed	■ If a particle undergoes angular displacement $\Delta\theta$ in time Δt, its **average angular velocity is a vector** with its **direction** defined by the **right-hand rule** and magnitude, that is, **angular speed**, given by $$\omega_{avg} = \frac{\Delta\theta}{\Delta t}$$ The **direction** of **average angular velocity** is **positive** for **counter-clockwise rotation** and **negative** for **clockwise rotation**.
Instantaneous angular velocity	■ The magnitude of the **instantaneous angular velocity** (generally **called the angular velocity**) of the particle is given by: $$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$ The **direction** of the angular velocity is defined by the **right-hand rule**. It is positive for counter-clockwise rotation and negative for clockwise rotation.

Average angular acceleration

- If the angular speed of a particle changes from ω_1 to ω_2 in the time interval $\Delta t = t_2 - t_1$, its **average angular acceleration** is given by

$$\alpha_{avg} = \frac{\omega_2 - \omega_1}{t_2 - t_1} = \frac{\Delta\omega}{\Delta t}$$

Instantaneous angular acceleration

- The **instantaneous angular acceleration** of a particle is equal to the rate of change of its angular velocity:

$$\alpha = \frac{d\omega}{dt}$$

Equations of kinematics for constant angular acceleration

- The kinematic equations for **constant angular acceleration** are given as follows:

$$\omega = \omega_0 + \alpha t$$

$$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$$

$$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$$

Relation between kinematical variables for angular and translational motion

- For a particle executing **non-uniform circular motion** in a circle of radius r, the relations between the kinematical variables for angular and translational motion are given as follows:

$$s = r\theta \quad (\theta \text{ in radians})$$

$$v = r\omega \ (\omega \text{ in rad}\,\text{s}^{-1})$$

The velocity of the particle is tangential to the circle pointing towards the direction in which the particle moves in the circle. The acceleration of the particle has two components: **radial or centripetal component** and **tangential component**. The **radial component** is **directed along the radius towards the centre of the circle** with **magnitude** given by

$$a_r = \frac{v^2}{r} = \omega^2 r \quad (\text{where } \omega \text{ is in units of rad}\,\text{s}^{-1})$$

The **tangential component** is **directed along the tangent to the circle** pointing towards the direction of **change** (increasing or decreasing) **in speed** with magnitude given by

$$a_t = \alpha r \quad (\alpha \text{ in rad}\,\text{s}^{-2})$$

For **uniform circular motion**, ω is constant and α is zero. The variables for angular and translational motion are related by

$$s = r\theta, \qquad v = r\omega \qquad \text{and} \qquad a_r = \frac{v^2}{r} = \omega^2 r$$

The acceleration vector

- The **acceleration vector for non-uniform circular motion** is given as follows:

$$\vec{\mathbf{a}} = \vec{\mathbf{a}}_r + \vec{\mathbf{a}}_t = -\omega^2 r\,\hat{\mathbf{r}} + \alpha r\,\hat{\boldsymbol{\theta}}$$

where $\vec{\mathbf{a}}_r = -\omega^2 r\,\hat{\mathbf{r}}$ and $\vec{\mathbf{a}}_t = \alpha r\,\hat{\boldsymbol{\theta}}$

$\hat{\mathbf{r}}$: the **unit vector in the radial direction pointing away** from the centre of the circle.

$\hat{\boldsymbol{\theta}}$: the **unit vector perpendicular to $\hat{\mathbf{r}}$ in the tangential direction** pointing towards the direction of increasing angle θ.

Also
$$a = \sqrt{a_r^2 + a_t^2} = \sqrt{(\omega^2 r)^2 + (\alpha r)^2}$$

$$\tan\beta = \frac{a_t}{a_r} = -\frac{\alpha}{\omega^2}$$

Here β is the angle that the acceleration vector $\vec{\mathbf{a}}$ makes with the radial direction.

Time period

- The **time period** or the time taken by the particle to move once in the circle is given by

$$T = \frac{2\pi r}{v} = \frac{2\pi}{\omega}$$

11.6 TERMINAL QUESTIONS

1. A bicycle travels 141 m along a circular track of radius 15 m. What is the angular displacement in radians of the bicycle from its starting position?
2. A merry-go-round makes 2 complete revolutions every 6 minutes. What is its angular speed in radian per second?
3. The initial angular speed of a fan is 12 rad s^{-1}. The power to the fan is increased so that it acquires a constant angular acceleration of 2.0 rad s^{-2}. What is the angular displacement of a point on it after 3.0 s?
4. A girl is sitting on a giant wheel that makes 1 revolution every 5 seconds. To bring the giant wheel to a stop, the operator puts on a brake that produces a constant acceleration of -1.0 rad s^{-2}.

   a) If the girl is sitting at a distance of 4.0 m from the centre, what is her speed when the wheel is turning at the rate of 1 revolution every 5 s?

   b) How long does it take before the giant wheel comes to a stop?

   c) How many revolutions does the giant wheel make by the time it comes to a stop?

   d) How far does she travel in the time that the wheel takes to stop?
5. A bicycle wheel of radius $r = 2.0$ m starts from rest and a particle on its rim moves a distance of 20 m in 20 s. Calculate

   a) the angular displacement of the particle, and

   b) its average angular velocity in the time interval of 20 s.
6. A merry-go-round starts from rest and attains an angular speed of 0.5 rpm in 2 minutes. What is its angular acceleration in rad s^{-2}?

7. A ball attached to a string starts from rest and undergoes a constant angular acceleration as it travels in a horizontal circle of radius 0.30 m. After 0.65 s, the angular speed of the ball is 9.7 rad s^{-1}. What is the tangential acceleration of the ball?

8. The Earth rotates about its axis once in approximately 24 hours and orbits the Sun once a year (or $365\frac{1}{4}$ days) in a nearly circular orbit. What is the average angular speed of a particle on the Earth's surface as it (a) rotates on its axis and (b) orbits the Sun? In each case take the direction of the Earth's rotation to be the positive direction of angular displacement.

9. A ball has an angular velocity of 5.0 rad s^{-1} counterclockwise. After the ball rotates by an angle of 4.5 rad, it has an angular velocity of 1.5 rad s^{-1} clockwise. Determine the angular acceleration of the ball and the time it takes to attain this angular velocity.

10. A grinding wheel starts from rest and attains a constant angular acceleration of 2.0 rad s^{-2}. What is the magnitude of the acceleration of a particle situated 1.0 m from the axis at $t = 2.0\text{ s}$?

11.7 SOLUTIONS AND ANSWERS

Self-Assessment Questions

1. We use Eq. (11.3) to find the angular displacement $\Delta\theta$.

θ_1 rad	+ 5	+ 7	– 5	– 3	– 5
θ_2 rad	+ 12	– 2	+ 2	– 4	– 1
$\Delta\theta = \theta_2 - \theta_1$ rad	7	– 9	7	– 1	4

2. a) The centre does not move. Distance travelled by it is zero.

   b) We use Eq. (11.8a) with $\omega_0 = 0 \text{ rad s}^{-1}$, $t = 2.0\text{ s}$ and $\alpha = 150 \text{ rad s}^{-2}$

   The final angular speed is $\omega = 150 \text{ rad s}^{-2} \times 2.0\text{ s} = 3.0\times 10^2 \text{ rad s}^{-1}$

3. In the answer to SAQ 2b, we have found the angular speed at $t = 2.0$ s to be $3.0 \times 10^2 \text{ rads}^{-1}$. Therefore, its speed after 2.0 s is

$$v = \omega r = 0.10\text{m} \times 3.0 \times 10^2 \text{ rads}^{-1} = 30 \text{ ms}^{-1}$$

The acceleration has both radial and tangential component and from Eqs. (11.15c, 11.15d and 11.16a), these are:

$$a_r = \omega^2 r = (3.0 \times 10^2 \text{ rads}^{-1})^2 \times 0.10\text{m} = 9.0 \times 10^3 \text{ms}^{-2}$$

$$a_t = \alpha r = 150 \text{ rads}^{-2} \times 0.10\text{m} = 15 \text{ms}^{-2}$$

The acceleration is

$$\vec{\mathbf{a}} = \vec{\mathbf{a}}_r + \vec{\mathbf{a}}_t = -\omega^2 r\,\hat{\mathbf{r}} + \alpha r\,\hat{\boldsymbol{\theta}}$$

$$= -(9.0 \times 10^3\ \text{ms}^{-2})\,\hat{\mathbf{r}} + (15\ \text{ms}^{-2})\,\hat{\theta}$$

Terminal Questions

1. From Eq. (11.1) the angle is $\theta = \dfrac{s}{r} = \dfrac{141\text{m}}{15\text{m}} = 9.4\,\text{rad}$

2. From Eq. (11.4), the angular speed is

$$\omega = \frac{\Delta\theta}{\Delta t} = \frac{2\times(2\pi)\text{rad}}{6\times 60\,\text{s}} = \frac{\pi}{90}\text{rad}\,\text{s}^{-1}$$

3. We use Eq. (11.8b) with $\omega_0 = 12\ \text{rad}\,\text{s}^{-1}$, $\alpha = 2.0\ \text{rad}\,\text{s}^{-1}$ and $t = 3.0\,\text{s}$ to get the angular displacement in 3.0 s as

$$\theta - \theta_0 = (12\,\text{rad}\,\text{s}^{-1})\times(3.0\ \text{s}) + \frac{1}{2}\times(2.0\,\text{rad}\,\text{s}^{-2})\times(3.0\ \text{s})^2 = 45\,\text{rad}$$

4. The initial angular speed of the girl is $\omega_0 = \dfrac{2\pi}{5}\text{rad}\,\text{s}^{-1} = 1.26\,\text{rad}\,\text{s}^{-1}$

   a) Using Eq. (11.9b) with $\omega_0 = 1.26\,\text{rad}\,\text{s}^{-1}$ and $r = 4.0\,\text{m}$, the speed of the girl is

$$v = (4.0\,\text{m})\times(1.26\,\text{rad}\,\text{s}^{-1}) = 5.0\,\text{ms}^{-1}$$

   b) Using Eq. (11.8a) with $\omega_0 = 1.26\,\text{rad}\,\text{s}^{-1}$, $\alpha = -1.0\,\text{rad}\,\text{s}^{-2}$ and $\omega = 0$ we get,

$$t = 1.26\ \text{s}$$

   c) We calculate the angular displacement using Eq. (11.8c) with $\omega_0 = 1.26\ \text{rad}\,\text{s}^{-1}$ and $\alpha = -1.0\,\text{rad}\,\text{s}^{-2}$.

$$\therefore \qquad \theta - \theta_0 = \frac{(1.26\,\text{rad}\,\text{s}^{-1})^2}{2\times(1.0\,\text{s}\,\text{rad}\,\text{s}^{-2})} = 0.79\,\text{rad}$$

   The number of revolutions in this time interval is $n = \dfrac{0.79}{2\pi} = 0.13$

   d) The distance travelled by the girl can be determined using Eq. (11.15a) with $\theta = 0.79\ \text{rad}$ and $r = 4.0\,\text{m}$ to get

$$s = (4.0\,\text{m})\times(0.79\,\text{rad}) = 3.2\,\text{m}$$

5. a) Using Eq. (11.1) with $s = 20\,\text{m}$ and $r = 2.0\,\text{m}$, the angular displacement of the particle is

$$\theta = \frac{20\,\text{m}}{2.0\,\text{m}} = 10\,\text{rad}$$

b) The particle's average angular velocity in 20 s is

$$\omega = \frac{10\,\text{rad}}{20\,\text{s}} = 0.50\,\text{rad}\,\text{s}^{-1}$$

6. The angular speed of the merry-go-round is

$$\omega = 0.5\ \text{rpm} = \frac{0.5 \times 2\pi}{60}\text{rad}\,\text{s}^{-1} = \frac{\pi}{60}\text{rad}\,\text{s}^{-1}$$

From Eq. (11.8a), its angular acceleration is

$$\alpha = \frac{\omega}{t} = \left(\frac{\pi}{60}\right) \times \left(\frac{1}{2 \times 60}\right)\text{rad}\,\text{s}^{-2} = \frac{\pi}{7200}\text{rad}\,\text{s}^{-2}$$

7. From Eq. (11.8a), the angular acceleration of the ball is

$$\alpha = \frac{\omega - \omega_0}{t} = \frac{9.7\,\text{rad}\,\text{s}^{-1}}{0.65\,\text{s}} = 14.9\,\text{rad}\,\text{s}^{-2}$$

From Eq. (11.16c), its tangential acceleration is

$$a_t = (14.9\,\text{rad s}^{-2}) \times (0.30\,\text{m}) = 4.5\ \text{ms}^{-2}$$

8. a) For the Earth's rotation on its axis, $\Delta\theta$ for the particle in one rotation of the Earth is 2π rad, and the time taken is 24 hours.

$$\therefore\ \omega = \frac{2\pi}{(24 \times 60 \times 60)\,\text{s}} = \frac{2\pi}{86400\,\text{s}} = 7.28 \times 10^{-5}\ \text{rad}\,\text{s}^{-1} \approx 7.3 \times 10^{-5}\ \text{rad}\,\text{s}^{-1}$$

b) For the Earth orbiting around the Sun, its angular displacement is 2π rad in 365¼ days.

$$\therefore\ \omega = \frac{2\pi}{(365\,\tfrac{1}{4} \times 24 \times 60 \times 60)\,\text{s}} = 1.99 \times 10^{-7}\ \text{rad}\,\text{s}^{-1} \approx 2.0 \times 10^{-7}\ \text{rad}\,\text{s}^{-1}$$

9. The initial angular speed of the ball is $\omega_0 = +5.0\,\text{rad}\,\text{s}^{-1}$.

The final angular speed of the ball is $\omega = -1.5\,\text{rad}\,\text{s}^{-1}$. Using Eq. (11.8c) with the angular displacement $\theta - \theta_0 = 4.5\,\text{rad},$ we get

$$\alpha = \frac{\omega^2 - \omega_0^2}{2(\theta - \theta_0)} = \frac{(-1.5\,\text{rad}\,\text{s}^{-1})^2 - (5.0\,\text{rad}\,\text{s}^{-1})^2}{2 \times (4.5)\,\text{rad}} = -2.5\,\text{rad}\,\text{s}^{-2}$$

The time taken for the ball to attain this angular velocity is calculated using Eq. (11.8a):

$$t = \frac{\omega - \omega_0}{\alpha} = \frac{(-1.5\,\text{rad}\,\text{s}^{-1}) - (5.0\,\text{rad}\,\text{s}^{-1})}{-2.5\,\text{rad}\,\text{s}^{-2}} = 2.6\,\text{s}$$

10. We use Eq. 11.8a to calculate the angular speed ω at $t = 2.0\,\text{s}$ when $\omega_0 = 0$

$$\omega = 0 + (2.0\ \text{rad s}) \times (2.0\ \text{s}) = 4.0\ \text{rad s}$$

From Eqs. (11.15c and d), we have

$$a_r = \omega^2 r = (4.0\,\text{rad}\,\text{s}^{-1})^2 \times 1.0\,\text{m} \quad = 16\,\text{ms}^{-2}$$

$$a_t = \alpha r = (2.0\,\text{rad}\,\text{s}^{-2}) \times 1.0\,\text{m} \;= 2.0\,\text{ms}^{-2}$$

From Eq. (11.16d), the magnitude of the acceleration *a* is

$$a = \sqrt{(16\,\text{ms}^{-2})^2 + (2\,\text{ms}^{-2})^2} \quad = 16\,\text{ms}^{-2}$$

Why do we attach a long handle to the car jack to lift the jack and the car with it? This unit will help you answer this question!

UNIT 12

DYNAMICS OF ROTATIONAL MOTION

Structure

STUDY GUIDE

In this unit, you will study the **dynamics of rotational motion** with a special focus on **non-uniform circular motion**. You will also study about **torque** and **angular momentum**. These concepts are involved in many of our common daily experiences, right from opening taps and the caps of toothpastes in the morning to turning of clock-hands and pages in books while studying, and turning in our beds while sleeping! There are many technological applications of these concepts around us, the most common being that of the wheels. These concepts are *analogous* to force and linear momentum and so you need not think of these concepts as difficult. In order to understand these concepts, you need to know the concept of the **cross product** or the **vector product** which you should revise from Units 1 and 2. You will also study the **work-energy theorem for rotational motion** and so you must know the concepts of **integral calculus** which you have studied in school. You will also study the law of conservation of angular momentum. Solve all examples, SAQs and Terminal Questions on your own!

"Everything should be made as simple as possible, but no simpler." ***Albert Einstein***

12.1 INTRODUCTION

Fig. 12.1: **The effect of torque is seen all around us.**

In Block 2 and Unit 11, you have studied **uniform** and **non-uniform circular motion** and learnt how to **describe** those using **kinematical angular variables**. You have solved problems related to **circular motion in the horizontal and vertical plane**. In this unit, you will begin your study with the **dynamics of angular motion** (Sec. 12.2). In this section, we shall focus on **dynamics of uniform** and **non-uniform circular motion**, establish the equations of motion and apply them to examples around us. In Sec. 12.3, we introduce the concept of **torque**, which plays the same role for angular and rotational motion as **force** does for **translational** motion. You must understand that **torque is as fundamental a concept as force**. **The concept of torque is useful in analysing rotational motion**.

The world around us is full of objects rotating or turning about some axis. Torque is involved when we drive bicycles, buses or cars, turn taps, open bottle caps, use a pair of scissors, move swings, play cricket or even turn our knee caps when we jump or squat on the floor (Fig. 12.1). Merry-go-rounds and see-saws are excellent examples of the importance of torque. Whenever you see an object in angular motion or rotating about an axis **with some angular acceleration or with a changing angular speed**, you must immediately understand that **torque is being exerted on it**.

The problems on the **dynamics of angular or rotational motion** are simpler to solve if we use the concept of torque. You will learn how to solve the equation of motion for various objects in circular motion. In the process, you will learn about **rotational inertia**, which is analogous to **mass**.

We apply the **work-energy theorem** to angular motion (Sec. 12.4) and determine the **rotational kinetic energy** of particles. In Sec. 12.5, you will learn the concept of **angular momentum** and its relation with torque. We discuss the law of **conservation of angular momentum** in Sec. 12.6 and apply it to a variety of simple situations. With this we have discussed all concepts related to the mechanics of a single particle. In the next three units, you will learn how to apply the laws you have studied so far to many-particle systems.

Expected Learning Outcomes

After studying this unit, you should be able to:

- solve problems on dynamics of uniform and non-uniform circular motion;
- determine torque being exerted on a particle in angular/rotational motion and explain the concept of rotational inertia;
- apply the rotational analogue of Newton's second law to solve problems on rotational motion;
- apply the work-energy theorem to rotational motion and calculate the rotational kinetic energy of a particle;
- determine the angular momentum of a particle undergoing angular/rotational motion; and
- apply the law of conservation of angular momentum to simple situations.

12.2 DYNAMICS OF ANGULAR MOTION

To start with, let us take up circular motion, which is quite common around us. We begin by revising what you have already studied about the dynamics of uniform circular motion in Unit 6.

12.2.1 Dynamics of Uniform Circular Motion

In Unit 6, you have learnt that a particle in uniform circular motion (motion in a circle with *constant angular speed*) has **centripetal acceleration**. You also know that a finite acceleration in an object means that a **net force** is being exerted on it. This force is called the **centripetal force**. Let us recall its definition for uniform circular motion.

CENTRIPETAL FORCE

The net force required to keep a particle of mass *m* in uniform circular motion with constant angular speed ω is called the centripetal force. Its magnitude is:

$$F_c = \frac{mv^2}{r} = m\omega^2 r \tag{12.1a}$$

where r is the radius of the circle in which the particle moves and v, its linear speed. The centripetal force is always directed towards the centre of the circle and its direction changes continuously as the particle moves. In the unit vector notation, we can express the vector $\vec{\mathbf{F}}_c$ as follows:

$$\vec{\mathbf{F}}_c = -\frac{mv^2}{r}\,\hat{\mathbf{r}} = -m\omega^2 r\,\hat{\mathbf{r}} \tag{12.1b}$$

Centripetal force

So far you have studied that it is the centripetal force, **which keeps an object moving uniformly in a circle**. The centripetal force is provided by any of the forces in nature about which you have studied in Unit 6. This force is always directed towards the centre of the circle in which the object is moving. However, you have also studied in Unit 11 that objects moving in a circle need not always move with the same angular speed. They may speed up or slow down while still on the circular path. So let us now ask: **Can an object at rest start moving in a circle *or* can its angular speed be changed, if we apply a force directed towards the circle's centre**?

You can find the answer if you do any of the following activities.

Activity

CHANGING THE ANGULAR SPEED IN CIRCULAR MOTION

Do any of the following:

- Try to open a book by applying a force $\vec{\mathbf{F}}_1$ in the plane of the book cover in a direction towards its bound edge (see Fig. 12.2a);

- Try opening a door by pushing it by a force $\vec{F}_1$ in a direction in the door's plane towards its hinges (see Fig. 12.2b);
- Try to set a merry-go-round in motion by pushing it in a direction towards its centre (see Fig. 12.2c).

What do you find? Now try the following:

- Try to open the book or the door by applying the force $\vec{F}_2$ in a direction perpendicular to their plane.
- Try to set the merry-go-round in motion or stopping it by applying the force along the tangential direction.

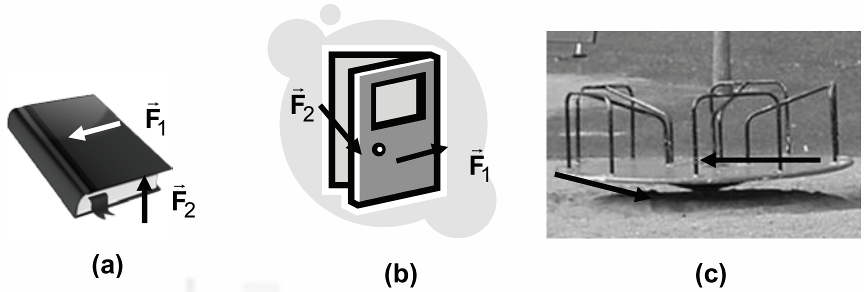


Fig. 12.2: **How can we <u>change</u> the angular speed in circular motion**?

What do you find? Is it easier to do all these things in this case? What is the difference between each set of activities? Did you note that the difference is in the **direction of the force** being applied? The force in the radial direction (the centripetal force) does not bring about a change in the angular speed of the object but the force in the tangential direction does.

You may like to know: What is the force law for this force? This brings us to the dynamics of non-uniform circular motion.

12.2.2 Dynamics of Non-uniform Circular Motion

You have studied in Sec. 11.4.2 about uniform circular motion (in which the angular speed of the object remains constant) and non-uniform circular motion (in which its angular speed changes). You have learnt in Sec. 12.2.1 that **the acceleration of an object in uniform circular motion is centripetal** (i.e., it is directed towards the centre of the circle) and a **centripetal force acts upon it to keep it moving in a circle**.

However, while doing the activity above, you have discovered that you could not change the angular speed of any object (from zero to some value) by applying a centripetal force (in the radial direction). You needed to **apply force in a tangential direction**. Thus, you have found that a force must act in the tangential direction on an object for it to execute non-uniform circular motion.

In general, we say that **the force being exerted on an object in circular motion must have a component in the tangential direction to bring about a change in its angular speed**. Once again, we get the force law for non-uniform circular motion by using Newton's second law.

FORCE LAW FOR NON-UNIFORM CIRCULAR MOTION

The net force on a particle of mass m moving with changing angular speed and angular acceleration α in a circular path of radius r is given by

$$\vec{\mathbf{F}} = m\vec{\mathbf{a}} = m\left(\vec{\mathbf{a}}_r + \vec{\mathbf{a}}_t\right) = \vec{\mathbf{F}}_r + \vec{\mathbf{F}}_t \qquad (12.2a)$$

where $\vec{\mathbf{F}}_r$ and $\vec{\mathbf{F}}_t$ are the **radial** (or **centripetal**) and **tangential** components of the force given by

$$\vec{\mathbf{F}}_r = -\frac{mv^2}{r}\hat{\mathbf{r}} = -m\omega^2 r\,\hat{\mathbf{r}} \qquad \text{(radial component)} \qquad (12.2b)$$

and

$$\vec{\mathbf{F}}_t = m\alpha r\,\hat{\boldsymbol{\theta}} \qquad \text{(tangential component)} \qquad (12.2c)$$

Radial and tangential components of force for non-uniform circular motion

The magnitude of the net force is given by

$$\left|\vec{\mathbf{F}}\right| = \sqrt{F_r^2 + F_t^2} = m\sqrt{(\omega^2 r)^2 + (\alpha r)^2} \qquad (12.2d)$$

and the angle which it makes with the radial direction is β, given by

$$\tan\beta = \frac{ma_t}{ma_r} = -\frac{\alpha}{\omega^2} \qquad (12.2e)$$

Let us take up an example to apply the force law for non-uniform circular motion.

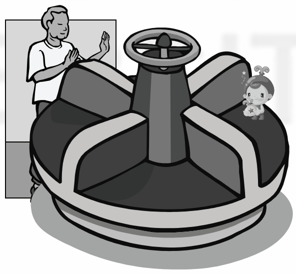
EXAMPLE 12.1: NON-UNIFORM CIRCULAR MOTION

A person applies force on a merry-go-round to set it rotating from rest with a constant angular acceleration of $0.10\,\text{rad}\,\text{s}^{-2}$ (Fig. 12.3a).

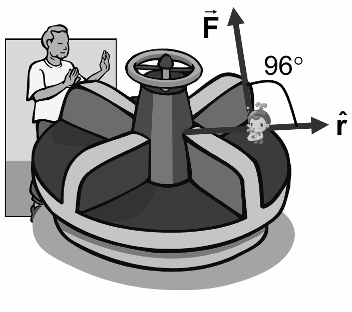
a) What is the net force on a child of mass 30 kg standing at a distance of 1.0 m from the centre of the merry-go-round after 1.0 s?

b) At $t = 5.0\,\text{s}$, the person pushing the merry-go-round steps back and the merry-go-round keeps rotating with a constant angular speed for the next 2.0 s. What is the net force on the child during this time interval?

SOLUTION ■ The **KEY IDEA** is that when the merry-go-round is **set rotating from rest**, its angular speed changes and, hence, it possesses a finite angular acceleration. Therefore, the net force on the child has both radial and tangential components. But **when it rotates with constant angular speed, the net force on the child has only a centripetal component**. We need to use Eqs. (12.2b to e) to solve this problem The kinematical equation (11.8a) can be used to determine the angular speed at any instant.



(a)



(b)

Fig. 12.3: **Non-uniform circular motion.**

a) At $t = 1.0\,\text{s}$, the angular speed of the child is given by Eq. (11.8a) as

$$\omega = \alpha t = 0.10\,\text{rad}\,\text{s}^{-2} \times 1.0\,\text{s} = 0.10\,\text{rad}\,\text{s}^{-1} \qquad (\because \omega_0 = 0)$$

The net force has both radial and tangential components at $t = 1.0\,\text{s}$:

$$\vec{\mathbf{F}}_r = -m\omega^2 r\,\hat{\mathbf{r}} = -(30\,\text{kg}) \times (0.10\,\text{rad}\,\text{s}^{-1})^2 \times (1.0\,\text{m})\,\hat{\mathbf{r}} = -0.30\,\text{N}\,\hat{\mathbf{r}}$$

and $\vec{\mathbf{F}}_t = m\alpha r\,\hat{\boldsymbol{\theta}} = (30\,\text{kg}) \times (0.10\,\text{rad}\,\text{s}^{-2}) \times (1.0\,\text{m})\,\hat{\boldsymbol{\theta}} = 3.0\,\text{N}\,\hat{\boldsymbol{\theta}}$

The magnitude of the net force is given by

$$\left|\vec{\mathbf{F}}\right| = \sqrt{F_r^2 + F_t^2} = \sqrt{(0.30)^2 + (3.0)^2}\,\text{N} = 3.01\,\text{N} \approx 3.0\,\text{N}$$

And its direction is given by $\tan\beta = \dfrac{F_t}{F_r} = \dfrac{3.0}{-0.30} = -10.0$

or $\qquad \beta = (180^\circ - 84.3^\circ) = 95.7^\circ \approx 96^\circ$

Note that β is the angle between $\vec{\mathbf{F}}$ and the radial direction (Fig. 12.2b).

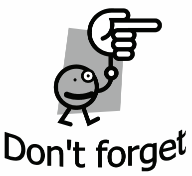
b) At $t = 5.0\,\text{s}$, $\omega = \alpha t = 0.10\,\text{rad}\,\text{s}^{-2} \times 5.0\,\text{s} = 0.50\,\text{rad}\,\text{s}^{-1}$

Since the angular speed is constant for the next 2.0 s, $\alpha = 0$, and only the centripetal force is exerted on the child in this time interval. It is given by:

$$\vec{\mathbf{F}}_r = -m\omega^2 r\,\hat{\mathbf{r}} = -(30\,\text{kg}) \times (0.50\,\text{rad}\,\text{s}^{-1})^2 \times (1.0\,\text{m})\,\hat{\mathbf{r}} = -7.5\,\text{N}\,\hat{\mathbf{r}}$$

Note: In deciding which value of θ to take, you have to account for the directions of $\vec{\mathbf{F}}_t$ and $\vec{\mathbf{F}}_r$ and remember that the angle θ is the angle between their resultant $\vec{\mathbf{F}}$ and the radial direction $\hat{\mathbf{r}}$. Also remember that the direction of $\hat{\mathbf{r}}$ is changing continuously in circular motion.

Let us summarise what you have learnt from Example 12.1.



- **The force being exerted on a particle executing non-uniform circular motion has a finite tangential component given by Eq. (12.2c) in addition to the centripetal component given by Eq. (12.2b).**
- **The tangential component of the force brings about a change in the particle's angular speed. It is zero when the particle moves with a constant angular speed.**
- **The centripetal component is needed to keep the particle moving in the circle with a constant angular speed.**

SAQ 1 – Dynamics of non-uniform circular motion

A particle starts from rest and moves in a circle with a constant angular acceleration. After it has moved through a certain angle, the magnitude of the centripetal force on it is twice the magnitude of the tangential force. Determine the angle.

In the next section we discuss the concept of torque which is very important in the analysis of any kind of rotational motion.

12.3 TORQUE

In Sec. 12.2.2, you have learnt that in order to change the angular speed of an object or to rotate it from rest, we need to apply a force on it having a tangential component. We now introduce some other factors in the dynamics of angular motion: the **distance** of the point, at which the force is applied, from the axis of rotation and **the angle** at which force is applied to the object. Let us begin the discussion by asking: **Why is the distance of the point at which the force is applied important in angular motion**? To find an answer, you may like to think about the following questions and then do an activity as suggested.

- Why is a door knob located as far as possible from the door hinge? Can we open a door if we push it at its hinges even in a direction perpendicular to the plane of the door?
- Why do we find it easier to open a bolt with a spanner that has a long handle than with one that has a short handle?
- Why is the rod connecting the pedal of a bicycle perpendicular to the wheel's plane?
- Why is the handle of the grinding stone perpendicular to the grinding stone's plane and far from the centre of the stone?

Activity

DOES THE POINT, AT WHICH FORCE IS APPLIED, MATTER?

a) Take a metre stick and pivot it at its centre so that it is free to rotate in the vertical plane (Fig. 12.4a). Hang a weight on one side, at some distance from its centre so that the stick turns about its axis. Then hang another weight at a suitable distance on the other side to bring the stick back to its original horizontal position.

   You could also vary the weights you hang. Try to guess what would happen if you hung a particular weight at some distance on the other side to balance the first weight: Would the scale return to its original position or not? ***See*** what actually happens. Try to correlate the weights and their distances from the centre of the stick while rotating and balancing it.

b) You could do the same activity on a see-saw in a much more enjoyable way! You must have played on a see-saw in your childhood. How did you balance it? Go to the park with your friends and have a game of balancing the see-saw with a thin friend on one side of the see-saw and a not so thin friend on the other side (Figs. 12.4b and c). Where would the two persons need to sit on the see-saw to balance it in a horizontal position?

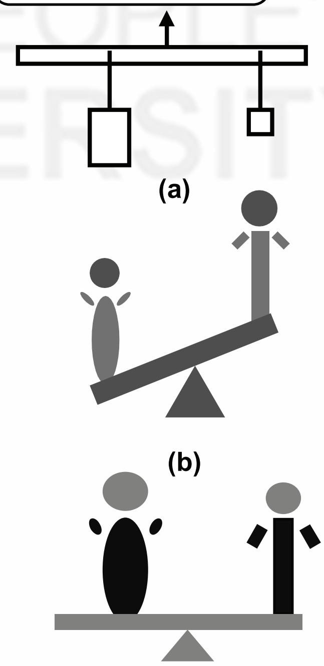
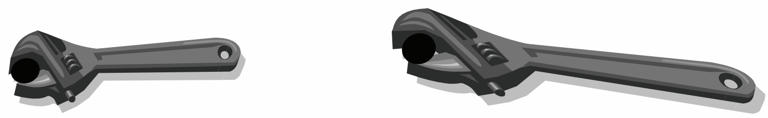


Fig. 12.4: The point at which force is applied matters when we wish to turn an object.

c) Try to open a rusted bolt with a short-handled spanner and then with a long-handled spanner. In which case is it easier to turn it?



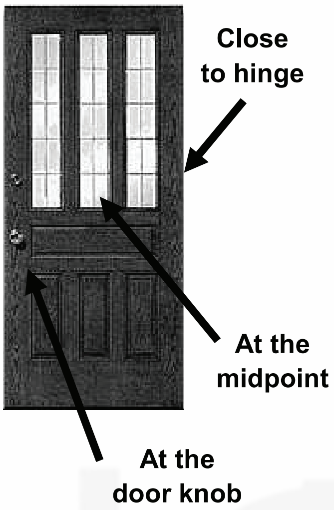


Fig. 12.5: **Understanding torque**.

While doing any of the activities above, did you note that the point at which the force was applied was equally important if you wanted to turn an object? **Is there any other factor which is important?**

You may like to follow the steps listed below to answer this question. Refer to Fig. 12.5.

- Open a door in the usual way. Now try to open it by applying some force at its hinges. Can you open it?

- Next apply **force** at the **midpoint** of the door and **finally at the door knob**. Make sure that all pushes are about equally hard and last for the same time. Also, in all cases, you should apply the force in a direction perpendicular to the plane of the door.

  Now try to answer the following questions:

  a) Is there any difference in the door's motion in the three cases?

  b) In which case does the door turn by a greater angle?

- Next, **apply the same force twice at the same point**, say, at the door knob but at two different angles. For example,

  – force 1 perpendicular to the door, and

  – force 2 at some angle to the plane of the door (say, less than 45°).

  Which force turns the door by a greater angle? Is there an angle at which you can push the door so that it does not turn at all?

Do you recognise that the application of force with a tangential component is only one of the factors when we **set an object rotating**? (In many cases, there may be more than one force being exerted on the object. Then we take the net force.) You may have noted that

Three factors are involved if we wish to rotate or turn an object:

i) the **net force** applied on the object,

ii) the **distance of the point of application of the force from the axis of rotation**. The larger the distance, the easier it is to **set an object rotating** with the same force.

iii) the **angle** at which the force is applied.

Thus, we need to define a new physical quantity other than force which includes all these factors. This new quantity is termed **torque**. A finite torque is exerted on every rotating object which has a finite angular acceleration. Torque can be thought of as representing the '**turning effect of force**'. You may like to know: What is the **difference between force and torque**?

> **Force is involved when an object is accelerated in <u>translational</u> motion. Torque is involved when an object <u>rotates</u> with finite angular acceleration or turns faster or slower. Torque is the tendency of a force to change the state of rotational motion of an object about some axis. It is also called the turning effect of force.**

An object is said to be in **translational motion** when all points on it have the same velocity. If you slide a coin on your desk, it is in translational motion. If you spin it at one place, it executes rotational motion.

Now that you have understood that the concept of force alone is not enough to analyse rotational motion, let us ask another question: **How do we define torque mathematically**? Study Fig. 12.6a, which shows a bolt being opened by a spanner by applying a force $\vec{\mathbf{F}}$ at point *P*. The distance of *P* from the bolt at *O* is *r*.

Let us take the origin *O* to be at the bolt. Let $\vec{\mathbf{r}}$ denote the position vector of the point *P* with respect to *O*. Then we define torque at point *P* as the **vector product** of $\vec{\mathbf{r}}$ and $\vec{\mathbf{F}}$:

$$\vec{\tau} = \vec{\mathbf{r}} \times \vec{\mathbf{F}} \qquad (12.3a)$$

Torque

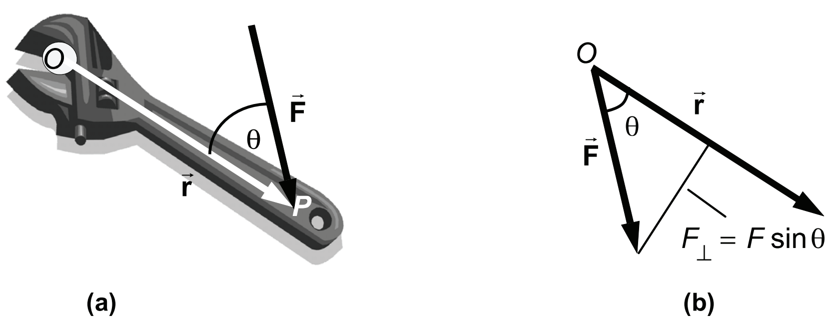


Fig. 12.6: **Defining torque mathematically**.

Torque is a vector and its magnitude is given by

$$\tau = r F \sin\theta \qquad (12.3b)$$

where θ is the angle between $\vec{\mathbf{r}}$ and $\vec{\mathbf{F}}$ when these vectors are placed tail to tail as shown in Fig. 12.6b. The **direction of torque** is given by the right-hand rule and is perpendicular to the plane containing $\vec{\mathbf{r}}$ and $\vec{\mathbf{F}}$. Thus, $\vec{\tau}$ is always perpendicular to both $\vec{\mathbf{r}}$ and $\vec{\mathbf{F}}$. In Fig. 12.6b, it points into the page, perpendicular to both $\vec{\mathbf{r}}$ and $\vec{\mathbf{F}}$. For circular or rotational motion about a fixed axis of rotation passing through the object, the **vector** $\vec{\tau}$ is directed along the axis of rotation (Fig. 12.7). Using the right-hand rule, you can see that it points upwards for counter-clockwise rotation and downwards for clockwise rotation.

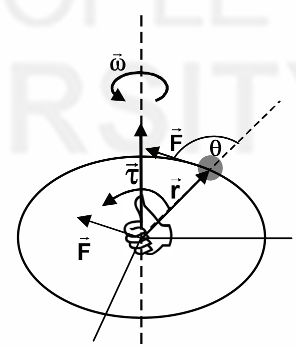
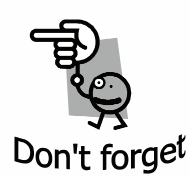


Fig. 12.7: **The direction of torque is given by the right-hand rule for a vector product**.

> **Torque is not the same as work done by a force even though the dimensions of both quantities are the same. Torque is a force-like quantity in rotational motion and its units are Nm in the SI system.**



To check whether this mathematical definition of torque includes the three factors involved in turning or rotating an object, recall your activity with the door. You can see that when the force is applied parallel to the plane of the door, that is, $\vec{\mathbf{r}} \parallel \vec{\mathbf{F}}$, $\theta = 0^\circ$ and from Eq. (12.3b), $|\vec{\boldsymbol{\tau}}| = 0$. Thus, no matter how hard you push or pull in this direction, you cannot rotate the door because the torque is zero.

When the force is perpendicular to the plane of the door, $\theta = 90^\circ$. From Eq. (12.3b), you can see that the torque is maximum, and it is easier to rotate the door. The magnitude of torque depends both on the force applied and the distance of the point at which it is applied from the origin.

To sum up, there can be three ways of changing torque in a rotating system:

- By changing the applied force $\vec{\mathbf{F}}$,
- By changing the length of $\vec{\mathbf{r}}$,
- By changing the angle between $\vec{\mathbf{r}}$ and $\vec{\mathbf{F}}$.

Let us now sum up the discussion so far and present the formal concept of torque.

Recap

TORQUE

If a net force $\vec{\mathbf{F}}$ is exerted on a particle situated at a point *P*, which has position vector $\vec{\mathbf{r}}$ with respect to an origin *O*, the **torque** exerted on the particle with respect to *O* is defined as

$$\vec{\boldsymbol{\tau}} = \vec{\mathbf{r}} \times \vec{\mathbf{F}} \tag{12.3a}$$

Torque

Its magnitude is given by

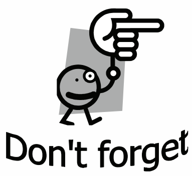
$$\tau = rF \sin\theta \tag{12.3b}$$

where θ is the angle between $\vec{\mathbf{r}}$ and $\vec{\mathbf{F}}$ when these vectors are placed tail to tail. The direction of torque is given by the right-hand rule and is perpendicular to the plane containing $\vec{\mathbf{r}}$ and $\vec{\mathbf{F}}$. The SI unit of torque is Nm.

The word torque comes from the Latin word meaning "twist" and may be identified as the turning action of force.

Just as the net external force brings about ***change*** in the state of translational motion, the net external torque brings about **<u>change</u> in the state of rotational motion**.

The net torque on a particle at rest or in a state of uniform rotational motion is zero. The particle will continue to remain in this state until and unless a **net external torque** is exerted on it. The particle's state of rotation **changes** only when a **net external torque** is exerted on it.



TORQUE and force are <u>NOT</u> the same quantities. Torque gives the relationship between the force exerted and the tendency of a body to rotate.

You may now like to determine the torque acting on some objects. We take up an example to apply Eqs. (12.3a and b) and then you can solve an SAQ.

EXAMPLE 12.2: TORQUE

Fig. 12.8a shows a common exercise for strengthening the muscles of the arm. A dumb-bell of mass 1.0 kg is lifted from the horizontal position and rotated about the elbow joint by an angle of 60°. What is the torque exerted by the fore-arm muscles about the elbow joint

a) when the arm is horizontal, and

b) when it makes an angle of 60° with the horizontal?

Take the distance between the palm holding the dumb-bell and the elbow joint to be 30 cm and $g = 9.8\ \text{ms}^{-2}$.

SOLUTION ■ The **KEY IDEA** is that the elbow joint (J in Fig. 12.8b) is the point about which the torque has to be exerted by the muscles of the fore-arm to rotate the dumb-bell. **The fore-arm muscles have to exert a force $(-\vec{\mathbf{F}})$ at least equal and opposite to the weight of the dumb-bell to produce this torque**. The torque is given by Eqs. (12.3a and b). It is given that $r = 0.30\,\text{m}$ and $F = (1.0\ \text{kg})(9.8\,\text{ms}^{-2}) = 9.8\,\text{N}$.

a) When the dumb-bell is held in a horizontal position (Fig. 12.8b), the angle between vectors $\vec{\mathbf{r}}$ and $(-\vec{\mathbf{F}})$ (with their tails at a common point) is 90° (see Fig. 12.8c). Therefore, the torque in this case is given by

$$\tau = rF\sin\theta = (0.30\,\text{m}) \times (9.8\,\text{N}) \times (\sin 90^\circ) = 2.9\,\text{Nm}$$

The direction of torque is given by the right-hand rule and is perpendicular to the plane containing $\vec{\mathbf{r}}$ and $(-\vec{\mathbf{F}})$ and pointing out of the page.

b) In this case the dumb-bell is rotated about the elbow joint by an angle of 60° (Fig. 12.8d). When the tails of vectors $\vec{\mathbf{r}}$ and $\left(-\vec{\mathbf{F}}\right)$ are placed at a common point, the angle between them is 30° (see Fig. 12.8e). Therefore, the torque in this case is given by

$$\tau = (0.30\,\text{m}) \times (9.8\,\text{N}) \times (\sin 30^\circ) = (0.30\,\text{m}) \times (9.8\,\text{N}) \times 0.50 = 1.5\,\text{Nm}$$

up to two significant figures. The direction of torque is given by the right-hand rule and is perpendicular to the plane containing $\vec{\mathbf{r}}$ and $(-\vec{\mathbf{F}})$ and pointing out of the page.

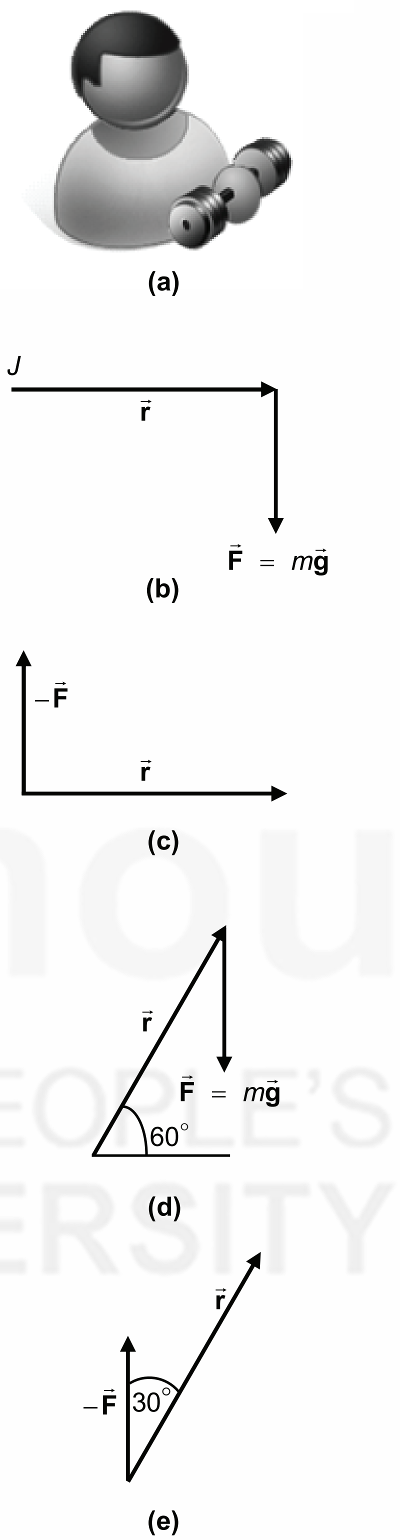


Fig. 12.8: Torque exerted by fore-arm muscles.

Would you like to calculate torque using Eq. (12.3b)? Answer SAQ 2!

SAQ 2 – Determining torque

A child swings on a rope attached to a tree at point P and can go up to a maximum angle of 30° with the vertical (Fig. 12.9). What is the torque exerted by the child about P given that the weight of the child is 40 N and the length of the rope from the point P to the child is 5.0 m?

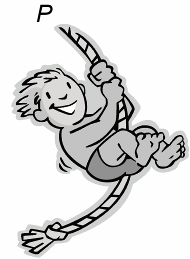


Fig. 12.9: Diagram for SAQ 2; not to scale.

12.3.1 Some Features of Torque

Eqs. (12.3a and b) that define torque give us some interesting information about torque, which we now describe.

1. We can express the force exerted on a particle executing angular motion in a plane in terms of its components as

$$\vec{\mathbf{F}} = \vec{\mathbf{F}}_r + \vec{\mathbf{F}}_t \tag{12.4}$$

where $\vec{\mathbf{F}}_r$ is the component of $\vec{\mathbf{F}}$ in the radial direction (that is, along $\hat{\mathbf{r}}$ or $\vec{\mathbf{r}}$) and $\vec{\mathbf{F}}_t$, the component of $\vec{\mathbf{F}}$ in a direction perpendicular to the radial direction. Substituting $\vec{\mathbf{F}}$ from Eq. (12.4) in Eq. (12.3a), we get

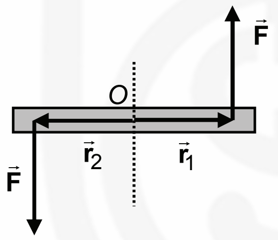
$$\vec{\tau} = \vec{\mathbf{r}} \times \vec{\mathbf{F}} = \vec{\mathbf{r}} \times \vec{\mathbf{F}}_r + \vec{\mathbf{r}} \times \vec{\mathbf{F}}_t \tag{12.5a}$$

Since $\vec{\mathbf{F}}_r$ is parallel to $\vec{\mathbf{r}}$, the angle between them is zero and $\vec{\mathbf{r}} \times \vec{\mathbf{F}}_r = \vec{\mathbf{0}}$. Hence,

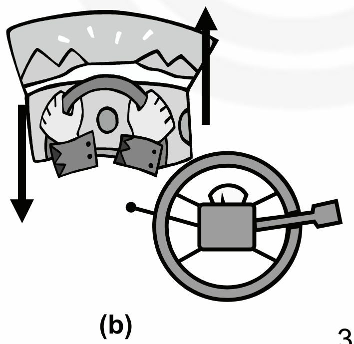
$\vec{\mathbf{F}}_t$ is also called the **transverse** component of force.

$$\boxed{\vec{\tau} = \vec{\mathbf{r}} \times \vec{\mathbf{F}}_t} \tag{12.5b}$$

Thus, **a net torque acts on a particle only if the force has a finite component perpendicular to the radial direction**. We call this component, the **tangential component** or the **transverse component**. As you know from Eq. (12.2c), it is given by $\vec{\mathbf{F}}_t = m\vec{\mathbf{a}}_t$, where $\vec{\mathbf{a}}_t = \alpha r\,\hat{\theta}$.



(a)



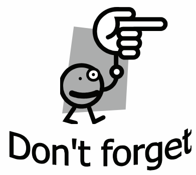
(b)

Fig. 12.10: **Torque can be non-zero, even if the net force on an object is zero.**

2. However, in an interesting case, the **torque can be non-zero**, **even if the net force on an object is zero** as you can see from Fig. 12.10a. The net force on the scale is zero since the two forces are of equal magnitude but directed opposite to each other. But the torque due to each force is directed out of the page towards you and is non-zero. The scale will rotate counter-clockwise due to the net non-zero torque.

Forces that are parallel to each other (but do not act along the same line), **equal in magnitude but opposite in direction**, constitute a **couple**. A couple does not produce translation; it only produces rotation. For example, the forces that two hands apply to turn a steering wheel should be a couple (Fig. 12.10b). Each hand grips the wheel at points on opposite sides of the shaft. When the hands apply such a couple on the steering wheel, it rotates.

3. Torque on an object will be zero if the tangential (transverse) component of the force is zero and there is no couple acting on it. However, it is possible that the torque on a particle is zero even if the net force on it is non-zero.



Torque is zero if

- $\vec{\mathbf{F}}$ is parallel to $\vec{\mathbf{r}}$,
- $\vec{\mathbf{r}}$ is zero, and
- $\vec{\mathbf{F}}$ is **zero** and no couple is acting on the object.

4. Torques obey the **principle of superposition**.

PRINCIPLE OF SUPERPOSITION FOR TORQUE

When several torques are exerted on a particle, the **net torque** or **resultant** torque is the vector sum of individual torques:

$$\vec{\tau}_{net} = \sum_i \vec{\tau}_i \qquad (12.6)$$

Let us apply Eqs. (12.5b and 12.6) to determine torque for simple cases.

EXAMPLE 12.3: TORQUE

a) What is the torque on the Earth due to the force of gravitation exerted by the Sun on it as it goes around the Sun in a nearly circular orbit (Fig. 12.11)?

b) Forces of equal magnitudes are being exerted on two discs, each of radius r in the directions shown in Fig. 12.12. Which of the disc(s) would rotate about an axis passing through its centre and perpendicular to its plane? What is the magnitude of the net torque on each disc?

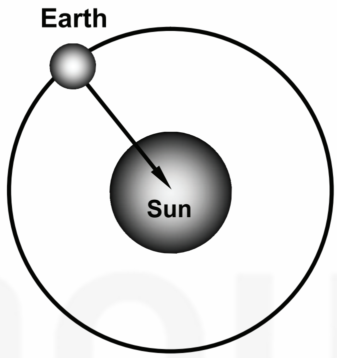


Fig. 12.11

SOLUTION ■ The **KEY IDEA** is to use the definition of torque.

a) The force of gravitation on the Earth due to the Sun points towards the Sun along the radial direction (see Fig. 12.11) and is given by

$$\vec{\mathbf{F}} = -\frac{GM_{Sun}M_{Earth}}{r^2}\,\hat{\mathbf{r}}$$

where r is the distance between the Earth and the Sun and $\hat{\mathbf{r}}$ is the unit vector pointing from the Sun to the Earth. By definition, $\vec{\tau} = \vec{\mathbf{r}} \times \vec{\mathbf{F}}$ and using the expression for $\vec{\mathbf{F}}$ above, we get

$$\vec{\tau} = \vec{\mathbf{r}} \times \vec{\mathbf{F}} = \vec{\mathbf{r}} \times \left(-\frac{GM_{Sun}M_{Earth}}{r^2}\right)\hat{\mathbf{r}} = \vec{\mathbf{0}}$$

The vector product is a null vector since $\hat{\mathbf{r}}$ is a unit vector in the direction of vector $\vec{\mathbf{r}}$.

b) In Fig. 12.12a, the forces are being exerted in the same direction and at the same distance from the axis of rotation. The torques due to these forces about the centre of the disc, are equal and opposite. Hence, the net torque is zero: $\tau_{net} = (-rF) + rF = 0$

Since the net torque on the disc is zero, it will not rotate. In Fig. 12.12b, the forces being exerted are in the opposite directions and at the same distance from the axis of rotation. The torques due to these forces about the centre of the disc, are equal and point in the same direction. Hence, the magnitude of the net torque is: $\tau_{net} = (rF) + (rF) = 2rF$

The disc will rotate in the counter-clockwise direction.

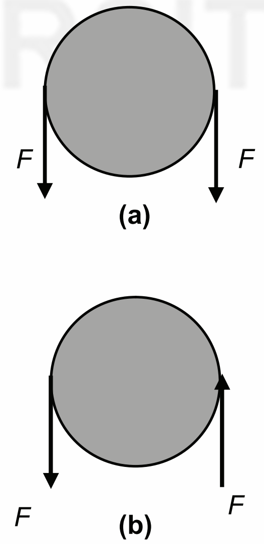


Fig. 12.12: Diagram for Example 12.3b.

You may like to try an SAQ before studying further.

SAQ 3 **– Torque**

Suppose you exert a force of 50 N on a frictionless door as you rush out of a building. What is the torque on the door if you apply the force at the edge of the door in a direction perpendicular to the plane of the door? It is given that the door is 1.0 m wide.

You have just studied that when an object is executing angular motion and its angular speed is changing, a torque is being exerted on it. Let us take up the special case of non-uniform circular motion and determine the torque.

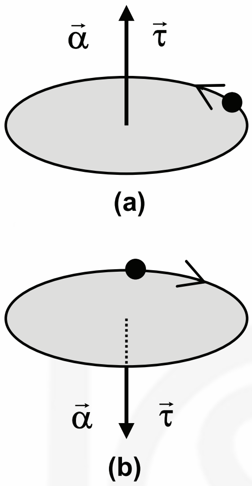


Fig. 12.13: **Torque on a particle in non-uniform circular motion**: **a) counter-clockwise motion**; **b) clockwise motion**.

12.3.2 Torque for Non-uniform Circular Motion

Consider a particle of mass *m* moving in a circle of radius *r* with angular acceleration α (Fig.12.13). Eq. (12.2c) gives the tangential component of force on a particle in non-uniform circular motion as

$$\vec{\mathbf{F}}_t = mr\alpha\hat{\boldsymbol{\theta}} \tag{12.7a}$$

Substituting this expression in Eq. (12.5b), we get

$$\vec{\boldsymbol{\tau}} = \vec{\mathbf{r}} \times \vec{\mathbf{F}}_t = \vec{\mathbf{r}} \times mr\alpha\hat{\boldsymbol{\theta}} = mr^2\alpha(\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}}) \tag{12.7b}$$

Using the definition of cross product as given in Eq. (1.12) of Unit 1, we can determine the direction of $(\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}})$. It is a unit vector perpendicular to the plane of the circle (containing both these vectors). In this case, it is a unit vector along the axis of rotation.

As per the right-hand rule, the directions of this unit vector for counter-clockwise and clockwise rotations are as shown in Figs. 12.13a and b. **Note** that in each case, **it is along the direction of the angular acceleration vector**. Then we can write

$$\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\alpha}} \tag{12.7c}$$

where $\hat{\alpha}$ is a unit vector in the direction of the angular acceleration vector. Thus,

$$\vec{\boldsymbol{\tau}} = mr^2\alpha\,\hat{\boldsymbol{\alpha}} \tag{12.8}$$

Since α is the magnitude of angular acceleration, we can write the angular acceleration vector as

$$\vec{\boldsymbol{\alpha}} = \alpha\hat{\boldsymbol{\alpha}} \tag{12.9}$$

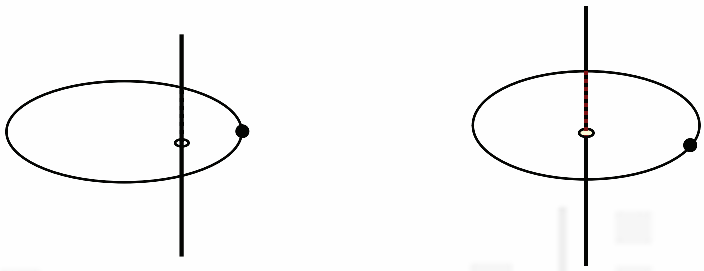
Substituting Eq. (12.9) in Eq. (12.8), we get the expression for torque exerted on a particle in non-uniform circular motion in terms of the angular acceleration vector:

$$\boxed{\vec{\boldsymbol{\tau}} = mr^2\vec{\boldsymbol{\alpha}}} \tag{12.10}$$

Let us compare Eq. (12.10) with Newton's second law of motion for constant mass: $\vec{\mathbf{F}} = m\vec{\mathbf{a}}$. From Eq. (12.10), you can see that torque on a particle in non-uniform circular motion is the product of the angular acceleration and a quantity mr^2, which is analogous to mass m. We call this quantity the **rotational inertia** or the **moment of inertia** and denote it by I:

$$I = mr^2 \qquad (12.11)$$

The SI unit of rotational inertia is kg m^2. From Eq. (12.11), you can see that **rotational inertia depends on both the mass of the particle and its location with respect to the axis of rotation**. Note that for a particle having constant mass, I would change if we changed the axis of rotation or its distance r from the axis of rotation (Fig. 12.14).



(a) (b)

Fig. 12.14: **The rotational inertia or the moment of inertia changes if we change the axis of rotation or** r.

Substituting I from Eq. (12.11) in Eq. (12.10), we can write

$$\vec{\tau} = I\vec{\alpha} \qquad (12.12)$$

Torque for non-uniform circular motion

Eq. (12.12) is the **equation of motion for a particle possessing angular acceleration** $\vec{\alpha}$ and rotational inertia I about a **fixed axis of rotation**. It is also called the **rotational analogue of Newton's second law of motion**. For a particle of **mass** m **moving in a circle of radius** r, rotational inertia I is given by Eq. (12.11).

You can see that Eq. (12.12) is the analogue of Newton's second law of motion. In some textbooks it is also called **Newton's second law for rotational motion**.

Notice that Eq. (12.12) is of the form $\vec{\mathbf{F}} = m\vec{\mathbf{a}}$. It tells us that **torque on an object is proportional to its angular acceleration and is a product of its rotational inertia and angular acceleration**. Although we have derived this equation for a special case of circular motion, it is quite general and applies to a variety of rotational motion.

Of course, **rotational inertia is different for different objects** (Fig. 12.15).

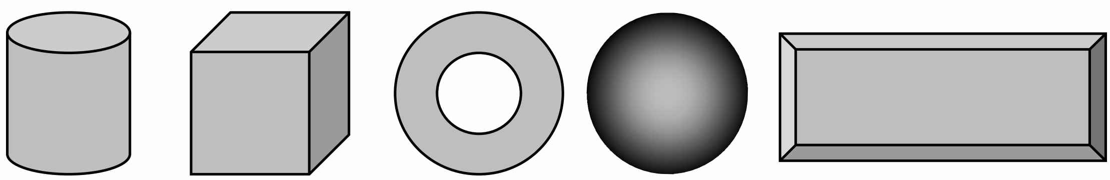


Fig. 12.15: **The rotational inertia of each of these objects is different.**

Let us further understand these ideas with the help of an example.

EXAMPLE 12.4: TORQUE IN CIRCULAR MOTION

A merry-go-round carries children around in a horizontal circle of radius 5.0m (Fig. 12.16). It is set rotating from rest in a clockwise direction and attains an angular speed of $0.30\ \text{rad}\,\text{s}^{-1}$ in 60 s. (a) What is the torque experienced by a child of mass 25 kg (about the centre of the circle) if she is sitting at a distance of 2.0 m from the centre? (b) What would the torque be if the child were sitting at the edge of the merry-go-round?

SOLUTION ■ The **KEY IDEA** is that since the motion is non-uniform and circular, we determine the torque using Eq. (12.8).

From Eq. (12.8), $\vec{\tau} = mr^2\alpha\,\hat{\alpha}$ and α can be obtained from the kinematical equation Eq. (11.8a). It is given by

$$\alpha = \frac{\omega - \omega_0}{t} = \frac{0.30\,\text{rad}\,\text{s}^{-1}}{60\,\text{s}} = 5.0 \times 10^{-3}\,\text{rad}\,\text{s}^{-2} \quad (\because \omega_0 = 0)$$

Therefore, the torque on the child in both cases is given as follows:

(i) $$\vec{\tau} = (25\,\text{kg}) \times (2.0\,\text{m})^2 \times (5.0 \times 10^{-3}\,\text{rad}\,\text{s}^{-2})\,\hat{\alpha} = 0.50\,\text{N}\,\text{m}\,\hat{\alpha}$$

(ii) $$\vec{\tau} = (25\,\text{kg}) \times (5.0\,\text{m})^2 \times (5.0 \times 10^{-3}\,\text{rad}\,\text{s}^{-2})\,\hat{\alpha} = 3.1\,\text{N}\,\text{m}\,\hat{\alpha}$$

The direction of the torque is shown in Fig. 12.16.

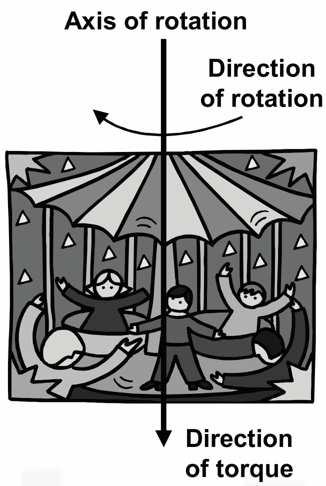


Fig. 12.16: Torque on a child in circular motion. The merry-go-round rotates about an axis passing through its centre. Since rotation is clockwise, the torque points downward along the axis of rotation.

The following SAQ will help you check whether you have understood the concept of torque for circular motion.

SAQ 4 – Torque for circular motion

Old record players had turn tables which could rotate at 16, 33-1/3, 45 or 78 rpm. Will the torque required to get a record rotating from rest to each of these angular speeds in the same time interval, increase or decrease with angular speed? Assume that the rotational inertia of all records is the same.

Let us now understand the physical meaning of rotational inertia.

12.3.3 Physical Meaning of Rotational Inertia

Eq. (12.12) tells us the **physical meaning of rotational inertia or moment of inertia**. Compare it with the equation $\vec{\mathbf{F}} = m\vec{\mathbf{a}}$. You can see that **rotational inertia plays the same role in rotational motion that mass plays in translational motion**. We say that rotational inertia I is the **rotational analogue of mass** m. In Unit 5, you have learnt that inertial mass is a measure of a particle's resistance to change in its state of translational motion.

In the same way, **rotational inertia is a measure of the particle's resistance to change in its rotational motion** (see Figs. 12.17 and 12.18). Thus,

- For constant I, the angular acceleration of a particle is proportional to the torque applied on it:

$$\vec{\tau} \propto \vec{\alpha} \quad \text{for constant } I$$

- The same torque would produce smaller angular acceleration in a particle possessing greater rotational inertia (Fig. 12.17) :

$$\textbf{Same torque:} \quad \vec{\tau} = I_1\vec{\alpha}_1 = I_2\vec{\alpha}_2 \tag{12.13a}$$

$$I_1 > I_2 \quad \Rightarrow \quad \alpha_1 < \alpha_2 \tag{12.13b}$$

- A particle possessing greater rotational inertia would require larger torque to produce the same angular acceleration as that of a particle possessing smaller rotational inertia (Fig. 12.18):

$$\textbf{Same angular acceleration:} \quad \vec{\tau}_1 = I_1\vec{\alpha} \quad \text{and} \quad \vec{\tau}_2 = I_2\vec{\alpha} \tag{12.14a}$$

$$I_1 > I_2 \quad \Rightarrow \quad \tau_1 > \tau_2 \tag{12.14b}$$

- When the torque on a particle is zero, it moves with a constant angular speed:

$$\vec{\tau} = \vec{0} \Rightarrow \vec{\alpha} = \vec{0} \Rightarrow \vec{\omega} = \text{constant} \tag{12.15}$$

You may like to determine the rotational inertia of a particle in circular motion to understand its analogy with mass.

Same τ: $\alpha_A > \alpha_B$
since $I_A < I_B$

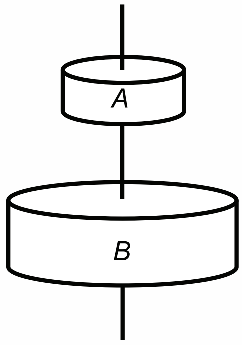


Fig. 12.17: **Torque is same but moment of inertia of the two objects is different**.

Same α: $\tau_B > \tau_A$
since $I_B > I_A$

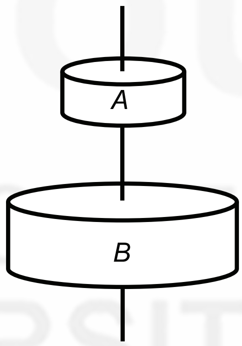


Fig. 12.18: **Angular acceleration is same but moment of inertia of the two objects is different**.

SAQ 5 – **Rotational inertia of a particle**

What is the rotational inertia of a bead of mass 0.50 kg situated at the edge of a rotating wheel of radius 1.0 m? What is the angular acceleration of the bead if the torque exerted on it is (i) 2.5 Nm and (ii) 5.0 Nm?

Now that you have studied about torque, we can extend the concepts of work and work-energy theorem to rotational motion. When we do this, we arrive at an expression for the **kinetic energy of rotation**.

12.4 WORK-ENERGY THEOREM AND KINETIC ENERGY OF ROTATION

Consider a particle of mass m moving from point A to point B under the influence of force $\vec{\mathbf{F}}$ (Fig. 12.19). Recall the work-energy theorem, which relates work done by the force on the particle to the change in its kinetic energy:

$$W = \Delta K = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 \tag{12.16}$$

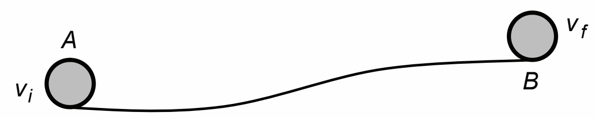


Fig. 12.19: **Work done on a particle results in a change in its kinetic energy**.

We can extend this theorem to angular motion. When a net torque acts on a particle in angular motion about a fixed axis, it does work on the particle. We now show that the **work done by the torque results in a change in the rotational kinetic energy of the particle**. We consider circular motion to keep the mathematics simple.

Consider a particle of mass *m* moving counter-clockwise in a circle of radius *r* under the influence of a net torque $\vec{\tau}$ which is responsible for the angular acceleration of the particle. You have learnt in Sec. 12.3.2 (Eq. 12.7b) that only the tangential component F_t of the force gives rise to torque. Therefore, only F_t does work on the particle.

What is the work dW done on the particle as it moves along an arc of length ds by an angle $d\theta$ in the circle (Fig. 12.20)? We use the definition of work done and the relation $s = r\theta$ (Eq. 11.1) and get:

$$dW = F_t\, ds = F_t\, r\, d\theta \qquad (\because\ ds = r\, d\theta) \tag{12.17a}$$

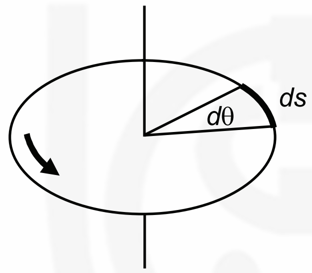


Fig. 12.20: **Work done in circular motion**.

Now for circular motion, we have

$$|\vec{\tau}| = |\vec{\mathbf{r}} \times \vec{\mathbf{F}}_t| = r F_t \sin 90^\circ = r F_t \tag{12.17b}$$

Using Eq. (12.17b), we can write Eq. (12.17a) as:

$$dW = \tau\, d\theta \tag{12.17c}$$

We can now determine the work done on a particle moving from an angular position θ_1 to θ_2 by integrating Eq. (12.17c).

NOTE

Always express θ in radians.

Work done by torque

WORK DONE BY TORQUE

The work done by the **net torque** in turning an object from an angle θ_1 to θ_2 is given by

$$W = \int_{\theta_1}^{\theta_2} \tau\, d\theta \tag{12.18}$$

The work done by **constant torque** τ**,** which turns an object by an angle θ, is given by

$$W = \tau\, \theta \tag{12.19}$$

The work done by a torque in turning an object by some angle brings about a change in its kinetic energy. This is called the object's **rotational kinetic energy**. Let us determine its expression for circular motion. What should we do to arrive at a general result for kinetic energy which holds for rotational motion of particles? We should make use of the fact that the angular speed of particles on a rotating object remains the same. Therefore, for angular motion, we express the kinetic energy in terms of the angular speed of a particle. So let us express the kinetic energy of an object in terms of the rotational variables I and ω. You know that the speed v of a particle in circular motion depends on its distance from the axis of rotation: $v = r\omega$. Therefore,

$$\text{K.E.} = \frac{1}{2}mv^2 = \frac{1}{2}mr^2\omega^2 = \frac{1}{2}I\omega^2 \qquad (12.20)$$

Although we have derived this result for circular motion of a particle, **it applies to all objects in rotational motion**. **We now state the general definition of rotational kinetic energy**.

ROTATIONAL KINETIC ENERGY

The **rotational kinetic energy** of an object rotating with an angular speed of ω about a fixed axis and having rotational inertia I is given by

$$\text{K.E.} = \frac{1}{2}I\omega^2 \qquad (12.20)$$

The SI unit of rotational kinetic energy is joule (J).

Rotational kinetic energy

We can also express the work-energy theorem for angular motion in terms of rotational kinetic energy. Suppose the angular speed of the particle as it moves in the circle from θ_1 to θ_2 changes from ω_1 to ω_2. Its kinetic energy changes as its speed changes from $r\omega_1$ to $r\omega_2$. The change in the rotational kinetic energy of the particle is given by

$$\Delta\text{K.E.} = \frac{1}{2}I\omega_2^2 - \frac{1}{2}I\omega_1^2 \qquad (12.21)$$

Thus, the **work-energy theorem for angular motion** takes the form:

$$W = \int_{\theta_1}^{\theta_2} \tau\, d\theta = \frac{1}{2}I\omega_2^2 - \frac{1}{2}I\omega_1^2 \qquad (12.22)$$

Work-energy theorem for angular motion

You may now like to determine the rotational kinetic energy of a particle.

SAQ 6 – Rotational kinetic energy

a) What is the kinetic energy of rotation of a particle of mass 1 kg moving in a circle of radius 1 m with an angular speed of $1\,\text{rad}\,\text{s}^{-1}$?

b) The rotational kinetic energy of a particle in circular motion is 50.0 J. It moves once in the circle in 22.0 s. What is the rotational inertia of the particle?

In Unit 11 and so far in Unit 12, you have studied the kinematics and dynamics of angular motion with a special focus on circular motion. You have learnt many new concepts such as angular position, angular displacement, angular velocity, angular acceleration, torque, rotational inertia and kinetic energy of rotation that have analogues in translational motion.

You may like to know: **Is there an analogue of linear momentum for rotational motion**?

The answer is that there is, indeed, a quantity defined as **angular momentum**. It plays the same role in angular motion as linear momentum plays in translational motion.

The concept of angular momentum is also important because the **law of conservation of angular momentum** is a fundamental law of nature and as important as the other two laws, namely, the laws of conservation of linear momentum and energy that you have studied so far.

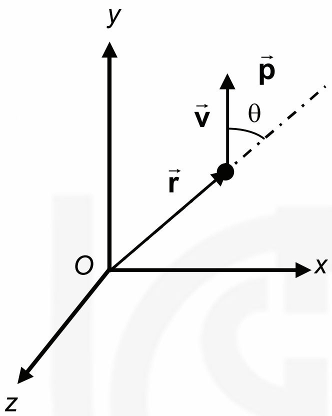


Fig. 12.21: **Angular momentum of a particle**.

Many phenomena in the universe, which involve angular motion, are governed by this law. For example, the fact that almost all planets in the solar system and the Sun lie in a plane is explained using the conservation of angular momentum. The principle of conservation of angular momentum is used to keep the artificial satellites stable in their orbits. There are many other practical applications of conservation of angular momentum as you will now study.

12.5 ANGULAR MOMENTUM

Let us begin by defining angular momentum. Consider a particle *P* of mass *m* having position vector $\vec{\mathbf{r}}$ with respect to the origin *O* (Fig. 12.21). Let it move with a velocity $\vec{\mathbf{v}}$ along some path. You know that its linear momentum is $\vec{\mathbf{p}} = m\vec{\mathbf{v}}$. Then, we define the angular momentum $\vec{\mathbf{L}}$ of the particle as follows:

Angular momentum

$$\vec{\mathbf{L}} = \vec{\mathbf{r}} \times \vec{\mathbf{p}} \tag{12.23a}$$

The magnitude of the angular momentum is given by

$$L = r\,p\sin\theta \tag{12.23b}$$

where θ is the angle between $\vec{\mathbf{r}}$ and $\vec{\mathbf{p}}$.

The direction of $\vec{\mathbf{L}}$ is given by the **right-hand rule** for the vector product and is perpendicular to the plane containing the vectors $\vec{\mathbf{r}}$ and $\vec{\mathbf{p}}$.

$\vec{\mathbf{L}}$ is zero if $\vec{\mathbf{r}}$ and $\vec{\mathbf{p}}$ are in the same direction since $\sin 0^\circ = 0$.

Note that $\vec{\mathbf{L}}$ is defined with respect to some reference point, which we take as the origin. Its value changes if we choose a different point as the reference point or the origin.

Let us explain this point with the help of an example.

EXAMPLE 12.5: ANGULAR MOMENTUM

A particle of mass m moves in the x-direction with velocity $\vec{\mathbf{v}} = v\,\hat{\mathbf{i}}$ (Fig. 12.22a). a) What is its angular momentum with respect to the origin O? b) What is its angular momentum with respect to a point P on the y-axis?

SOLUTION ■ The **KEY IDEA** is to determine the position vector of the particle with respect to both the points O and P (Figs. 12.22a and b) and then apply Eq. (12.23a) to determine the angular momentum.

a) Since the particle moves in the x-direction, its position vector **with respect to** O is given by $\vec{\mathbf{r}} = r_{Ox}\,\hat{\mathbf{i}}$ (Fig. 12.22a). From Eq. (12.23a),

$$\vec{\mathbf{L}} = \vec{\mathbf{r}} \times \vec{\mathbf{p}} = r_{Ox}\,\hat{\mathbf{i}} \times mv\,\hat{\mathbf{i}} = \vec{\mathbf{0}} \quad (\because\ \hat{\mathbf{i}} \times \hat{\mathbf{i}} = \vec{\mathbf{0}} \text{ or } \theta = 0)$$

b) In this case, the position vector $\vec{\mathbf{r}}_P$ of the particle with P as the origin is shown in Fig. 12.22b. Let us resolve this vector in terms of its components r_{Px} and r_{Py}, respectively along the x and y-axes.

$$\vec{\mathbf{r}}_P = r_{Px}\,\hat{\mathbf{i}} - r_{Py}\,\hat{\mathbf{j}}$$

You can see that the x-component of the position vector $\vec{\mathbf{r}}_P$ is parallel to the velocity $\vec{\mathbf{v}}$ and so its vector product with velocity will be zero:

$$r_{Px}\,\hat{\mathbf{i}} \times mv\,\hat{\mathbf{i}} = \vec{\mathbf{0}} \quad (\because\ \hat{\mathbf{i}} \times \hat{\mathbf{i}} = \vec{\mathbf{0}} \text{ or } \theta = 0)$$

The y-component of the position vector is perpendicular to the velocity $\vec{\mathbf{v}}$ and hence, we get

$$\vec{\mathbf{L}}_P = -r_{Py}\,\hat{\mathbf{j}} \times mv\,\hat{\mathbf{i}} = -mv\,r_{Py}\,(\hat{\mathbf{j}} \times \hat{\mathbf{i}})$$

The vector product $(\hat{\mathbf{j}} \times \hat{\mathbf{i}})$ is a vector of unit magnitude and its direction is given by the right-hand rule to be perpendicular to both $\hat{\mathbf{j}}$ and $\hat{\mathbf{i}}$ pointing into the page. Hence, the angular momentum vector is directed perpendicular to the xy plane, pointing outside the page towards us. The magnitude of the angular momentum vector $\vec{\mathbf{L}}_P$ is

$$|\vec{\mathbf{L}}_P| = mv\,r_{Py}$$

NOTE that only the component of the vector $\vec{\mathbf{r}}$ perpendicular to the velocity contributes to the angular momentum. This is also the **perpendicular distance** from the origin (P) to the direction of the velocity or the straight line along which the particle moves.

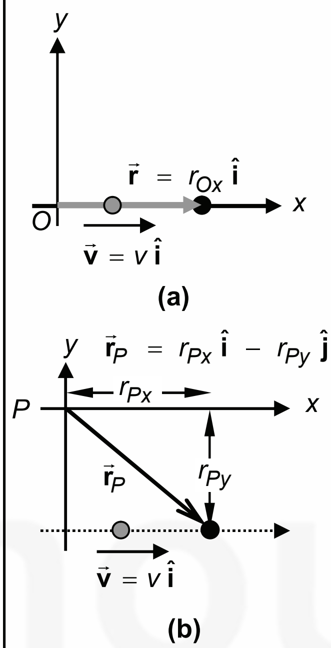


Fig. 12.22: **Angular momentum of a particle is different with respect to different points.**

SAQ 7 – Angular momentum

A particle of mass 5.0 kg is moving in a straight line with a velocity of 2.0 ms^{-1} as shown in Fig. 12.23. What is the angular momentum of the particle with respect to the point O if the perpendicular distance between O and the line is 2.0 m?

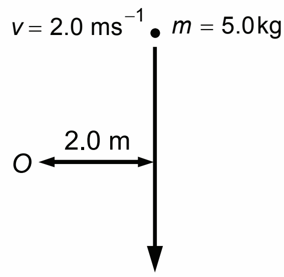


Fig. 12.23

12.5.1 Angular Momentum for Circular Motion

We can express $\vec{\mathbf{L}}$ in terms of the rotational inertia and the angular velocity of an object. To do so, we consider the simple case of a particle of mass *m* moving in a circle of radius *r*. By definition, its angular momentum is given by

$$\vec{\mathbf{L}} = \vec{\mathbf{r}} \times \vec{\mathbf{p}} = \vec{\mathbf{r}} \times m\vec{\mathbf{v}} \tag{12.24a}$$

Now, let $\hat{\mathbf{r}}$ be the unit vector in the direction of the vector $\vec{\mathbf{r}}$. Noting that the vector $\vec{\mathbf{v}}$ is along the tangent to the circle, we can write vectors $\vec{\mathbf{r}}$ and $\vec{\mathbf{v}}$ as

$$\vec{\mathbf{r}} = r\hat{\mathbf{r}} \text{ and } \vec{\mathbf{v}} = v\hat{\boldsymbol{\theta}} = r\omega\,\hat{\boldsymbol{\theta}} \tag{12.24b}$$

Substituting Eq. (12.24b) in Eq. (12.24a), we get

$$\vec{\mathbf{L}} = r\hat{\mathbf{r}} \times mv\hat{\boldsymbol{\theta}} = mvr\,(\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}}) \tag{12.24c}$$

Note once again that $(\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}})$ is a unit vector perpendicular to the plane of the circle (containing both these vectors). In this case, it is a unit vector along the axis of rotation. The directions of this unit vector for counter-clockwise and clockwise rotations are given by the right-hand rule. In each case, it is along the direction of the angular velocity vector $\vec{\omega}$ (see Fig. 12.24). Substituting $v = r\omega$ in Eq. (12.24c), we get

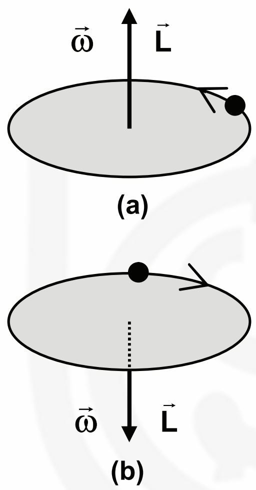


Fig. 12.24: Angular momentum of a particle in circular motion is directed along the angular velocity for both a) counter-clockwise and b) clockwise rotation.

$$\vec{\mathbf{L}} = mr^2\omega\,(\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}}) = mr^2\vec{\omega} \tag{12.24d}$$

where $\vec{\omega}$ is the angular velocity vector. You can recognize the quantity mr^2 as the rotational inertia *I* for this particle. Hence, Eq. (12.24d) becomes

$$\vec{\mathbf{L}} = I\vec{\omega} \tag{12.24e}$$

Notice that this equation is an analogue of the relation $\vec{\mathbf{p}} = m\vec{\mathbf{v}}$. Further, although we have derived Eq. (12.24e) for a particle in circular motion, it holds for any object rotating with an angular velocity $\vec{\omega}$ and having rotational inertia *I*.

Angular momentum for circular motion

ANGULAR MOMENTUM FOR CIRCULAR MOTION

The angular momentum of an object rotating with an angular velocity $\vec{\omega}$ about a fixed axis and having rotational inertia *I* is given by

$$\vec{\mathbf{L}} = I\vec{\omega} \tag{12.25a}$$

and

$$L = I\omega \tag{12.25b}$$

The SI unit of angular momentum is $\text{kg m}^2\,\text{s}^{-1}$.

You have learnt that the angular momentum is an analogue of linear momentum. You may wonder: **Is there a relationship between torque and angular momentum which is an analogue of Newton's second**

law: $\vec{\mathbf{F}} = \dfrac{d\vec{\mathbf{p}}}{dt}$? Let us see what it is.

12.5.2 Relation between Torque and Angular Momentum

Let us begin from the definition of torque as given by Eq. (12.3a) and substitute $\vec{\mathbf{F}} = \dfrac{d\vec{\mathbf{p}}}{dt}$ in it. We get

$$\vec{\tau} = \vec{\mathbf{r}} \times \vec{\mathbf{F}} = \vec{\mathbf{r}} \times \frac{d\vec{\mathbf{p}}}{dt} \tag{12.26}$$

Since $\vec{\mathbf{F}}$ is the net external force, the torque in Eq. (12.26) is the **net external torque**. Let us now **differentiate the angular momentum** [defined by Eq. (12.23a)] **with respect to time** and see what we get.

$$\frac{d\vec{\mathbf{L}}}{dt} = \frac{d}{dt}\left(\vec{\mathbf{r}} \times \vec{\mathbf{p}}\right) = \left(\frac{d\vec{\mathbf{r}}}{dt} \times \vec{\mathbf{p}}\right) + \left(\vec{\mathbf{r}} \times \frac{d\vec{\mathbf{p}}}{dt}\right) \tag{12.27a}$$

Since $\vec{\mathbf{p}} = m\vec{\mathbf{v}}$, Eq. (12.27a) becomes

$$\frac{d\vec{\mathbf{L}}}{dt} = \left(\frac{d\vec{\mathbf{r}}}{dt} \times m\vec{\mathbf{v}}\right) + \left(\vec{\mathbf{r}} \times \frac{d(m\vec{\mathbf{v}})}{dt}\right) \tag{12.27b}$$

For an object of constant mass, we can write Eq. (12.27b) as

$$\frac{d\vec{\mathbf{L}}}{dt} = m(\vec{\mathbf{v}} \times \vec{\mathbf{v}}) + m\left(\vec{\mathbf{r}} \times \frac{d\vec{\mathbf{v}}}{dt}\right) \tag{12.27c}$$

Since the velocity vector is parallel to itself, the term $(\vec{\mathbf{v}} \times \vec{\mathbf{v}})$ is a null vector and we get

$$\frac{d\vec{\mathbf{L}}}{dt} = m\left(\vec{\mathbf{r}} \times \frac{d\vec{\mathbf{v}}}{dt}\right) = \vec{\mathbf{r}} \times \frac{d\vec{\mathbf{p}}}{dt} \tag{12.27d}$$

On comparing Eqs. (12.27d) and (12.26), we get the relation between torque and angular momentum.

TORQUE AND ANGULAR MOMENTUM

The net external torque exerted on an object is related to its angular momentum by the following equation:

$$\vec{\tau} = \frac{d\vec{\mathbf{L}}}{dt} \tag{12.28}$$

Thus, **the net external torque exerted on an object is equal to the rate of change of its angular momentum**.

Torque and angular momentum

You have learnt that a net external torque exerted on an object brings about a change in its angular momentum. We now ask: **What happens to the angular momentum of an object if the net torque on it is zero?** We get the **law of conservation of angular momentum**. This is what you will study in the last section of this unit. But before studying it you may like to attempt an SAQ on determining angular momentum for different cases.

SAQ 8 – **Angular momentum**

A child of mass 25 kg is sitting on the edge of a merry-go-round of radius 2.5 m. The merry-go-round is rotating clockwise. What is the angular momentum of the child at the instant at which the angular speed of the merry-go-round is 4.0 rpm? What is the magnitude of the torque exerted on the child if the angular speed of the merry-go-round changes with time as $\omega = 0.80t \text{ rad s}^{-1}$?

12.6 CONSERVATION OF ANGULAR MOMENTUM

When the net external torque on the object becomes zero, Eq. (12.28) becomes

$$\vec{\tau} = \frac{d\vec{\mathbf{L}}}{dt} = \vec{\mathbf{0}}$$

or

$$\boxed{\vec{\mathbf{L}} = \text{constant}} \qquad (12.29)$$

Thus, we get the law of conservation of angular momentum.

Angular momentum conservation

CONSERVATION OF ANGULAR MOMENTUM

The **total angular momentum of a system remains constant** (is conserved), if the net external torque exerted on the system is zero:

$$\vec{\mathbf{L}} = \text{constant} \quad \text{if} \quad \vec{\tau} = \vec{\mathbf{0}} \qquad (12.29)$$

We now take up some examples of conservation of angular momentum.

EXAMPLE 12.6: **ANGULAR MOMENTUM CONSERVATION**

Our solar system was formed from a huge rotating cloud of gas. It is known that the planets (with the exception of Pluto) and the Sun lie in a plane (Fig. 12.25). How can we explain the solar system's shape on the basis of the conservation of angular momentum?

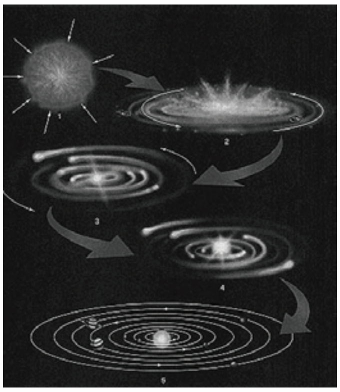


Fig. 12.25

SOLUTION ■ The **KEY IDEA** is to determine the force and the torque on the gas cloud and see whether angular momentum is conserved. If it is so, what does it imply for the shape of the solar system?

Since the force of gravitation is the only force between the particles of the cloud and it would be in the radial direction, the net torque on the system would be zero. Hence, the angular momentum of the cloud would be constant. This means that **both the magnitude and the direction of the angular momentum are constant**.

By definition, $\vec{\mathbf{L}} = \vec{\mathbf{r}} \times \vec{\mathbf{p}} = \vec{\mathbf{r}} \times m\vec{\mathbf{v}}$
that is, $\vec{\mathbf{L}}$ is a vector product of $\vec{\mathbf{r}}$ and $\vec{\mathbf{v}}$ and $\vec{\mathbf{L}}$ is perpendicular to the plane containing $\vec{\mathbf{r}}$ and $\vec{\mathbf{v}}$. Since the direction of $\vec{\mathbf{L}}$ is constant, this means that $\vec{\mathbf{r}}$ and $\vec{\mathbf{v}}$ would always lie in a **fixed plane** perpendicular to the constant vector $\vec{\mathbf{L}}$ (see Fig. 12.25). Thus, the particles in the cloud of gas would rotate in a fixed plane. Therefore, when the planets and the Sun were formed from this cloud of gas, they also moved in the same plane giving rise to the flat shape of the solar system.

In Example 12.7, we present an interesting application of the conservation of angular momentum.

EXAMPLE 12.7: ANGULAR MOMENTUM CONSERVATION

In an entertainment show on ice, two performers of equal mass hold two ends of a long pole of negligible mass and skate in a circle centred at the pole (Fig. 12.26). During the act, each of the skaters pulls along the pole so that their separation is reduced to half its initial value. Determine their final tangential speed if the length of the pole is 2.0 m, the mass of each skater is 45.0 kg and they have initial opposite tangential velocities of $1.5\,\text{ms}^{-1}$. Neglect friction.

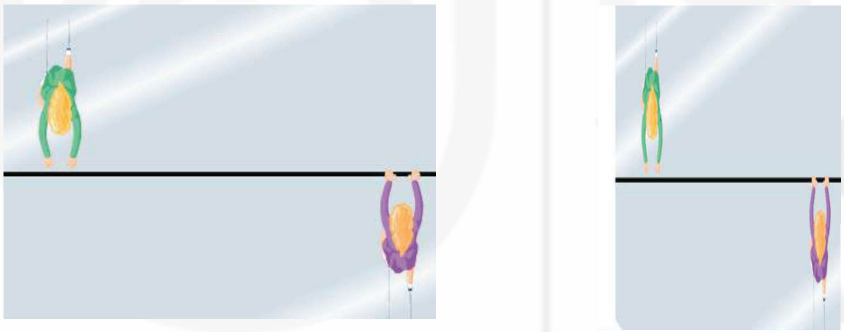


Fig. 12.26

SOLUTION ■ The **KEY IDEA** here is that there is no torque due to friction and the torques due to the force of gravity on both skaters are equal and opposite, since they are on opposite ends of the pole ($\vec{\mathbf{r}}$ for the respective skater is equal and opposite). Therefore, their net external torque about the centre of the pole is zero and hence, angular momentum is conserved. Since the skaters are in circular motion, we apply Eqs. (12.25b and 12.29) treating both skaters as particles.

Since both skaters are moving in the same direction in the circle, the angular momentum of each skater will be in the same direction. Since their masses, tangential speed and the distance from the centre of the circle are the same, the total initial angular momentum will be

$$L_i = mvr + mvr = 2mvr = 2(45.0\,\text{kg})(1.5\,\text{ms}^{-1})(1.0\,\text{m}) = 135\,\text{kgm}^2\text{s}^{-1}$$

Let the final tangential speed be V. Then their final angular momentum is

$$L_f = \frac{mVr}{2} + \frac{mVr}{2} = mVr$$

Since $L_i = L_f$, we have,

$$mVr = 135\,\text{kg}\,\text{m}^2\text{s}^{-1}$$

or $$V = \frac{135\,\text{kg}\,\text{m}^2\text{s}^{-1}}{45.0\text{ kg} \times 1.0\,\text{m}} = 3.0\text{ ms}^{-1}$$

Thus, the speed of the skaters is doubled when they reduce the distance between them to half the initial distance. Their kinetic energy would increase four times.

In Example 12.7, speeds of the objects increased when their rotational inertia was reduced. This happened because angular momentum was conserved. This application of conservation of angular momentum is seen in many instances, which we describe below.

1. **Dancers or skaters on ice** can increase their angular speed by pulling in their extended arms as shown in Fig. 12.27. They can also slow down when they extend their arms. Thus, they control their angular speed by changing their rotational inertia.

(a) (b)

Fig. 12.27: A skater on ice starts to spin slowly with both her arms and legs outstretched. As she pulls in her arms and legs, her rotational inertia decreases and her angular speed increases.

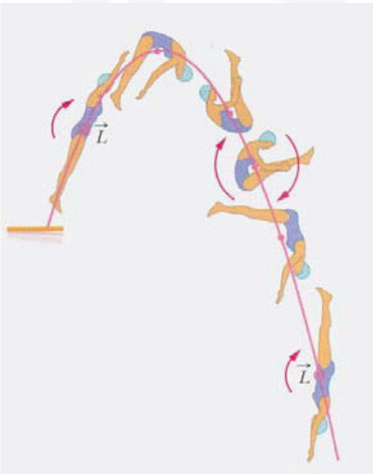


Fig. 12.28: The angular momentum of the swimmer is conserved as she dives into the pool. When she pulls in her arms and legs, her angular speed increases. When she extends them again, her angular speed decreases.

2. A **swimmer diving in the swimming pool** executes somersaults before entering the swimming pool by changing her rotational inertia. She leaves the springboard with a definite angular momentum $\vec{\mathbf{L}}$, represented by a vector pointing into the plane of Fig. 12.28, perpendicular to the page. When she is in the air, no net external torque is exerted on her (assuming air drag is negligible). So, her angular momentum is conserved.

   By pulling her arms and legs into the closed position, she reduces her rotational inertia and thus, increases her angular speed. Pulling out of the closed position (and back into the *open layout position)* at the end of the dive increases her rotational inertia. This slows her rotation rate so that she can enter the water with little splash. Even in a more complicated dive involving both twisting and somersaulting, the rotational momentum of the diver must be conserved, in both magnitude and direction, throughout the dive.

3. **Stars** that have exhausted their nuclear fuel begin to collapse depending on their mass. A star can shrink so much that its radius is reduced from something like that of the Sun (695 500 km) to a few kilometres. The star then becomes a **neutron star** (its material has been compressed to an extremely dense gas of neutrons). During the shrinking process, the star is an isolated system and its angular momentum $\vec{\mathbf{L}}$ is conserved. Since the rotational inertia of the **shrinking star** is greatly reduced, its angular speed is greatly increased.

You may like to do a calculation yourself to see what is the angular speed of a shrinking star about an axis passing through it and compare it with the angular speed of the Sun, which is about one revolution per month. You may like to calculate the change in angular speed of the neutron star.

SAQ 9 – Conservation of angular momentum

A star having rotational inertia of 8.6×10^{48} kgm^2 is rotating at an angular speed of 1 revolution per month about its axis. The only force on it is the force of gravitation. When its nuclear fuel is exhausted, it shrinks to a neutron star having rotational inertia of 4.5×10^{37} kgm^2. Determine the angular speed of the neutron star in revolutions per month.

So far, you have seen some applications of conservation of angular momentum in which the rotational inertia of an object can be changed to alter its angular speed.

> The satellites in space which rotate about an axis are also kept stable using the same law. You use conservation of angular momentum while riding a bicycle or motorcycle to keep their direction steady. A spinning top and a spinning Frisbee remain stable as long as their angular momentum is conserved.

Conservation of angular momentum is also useful for objects which need to be kept stable while rotating about an axis. This happens because when angular momentum is conserved, the *direction of the angular momentum vector is constant*. Therefore,

The axis about which the object spins remains fixed and the rotating object does not topple over: it remains stable while rotating.

In Units 11 and 12, we have introduced many new physical quantities related to angular motion such as angular position, angular displacement, angular velocity, angular acceleration, rotational inertia, torque, rotational kinetic energy, work-energy theorem for rotational motion, rotational analogue of Newton's second law of motion and angular momentum.

While introducing them, we have pointed out that each of these physical quantities has an analogue in translational motion. We list the corresponding quantities and relations for both types of motion in Table 12.1 for ready reference.

Table 12.1: Analogues for translational and rotational motion.

Pure translational motion (fixed direction)		Pure rotational motion (fixed axis)	
Position	x	Angular position	θ
Displacement	Δx	Angular displacement	$\Delta\theta$
Velocity	$v = \frac{dx}{dt}$	Angular velocity	$\omega = \frac{d\theta}{dt}$
Acceleration	$a = \frac{dv}{dt}$	Angular acceleration	$\alpha = \frac{d\omega}{dt}$
Mass	m	Rotational inertia	I
Force	$\vec{\mathbf{F}}$	Torque	$\vec{\tau}$
Newton's second law:	$\vec{\mathbf{F}}_{net} = m\vec{\mathbf{a}}$	Newton's second law:	$\vec{\tau}_{net} = I\vec{\alpha}$
Work done	$W = \int F\,dx$	Work done	$W = \int \tau\,d\theta$
Kinetic energy	$\text{K.E.} = \frac{1}{2}mv^2$	Rotational kinetic energy	$\text{K.E.} = \frac{1}{2}I\omega^2$
Linear momentum	$\vec{\mathbf{p}} = m\vec{\mathbf{v}}$	Angular momentum	$\vec{\mathbf{L}} = \vec{\mathbf{r}} \times \vec{\mathbf{p}}$

With this, we end the discussion on dynamics of angular/rotational motion and summarise what you have studied in the unit.

12.7 SUMMARY

Concept	Description
Dynamics of Uniform Circular Motion	■ The net force required to keep a particle of mass m in uniform circular motion with constant angular speed ω is called the **centripetal force**. Its **magnitude** is: $$F_c = \frac{mv^2}{r}$$ where r is the radius of the circle in which the particle moves and v, its speed. The centripetal force is always directed towards the centre of the circle and its direction changes continuously as the particle moves. In the unit vector notation, we can express the centripetal force as follows: $$\vec{\mathbf{F}}_c = -\frac{mv^2}{r}\hat{\mathbf{r}} = -m\omega^2 r\,\hat{\mathbf{r}}$$

Dynamics of Non-uniform Circular Motion

- The net force on a particle of mass *m* moving with changing angular speed and angular acceleration α in a circular path of radius *r* is given by

$$\vec{\mathbf{F}} = m\vec{\mathbf{a}} = m(\vec{\mathbf{a}}_r + \vec{\mathbf{a}}_t) = \vec{\mathbf{F}}_r + \vec{\mathbf{F}}_t$$

where $\vec{\mathbf{F}}_r$ and $\vec{\mathbf{F}}_t$ are the **radial** (or **centripetal**) and **tangential** components of the force given by

$$\vec{\mathbf{F}}_r = -\frac{mv^2}{r}\hat{\mathbf{r}} = -m\omega^2 r\,\hat{\mathbf{r}} \qquad \text{(radial component)}$$

and $$\vec{\mathbf{F}}_t = m\alpha r\,\hat{\boldsymbol{\theta}} \qquad \text{(tangential component)}$$

The **magnitude** of the net force is given by

$$|\vec{\mathbf{F}}| = \sqrt{F_r^2 + F_t^2} = m\sqrt{(\omega^2 r)^2 + (\alpha r)^2}$$

and the **angle** which it makes with the radial direction is β, given by

$$\tan\beta = \frac{ma_t}{ma_r} = -\frac{\alpha}{\omega^2}$$

Torque

- If a net force $\vec{\mathbf{F}}$ is exerted on a particle situated at a point *P*, which has a position vector $\vec{\mathbf{r}}$ with respect to an origin *O*, the **torque** exerted on the particle with respect to *O* is defined as

$$\vec{\boldsymbol{\tau}} = \vec{\mathbf{r}} \times \vec{\mathbf{F}}$$

Its magnitude is given by $\tau = rF\sin\theta$, where θ is the angle between $\vec{\mathbf{r}}$ and $\vec{\mathbf{F}}$ when these vectors are placed tail to tail. The direction of torque is given by the right-hand rule and is perpendicular to the plane containing $\vec{\mathbf{r}}$ and $\vec{\mathbf{F}}$. Torque is the tendency of a force to change the state of rotational motion of an object about some axis. It is also called the turning effect of force. When several torques act on a particle, the **net torque** or **resultant** torque is the vector sum of individual torques:

$$\vec{\boldsymbol{\tau}}_{net} = \sum_i \vec{\boldsymbol{\tau}}_i$$

Rotational analogue of Newton's second law of motion

- The **rotational analogue of Newton's second law of motion** is

$$\vec{\boldsymbol{\tau}}_{net} = I\vec{\boldsymbol{\alpha}}$$

where $\vec{\boldsymbol{\tau}}_{net}$ is the net external torque on the particle, *I* is the rotational inertia and $\vec{\boldsymbol{\alpha}}$ is the angular acceleration of the particle.

Rotational inertia

- **Rotational inertia** *I* is the **rotational analogue of mass** *m.* It plays the same role in rotational motion that mass plays in translational motion. Just as inertial mass is a measure of a particle's resistance to change in its state of translational motion, the rotational inertia of the particle is a measure of its resistance to change in its rotational motion. The rotational inertia of a particle of mass *m* situated at a distance *r* from the axis of rotation is given by

$$I = mr^2$$

Work done by torque

- The **work done by the net torque** in turning an object from an angle θ_1 to θ_2 is given by

$$W = \int_{\theta_1}^{\theta_2} \tau \, d\theta$$

The work done by a **constant torque** τ which turns an object by an angle θ, is given by

$$W = \tau\,\theta$$

Work-energy theorem and rotational kinetic energy

- The **work-energy theorem for angular motion** takes the form

$$W = \int_{\theta_1}^{\theta_2} \tau \, d\theta = \Delta \text{K.E.} = \frac{1}{2} I \omega_2^2 - \frac{1}{2} I \omega_1^2$$

where $\text{K.E.} = \frac{1}{2} I \omega^2$ is the **rotational kinetic energy** of an object rotating with an angular speed of ω about a fixed axis and having a rotational inertia *I*.

Angular momentum of a particle

- The **angular momentum** $\vec{\mathbf{L}}$ of a particle of mass *m* having position vector $\vec{\mathbf{r}}$ with respect to some origin *O* and linear momentum $\vec{\mathbf{p}} = m\vec{\mathbf{v}}$, is defined as

$$\vec{\mathbf{L}} = \vec{\mathbf{r}} \times \vec{\mathbf{p}}$$

The magnitude of the angular momentum is given by $L = r\,p \sin\theta$ where θ is the angle between $\vec{\mathbf{r}}$ and $\vec{\mathbf{p}}$. The direction of $\vec{\mathbf{L}}$ is given by the right-hand rule and is perpendicular to the plane containing the vectors $\vec{\mathbf{r}}$ and $\vec{\mathbf{p}}$. $\vec{\mathbf{L}}$ is defined with respect to some reference point and its value changes if a different point is chosen as the reference point. The angular momentum of an object rotating with an angular velocity $\vec{\omega}$ about a fixed axis and having a rotational inertia *I* is given by

$$\vec{\mathbf{L}} = I\vec{\omega} \qquad \text{and} \qquad L = I\omega$$

Torque and angular momentum

- The **net torque $\vec{\tau}$ on a particle is equal to the rate of change of angular momentum of the particle:**

$$\vec{\tau} = \frac{d\vec{\mathbf{L}}}{dt}$$

Conservation of angular momentum

- The total angular momentum of a system is **conserved** or remains **constant**, if the net external torque acting on the system is zero:

$$\vec{\mathbf{L}} = \text{constant} \quad \text{if} \quad \vec{\tau} = \vec{\mathbf{0}}$$

12.8 TERMINAL QUESTIONS

1. A stone of mass 5.0×10^{-3} kg is lodged in a bus tyre. The coefficient of static friction between the stone and the tyre treads is 0.65. The stone flies out of the tyre when the speed of the tyre surface is $1.5\ \text{ms}^{-1}$. Assuming that only the force of static friction provides the centripetal force, calculate the radius of the tyre. Take $g = 9.8\ \text{ms}^{-2}$.

2. A cyclist starts from rest and reaches a speed of $5.0\ \text{ms}^{-1}$ in 2.5 s. The radius of the bicycle wheel is 30 cm. Determine the centripetal and tangential components of the net force on a particle of mass 10 g situated at the edge of the tyre. Hence, calculate the net force on the particle.

3. A wheel starts from rest and rotates through 360 radians in 4.0 s under the action of a constant torque. If the rotational inertia of the wheel is $10\,\text{kg}\,\text{m}^2$, what is the torque acting on the wheel?

4. At a given instant of time, the position vector of an object of mass 5.0 kg is defined (in m) by the coordinates $(2t^2, 3t, 4)$. Determine the velocity and acceleration of the object, the angular momentum of the object about the origin and the torque about the origin being exerted on the object.

5. Fig. 12.29 shows a meter stick that is pivoted at the point *O*. The torque for which of the following five forces (of equal magnitude *F*) has the largest magnitude with respect to *O*?

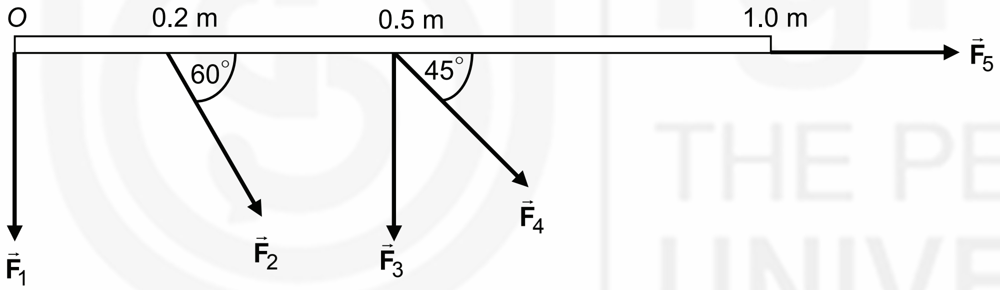


Fig. 12.29

6. A boat of mass 200 kg turns in a circle of radius 30.0 m on a lake. While it is turning, a tangential force of magnitude 500 N is applied on the boat due to the engine thrust. The initial tangential speed of the boat as it starts turning is $5.00\ \text{ms}^{-1}$.

   a) Determine the tangential acceleration of the boat. What is its centripetal acceleration 2.00 s later?

   b) Write the expression of the net force in terms of the unit vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$.

   c) Calculate the magnitude of the net force and the angle the force makes with the radial direction.

7. The moment of inertia of a wheel about its axis of rotation is $3.0\ \text{kg}\,\text{m}^2$. It is initially at rest and then connected to a motor which delivers a constant torque of 30 Nm about its axis. How much work has been done by the motor on the wheel after the wheel has gone through 8.0 revolutions? What is the angular speed of the wheel at that time?

8. A light rod of length 1.0 m rotates in the *xy* plane about a pivot through the rod's centre. Two particles of mass 2.0 kg and 3.0 kg are connected to its ends. Determine the angular momentum of the system at the instant the speed of each particle is $10\ \text{ms}^{-1}$.

9. A girl of mass 20 kg stands at the edge of a merry-go-round having a moment of inertia of $500\,\text{kg}\,\text{m}^2$, and a radius of $5.0\,\text{m}$. The merry-go-round is initially at rest. The girl then starts walking clockwise around the edge of the merry-go-round at a constant speed of $1.5\ \text{ms}^{-1}$.

   a) In what direction and with what angular speed does the merry-go-round rotate?

   b) How much work does the girl do to set herself and the merry-go-round in motion?

10. A merry-go-round possessing rotational inertia of $4500\ \text{kg}\,\text{m}^2$ is mounted on a frictionless vertical axle and is initially rotating at an angular speed of 1 revolution per minute. A girl jumps onto the platform in the radial direction. If the rotational speed of the merry-go-round reduces to 0.9 rpm, calculate the girl's rotational inertia.

12.9 SOLUTIONS AND ANSWERS

Self-Assessment Questions

1. When the magnitude of the centripetal force is twice the magnitude of the tangential force, using Eqs. (12.2b) and (12.2c) we can write,

$$\omega^2 r = 2\alpha r \quad \Rightarrow \quad \omega^2 = 2\alpha \qquad \text{(i)}$$

We now use Eq. (11.8c) with $\omega_0 = 0$ and $\theta_0 = 0$ to find the angle θ through which the particle has moved.

$$\therefore \quad \omega^2 = 2\alpha\theta \quad \Rightarrow \quad \theta = \frac{\omega^2}{2\alpha} \qquad \text{(ii)}$$

Substituting the value of ω^2 from Eq. (i) in Eq. (ii), we get $\theta = \dfrac{2\alpha}{2\alpha} = 1\ \text{rad}$

2. We use Eqs. (12.3a and b) with $r = 5.0\,\text{m}$, $\theta = 30^\circ$ and $F = 40\ \text{N}$:

$$\tau = (5.0\,\text{m}) \times (40\ \text{N}) \times \sin 30^\circ = 1.0 \times 10^2\ \text{Nm}$$

The direction of torque is perpendicular to the plane containing $\vec{\mathbf{r}}$ and $\vec{\mathbf{F}}$ and directed into the plane of the paper.

3. From Eq. (12.3b) with $r = 1.0\,\text{m}$, $\theta = 90^\circ$ and $F = 50\,\text{N}$:

$$\tau = (1.0\,\text{m}) \times (50\,\text{N}) \times \sin 90^\circ = 50\ \text{Nm}$$

From the right-hand rule, it is directed vertically upward, perpendicular to the plane containing $\vec{\mathbf{r}}$ and $\vec{\mathbf{F}}$.

4. It is given that the time interval *t* in which the final angular speed is attained is the **same**. Thus, from Eq. (11.8a), for zero initial angular speed $(\omega_0 = 0)$, the angular acceleration $\alpha = \dfrac{\omega}{t}$ is larger for the higher final

angular speed. Hence from Eq. (12.12), for the same rotational inertia, the **torque increases with increasing angular speed**.

5. The rotational inertia of the bead is given by Eq. (12.11):

$$I = (0.50\,\text{kg}) \times (1.0\,\text{m})^2 = 0.50\,\text{kg}\,\text{m}^2$$

Its angular acceleration is given by Eq. (12.12):

(i) Hence for $\tau = 2.5\,\text{Nm}$, $\alpha = \dfrac{2.5}{0.50}\,\text{rad}\,\text{s}^{-2} = 5.0\,\text{rad}\,\text{s}^{-2}$

(ii) For $\tau = 5.0\,\text{Nm}$, $\alpha = \dfrac{5.0}{0.50}\,\text{rad}\,\text{s}^{-2} = 10\,\text{rad}\,\text{s}^{-2}$

6. a) From Eq. (12.20) and with I given by Eq. (12.11)

$$\text{K.E.} = \frac{1}{2} \times (1\,\text{kg}) \times (1\,\text{m})^2 \times (1\,\text{rad}\,\text{s}^{-1})^2 = 0.5\ \text{J}$$

b) The angular speed of the particle is $\omega = \dfrac{2\pi}{(22.0\,\text{s})} = 0.286\ \text{rad}\text{s}^{-1}$

From Eq. (12.20), its rotational inertia is $I = \dfrac{2(\text{K.E.})}{\omega^2}$

$$\therefore\ I = \frac{2 \times (50.0\,\text{J})}{(0.286\,\text{rad}\,\text{s}^{-1})^2} = 1222.6\,\text{kgm}^2 \approx 1.22 \times 10^3\ \text{kgm}^2$$

7. From Eq. (12.23b), the magnitude of the angular momentum of the particle is $L = mvr\sin\theta$, where θ is the angle between the vectors $\vec{\mathbf{r}}$ and $\vec{\mathbf{p}}$. From Fig. 12.23, the perpendicular distance between O and the line is simply $r\sin\theta$. Thus, the magnitude of the angular momentum is

$$L = (5.0\ \text{kg}) \times (2.0\,\text{ms}^{-1}) \times (2.0\,\text{m}) = 20\ \text{kgm}^2\text{s}^{-1}$$

Its direction is given by the right-hand rule and is perpendicular to the plane containing $\vec{\mathbf{r}}$ and $\vec{\mathbf{p}}$, pointing into the plane of the paper.

8. From Eq. (12.24d), the magnitude of the angular momentum of the child is

$$L = (25\,\text{kg}) \times (2.5\,\text{m})^2 \times \left(\frac{4.0 \times 2\pi}{60}\,\text{rad}\,\text{s}^{-1}\right) = 65.4\,\text{kgm}^2\text{s}^{-1} \approx 65\,\text{kgm}^2\text{s}^{-1}$$

Since the merry-go-round is rotating clockwise, the direction of the angular momentum is downwards along the axis of rotation. From Eq. (12.12), the magnitude of torque is $\tau = I\alpha$, where $\alpha = \dfrac{d\omega}{dt}$.

$$\therefore \tau = (25\,\text{kg}) \times (2.5\,\text{m})^2 \times 0.80\,\text{rad}\,\text{s}^{-2} = 125\ \text{Nm} \approx 1.3 \times 10^2\ \text{Nm}$$

9. The only force is the force of gravitation, for which, as we have seen in Example 12.6, the angular momentum is conserved. The initial angular momentum of the star is $L_i = (8.6 \times 10^{48}\,\text{kgm}^2) \times 1$ revolution per month

The final angular momentum is $L_f = (4.5 \times 10^{37}\,\text{kgm}^2) \times \omega$ where ω is the angular speed of the neutron star.

$$\because L_i = L_f,\ \omega = \frac{8.6 \times 10^{48}\,\text{kgm}^2}{4.5 \times 10^{37}\,\text{kgm}^2}\ \text{rev per month}$$

$$= 1.9 \times 10^{11}\ \text{revolutions per month}$$

Terminal Questions

1. The force of static friction on the stone is

   $F_S = 0.65 \times (5.0 \times 10^{-3}\ \text{kg}) \times (9.8\ \text{ms}^{-2}) = 3.18 \times 10^{-2}\ \text{N}$

   If this force provides the centripetal force, then for $v = 1.5\ \text{ms}^{-1}$, we have

   $$F_S = \frac{mv^2}{R} \quad \Rightarrow \quad R = \frac{mv^2}{F_S},$$ where R is the radius of the tyre.

   $$\therefore \quad R = \frac{(5.0 \times 10^{-3}\ \text{kg}) \times (1.5\ \text{ms}^{-1})^2}{3.18 \times 10^{-2}\ \text{N}} = 0.35\ \text{m}$$

2. We first determine the angular speed and then the angular acceleration of the particle situated at the edge of the tyre in order to calculate the centripetal and tangential components of the net force on it. To determine the angular speed corresponding to the speed $5.0\ \text{ms}^{-1}$, we use the result $v = \omega r$ with $r = 30\ \text{cm} = 0.30\,\text{m}$. Therefore,

   $$\omega = \frac{5.0\ \text{ms}^{-1}}{0.30\,\text{m}} = 16.7\ \text{rads}^{-1}$$

   From Eq. (11.8a), with $\omega_0 = 0,\ t = 2.5\ \text{s}$ and $\omega = 16.7\ \text{rads}^{-1}$

   $$\alpha = \frac{16.7}{2.5}\ \text{rads}^{-2} = 6.68\ \text{rads}^{-2} \approx 6.7\ \text{rads}^{-2}$$

   From Eq. (12.2b), centripetal component of the net force on the particle is

   $$\vec{\mathbf{F}}_r = -(1.0 \times 10^{-2}\ \text{kg}) \times (16.7\ \text{rads}^{-1})^2 \times (0.30\ \text{m})\ \hat{\mathbf{r}} = -0.84\ \text{N}\ \hat{\mathbf{r}}$$

   From Eq. (12.2c), tangential component of the net force on the particle is

   $$\vec{\text{F}}_t = (1.0 \times 10^{-2}\ \text{kg}) \times (6.7\ \text{rads}^{-2}) \times (0.30\ \text{m})\ \hat{\theta} = 2.0 \times 10^{-2}\ \text{N}\ \hat{\theta}$$

   From Eq. (12.2d), the magnitude of the net force on the particle is

   $$F = \sqrt{(-0.84)^2 + (0.02)^2}\ \text{N} = 0.84\ \text{N}$$

   The direction of the force is given by the angle the force makes with the radial direction, which is given by Eq. (12.2e) as

   $$\theta = \tan^{-1} - \frac{6.7\,\text{rads}^{-2}}{(16.7\,\text{rads}^{-1})^2} = \tan^{-1}(-0.024) = 178.6^\circ \approx 179^\circ$$

3. From Eq. (11.8b), the angular acceleration of the wheel is

   $$\alpha = \frac{2 \times 360\,\text{rad}}{(4.0\,\text{s})^2} = 45\ \text{rads}^{-2}$$

   From Eq. (12.12) $\tau = (10\,\text{kgm}^2) \times 45\,\text{rads}^{-2} = 450\ \text{Nm}$

4. The position vector of the object is $\vec{\mathbf{r}} = (2t^2)\text{m}\,\hat{\mathbf{i}} + (3t)\text{m}\,\hat{\mathbf{j}} + 4\text{m}\,\hat{\mathbf{k}}$

   The velocity and the acceleration of the object are

   $$\vec{\mathbf{v}} = \frac{d\vec{\mathbf{r}}}{dt} = \left[\frac{d}{dt}(2t^2)\ \hat{\mathbf{i}} + \frac{d}{dt}(3t)\,\hat{\mathbf{j}} + \frac{d}{dt}(4)\ \hat{\mathbf{k}}\ \right]\text{ms}^{-1} = (4t)\text{ms}^{-1}\,\hat{\mathbf{i}} + 3\,\text{ms}^{-1}\,\hat{\mathbf{j}}$$

   $$\vec{\mathbf{a}} = \frac{d\vec{\mathbf{v}}}{dt} = \frac{d}{dt}(4t)\text{ms}^{-2}\,\hat{\mathbf{i}} + \frac{d}{dt}(3)\text{ms}^{-2}\,\hat{\mathbf{j}} = 4\,\text{ms}^{-2}\,\hat{\mathbf{i}}$$

   The angular momentum of the object is given by Eq. (12.23a).

$$\vec{\mathbf{L}} = (5.0\,\text{kg})[(2t^2\,\text{m}\,\hat{\mathbf{i}} + 3t\,\text{m}\,\hat{\mathbf{j}} + 4\,\text{m}\,\hat{\mathbf{k}}) \times (4t\,\text{ms}^{-1}\,\hat{\mathbf{i}} + 3\,\text{ms}^{-1}\,\hat{\mathbf{j}})]$$

$$= -\,60\,\text{kgm}^2\text{s}^{-1}\,\hat{\mathbf{i}} + 80\,t\,\text{kgm}^2\text{s}^{-1}\,\hat{\mathbf{j}} - 30\,t^2\,\text{kgm}^2\text{s}^{-1}\,\hat{\mathbf{k}}$$

The torque is given by Eq. (12.28): $\vec{\boldsymbol{\tau}} = \dfrac{d\vec{\mathbf{L}}}{dt} = (80\ \text{Nm}\,\hat{\mathbf{j}} - 60t\ \text{Nm}\,\hat{\mathbf{k}})$

5. Torque due to $\vec{\mathbf{F}}_1$ is $\vec{\tau}_1 = \vec{\mathbf{r}}_1 \times \vec{\mathbf{F}}_1 = \vec{\mathbf{0}}$ $(\because\ \vec{\mathbf{r}}_1 = \vec{\mathbf{0}})$

   Magnitude of torque due to $\vec{\mathbf{F}}_2$ is $|\vec{\tau}_2| = |\vec{\mathbf{r}}_2 \times \vec{\mathbf{F}}_2|$

   $$= (0.2\,\text{m}) \times (F \sin 60°\,\text{N}) = 0.17\,F\,\text{Nm} \approx 0.2\,F\,\text{Nm}$$

   Magnitude of torque due to $\vec{\mathbf{F}}_3$ $= (0.5\,\text{m}) \times (F \sin 90°\,\text{N}) = 0.5\,F\,\text{Nm}$

   Magnitude of torque due to $\vec{\mathbf{F}}_4$: $= (0.5\,\text{m}) \times (F \sin 45°\,\text{N}) = 0.35\,F\,\text{Nm}$

   $\vec{\tau}_5 = \vec{\mathbf{r}}_5 \times \vec{\mathbf{F}}_5 = \vec{\mathbf{0}}$ since $\vec{\mathbf{r}}_5$ is parallel to $\vec{\mathbf{F}}_5$. Therefore, the torque due to $\vec{\mathbf{F}}_3$ has the largest magnitude.

6. a) From Eq. (12.2a), the tangential acceleration of the boat is

   $$a_t = \frac{F_t}{m} = \frac{500\,\text{N}}{200\,\text{kg}} = 2.50\,\text{ms}^{-2}$$

   To determine the centripetal acceleration we first need to obtain the angular velocity at 2.0 s. From Eq. (11.16c), the angular acceleration

   α is $\alpha = \dfrac{a_t}{r} = \dfrac{2.50\,\text{ms}^{-2}}{30.0\,\text{m}} = 8.33 \times 10^{-2}\,\text{rad}\,\text{s}^{-2}$

   The initial angular speed $\omega_0 = \dfrac{v_t}{r} = \dfrac{5.00\,\text{ms}^{-1}}{30.0\,\text{m}} = 0.167\,\text{rad}\,\text{s}^{-1}$

   From Eq. (11.8a), we get the angular speed at 2.00 s:

   $$\omega = 0.167\,\text{rad}\,\text{s}^{-1} + 8.33 \times 10^{-2}\,\text{rad}\,\text{s}^{-2} \times 2.00\,\text{s} = 0.334\,\text{rad}\,\text{s}^{-1}$$

   From Eq. (11.16c), the centripetal acceleration is:

   $$\vec{\mathbf{a}}_r = -(0.334\,\text{rad}\,\text{s}^{-1})^2 \times (30.0\,\text{m})\,\hat{\mathbf{r}} = -\,3.35\,\text{ms}^{-2}\,\hat{\mathbf{r}}$$

   The centripetal force is given by:

   $$\vec{\mathbf{F}}_r = m\,\vec{\mathbf{a}}_r = -(200)\text{kg} \times (3.35\,\text{ms}^{-2})\,\hat{\mathbf{r}} = -\,670\,\text{N}\,\hat{\mathbf{r}}$$

   b) The net force is the vector sum of the centripetal and the tangential force: $\vec{\mathbf{F}} = -670\ \text{N}\hat{\mathbf{r}} + 500\text{N}\hat{\boldsymbol{\theta}}$

   c) The magnitude of the net force is $F = \sqrt{(670)^2 + (500)^2}\ \text{N} = 836\ \text{N}$

   From Eq. (12.2e), the angle that the net force makes with the radial direction is

   $$\beta = \tan^{-1}\left(\frac{500\,\text{N}}{-670\,\text{N}}\right) = \tan^{-1}[-0.746] = 143.2° \approx 143°$$

7. The work done by the motor on the wheel is given by Eq. (12.19):

   $$W = 30\text{Nm} \times 8.0 \times 2\pi\ \text{rad} = 1.5 \times 10^3\ \text{J}$$

   To find the angular speed ω of the wheel, we use Eq. (12.22) with

   $W = 1.5 \times 10^3$ J, $I = 3.0$ kgm^2, $\omega_1 = 0$ and $\omega_2 = \omega$.

   $$\frac{1}{2} \times (3.0\,\text{kgm}^2)\omega^2 = 1.5 \times 10^3\ \text{J} \quad \Rightarrow \quad \omega = 31.6\,\text{rads}^{-1} \approx 32\,\text{rads}^{-1}$$

8. The total angular momentum $= (\vec{\mathbf{r}}_1 \times m\vec{\mathbf{v}}_1) + (\vec{\mathbf{r}}_2 \times m\vec{\mathbf{v}}_2)$

$$= (0.5\,\text{m}\,\hat{\mathbf{i}}) \times (2.0\ \text{kg} \times 10\,\text{ms}^{-1}\,\hat{\mathbf{j}}) + (-0.5\,\text{m}\,\hat{\mathbf{i}}) \times (3.0\ \text{kg} \times [-10\,\text{ms}^{-1}\,\hat{\mathbf{j}}])$$

$$= (10\ \text{kgm}^2\,\text{s}^{-1})(\hat{\mathbf{i}} \times \hat{\mathbf{j}}) + (15\ \text{kgm}^2\,\text{s}^{-1})(\hat{\mathbf{i}} \times \hat{\mathbf{j}}) = (25\ \text{kgm}^2\,\text{s}^{-1})\hat{\mathbf{k}}$$

9. a) We apply the principle of conservation of angular momentum.
Initial angular momentum of the system = $\vec{\mathbf{L}}_i = \vec{\mathbf{0}}$. From Eq. (12.24e), the final angular momentum is

$\vec{\mathbf{L}}_f =$ angular momentum of the merry-go-round + angular momentum of the girl

$$= 500\,\text{kgm}^2\,\vec{\omega} \;\; - 20\,\text{kg} \times (5.0\,\text{m})^2 \;\; \frac{1.5}{5.0}\text{s}^{-1}\;\; \hat{\mathbf{k}} \quad (\because v = r\omega) \qquad \text{(i)}$$

Here $\hat{\mathbf{k}}$ is the unit vector along the direction of the axis of the merry-go-round. Since $\vec{\mathbf{L}}_i = \vec{\mathbf{L}}_f$,

$$500\,\text{kgm}^2\,\vec{\omega} \;\; - 150\,\text{kgm}^2\,\text{s}^{-1}\hat{\mathbf{k}} = \vec{\mathbf{0}} \qquad \vec{\omega} \quad 0.30\ \text{rad}\,\text{s}^{-1}\hat{\mathbf{k}}$$

The merry-go-round rotates in the direction opposite to the girl, i.e., in the counter-clockwise direction.

b) From the work-energy theorem, the work done by the girl is equal to the change in the kinetic energy of the merry-go-round and the girl. Since initially both were at rest, from Eq. (12.20) for the rotational kinetic energy of the merry-go-round, we get,

$$\Delta\text{K.E.} = \frac{1}{2} m_g v_g^2 + \frac{1}{2} I_m \omega^2 = W$$

Here m_g and v_g are the mass and the speed of the girl, I_m is the moment of inertia of the merry-go-round and ω its angular speed. Thus,

$$W = \frac{1}{2} \times (20\,\text{kg}) \times (1.5\,\text{ms}^{-1})^2 + \frac{1}{2} \times (500\,\text{kgm}^2) \times (0.30\,\text{rad}\,\text{s}^{-1})^2 = 45\,\text{J}$$

10. Since the axle is frictionless, the net external torque on the merry-go-round is zero and its angular momentum is conserved: $L_i = L_f$. Here,

$$L_i = (4500\ \text{kgm}^2) \times \left(\frac{2\pi}{60}\text{rad}\,\text{s}^{-1}\right)$$

Let the moment of inertia of the girl be I_g. Then

$$\text{L}_\text{f} = (4500\ \text{kg m}^2 + \text{I}_\text{g}) \times \;\; 0.9 \times \frac{2\pi}{60}\text{rad}\,\text{s}^{-1}$$

$$\therefore (4500\ \text{kg m}^2 + I_g) \times \;\; 0.9 \times \frac{2\pi}{60}\text{rad}\,\text{s}^{-1} \quad = 4500\,\text{kgm}^2 \times \left(\frac{2\pi}{60}\text{rad}\,\text{s}^{-1}\right)$$

$$I_g = \frac{4500}{0.9}\text{kgm}^2 - 4500\,\text{kgm}^2 = 500\,\text{kgm}^2 = 5 \times 10^2\,\text{kgm}^2$$

What is the maximum distance of comet Halley, which returns to the Earth once in 76 years on an average, from the Sun? You will learn the answer in this unit!

UNIT 13

MOTION UNDER CENTRAL FORCES

Structure

STUDY GUIDE

In this unit, you will learn the concepts of **central force** and **central conservative force** and study the **general features of motion** under central conservative forces. You will obtain the **equation of motion** for an object moving under such forces. You are not expected to solve this equation but we expect you to know its **general solution for the force of gravitation** given in this unit. You may be learning the concepts discussed in this unit **for the first time**. We have tried to keep the mathematics simple and given all steps. For understanding the concepts in this unit well, you should know the concepts of work, energy, conservative force and conservation of mechanical energy explained in Units 9 and 10 of Block 2. You should revise the concepts of angular momentum and torque explained in Unit 12, particularly for forces directed along the radial direction. You should also revisit the concepts of **scalar** and **vector products** given in Units 1 and 2 in Block 1. We advise you once again to solve all examples, SAQs and Terminal Questions on your own!

"The diversity of the phenomena of nature is so great, and the treasures hidden in the heavens so rich, precisely in order that the human mind shall never be lacking in fresh nourishment."

Johannes Kepler

13.1 INTRODUCTION

In Block 2 and Units 11 and 12 of this block, you have studied Newtonian mechanics including concepts related to work, energy and angular motion. **Newtonian mechanics** gives us both the physical insight and the mathematical tools required to study the **motion of all objects in the Universe**. You have learnt that Newtonian mechanics applies to the motion of macroscopic objects around us and in the entire universe. In this unit, you will study the motion of objects under a special category of forces called the **central forces** or **centre seeking** forces. We have already introduced two such forces in Units 6 and 7 of Block 2: the **force of gravitation** and the **spring force**, which are also conservative. In Secs. 13.2 and 13.3 of this unit, you will learn the **formal definition of central forces and central conservative forces**.

In Sec. 13.4, we discuss the special **properties of motion under central forces**. These properties give us a qualitative understanding of the motion of objects under any type of central force. We also discuss the motion of objects under a certain type of central force: the **central conservative force**. You know that the force of gravitation is a conservative force. It is also a central force. Therefore, it is a central conservative force. In Sec. 13.5, you will study in greater detail, **the motion of objects under inverse square forces with special focus on the force of gravitation**. You will learn how to calculate the orbits of objects in the solar system. It will also become clear to you how Kepler's laws of planetary motion are a simple consequence of Newton's laws of motion and gravitation. Finally, you will apply these concepts to objects moving around the Earth or other planets. In the next unit, you will study the motion of many-particle systems.

Expected Learning Outcomes

After studying this unit, you should be able to:

- ❖ define and identify central forces and central conservative forces;
- ❖ explain the properties of motion under central conservative forces;
- ❖ derive the law of equal areas for central forces;
- ❖ write down the equation of motion for an object moving under the force of gravitation, and its general solution;
- ❖ write down the conditions on eccentricity for which the path of an object moving under an inverse square force is a circle, ellipse, hyperbola or parabola;
- ❖ determine the total mechanical energy and time period of an object moving in an elliptical orbit under the force of gravitation;
- ❖ apply Kepler's laws of planetary motion; and
- ❖ calculate orbit parameters like the eccentricity, lengths of semi-major and semi-minor axes, aphelion (apogee) and perihelion (perigee) for an object in the solar system moving under the gravitational force of the Sun (Earth).

13.2 WHAT IS A CENTRAL FORCE?

Let us consider the following forces, which you know very well from your school physics and Block 2:

- The **force of gravitation** due to a point mass m_1 on another point mass m_2 separated from it by a distance r (Fig. 13.1). It has a magnitude of $\dfrac{Gm_1m_2}{r^2}$ and is directed towards m_1 along the line joining the two masses.
- The **restoring force** $\vec{\mathbf{F}}$ on a mass due to a stretched spring. It is proportional to the length x by which the spring is stretched (Fig. 13.2) and directed along the spring opposite to the displacement $\vec{\mathbf{x}}$ of the mass.
- The **electrostatic force** due to a point charge q on another point charge Q at a distance r (Fig. 13.3). It has magnitude $k\dfrac{qQ}{r^2}$, where k is a constant.

  This is also called the **Coulomb force**. From Fig. 13.3a, you can see that the force is directed towards q when it is attractive (between unlike charges, that is one of the charges is positive and the other negative). Fig.13.3b shows that it is directed away from q when it is repulsive (between like charges).

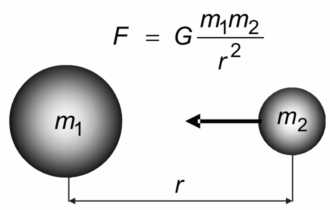


Fig. 13.1: **The attractive force of gravitation on a point mass m_2 due to another point mass m_1 separated by distance r from it.**

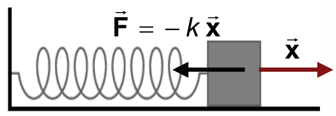


Fig. 13.2: **The restoring force $\vec{\mathbf{F}}$ on a mass attached to a spring.**

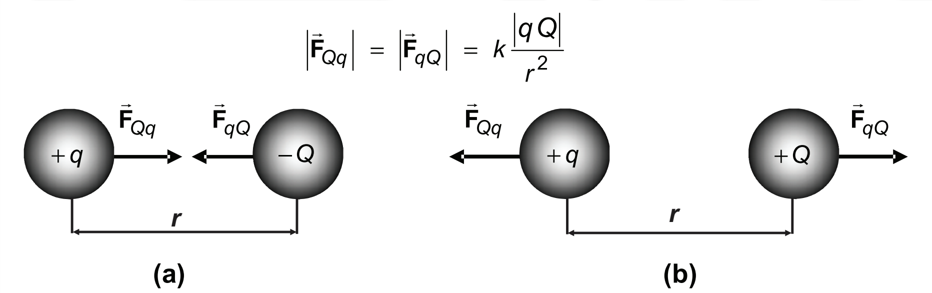


Fig. 13.3: **a) The forces $\vec{\mathbf{F}}_{Qq}$ and $\vec{\mathbf{F}}_{qQ}$ between two unlike charges are attractive; b) the forces $\vec{\mathbf{F}}_{Qq}$ and $\vec{\mathbf{F}}_{qQ}$ between two like charges are repulsive.**

NOTE

The notation for force adopted here is:

$\vec{\mathbf{F}}_{qQ}$ is the force on Q due to q.

$\vec{\mathbf{F}}_{Qq}$ is the force on q due to Q.

You can see that these forces describe very different physical situations. Yet, **do you think that they have something in common**?

To find the answer, consider the **direction of the force** in each of these examples.

Did you notice that in each case, **the force is always directed towards or away from a particular point**? **Also this point remains fixed, that is, it remains the same for a given force**.

For example, the gravitational force on m_2 due to m_1 is **always directed towards the position of the mass** m_1. This is an example of a force called the **central force**.

From Figs. 13.2 and 13.3, you can see that the **spring force** and the **electrostatic** forces are also examples of **central forces**.

Let us now give a formal definition of the central force.

CENTRAL FORCE

A central force is a force which is always directed towards or away from a particular point, which remains fixed. The fixed point is called the centre of force. The force is always directed towards or away from the centre of force.

Let us now write down the mathematical expression for the central force $\vec{\mathbf{F}}$ of magnitude F acting on a particle:

Central force

$$\vec{\mathbf{F}} = F\,\hat{\mathbf{r}} \tag{13.1}$$

NOTE

When we say that the centre of force is fixed, we mean that it remains the same for a given force. It may or may not move in space. For example, the gravitational force on the Earth or any other planet due to the Sun always points towards the Sun. The centre of force is at the position of the Sun. So, even though the Sun moves in space, the centre of the force remains fixed at the Sun for the force of gravitation exerted by the Sun on the planets.

Here $\hat{\mathbf{r}}$ **is the unit vector along the radial direction** (see Fig. 13.4a). **It lies along the line joining the particle to the centre of the force and is directed away from the centre of force** O **towards the particle**. Thus, a **central force is always directed along the radial direction**. **For systems of two particles** (as in the case of gravitational or electrostatic forces between two point masses or two point charges), **it is always directed along the line joining the particles**. That is why it is also called the **radial force**.

Note that the direction of the unit vector $\hat{\mathbf{r}}$ would change as the particle moves under the influence of the force. Recall Example 7.1 of a geosynchronous satellite held in its orbit by the force of gravitation, which you have studied in Unit 7. The force on the satellite is always directed towards the centre of the Earth. Thus, the direction of the unit vector along the force changes at every point of its orbit (see Fig. 13.4b).

In general, the function F of Eq. (13.1) depends on the spherical polar coordinates (r, θ, ϕ) in the three-dimensional space. This discussion is being kept beyond the scope of this course.

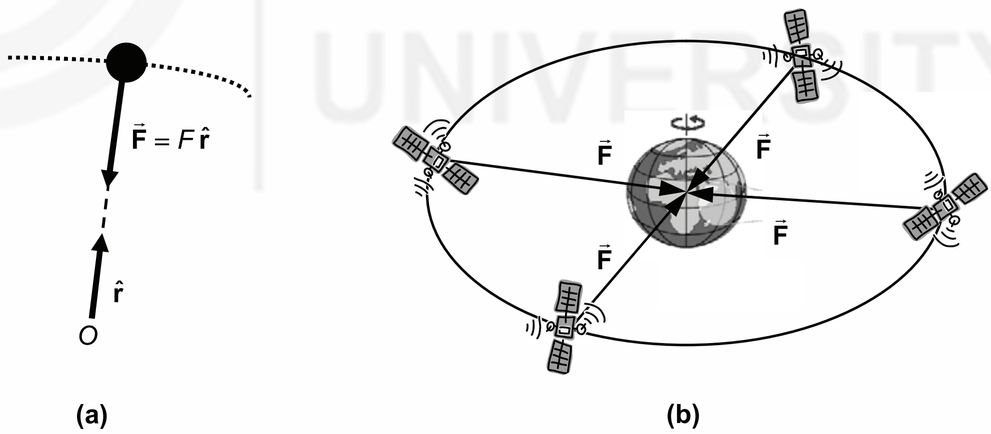


Fig. 13.4: **a) A central force $\vec{\mathbf{F}}$ with O as the centre of force. Note that $\hat{\mathbf{r}}$ is the unit vector along the radial direction. It points away from the centre of force; b) the force of gravitation $\vec{\mathbf{F}}$ on a satellite due to the Earth is directed towards the centre of the Earth as it orbits the Earth. It is a central force.**

What did you notice about the **magnitude of the force** in the examples of the central forces you have studied so far? In all these examples, **the magnitude of the force on a particle depends only on its distance from the centre of force** (apart from other constant physical quantities such as the mass, the charge or the spring constant, etc.). Therefore, in Eq. (13.1), we can substitute the expression $F = f(r)$ for the magnitude of the force and write the force as

$$\vec{\mathbf{F}} = f(r)\,\hat{\mathbf{r}} \qquad (13.2)$$

Central conservative force

Eq. (13.2) tells us that the **magnitude of $\vec{\mathbf{F}}$ given by $f(r)$ depends only on the <u>distance</u> of the particle from the centre of the force**. The forces defined by Eq. (13.2) are called **central conservative forces**. Why they are called so will become clear to you when you study the next section. But before studying further, you may like to identify central forces using Eq. (13.1).

SAQ 1 – **Identifying central forces**

Which of the following forces are central? (**Hint**: You have to determine which of the forces are directed only along the radial direction $\hat{\mathbf{r}}$.)

a) The force $\vec{\mathbf{F}} = -k\vec{\mathbf{r}}$ for a simple harmonic oscillator oscillating in space, where $\vec{\mathbf{r}}$ is the position vector of the oscillator and *k* is a constant.

b) Force $\vec{\mathbf{F}} = ar\,\hat{\mathbf{r}} + br\,\hat{\boldsymbol{\theta}}$ on a particle rotating with changing angular velocity. Here *a* and *b* are constants.

c) Force $\vec{\mathbf{F}} = -(k/r^3)\,\hat{\mathbf{r}}$, where *k* is a constant.

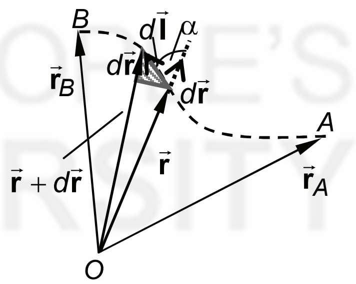
13.3 WHAT IS A CENTRAL CONSERVATIVE FORCE?

Consider a particle moving along the path *AB* (Fig. 13.5a) in space under the influence of the force $\vec{\mathbf{F}} = f(r)\hat{\mathbf{r}}$. Recall Eq. (9.11) from Sec. 9.4 of Unit 9, which defines the work done by a force $\vec{\mathbf{F}}$ in moving the particle from any given point to another. From Eq. (9.11), the work done by any force $\vec{\mathbf{F}}$ in moving a particle from a point *A* to point *B* along its path is given by

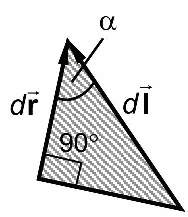
$$W = \int_A^B \vec{\mathbf{F}}.d\vec{\mathbf{l}} \qquad (13.3)$$

where $d\vec{\mathbf{l}}$ is the differential element of displacement along the path. Note that we have used a different symbol for the element of displacement in Eq. (13.3). Let us substitute the expression of the force from Eq. (13.2) in Eq. (13.3). Then the work done by the force $\vec{\mathbf{F}} = f(r)\hat{\mathbf{r}}$ in taking a particle from the point *A* to the point *B* is given by

$$W_{AB} = \int_A^B \vec{\mathbf{F}}.d\vec{\mathbf{l}} = \int_A^B f(r)\hat{\mathbf{r}}.d\vec{\mathbf{l}} \qquad (13.4a)$$



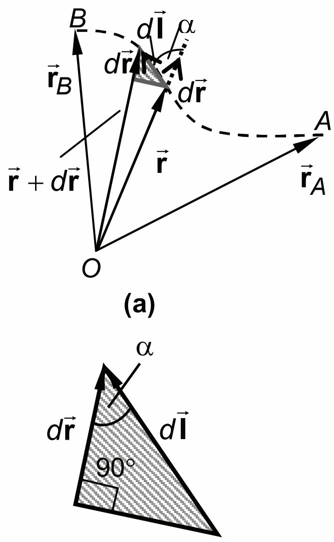
(a)



$$\cos\alpha = \frac{dr}{dl}$$
$$\therefore\ dr = dl\cos\alpha$$

(b)

Fig. 13.5: **Work done by a central conservative force.**



$$\cos\alpha = \frac{dr}{dl}$$

$$\therefore dr = dl \cos\alpha$$

(b)

Fig. 13.5: **Work done by a central conservative force**.

Let us now obtain the value of $\hat{\mathbf{r}}.d\vec{\mathbf{l}}$. From Fig. 13.5a, note that $d\vec{\mathbf{l}}$ is a very small displacement along the path of the particle and $d\vec{\mathbf{r}}$ is a small element along the radius vector $\vec{\mathbf{r}} + d\vec{\mathbf{r}}$. Let α be the angle between the element $d\vec{\mathbf{l}}$ and $\vec{\mathbf{r}} + d\vec{\mathbf{r}}$. Since $d\vec{\mathbf{l}}$ is very small, α is approximately equal to the angle between the element $d\vec{\mathbf{l}}$ and the radius vector $\vec{\mathbf{r}}$ or the unit vector $\hat{\mathbf{r}}$ along $\vec{\mathbf{r}}$.

Using the definition of the scalar product, we can write

$$\hat{\mathbf{r}}.d\vec{\mathbf{l}} = dl \cos\alpha \tag{13.4b}$$

Now study Fig. 13.5b, which magnifies the shaded area of Fig. 13.5a containing the elements $d\vec{\mathbf{l}}$ and $d\vec{\mathbf{r}}$ along with the angle α between them. Let *dl* and *dr* be the magnitudes of vectors $d\vec{\mathbf{l}}$ and $d\vec{\mathbf{r}}$, respectively. From the right-angled triangle in Fig. 13.5b, you can see that $dl \cos\alpha = dr$ and hence

$$\hat{\mathbf{r}}.d\vec{\mathbf{l}} = dl \cos\alpha = dr \tag{13.4c}$$

Substituting Eq. (13.4c) in Eq. (13.4a), we get

$$W_{AB} = \int_{r_A}^{r_B} f(r)\,dr \tag{13.5}$$

Eq. (13.5) tells us that the work done by a force of the form $\vec{\mathbf{F}} = f(r)\hat{\mathbf{r}}$ on a particle moving between any two points *A* and *B* depends only on the **distance of these points from the centre of force**. It is **<u>independent of the actual path followed by the particle between these two points</u>**. This is just the definition of a conservative force that you have learnt in Sec. 10.2 of Unit 10. Thus, Eq. (13.5) along with the definition of the conservative force tells us that **forces of the form** $\vec{\mathbf{F}} = f(r)\hat{\mathbf{r}}$ are **conservative**.

You know from Eq. (13.1) that these are also **central**. Hence such **<u>central forces, which are also conservative, are called central conservative forces</u>**. Let us now give a formal definition of a central conservative force.

NOTE

Note that in Eq. (13.5) we have written the limits of integration from point *A* to point *B* as r_A and r_B since now the variable of integration is *r*.

Eq. (13.5) is another way of defining a conservative force. But Eq. (13.5) also tells us that central forces of the form $\vec{\mathbf{F}} = f(r)\hat{\mathbf{r}}$ are conservative.

Recall from Unit 10 that conservative forces are called so because the total mechanical energy of the objects moving under such forces is conserved.

CENTRAL CONSERVATIVE FORCE

A <u>central conservative force</u> is always directed towards or away from a particular point, which remains fixed. Its magnitude depends only on the <u>distance</u> of the particle from the centre of the force. Thus, all forces of the form

$$\vec{\mathbf{F}} = f(r)\hat{\mathbf{r}}$$

are **central conservative forces**. For such forces, the work done in taking a particle from one point to another is given by Eq. (13.5). It depends only on the **<u>distance</u> of the particle from the centre of force at those points** and **<u>not</u> on the path followed** by the particle. **<u>All conservative forces are central.</u>**

Before studying the motion of objects under central conservative forces, you may like to identify some such forces using Eq. (13.2) in the following SAQ.

SAQ 2 – **Identifying central conservative forces**

Which of the following forces are central conservative forces? (**Hint**: You have to determine which of the forces have magnitudes depending only on r and are directed only along the radial direction $\hat{\mathbf{r}}$.)

a) The force $\vec{\mathbf{F}} = -kx\,\hat{\mathbf{x}} - b\vec{\mathbf{v}}$ on a damped harmonic oscillator moving along the x-axis, where $\vec{\mathbf{v}}$ is the velocity of the oscillator.

b) The force $\vec{\mathbf{F}} = -\left(\dfrac{K}{r^3}\right)\hat{\mathbf{r}}$, where K is a constant.

c) The force $\vec{\mathbf{F}} = k\dfrac{\cos\theta}{r^3}\hat{\mathbf{r}}$, where k is a constant.

13.4 MOTION UNDER CENTRAL CONSERVATIVE FORCES

Let us now study motion under central conservative forces. We can obtain the positions and velocities of the particles moving under any central conservative force by solving the **equation of motion**. Let us write down the **equation of motion** of a particle of mass m moving with acceleration $\vec{\mathbf{a}}$ under the influence of a central conservative force. For this, we substitute the expression of $\vec{\mathbf{F}}$ given by Eq. (13.2) in Newton' second law of motion and get

$$m\vec{\mathbf{a}} = f(r)\hat{\mathbf{r}} \qquad \text{or} \qquad m\frac{d^2\vec{\mathbf{r}}}{dt^2} = f(r)\hat{\mathbf{r}} \tag{13.6}$$

Now, the equation of motion would be different for different forms of the function $f(r)$, that is, for different central conservative forces. Hence, the solutions of Eq. (13.6) would also be different. However, motion under such forces has some **general features** that make it easier for us to solve the equation of motion. We begin our study of motion under a central conservative force by understanding these general features.

NOTE

For example, for the force of gravitation between particles of mass *m* and *M*, respectively situated at a distance of *r* from each other,

$$f(r) = \frac{GMm}{r^2}$$

and for the electrostatic force between two charges *q* and *Q* situated at a distance of *r* from each other

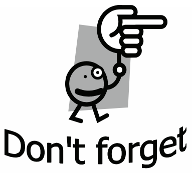
$$f(r) = k\frac{|qQ|}{r^2}$$

13.4.1 Features of Motion under Central Conservative Forces

Even without knowing the actual form of the function $f(r)$, it is possible to deduce certain properties of the motion of objects under central conservative forces. We now explain these features.

1. The Total Mechanical Energy is Constant.

You have learnt in Unit 10 that the total mechanical energy of a particle moving under the influence of a conservative force is conserved. This result is also true for central forces that are conservative. Therefore,



For motion under central conservative forces, the total mechanical energy is conserved.

2. The Angular Momentum is Constant.

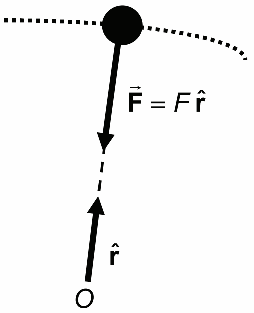


Fig. 13.6: **Torque due to central force about the centre of force is zero because the force $\vec{\mathbf{F}}$ is parallel to $\vec{\mathbf{r}}$.**

Recall the definition of a central force – it is always directed towards or away from a fixed point. This results in an interesting property. Let us find the torque $\vec{\tau}$ due to a central force $\vec{\mathbf{F}} = F\hat{\mathbf{r}}$ about the centre of force (see Fig. 13.6). Since $\hat{\mathbf{r}}$ is a unit vector along the vector $\vec{\mathbf{r}}$, we have:

$$\vec{\tau} = \vec{\mathbf{r}} \times \vec{\mathbf{F}} = \vec{\mathbf{r}} \times F\hat{\mathbf{r}} = r\hat{\mathbf{r}} \times F\hat{\mathbf{r}} \tag{13.7a}$$

where we have substituted $\vec{\mathbf{r}} = r\hat{\mathbf{r}}$ in Eq. (13.7a). Here r is the magnitude of the vector $\vec{\mathbf{r}}$. Now even without knowing what kind of a function F is, can you say what the value of $\vec{\tau}$ is in Eq. (13.7a)? It is the **zero vector**. Can you say why? This is because the vector product of $\hat{\mathbf{r}}$ with itself is zero:

$$\hat{\mathbf{r}} \times \hat{\mathbf{r}} = \vec{\mathbf{0}} \qquad (\because |\hat{\mathbf{r}} \times \hat{\mathbf{r}}| = |\hat{\mathbf{r}}||\hat{\mathbf{r}}| \sin 0^\circ = 0) \tag{13.7b}$$

Substituting Eq. (13.7b) in Eq. (13.7a), we get

$$\vec{\tau} = \vec{\mathbf{0}} \qquad \textbf{for central forces} \tag{13.7c}$$

Thus, **the torque due to a central force about the centre of force is always zero**. Notice that this is true for any type of central force and not just a central conservative force, whatever **its magnitude** F **may be**.

Now recall from Eq. (12.28) of Unit 12 that the **torque due to a force about a point is equal to the rate of change of angular momentum $\vec{\mathbf{L}}$ about that point**:

$$\vec{\tau} = \frac{d\vec{\mathbf{L}}}{dt} \tag{13.7d}$$

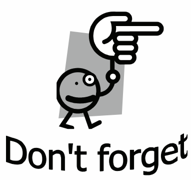
Hence, for central forces:

$$\vec{\tau} = \vec{\mathbf{0}} \quad \text{or} \quad \frac{d\vec{\mathbf{L}}}{dt} = \vec{\mathbf{0}} \quad \Rightarrow \quad \vec{\mathbf{L}} = \textbf{constant} \tag{13.7e}$$

Thus, we get the result that

$$\vec{\mathbf{L}} = \textbf{constant} \quad \textbf{for central and central conservative forces} \tag{13.8}$$

Here $\vec{\mathbf{L}}$ is a vector quantity. Therefore, you can say that



For central and central conservative forces, the angular momentum about the centre of force is constant both in magnitude and direction.

The fact that the angular momentum is conserved in central force motion, leads to another interesting feature of motion under central forces.

3. Particles Acted upon by Central Forces Move only in a Plane.

Recall Eq. (12.23a) of Unit 12 which defines angular momentum. You know that we can write the angular momentum of a particle of mass *m* about a point *O* as,

$$\vec{\mathbf{L}} = \vec{\mathbf{r}} \times \vec{\mathbf{p}} = \vec{\mathbf{r}} \times m\vec{\mathbf{v}} \qquad (13.9)$$

where $\vec{\mathbf{p}} = m\vec{\mathbf{v}}$ is the momentum of the particle, $\vec{\mathbf{v}}$ its velocity and $\vec{\mathbf{r}}$, its position vector with respect to *O*. Eq. (13.9) tells us that $\vec{\mathbf{L}}$ is always perpendicular to the plane containing $\vec{\mathbf{r}}$ and $\vec{\mathbf{v}}$ (see Fig. 13.7). Why is that so?

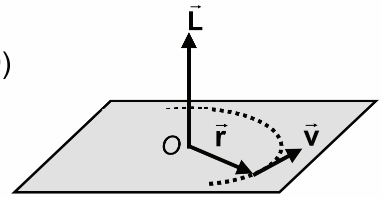
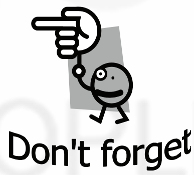


Fig. 13.7: **A particle having constant angular momentum $\vec{\mathbf{L}}$ moves in a fixed plane perpendicular to $\vec{\mathbf{L}}$.**

Recall Example 12.6 from Unit 12 as well as the definition of the cross product of vectors. You have learnt that the direction of the vector $\vec{\mathbf{c}} = (\vec{\mathbf{a}} \times \vec{\mathbf{b}})$ is given by the right-hand rule and is perpendicular to the plane containing the vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$. You know from Eq. (13.8) that for a central force, the direction of $\vec{\mathbf{L}}$ is fixed. This means that **the plane in the direction perpendicular to it is fixed** (see Fig. 13.7). That is, **the plane containing $\vec{\mathbf{r}}$ and $\vec{\mathbf{v}}$ is fixed**. Since $\vec{\mathbf{r}}$ and $\vec{\mathbf{v}}$ determine the path of the particle, the plane containing $\vec{\mathbf{r}}$ and $\vec{\mathbf{v}}$ is the plane in which the particle moves. Therefore,

> **Under a central force, a particle always moves in a fixed plane perpendicular to the direction of its angular momentum $\vec{\mathbf{L}}$.**



Once again, this feature of motion under central forces holds for any type of central force. It does not depend upon the nature of the central force as described by *F*.

Let us now summarise these features of motion under central conservative forces.

Recap

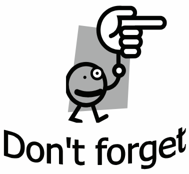
FEATURES OF MOTION UNDER CENTRAL CONSERVATIVE FORCES

For motion under central conservative forces,

1. **The total mechanical energy is constant**.
2. **The angular momentum about the centre of force is constant both in magnitude and direction**.
3. **The particle always moves in a plane perpendicular to the direction of the angular momentum**.

The last two features also hold for motion under those central forces, which are not conservative.

The physical quantities conserved in any motion, are called the **constants of motion.** We have, so far, identified two constants of motion.



Two constants of motion for motion under central conservative forces: the total mechanical energy and the angular momentum.

CONSTANTS OF MOTION FOR CENTRAL CONSERVATIVE FORCES

The constants of motion for a central conservative force are

- **the angular momentum about the centre of force and perpendicular to the path of the particle, and**
- **the total mechanical energy.**

We now take up an interesting application of the fact that angular momentum is conserved for motion under central and central conservative forces. This is the **law of equal areas**, which you already know in a different form as one of Kepler's laws of planetary motion. But before studying further, you should check whether you have understood the features of motion under central conservative forces.

SAQ 3 – **Motion under central conservative forces**

For each of the following forces, identify the constants of motion. For which forces would the particle move in a plane? (**Hint**: You have to determine which of the forces are central and which ones central conservative.)

a) The force $\vec{\mathbf{F}} = -k\,x\,\hat{\mathbf{x}} - b\,\vec{\mathbf{v}}$ of a damped harmonic oscillator moving along the *x*-axis, where $\vec{\mathbf{v}}$ is the velocity of the oscillator.

b) The force $\vec{\mathbf{F}} = -\left(\dfrac{k\cos\theta}{r^3}\right)\hat{\mathbf{r}}$, where k is a constant.

c) The force $\vec{\mathbf{F}} = -\dfrac{GmM}{r^2}\hat{\mathbf{r}}$, where G, m and M are constants.

In order to arrive at the law of equal areas, we first need to determine the expression for the **angular momentum of a particle moving in the *xy* plane under the influence of a central force**.

13.4.2 Angular Momentum for Motion under Central Force

Consider a particle of mass m moving in a plane under the influence of a central force. Since the particle always moves in a plane, we can describe its motion using a **two-dimensional coordinate system** such that the **centre of the force is at the origin**. We can study the motion using **plane polar coordinates** r and θ of a given point. We start from the definition of the angular momentum and use Eq. (12.23a) to get the expression for $\vec{\mathbf{L}}$ about the centre of the force (which we have taken at the origin). So we have

$$\vec{\mathbf{L}} = \vec{\mathbf{r}} \times \vec{\mathbf{p}} = \vec{\mathbf{r}} \times m\vec{\mathbf{v}} = \vec{\mathbf{r}} \times m\frac{d\vec{\mathbf{r}}}{dt} \qquad (13.10)$$

Substituting $\vec{\mathbf{r}} = r\hat{\mathbf{r}}$ in Eq. (13.10), we can write

$$\vec{\mathbf{L}} = r\hat{\mathbf{r}} \times m\frac{d(r\hat{\mathbf{r}})}{dt} = r\hat{\mathbf{r}} \times m\left(\hat{\mathbf{r}}\frac{dr}{dt} + r\frac{d\hat{\mathbf{r}}}{dt}\right)$$

or $$\vec{\mathbf{L}} = mr\frac{dr}{dt}(\hat{\mathbf{r}} \times \hat{\mathbf{r}}) + mr^2\left(\hat{\mathbf{r}} \times \frac{d\hat{\mathbf{r}}}{dt}\right) = \vec{\mathbf{0}} + mr^2\left(\hat{\mathbf{r}} \times \frac{d\hat{\mathbf{r}}}{dt}\right) \qquad (13.11)$$

The first term in the RHS of Eq. (13.11) is a null vector because $\vec{\mathbf{r}} \times \vec{\mathbf{r}} = \vec{\mathbf{0}}$.

We can now use the result $\dfrac{d\hat{\mathbf{r}}}{dt} = \dot{\theta}\hat{\boldsymbol{\theta}}$ (read the margin remark) in Eq. (13.11) to obtain the expression for angular momentum:

$$\vec{\mathbf{L}} = mr^2(\hat{\mathbf{r}} \times \dot{\theta}\hat{\boldsymbol{\theta}}) = mr^2\dot{\theta}(\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}}) \qquad (13.12)$$

Since the unit vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ are in the *xy* plane, their vector product will be the unit vector perpendicular to the *xy* plane. Its direction will be given by the right-hand rule and will be along the *z*-axis. Denoting this unit vector by $\hat{\mathbf{k}}$, we have $\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}} = \hat{\mathbf{k}}$, and hence

$$\boxed{\vec{\mathbf{L}} = mr^2\dot{\theta}\hat{\mathbf{k}}} \qquad (13.13)$$

This is the expression of the **angular momentum of a particle of mass *m* moving in the *xy* plane under the influence of a central force**, **about the centre of force**. We now derive the law of equal areas, which is just the conservation of angular momentum in a central force field stated in a different form.

13.4.3 The Law of Equal Areas

Study Fig. 13.8a. It shows the path *PAB* of a particle moving under a central force $\vec{\mathbf{F}}$. Let $\vec{\mathbf{r}}$ be the position vector of the particle at an instant of time *t*, at the point *A* of its path. Since the force is central, the particle moves only in a plane. So we need to use only two coordinates to describe its motion. Let the coordinates of the point *A* be (r, θ). Suppose the particle moves from *A* to *B* in time Δt, where its position vector is $\vec{\mathbf{r}} + \Delta\vec{\mathbf{r}}$. Let the coordinates of *B* be $(r + \Delta r, \theta + \Delta\theta)$.

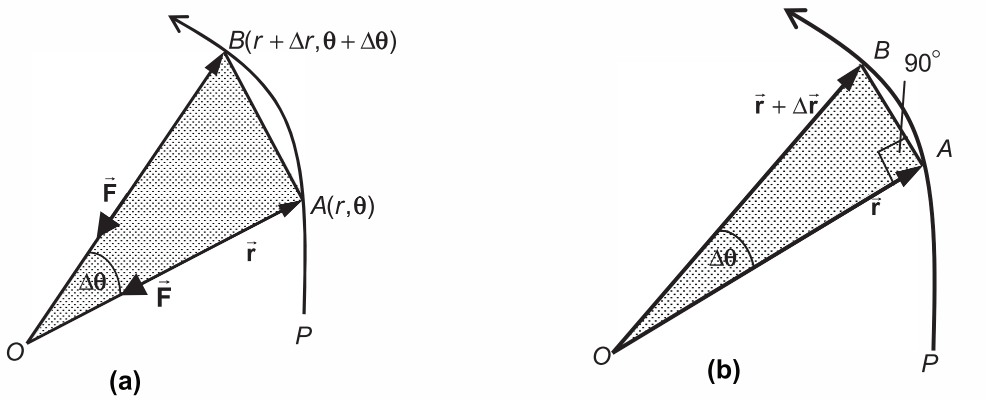
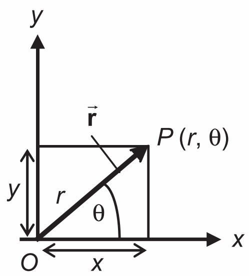


Fig. 13.8: **Arriving at the law of equal areas. a)** *PAB* **is the path of the particle during the time interval** Δt; **b) the path for very small values of** $\Delta\theta$.



The plane polar coordinates *r* and θ of the point *P* are given by:

$x = r\cos\theta$; $y = r\sin\theta$

$r = \sqrt{x^2 + y^2}$; $\tan\theta = y/x$

The position vector $\vec{\mathbf{r}}$ is then:

$\vec{\mathbf{r}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$

$\quad = r\cos\theta\,\hat{\mathbf{i}} + r\sin\theta\,\hat{\mathbf{j}}$

The unit vector along $\vec{\mathbf{r}}$ is:

$\hat{\mathbf{r}} = \dfrac{\vec{\mathbf{r}}}{r} = \cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}$ and

$\hat{\boldsymbol{\theta}} = -\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}}$

so that

$$\frac{d\hat{\mathbf{r}}}{dt} = \frac{d}{dt}(\cos\theta)\,\hat{\mathbf{i}} + \frac{d}{dt}(\sin\theta)\,\hat{\mathbf{j}}$$

$$= -\sin\theta\frac{d\theta}{dt}\,\hat{\mathbf{i}} + \cos\theta\frac{d\theta}{dt}\,\hat{\mathbf{j}}$$

$$= \dot{\theta}\left(-\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}}\right) = \dot{\theta}\hat{\boldsymbol{\theta}}$$

where we have written $\dfrac{d\theta}{dt}$ as $\dot{\theta}$. Also in arriving at Eq. (13.12), we have used the following formula of differential calculus:

$$\frac{d}{dt}(fg) = f\frac{dg}{dt} + g\frac{df}{dt}$$

where *f* and *g* are functions of *t*.

Note from Fig. 13.8a that in the time interval Δt, the position vector $\vec{\mathbf{r}}$ of the particle sweeps out the area *OAB* in the plane of the motion (shown by the shaded part). Let us denote the area *OAB*, by ΔA. Note from Fig. 13.8b that for very small values of $\Delta\theta$, the point *A* is very close to *B*. Hence, the area ΔA is approximately equal to the area of the triangle *OAB* (shown by the shaded part). Note also that for very small values of $\Delta\theta$, the arc length *AB* $(= r\Delta\theta)$ is a straight line equal to the height of the triangle *OAB*. Therefore, for very small values of $\Delta\theta$, the area of the triangle *OAB* is given by

$$\Delta A = \frac{1}{2}\left[\text{base } (OA) \times \text{ height } (AB)\right] = \frac{1}{2} r \times (r\,\Delta\theta) \tag{13.14}$$

or $$\Delta A = \frac{1}{2} r^2\,\Delta\theta \tag{13.15}$$

We define the **areal velocity as the <u>rate</u> at which area is swept out by the position vector**. Thus, it is simply the derivative of the area ΔA given by Eq. (13.15) with respect to time. You know from differential calculus that it is given by

$$\frac{dA}{dt} = \lim_{\Delta t \to 0} \frac{\Delta A}{\Delta t} = \frac{1}{2} r^2 \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{1}{2} r^2\,\dot{\theta} \tag{13.16}$$

Now compare Eq. (13.16) with Eq. (13.13). The magnitude of the angular momentum given by Eq. (13.13) is simply $mr^2\dot{\theta}$. Therefore, we can write the expression of the **areal velocity** as

$$\frac{dA}{dt} = \frac{1}{2} r^2\dot{\theta} = \frac{mr^2\dot{\theta}}{2m} = \frac{L}{2m} \tag{13.17}$$

Since **the angular momentum is constant for objects of constant mass moving under central forces**, **their areal velocity will also be constant**. This is the **law of equal areas**.

Law of equal areas

LAW OF EQUAL AREAS FOR CENTRAL FORCES

For any central force, the radial vector connecting the particle to the centre of force sweeps out <u>equal areas in equal intervals of time</u>.

$$\frac{dA}{dt} = \frac{1}{2} r^2\dot{\theta} = \frac{L}{2m} = \text{constant} \tag{13.17}$$

Angular velocity $\omega = \dfrac{d\theta}{dt}$ which is the same as $\dot{\theta}$.

What does the law of equal areas [Eq. (13.17)] tell us about the motion of the object? In general terms, it tells us the following: **The object would move faster when it is closer to the centre of force**. Can you explain why? From Eq. (13.17), you can see that the smaller *r* is, the greater would $\dot{\theta}$ (the **angular velocity** of the object) be. This is because the areal velocity $\left(\dfrac{1}{2} r^2\,\dot{\theta}\right)$ is constant. Thus, the object would move faster when *r* is smaller (that is, when the object is closer to the centre of force).

What will the case for larger values of r be, that is, when the object is farther from the centre of force? You may like to answer this question yourself.

SAQ 4 – **Law of equal areas**

Using the law of equal areas, explain why an object would move slower when it is farther away from the centre of force.

Kepler's second law of planetary motion is a special case of the law of equal areas. When we apply the law of equal areas to the motion of the planets around the Sun under its gravitational force, we get Kepler's second law. You will study it in the next section.

Let us now study **motion under an inverse square force**, which is the most familiar central conservative force around us. This is an important problem because the gravitational force between point masses and the electrostatic force between charges are inverse square forces. In Sec. 13.5, we shall focus on the **gravitational force between objects** and obtain the orbits of various objects in the solar system.

13.5 MOTION UNDER AN INVERSE SQUARE FORCE

The general form of an inverse square force is,

$$\vec{\mathbf{F}} = \frac{k}{r^2}\hat{\mathbf{r}} \tag{13.18}$$

Inverse square force

If the constant k in Eq. (13.18) is negative, (that is, $k < 0$), the force is directed opposite to $\hat{\mathbf{r}}$ and towards the centre of force. Therefore, it is **attractive**. Recall from Figs 13.1 and 13.3a that the **gravitational force between two masses and the Coulomb force between two unlike charges are attractive inverse square forces**. On the other hand, if k is positive, ($k > 0$), it is directed away from the centre of force and is **repulsive**. The **Coulomb force between two like charges is a repulsive inverse square force**.

$$\vec{\mathbf{F}} = \frac{k}{r^2}\hat{\mathbf{r}} \quad \textbf{attractive} \text{ for } k < 0 \tag{13.19a}$$

Attractive inverse square force

$$\vec{\mathbf{F}} = \frac{k}{r^2}\hat{\mathbf{r}} \quad \textbf{repulsive} \text{ for } k > 0 \tag{13.19b}$$

Repulsive inverse square force

In the rest of this section, we shall discuss the motion of objects moving under the gravitational force of the Sun. Do remember that **the method and the steps of the analysis will be the same for other inverse square central conservative forces as well**.

13.5.1 Objects moving under the Sun's Gravitation

Let us write the equation of motion for an object of mass m moving under the gravitational force exerted by the Sun of mass M. The mass of the Sun is much greater than the mass of any object in the solar system: $M >> m$. We assume that **the Sun is stationary** and take the **origin of the coordinate system at the Sun**. Then r is the distance of the object from the Sun (see Fig. 13.9). Substituting the expression for the force of gravitation in Newton's second law, we get

$$m\vec{\mathbf{a}} = -\frac{GMm}{r^2}\hat{\mathbf{r}} \tag{13.20a}$$

Equation of motion for objects in solar system

Thus, $$\vec{\mathbf{a}} = \frac{d^2\vec{\mathbf{r}}}{dt^2} = -\frac{GM}{r^2}\hat{\mathbf{r}} \tag{13.20b}$$ **Equation of motion**

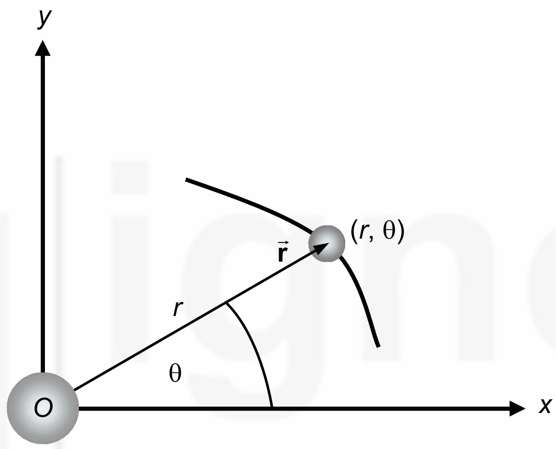


Fig. 13.9: **The Sun is at the origin and** (r, θ) **are the plane polar coordinates of any object in the solar system**.

We have used the following results to write Eq. (13.20b):

$$\vec{\mathbf{v}} = \frac{d\vec{\mathbf{r}}}{dt}$$

$$\vec{\mathbf{a}} = \frac{d\vec{\mathbf{v}}}{dt} = \frac{d}{dt}\left(\frac{d\vec{\mathbf{r}}}{dt}\right) = \frac{d^2\vec{\mathbf{r}}}{dt^2}$$

We can then solve Eq. (13.20b) using plane polar coordinates to determine the path followed by the object. **We shall not be solving Eq. (13.20b) in this course.**

Here we write the general solution and then discuss it qualitatively for different situations. The general solution of Eq. (13.20b) for any object (of mass m) moving under the gravitational force due to the Sun (of mass M) is given as

$$\frac{L^2}{GMm^2}\frac{1}{r} = (1 + e\cos\theta) \tag{13.21a}$$

where L is the angular momentum of the object about the Sun. From Eq. (13.13), $L = mr^2\dot{\theta}$, where $\dot{\theta}$ is the angular velocity of the object when it is at a distance r from the Sun. However, as we know, L is constant for motion under central force. Here

$$e = \sqrt{1 + \frac{2L^2E}{G^2M^2m^3}} \tag{13.21b}$$

and E is the total mechanical energy of the system, which is a constant. What we have in Eq. (13.21a) is a relation between r and θ, which determines the position of the object relative to the Sun. **Note** that while Eq. (13.20b) gives the variation of r with time t, Eq. (13.21a) is the **equation of the path** of the object around the Sun. We define a new constant $p = \dfrac{L^2}{GMm^2}$ to write Eq. (13.21a) as

$$\frac{p}{r} = (1 + e\cos\theta) \qquad \text{where } p = \frac{L^2}{GMm^2} \tag{13.22}$$

Equation of conic section of eccentricity *e*

We can now identify Eq. (13.22) as the equation of a **conic section** with an **eccentricity** of e. The value of the eccentricity decides the **shape** and the **type of the conic section** (see Figs. 13.10a to d).

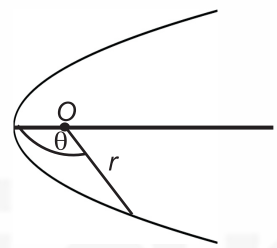
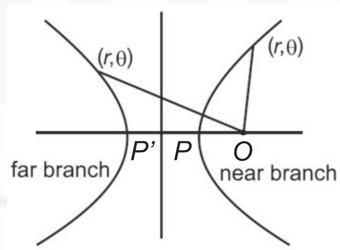
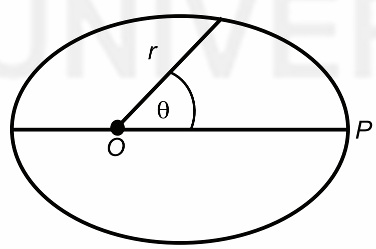
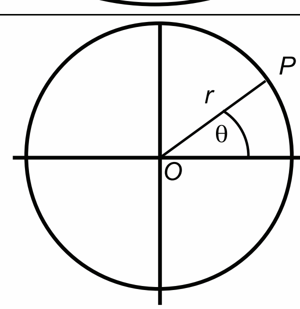
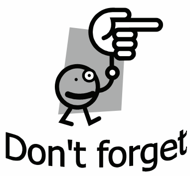
	Eccentricity	Conic Section	Shape
a)	$e = 1$	**Parabola**	 **(a)**
b)	$e > 1$	**Hyperbola**	 **(b)**
c)	$e < 1$	**Ellipse**	 **(c)**
d)	$e = 0$	**Circle**	 **(d)**

Fig. 13.10: **The shape and type of the conic section is determined by the value of the eccentricity. a) For a parabola**, $e = 1$; **b) for a hyperbola**, $e > 1$; **c) for an ellipse**, $e < 1$; **d) for a circle**, $e = 0$.

Notice from Figs. 13.10a and b that if the path followed by an object is a **parabola or** a **hyperbola**, it would approach the Sun (the centre of force) and go away without returning. Such orbits are called **open orbits**. If the path followed by an object is an **ellipse** or a **circle**, it keeps moving in its orbit around the Sun. Such orbits are called **closed orbits**. Let us summarise these points.



- **The trajectory or path of an object moving under the influence of the gravitational force exerted by the Sun is a conic section (circle, ellipse, parabola or hyperbola). This result is true for all inverse square forces.**
- **When the path or orbit of the object moving around the Sun is a parabola or a hyperbola, it is an open orbit. The object approaches the Sun and turns away after reaching the distance of closest approach without ever returning.**
- **When the orbit of an object moving around the Sun is a circle or an ellipse, it is a closed orbit. The object then keeps moving around the Sun in its orbit.**

REMEMBER: The solution given by Eq. (13.21a) has been written under the following simplifying assumptions:

(i) The Sun is stationary, and

(ii) The only force being exerted on the object is the gravitational force of the Sun. The gravitational forces exerted on the object by all other objects in the solar system are neglected.

We know that both these assumptions are not exactly true in the real solar system because the Sun is not stationary and all other members of the solar system exert gravitational forces on any object. However, these forces are negligible in comparison to the gravitational attraction of the Sun because the mass of the Sun is far greater than any other object. These assumptions are therefore reasonable for our solar system containing a huge Sun and a small number of other small objects like the planets, their satellites, comets and asteroids.

You may quickly like to apply Eq. (13.21a). Try the following SAQ.

SAQ 5 – Eccentricity and orbit

The orbit of a satellite about the Earth is given by

$$r = \frac{8000}{1 + 0.5\cos\theta}\ \text{km}$$

What are the eccentricity and shape of the orbit?

We now focus on elliptical orbits as these are the paths of the planets, satellites and many comets orbiting the Sun.

13.5.2 Elliptical Orbits in the Solar System

Most objects in the solar system move in closed elliptical orbits around the Sun, which is at one of the foci of the ellipses. **The orbits of most of the planets** (Fig. 13.11) **in the solar system are very nearly circular as the values of eccentricity are very small**. These are given in Table 13.1.

Table 13.1: Eccentricities of planets in the solar system.

Planet	e
Mercury	0.2056
Venus	0.0068
Earth	0.0167
Mars	0.0934
Jupiter	0.0483
Saturn	0.0560
Uranus	0.0461
Neptune	0.0100

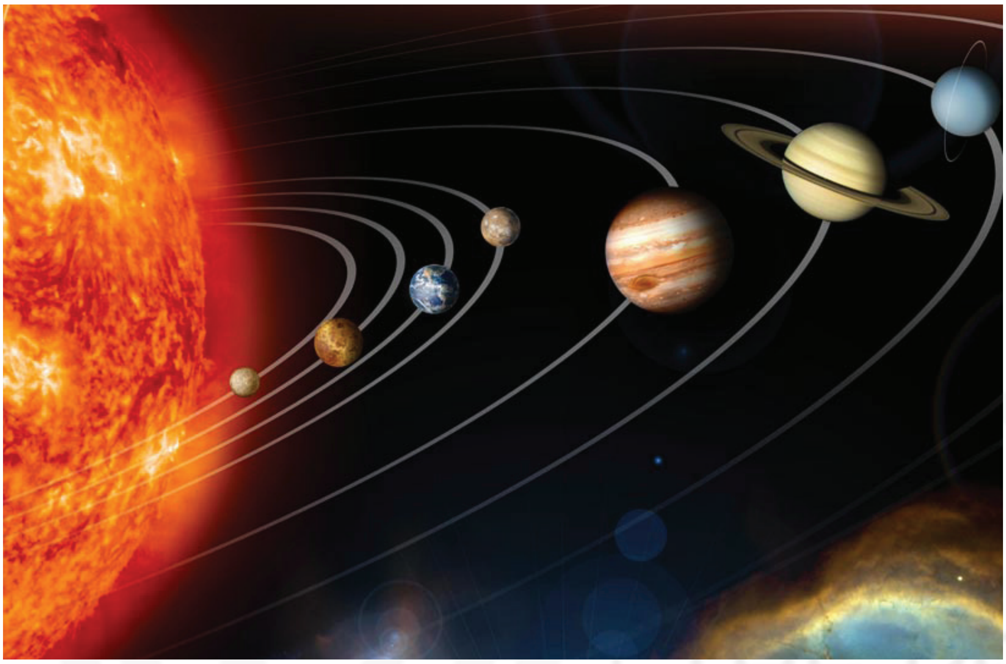


Fig. 13.11: **Model of planetary orbits in the solar system**. Source: www.jpl.nasa.gov

Let us now study the properties of the elliptical orbits in some more detail. We first define certain orbit parameters.

1. The **shape of the ellipse is determined by its eccentricity** as you can see in Fig. 13.12. When the value of e is close to zero, the ellipse is nearly circular, e.g., the orbit of planet Venus around the Sun. As it gets closer to 1, it becomes elongated, e.g., the orbits of comets.

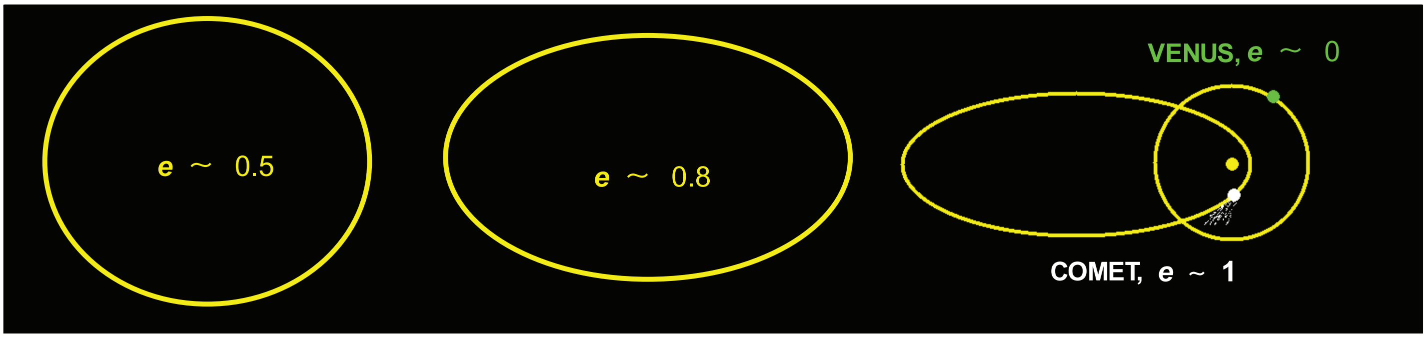


Fig. 13.12: **Ellipses of different eccentricities**.

2. Let us now obtain some other **orbital parameters**, **which determine the geometrical properties of elliptical orbits**.

   Recall from your school mathematics courses that an ellipse has two foci. In Fig. 13.13a, the point *O*, which is at the centre of the force, is at one of the foci of the ellipse. From the figure, you can see that the maximum value of *r* will occur at $\theta = \pi$ and its minimum value will occur at $\theta = 0$. Hence, from Eq. (13.22):

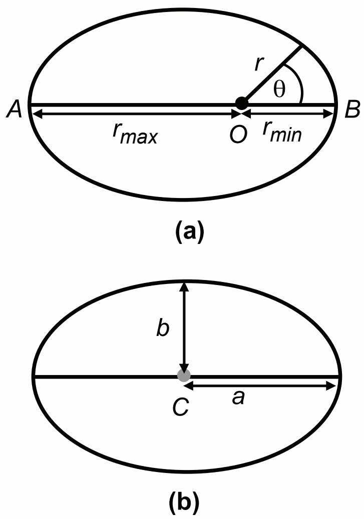


Fig. 13.13: **Parameters of the elliptical orbit**.

$$r_{max} = \frac{p}{1-e} \qquad \text{for} \qquad \theta = \pi \tag{13.23a}$$

$$\text{and} \qquad r_{min} = \frac{p}{1+e} \qquad \text{for} \qquad \theta = 0 \tag{13.23b}$$

At the point *A*, at which *r* is maximum, the object is farthest from the centre of the force. This point is called the **<u>aphelion</u> for objects orbiting the Sun** and **<u>apogee</u> for objects orbiting the Earth**. At the point *B*, *r* is minimum. It is the point of closest approach of the orbiting object to the centre of force. This point is called the **<u>perihelion</u> for objects orbiting the Sun** and **<u>perigee</u> for objects orbiting the Earth**.

From Fig. 13.13a, you can see that *AB* is the **major axis** of the ellipse and its length (*A*) is equal to the sum of the two distances r_{max} and r_{min} :

Major axis

$$A = r_{max} + r_{min} = \frac{p}{1-e} + \frac{p}{1+e} = \frac{2p}{1-e^2} \tag{13.24a}$$

The **lengths** of the **semi-major axis** (*a*) **and semi-minor axis** (*b*) of the ellipse (see Fig. 13.13b) are defined as:

Semi-major and semi-minor axes

$$a = \frac{A}{2} = \frac{p}{1-e^2} \qquad\qquad b = \frac{p}{\sqrt{1-e^2}} \tag{13.24b}$$

We can use Eq. (13.24b) to express the perihelion (or perigee) and aphelion (or apogee) distances in terms of the length of the semi-major axis.

PERIHELION/PERIGEE AND APHELION/APOGEE

The **perihelion** (or **perigee**) is the point of closest approach of the orbiting object to the centre of force. It is the point on the object's orbit at which *r* is minimum, and the **perihelion/perigee distance** is given by:

$$r_p = \frac{p}{1+e} = a(1-e) \tag{13.25a}$$

The **aphelion (**or **apogee**) is the point on the object's orbit which is farthest from the centre of force. It is the point at which *r* is maximum, and the **aphelion/apogee distance** is given by:

$$r_a = \frac{p}{1-e} = a(1+e) \tag{13.25b}$$

We can also express the total mechanical energy E in terms of a. Here we state the result without proof:

$$E = -\frac{GMm}{2a} \qquad (13.26)$$

We now summarise the results obtained for elliptical orbits in Table 13.2.

Table 13.2: Parameters for elliptical orbits.

Parameter	Expression
Length of semi-major axis, a	$a = \frac{p}{1-e^2}$
Length of semi-minor axis, b	$b = \frac{p}{\sqrt{1-e^2}}$
Perihelion distance, r_p	$r_p = a(1-e)$
Aphelion distance, r_a	$r_a = a(1+e)$
Energy	$E = -\frac{GMm}{2a}$

Let us now take up an example of the orbit of Halley's comet to show how we can determine the aphelion and perihelion distances given its orbital parameters.

EXAMPLE 13.1: ORBIT OF HALLEY'S COMET

Halley's comet (Fig. 13.14) moves in an elliptical orbit of eccentricity 0.967 about the Sun. The length of the orbit's semi-major axis is 2.7×10^{12} m. Determine the distance of Halley's comet from the Sun at perihelion and aphelion.

SOLUTION ■ The **KEY IDEA** here is to use Eqs. (13.25a and b) to obtain the results.

The perihelion distance of the orbit is

$$r_p = a(1-e) = 2.7 \times 10^{12}\,\text{m} \times (1-0.967) = 8.9 \times 10^{10}\,\text{m}$$

The aphelion distance of the orbit is

$$r_a = a(1+e) = 2.7 \times 10^{12}\,\text{m} \times (1+0.967) = 5.3 \times 10^{12}\,\text{m}$$

Fig. 13.14: **Halley's comet moves in a highly elliptical orbit and returns to the Earth once in every 76 years**.
Source: www.nasa.gov

You may now like to work out a problem based on these concepts.

SAQ 6 – Elliptical orbit

Pluto has an elliptical orbit with $a = 5.9 \times 10^{11}$ m and $e = 0.25$. Calculate the aphelion and perihelion distances and the total mechanical energy of the planet. It is given that the mass of Pluto is 1.3×10^{22} kg.

You have studied Kepler's laws of planetary motion in your school physics. We now relate what you have studied so far to these laws.

13.5.3 Kepler's Laws of Planetary Motion

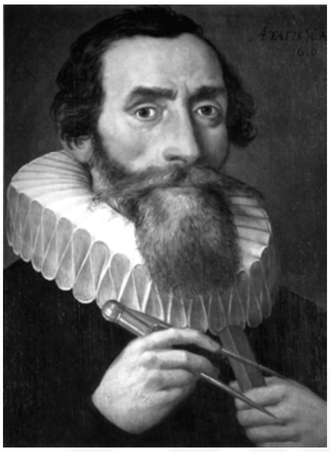


Fig. 13.15: **Johannes Kepler was a German mathematician and astronomer. He is best known for his laws of planetary motion.**

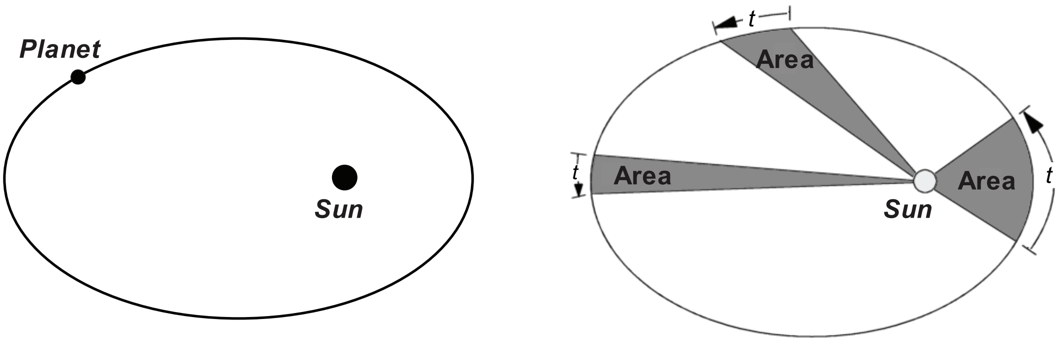
Johannes Kepler (Fig. 13.15) proposed the following three laws to explain the motion of planets round the Sun:

1. The paths of the planets about the Sun are elliptical in shape, with the centre of the Sun located at one of its foci. (**The Law of Ellipses**.)

2. An imaginary line drawn from the centre of the Sun to the centre of the planet will sweep out equal areas in equal intervals of time. (**The Law of Equal Areas**.)

3. The ratio of the squares of the periods of any two planets is equal to the ratio of the cubes of their average distances from the Sun. (**The Law of Harmonies.**)

You have learnt in Secs. 13.5.1 and 13.5.2 that the first two laws of Kepler (see Fig. 13.16) are a consequence of motion under the attractive inverse square gravitational force. You know that under the influence of the inverse square gravitational force exerted by the Sun, **the planets, asteroids and comets move in closed elliptical orbits with the Sun at one of its foci**. Kepler's first law of planetary motion which tells us about closed elliptical orbits is then a consequence of the attractive inverse square nature of the gravitational force.



(a) (b)

Fig. 13.16: **Kepler's first two laws of planetary motion. a) The orbits of planets are elliptical; b) the area swept out by the radius vector in equal times remains constant.**

The gravitational force of the Sun is central in nature and directed along the line joining the planet to the Sun. Refer to the law of equal areas that we have derived in Sec. 13.4.3. Let us apply it to the motion of planets around the Sun. It tells us that **the line joining the Sun to the planet sweeps out equal areas in equal intervals of time**, which is just Kepler's second law.

Using these two laws we can arrive at the third law of Kepler.

You know that the areal velocity of a planet or the **area swept out per unit time by the position vector of the planet** is constant. We have

Area swept out per unit time, $\frac{dA}{dt} = \frac{L}{2m}$

If the **time period** T of the planet is the **time taken by it to complete one elliptical orbit**, then

Area swept out in time T or one time period $= \left(\frac{L}{2m}\right)T$ (13.27a)

Also, the **area of the elliptical orbit** $= \pi a b$ (13.27b)

where a is the length of the semi-major axis and b is the length of the semi-minor axis of the ellipse. Using Eqs. (13.27a and b), we can write

$$\frac{LT}{2m} = \pi a b \tag{13.27c}$$

or
$$T^2 = \left(\frac{2m\pi}{L}\right)^2 a^2 b^2 \tag{13.28}$$

From Eq. (13.24b), you can show that the lengths of the semi-major and semi-minor axes of the ellipse are related as follows:

$$b^2 = a^2\,(1 - e^2) \tag{13.29a}$$

and since $p = a(1 - e^2)$ as given in Table 13.2, from Eq. (13.29a), we can write

$$b^2 = ap \tag{13.29b}$$

Upon substituting b^2 from Eq. (13.29b) and $p = \frac{L^2}{GMm^2}$ from Eq. (13.22) in Eq.(13.28), we get

$$T^2 = \left(\frac{2m\pi}{L}\right)^2 pa^3 = \frac{4\pi^2 a^3}{GM}$$

or
$$\boxed{T^2 = ka^3 \qquad \text{where } k = \frac{4\pi^2}{GM}} \tag{13.30}$$

Kepler's 3rd law

This is just **Kepler's third law**. This law tells us that **for all planets orbiting the same massive body, the ratio $\frac{T^2}{a^3}$ always remains the same**.

From Eq. (13.24b), we know

$b = \frac{p}{\sqrt{1-e^2}}$

So $b^2 = \frac{p^2}{(1-e^2)}$

We can also write this as

$b^2 = \left[\frac{p}{(1-e^2)}\right]p$

From Table 13.2 we know that

$a = \frac{p}{1-e^2}$

So, $b^2 = ap$

The **results derived so far** and Kepler's laws **can be applied not only to the motion of the planets around the Sun but also to the motion of satellites (natural and artificial) around the planets**. Satellites move under the gravitational force of the planet. Since the nature of the force is the same, the laws governing their motion are also the same. In the case of motion of objects around any planet, *M* would represent the mass of the planet.

Can you imagine a simple application in which the calculation for the distance of closest approach can become important? Imagine an asteroid approaching the Earth with a certain velocity. Would it hit the Earth? Obviously it could if its distance of closest approach were less than the radius of the Earth! You may like to apply Kepler's laws to determine the orbital parameters of comet Hale-Bopp.

SAQ 7 – **Time period of comet Hale-Bopp**

Calculate the time period of the comet Hale-Bopp about the Sun given that the length of the semi-major axis of its orbit is $2.79 \times 10^{13}\,\text{m}$.

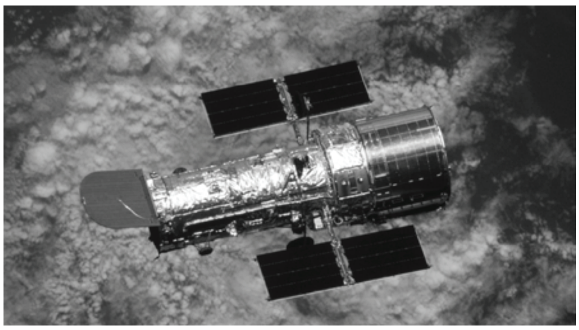
Let us now apply what we have discussed so far to artificial satellites orbiting the Earth or some other body in space.

13.5.4 Artificial Satellites

An artificial satellite is a manufactured object that continuously orbits the Earth or some other body in space. While most of the artificial satellites in use today orbit the Earth, there are some which have also orbited the Moon, the Sun, the asteroids, and the planets Venus, Mars and Jupiter.

Artificial satellites have several uses (see Fig. 13.17). They are used for weather forecasting, for transferring telephone calls and television signals over the oceans, assisting in the navigation of ships and aircraft, monitoring crops and other resources, and supporting military activities. There are also spacecrafts which carry astronauts. There are other artificial satellites like space capsules and space stations. The space telescopes gather information about the far reaches of the Universe.

(a)



(b)

(c)

Fig. 13.17: **Some artificial satellites**; **a) drawing of the Indian spacecraft Chandrayaan orbiting the Moon**; **b) the Hubble space telescope**; **c) EDUSAT, an Indian geostationary satellite**.

The motion of artificial satellites is also governed by the same laws that we have described in the previous section. Let us briefly discuss it. Consider an artificial satellite of mass m orbiting a celestial body of mass M in an elliptical orbit which has a semi-major axis a. The total mechanical energy of the satellite in the orbit, which is a constant, is given by Eq. (13.26) as

$$E = -\frac{GMm}{2a}$$

To understand the meaning of this equation, study Fig. 13.18. It shows the orbits of four satellites of equal mass around the Earth, which have the same semi-major axis. What can you say about the energy of the satellites?

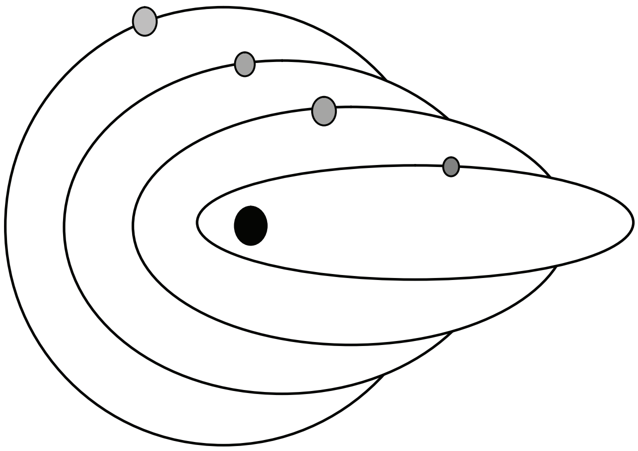


Fig. 13.18: Possible orbits of satellites having the same major axis and hence the same energy.

Different types of orbits are possible for artificial satellites. Some of these are **geostationary** and **polar** orbits.

- **Geostationary orbits**: You have learnt about these orbits in Example 7.1 of Unit 7 of Block 2. You have calculated the height of geosynchronous satellites from the surface of the Earth. You may know that communications satellites, such as those used to transmit satellite television and telephone signals are placed in geostationary orbits. They always appear in the same position when seen from the ground. Can you say why this is useful? This is why satellite television dishes that receive their signals can be fixed in their positions and do not need to move.

- **Polar orbits**: Satellites used for observation and monitoring are placed in polar orbits (Fig. 13.19). These orbits are inclined at 90° to the equator and take the satellites over both poles. Polar orbits are at lower heights from the surface of the Earth than geostationary orbits. So a complete orbit takes less than a day. As a result, the Earth spins beneath the satellite as it moves, and the satellite can scan the whole surface of the Earth.

Fig. 13.19: **Artificial satellites in polar orbits**. Source: http://www.enjoyspace.com

With this, we end the discussion on motion under central forces and summarise what you have studied in the unit.

13.6 SUMMARY

Concept	Description
Central force	■ A **central force** is always **directed towards or away from a particular point**, **which remains fixed**. The fixed point is called the **centre of force**. The central force can be expressed as $\vec{\mathbf{F}} = F\,\hat{\mathbf{r}}$
Central conservative force	■ When the magnitude of the central force, F, depends only on the distance of the object from the centre of the force, it is called a **central conservative force**. It can be written as $\vec{\mathbf{F}} = f(r)\,\hat{\mathbf{r}}$ It is conservative because the work done by the force between any two points on a path depends **only on the distance** of these points from the centre of force and **not on the path** followed by the particle.
Properties of motion under central and central conservative forces	■ For motion under **central and central conservative forces** i) **The angular momentum about the centre of force is constant both in magnitude and direction**. ii) Therefore, a particle on which a central force is exerted **moves only in a plane perpendicular to the direction of angular momentum**. In addition, for motion under **central conservative forces, the total mechanical energy is constant.**
Constants of motion	■ **The total mechanical energy and angular momentum are constants of motion for central conservative forces. The angular momentum is a constant of motion for central forces.**
Motion under central conservative forces	■ The **equation of motion for a central conservative force** is given by $\dfrac{d^2\vec{\mathbf{r}}}{dt^2} = \dfrac{f(r)}{m}\hat{\mathbf{r}}$ The solution of this equation gives the path followed by the particle. The paths or orbits of objects moving under a central conservative force may be classified by their energy into **open** and **closed** orbits. The **equation for the orbit for an inverse square central conservative force is that of a conic section**. The eccentricity and size of the conic section are completely determined by the angular momentum and the total mechanical energy of the object which are the **constants of motion**
Orbits under the force of gravitation	■ The **equation for the orbit** for **motion under the force of gravitation** is given by $\dfrac{p}{r} = (1 + e\cos\theta) \qquad \text{where } p = \dfrac{L^2}{GMm^2}$

Eccentricity and orbits

- The **eccentricity** of the orbit is related to the energy and angular momentum by

$$e = \sqrt{1 + \frac{2L^2E}{G^2M^2m^3}}$$

For $E > 0, \quad e > 1$ the orbit is a **hyperbola**.

For $E = 0, \quad e = 1$ the orbit is a **parabola**.

For $-\dfrac{L^2}{2mp^2} < E < 0, \quad 0 < e < 1$ the orbit is an **ellipse**.

For $E = -\dfrac{L^2}{2mp^2}, \quad e = 0$ the orbit is a **circle**.

Orbital parameters for elliptical orbits

- The eccentricity determines the shape of the **elliptical orbits** and the semi-major and semi-minor axes determine the size of the orbits. The **orbital parameters** are given as follows:

Length of semi-major axis $a = \dfrac{p}{1 - e^2}$

Length of semi-minor axis $b = \dfrac{p}{\sqrt{1 - e^2}}$

Perihelion/perigee distance $r_p = \dfrac{p}{1 + e} = a(1 - e)$

Aphelion/apogee distance $r_a = \dfrac{p}{1 - e} = a(1 + e)$

Energy and angular momentum for force of gravitation

- The **total mechanical energy** of the object moving in elliptical orbits under the force of gravitation is given by

$$E = -\frac{GMm}{2a}$$

The **magnitude** of the **angular momentum** of the object is given by

$$L = mr^2\dot{\theta}$$

Its **direction** is **perpendicular to the plane in which the object moves**.

13.7 TERMINAL QUESTIONS

1. Which of the following forces are central? Give reasons for your answer.

   a) $\vec{\mathbf{F}} = -4r^2\hat{\mathbf{r}} + k\dot{\theta}\hat{\boldsymbol{\theta}}$

   b) $\vec{\mathbf{F}} = \dfrac{\hat{\mathbf{r}}}{\sqrt{r}}$

   c) $\vec{\mathbf{F}} = -k\hat{\boldsymbol{\theta}}$

d) $\vec{\mathbf{F}} = \dfrac{(r-1)}{r^2+1}\,\hat{\mathbf{r}}$

2. Which of the following forces are central conservative? Give reasons for your answer.

   a) $\vec{\mathbf{F}} = -\,4r^2\hat{\mathbf{r}} + k\dot{\theta}\hat{\boldsymbol{\theta}}$

   b) $\vec{\mathbf{F}} = \dfrac{\hat{\mathbf{r}}}{\sqrt{r}}$

   c) $\vec{\mathbf{F}} = -\,k\hat{\boldsymbol{\theta}}$

   d) $\vec{\mathbf{F}} = \dfrac{(r-1)}{r^2+1}\,\hat{\mathbf{r}}$

3. State against each observation below whether it is true or false. Give reasons for your answer.

   a) The angular momentum of an artificial satellite orbiting the Earth under its gravitation varies with time.

   b) An alpha particle approaching a negative ion moves in a plane.

   c) An artificial satellite moves at greater speed when it is nearer to the Earth.

4. Select the physical quantities that are constants of motion for a central conservative force:

   a) Kinetic energy
   b) Potential energy
   c) Linear momentum
   d) Angular momentum
   e) Total mechanical energy
   f) Rotational energy

5. For a satellite moving about the Earth, $e = 0.15$. Determine its apogee and perigee for the following values of the semi-major axis of the orbit:

   a) $a = 7000\,\text{km}$

   b) $a = 36000\,\text{km}$

   Calculate the energy of the satellite in each case if its mass is 2000 kg.

6. What is the total mechanical energy of a satellite of mass 1000 kg moving about the Earth in an orbit with $a = 5000$ km? Determine the eccentricity and shape of the orbit for the following apogee distances:

   a) $r_a = 5000$ km

   b) $r_a = 7500$ km

7. A star moves in a circular orbit of radius 2, 500,000 km with a time period of 18 hours about another massive star of mass 1.00×10^{30} kg. How far apart are the two stars?

8. The planet Mercury has an elliptical orbit with $e = 0.2056$. Its mass is 3.3×10^{23} kg and the length of its semi-major axis is 57.9×10^6 km. Calculate

a) the aphelion and perihelion distances,

b) the total energy, and

c) the time period of its orbit.

9. Galileo discovered four moons of the planet Jupiter. One moon Io had an orbital period of 1.8 days. The distance of Io was measured to be 4.2 units from the centre of Jupiter. Another moon Ganymede was measured to be at a distance of 10.7 units from the centre of Jupiter. Using Kepler's third law, predict the orbital period of Ganymede.

10. Saturn has a mass of 95.18 Earth masses. One of its moons, Titan has an orbital period of 15.95 days. What is the distance of Titan from Saturn?

13.8 SOLUTIONS AND ANSWERS

Self-Assessment Questions

1. A central force $\vec{\mathbf{F}}$ can be expressed as $\vec{\mathbf{F}} = F\,\hat{\mathbf{r}}$ [Eq. (13.1)]. The forces given in (a) and (c) are both central forces because they are in the radial direction. However the force given in (b) is not a central force since it has not only a radial component which is $a\,r\,\hat{\mathbf{r}}$ but also an angular component at right angles to the radial direction given by $br\,\hat{\boldsymbol{\theta}}$.

2. Only the force (b) $F = -(K/r^3)\hat{\mathbf{r}}$ is a central conservative force because it is a radial force and its magnitude (K/r^3) is a function of only r. The force given in (a) is also radial, but the magnitude of the force depends on the velocity of the oscillator. Hence, it is not a conservative force. The force in (c) depends both on r and θ. Hence, it is not a conservative force.

3. To identify the constants of motion we first need to know which of these forces are central conservative forces. Forces (a) and (b) are central forces but not central conservative forces. The force (c) is a central conservative force. Therefore for a particle moving under the forces (a) and (b), the angular momentum is conserved and the particle moves in a plane. For the force (c) both angular momentum and the total mechanical energy are conserved. Also a particle moving under this force would be confined to move in a plane.

4. This is a consequence of the law of equal areas for central forces. From the law of equal areas given by Eq. (13.17), we have that

   $\frac{dA}{dt} = \frac{1}{2}r^2\dot{\theta} = \text{constant}$. Here r is the distance of the object from the centre of force and $\dot{\theta}$ is its angular speed. Since $r^2\dot{\theta}$ is a constant, the angular speed $\dot{\theta}$ is inversely proportional to the square of the distance of the object from the centre of force: $\dot{\theta} \propto \frac{1}{r^2}$. Therefore, when the object is closer to the centre of force, r is small and $\dot{\theta}$ is large. When r is large, $\dot{\theta}$ is small. Hence, the object moves slower when it is far from the centre of force.

5. Here we use the equation of the path of an object moving under the gravitational force of the Sun given by Eq. (13.22):

$$\frac{p}{r} = 1 + e\cos\theta \qquad \text{(i)}$$

Rewriting the given equation of the orbit $r = \dfrac{8000}{1 + 0.5\cos\theta}\,\text{km}$

as $$\frac{8000}{r} = 1 + 0.5\cos\theta \qquad \text{(ii)}$$

and comparing Eqs. (i) and (ii), we can identify the following parameters:

$p = 8000\,\text{km} = 8.0 \times 10^6\,\text{m}$ and $e = 0.5$

Therefore, the orbit is elliptical because its eccentricity is 0.5 and it satisfies the condition $0 < e < 1$. This is a closed orbit.

6. We make use of the equations given in Table (13.2) to find the required quantities. Using $a = 5.9 \times 10^{11}\,\text{m}$ and $e = 0.25$,

the perihelion distance is $r_p = a(1 - e) = 4.4 \times 10^{11}\,\text{m}$

and the aphelion distance is $r_a = a(1 + e) = 7.4 \times 10^{11}\,\text{m}$

The total mechanical energy is calculated from $E = -\dfrac{GMm}{2a}$.

Here M is the mass of the Sun $= 2.0 \times 10^{30}\,\text{kg}$, m is the mass of Pluto $= 1.3 \times 10^{22}\,\text{kg}$ and $G = 6.67 \times 10^{-11}\,\text{Nm}^2\,\text{kg}^{-2}$. Substituting the values of G, M, m and a, we get

$$E = -\frac{(6.67\times10^{-11}\,\text{Nm}^2\,\text{kg}^{-2})\times(2.0\times10^{30}\,\text{kg})\times(1.3\times10^{22}\,\text{kg})}{2\times(5.9\times10^{11}\,\text{m})} = -1.5\times10^{30}\,\text{J}$$

7. We use Eq. (13.30) with $a = 2.79\times10^{13}\,\text{m}$, $G = 6.67\times10^{-11}\,\text{Nm}^2\,\text{kg}^{-2}$ and the mass of the Sun $M = 2.00\times10^{30}\,\text{kg}$. From Eq. (13.30)

$$T = \sqrt{\frac{4\pi^2 a^3}{GM}} = \sqrt{\frac{4\pi^2(2.79\times10^{13}\,\text{m})^3}{6.67\times10^{-11}\,\text{Nm}^2\,\text{kg}^{-2}\times(2.00\times10^{30}\,\text{kg})}}$$

$$= 8.01\times10^{10}\,\text{s} \approx 2543\,\text{years}$$

Terminal Questions

1. The forces (b) and (d) are central forces because they are directed along the radial direction.

2. The forces (b) and (d) are central conservative forces because the magnitude of forces depends only on r, the distance from the centre of force. Forces (a) and (c) are not central as they have a component along $\hat{\boldsymbol{\theta}}$.

3. a) False. The angular momentum of the satellite is constant because the satellite moves under the gravitational force, which is a central force.

   b) True. The force between the negative ion and the alpha particle is the electrostatic force, which is a central force and hence, motion is confined to a plane.

   c) True. This follows from the law of equal areas and is a consequence of the fact that for motion under a central force, angular momentum is conserved. In this case, the artificial satellite is moving under the gravitational force due to the Earth, which is a central force.

4. The total mechanical energy and the angular momentum are constants of motion for a central conservative force.

5. We use Eqs. (13.25a and b) for calculating the apogee and perigee with $e = 0.15$ and a as given. For energy, we substitute the values of G, M (mass of the Earth) and m (mass of the satellite) in Eq. (13.26).

   a) For $a = 7000\,\text{km}$: $r_p = 5950\,\text{km}$, $r_a = 8050\,\text{km}$ and

$$E = -\frac{GMm}{2a} = -\frac{(6.67\times10^{-11}\,\text{Nm}^2\,\text{kg}^{-2})\times(5.97\times10^{24}\,\text{kg})\times(2000\,\text{kg})}{2\times7000\times10^3\,\text{m}}$$

$$= -5.70\times10^{10}\,\text{J}$$

   b) For $a = 36000\,\text{km}$: $r_p = 30600\,\text{km}$, $r_a = 41400\,\text{km}$ and $E = -1.11\times10^{10}\,\text{J}$

6. The total mechanical energy is calculated from Eq. (13.26): $E = -\dfrac{GMm}{2a}$

   with $G = 6.67\times10^{-11}\,\text{Nm}^2\,\text{kg}^{-2}$, mass of the Earth $M = 5.97\times10^{24}\,\text{kg}$, mass of the satellite $m = 1000\,\text{kg}$ and $a = 5000\,\text{km} = 5.00\times10^6\,\text{m}$.

$$\therefore E = -\frac{6.67\times10^{-11}\,\text{Nm}^2\,\text{kg}^{-2}\times5.97\times10^{24}\,\text{kg}\times1000\,\text{kg}}{2\times5.00\times10^6\,\text{m}} = -3.98\times10^{10}\,\text{J}$$

   To find the eccentricity and shape of the orbit, we use Eq. (13.25b) to write

$$e = \frac{r_a}{a} - 1 \qquad \text{(i)}$$

   Given r_a and $a = 5000\,\text{km}$, we can find e from (i).

   a) For $r_a = 5000\,\text{km}$ we get $e = 0$. The orbit of the satellite is a circle.

   b) For $r_a = 7500\,\text{km}$ we get $e = \dfrac{7500}{5000} - 1 = 0.5$ and the orbit of the satellite is an ellipse since $0 < e < 1$.

7. We use Eq. (13.30) to find the distance between the two stars. Here a is the radius of the circular orbit of the star, which is also the distance between the stars. It is given that $T = 18\,\text{hours} = 18\times3600\,\text{s} = 64800\,\text{s}$, mass of the star $M = 1.00\times10^{30}\,\text{kg}$ and $G = 6.67\times10^{-11}\,\text{Nm}^2\,\text{kg}^{-2}$. Thus,

$$a = \left[\frac{GMT^2}{4\pi^2}\right]^{1/3} = \left(\frac{6.67\times10^{-11}\,\text{Nm}^2\,\text{kg}^{-2}\times1.00\times10^{30}\,\text{kg}\times(64800\,\text{s})^2}{4\pi^2}\right)^{1/3}$$

$$= 1.92\times10^9\,\text{m}$$

8. a) To determine the aphelion and perihelion distances we use Eqs. (13.25a and b) with $a = 57.9\times10^6$ km and $e = 0.2056$.

$$r_a = 57.9\times10^6\,\text{km}\times(1+0.2056) = 69.8\times10^6\,\text{km}$$

$$r_p = 57.9\times10^6\,\text{km}\times(1-0.2056) = 46.0\times10^6\,\text{km}$$

b) The total energy is given by Eq. (13.26) with $M =$ mass of the Sun $= 2.0\times10^{30}$ kg, $m =$ mass of Mercury $= 3.3\times10^{23}$ kg and $a = 57.9\times10^6\,\text{km} = 57.9\times10^9\,\text{m}$

$$\therefore\ E = -\frac{6.67\times10^{-11}\,\text{Nm}^2\,\text{kg}^{-2}\times2.0\times10^{30}\,\text{kg}\times3.3\times10^{23}\,\text{kg}}{2\times57.9\times10^9\,\text{m}} = -3.8\times10^{32}\,\text{J}$$

c) To determine the time period of its orbit we use Eq. (13.30) with,

$a = 57.9\times10^9$ m, $G = 6.67\times10^{-11}\,\text{Nm}^2\,\text{kg}^{-2}$ and $M = 2.0\times10^{30}$ kg.

$$T = \left[\frac{4\pi^2(57.9\times10^9\,\text{m})^3}{6.67\times10^{-11}\,\text{Nm}^2\text{kg}^{-2}\times2.0\times10^{30}\,\text{kg}}\right]^{1/2}\,\text{s}$$

$$= 7.6\times10^6\,\text{s} \approx 88\ \text{Earth days}$$

9. Let T_1 and T_2 be the orbital periods of Jupiter's satellites Io and Ganymede, respectively, and a_1 and a_2 be their respective distances from the centre of Jupiter. If M_J is the mass of Jupiter, using Eq. (13.30) we can write,

$$T_1^2 = \frac{4\pi^2}{GM_J}a_1^3 \quad \text{and} \quad T_2^2 = \frac{4\pi^2}{GM_J}a_2^3$$

Given that, $T_1 = 1.8$ days, $a_1 = 4.2$ units and $a_2 = 10.7$ units, we can find the orbital period of Ganymede as follows:

$$T_2 = \left(\frac{T_1^2\,a_2^3}{a_1^3}\right)^{1/2} = \left[\frac{(1.8\ \text{days})^2\times(10.7\ \text{units})^3}{(4.2\ \text{units})^3}\right]^{1/2} = 7.3\ \text{days}$$

10. We use Kepler's third law as expressed in Eq. (13.30) to find the distance

$$a.\ a = \left(\frac{GMT^2}{4\pi^2}\right)^{1/3} \quad \text{with}\ \ M = 95.18\times5.97\times10^{24}\,\text{kg} = 5.68\times10^{26}\,\text{kg}$$

$$T = 15.95\times86400\,\text{s} = 1.38\times10^6\,\text{s} \ \text{and}\ \ G = 6.67\times10^{-11}\,\text{Nm}^2\,\text{kg}^{-2}$$

$$\therefore\ a = \left[\frac{6.67\times10^{-11}\,\text{Nm}^2\text{kg}^{-2}\times5.68\times10^{26}\,\text{kg}\times(1.38\times10^6\,\text{s})^2}{4\times3.14\times3.14}\right]^{1/3}$$

$$= 1.22\times10^9\,\text{m}$$

How can we describe the motion of particles in such fireworks? Picture source: http://www.colorado.edu/physics/
You will learn the answer in this unit!

UNIT 14

DYNAMICS OF MANY-PARTICLE SYSTEMS

Structure

STUDY GUIDE

In this unit, you will study the dynamics of **two-particle systems** and see how the coupled equations of motion for such systems can be replaced by two independent equations and solved. You will be using all the concepts you have learnt for single particle motion. You may be learning the mathematical techniques discussed in this unit for the first time. Therefore, we have tried to keep the mathematics simple and given all steps. You will also learn how to apply these concepts to **many-particle systems.**

For understanding the concepts in this unit well, you should know the concepts of Newtonian mechanics for a single particle explained in Block 2 of this course. So do revise the concepts of mechanics discussed in Block 2. You should also revise the mathematical concepts of **vector algebra** given in Units 1 and 2 and **calculus** from school mathematics. Do solve all examples, SAQs and Terminal Questions on your own!

"Scientific knowledge is a body of statements of varying degrees of certainty – some most unsure, some nearly sure, but none absolutely certain."

Richard Feynman

14.1 INTRODUCTION

Fig. 14.1: **Some examples of two-particle systems**.

In Block 2, you have essentially studied the **dynamics of single particles** with applications in many physical situations. However, all around us we see many examples of two-particle systems other than that of the Sun and a planet or the Earth and Moon in the solar system (Fig. 14.1). For example, a hockey stick or bat striking a ball, a diatomic molecule and binary star systems are two-body systems. Similarly, many-particle systems exist all around us. For example, the solar system made up of the Sun, planets, asteroids and comets is one such system. Gas filled in a cylinder, exploding stars and fireworks, a cup of tea, an acrobat, a car, a ball are all examples of many-particle systems.

How do we describe the motion of such systems? We can apply Newton's second law of motion to each particle in the system. For example, consider the Earth-Moon system. To determine the motion of the Moon and the Earth, we can write down the equations of motion for both the Earth and Moon and solve them to obtain the paths of both objects. But it turns out that the mathematics is quite complex. We find that the dynamics of such systems can be simplified by introducing the concept of the **centre of mass**.

Therefore, we begin our study of two-body systems by explaining this concept in Sec. 14.2 and introducing the **centre of mass** and **relative coordinates**. We then obtain the equation of motion for two-particle systems in terms of these coordinates in Sec. 14.3. In Sec. 14.4, we extend these concepts to many-particle systems. In the next unit, we obtain the expressions of the linear momentum, angular momentum and energy for two-particle and many-particle systems and revisit the conservation laws.

Expected Learning Outcomes

After studying this unit, you should be able to:

- ❖ define the centre of mass and determine its coordinates for two-particle systems;
- ❖ determine the relative coordinates for two-particle systems;
- ❖ reduce the equation of motion of two-particle systems to the equations of motion in centre of mass and relative coordinates of a single particle;
- ❖ determine the reduced mass of two-particle systems;
- ❖ determine the centre of mass coordinates of many-particle systems; and
- ❖ describe the centre of mass motion for many-particle systems under the force of gravity.

14.2 DYNAMICS OF TWO-PARTICLE SYSTEMS

Consider the following systems:

- The Earth-Moon system (or the Sun and any one of its planets);
- Rocket ejecting gas (where the ejected gas is treated as one body);

- Binary star systems;
- Hydrogen atom (made up of one electron and one proton);
- Diatomic molecule;
- Marble (or billiards ball) striking another marble (or ball);
- Elementary particle scattered from another particle, and so on.

What is special about these systems? In each of these examples, there are **two objects**, which interact with each other. These objects may also be interacting with other objects in the universe. However, for the time being, let us suppose that all other interactions except that between the two of them can be neglected. **Such two-body systems are also called binary systems**.

The question before us is: **How do we describe the motion of each object in this two-body system in a simple way**? The answer to this question leads us to the concept of **centre of mass**. So we relate the two and ask: **Why do we need to define the centre of mass**?

14.2.1 Why Define a Centre of Mass?

You can apply Newton's second law of motion to determine the path of each object in a binary system. For example, consider the Earth-Moon system (see Fig. 14.2). To determine the motion of the Moon and the Earth, you can write down the equations of motion for the Moon (mass m_1, position vector $\vec{\mathbf{r}}_1$) and the Earth (mass m_2, position vector $\vec{\mathbf{r}}_2$), respectively, as follows:

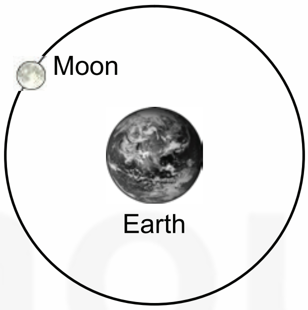


Fig. 14.2: **The Earth-Moon system**.

$$m_1 \vec{\mathbf{a}}_1 = m_1 \frac{d^2\vec{\mathbf{r}}_1}{dt^2} = -G \frac{m_1 m_2}{r_{12}^2} \hat{\mathbf{r}}_{12} \qquad (14.1a)$$

$$m_2 \vec{\mathbf{a}}_2 = m_2 \frac{d^2\vec{\mathbf{r}}_2}{dt^2} = -G \frac{m_1 m_2}{r_{12}^2} \hat{\mathbf{r}}_{21} \qquad (14.1b)$$

NOTE

We shall treat all objects as particles in this unit. Although the Earth and the Moon are extended objects you may **take them to be particles under the condition that the distance between them is much greater than their radii.**

Notice that Eq. (14.1a) contains the position vectors of both the Earth and the Moon, since $\hat{\mathbf{r}}_{12} = \hat{\mathbf{r}}_1 - \hat{\mathbf{r}}_2$. In the same way, Eq. (14.1b) contains the position vectors of both the Earth and the Moon. Also note that in writing Eqs. (14.1a and b), **we have treated the Earth and Moon as particles even though these are extended objects**. Why can we do so? Read the note in the margin. In general, the equations of motion for two-particle systems would be of the following form:

$$m_1 \vec{\mathbf{a}}_1 = m_1 \frac{d^2\vec{\mathbf{r}}_1}{dt^2} = \vec{\mathbf{F}}_1 \qquad (14.1c)$$

and

$$m_2 \vec{\mathbf{a}}_2 = m_2 \frac{d^2\vec{\mathbf{r}}_2}{dt^2} = \vec{\mathbf{F}}_2 \qquad (14.1d)$$

where $\vec{\mathbf{F}}_1$ and $\vec{\mathbf{F}}_2$ are the net forces on the respective particles. We can add Eqs. (14.1c and d) and write

$$m_1 \frac{d^2\vec{\mathbf{r}}_1}{dt^2} + m_2 \frac{d^2\vec{\mathbf{r}}_2}{dt^2} = \vec{\mathbf{F}} \qquad \text{where } \vec{\mathbf{F}} = \vec{\mathbf{F}}_1 + \vec{\mathbf{F}}_2 \qquad (14.1e)$$

Thus, $\vec{\mathbf{F}}$ is the total force on the system. Now, to determine the paths of the two particles, we need to solve Eqs. (14.1a and b, 14.1c and d or 14.1e) and obtain the solutions for $\vec{\mathbf{r}}_1$ and $\vec{\mathbf{r}}_2$. Since both these vectors may appear in both equations, equations of this type (called **coupled differential equations**) are difficult to solve. For example, if we consider the motion of the Moon-Earth system in space, we need to **use three coordinates for each object**. Then we would get **six coupled differential equations** involving the three coordinates of each object. Solving these six equations is definitely not an easy task!

So you see that **determining the motion of the particles in a two-particle system is much more complicated than determining the motion of a single particle system**. Let us now ask: **Is it possible to simplify the problem**? The answer is yes, and we can do it by introducing the concept of a **"centre of mass"**. Let us explain this concept.

14.2.2 What is the Centre of Mass?

Let us ask: Would it not be good if we could write Eqs. (14.1a, b, c or d) as the **total mass multiplied by some acceleration**? Indeed, we can do so. To do this, we take the sum of the two masses, i.e., the total mass of the two-particle system to be $M\ (= m_1 + m_2)$. Then if we define a certain vector $\vec{\mathbf{R}}_{cm}$ as

Centre of mass

$$\vec{\mathbf{R}}_{cm} = \frac{m_1\vec{\mathbf{r}}_1 + m_2\vec{\mathbf{r}}_2}{m_1 + m_2} \tag{14.2}$$

we can write Eq. (14.1e) as

Equation of motion of the c.m.

$$M\frac{d^2\vec{\mathbf{R}}_{cm}}{dt^2} = \vec{\mathbf{F}} \qquad \text{where } M = m_1 + m_2 \tag{14.3}$$

In writing Eq. (14.3), we also assume that M is a constant. Would you like to check the result in Eq. (14.3)? For this, let us rewrite Eq. (14.2) as follows:

$$M\vec{\mathbf{R}}_{cm} = m_1\vec{\mathbf{r}}_1 + m_2\vec{\mathbf{r}}_2 \tag{14.4}$$

Let us differentiate Eq. (14.4) twice with respect to time. We get

$$M\frac{d\vec{\mathbf{R}}_{cm}}{dt} = m_1\frac{d\vec{\mathbf{r}}_1}{dt} + m_2\frac{d\vec{\mathbf{r}}_2}{dt} \tag{14.5a}$$

and

$$M\frac{d^2\vec{\mathbf{R}}_{cm}}{dt^2} = m_1\frac{d^2\vec{\mathbf{r}}_1}{dt^2} + m_2\frac{d^2\vec{\mathbf{r}}_2}{dt^2} \tag{14.5b}$$

Substituting $\vec{\mathbf{F}}$ from Eq. (14.1e) in Eq. (14.5b), we get Eq. (14.3).

Thus we find that,

The net external force on the two-particle system is equal to just the total mass multiplied by the acceleration of an imaginary point whose position vector is given by $\vec{\mathbf{R}}_{cm}$. This point is called the centre of mass of

the system. Note that it is an imaginary point and a mathematical invention.

You may ask: **What is the advantage of introducing the centre of mass**? To answer this question, let us see what Eq. (14.3) tells us. It tells us the following things:

1. The **centre of mass of a system moves as if a particle of mass *M* (the total mass of the system) is situated at the point and acted on by the net force which is the resultant of all the external forces acting on the particles of the system.**

2. If the net external force on the system is zero, the particles in the system can move in any way but the **centre of mass will move with a constant velocity**. If it is initially at rest, it will remain at rest.

3. We can treat the motion of the centre of mass independently of the motion of the particles in the system.

Thus, by inventing a point called the centre of mass and defining it by Eq. (14.2), we can analyse the motion of the **entire** system rather than that of its individual parts. If we focus only on the motion of the centre of mass of a system of particles or an extended object with many particles in it, we can describe the complicated motion of the system in a simple way.

You may like to watch a video clip on this concept available at http://www.answers.com/topic/center-of-mass

As an example, consider the picture of a diver diving into the swimming pool (Fig. 14.3). Focus on the point *P* marked on her body. What path does this point follow as the diver jumps off the board? It is a parabola – the path followed by a particle in projectile motion. So there is a point on the body, which moves as the body would, if we took it to be a particle. This point is the **centre of mass of the body**. So we need to consider only the motion of the centre of mass if we are not interested in knowing how her arms and legs move when she dives.

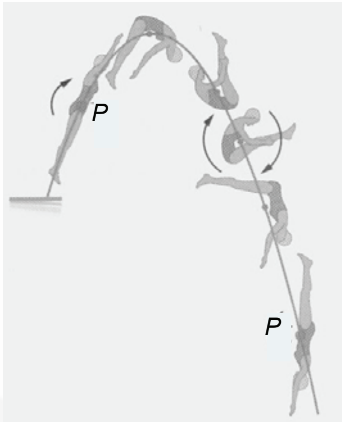


Fig. 14.3: A diver dives off a board. Even though most points on her body follow complex paths, a special point shown by dots follows a parabolic path. This point is her centre of mass. If we treated the diver as a particle, it would follow this path.

There is another advantage of defining the centre of mass. Study Eq. (14.5a). We can write it as

$$M\vec{\mathbf{V}}_{cm} = m_1\vec{\mathbf{v}}_1 + m_2\vec{\mathbf{v}}_2 \tag{14.6}$$

where $\vec{\mathbf{V}}_{cm}$ is the velocity of the centre of mass, $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$ are the velocities of the masses m_1 and m_2, respectively. What does Eq. (14.6) tell us? It tells us that the **linear momentum of the total mass located at the centre of mass is equal to the sum of the linear momenta of the particles in the two-particle system**. You will learn about the power of this equation in the next unit when you study the conservation of linear momentum.

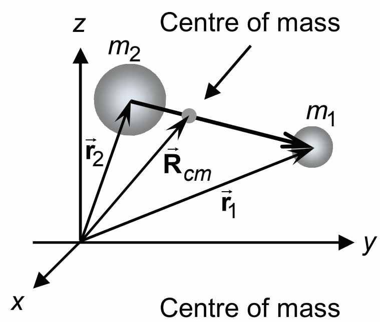


Fig. 14.4: Centre of mass of a two-particle system. It lies on the line joining the two masses.

Let us now state the concept of centre of mass of a two-particle system in a formal way.

Consider the system of two particles 1 and 2 in Fig. 14.4. Let the masses of the particles be m_1 and m_2, respectively. Their positions are given by $\vec{\mathbf{r}}_1$ and $\vec{\mathbf{r}}_2$, respectively. Then we define the centre of mass as follows.

Centre of mass coordinates

CENTRE OF MASS

The **centre of mass** of a material body or system of particles **is the point, which moves as though the system's total mass existed at that point and all external forces were applied at that point**. **Its position vector for a two-body system is defined as**

$$\vec{\mathbf{R}}_{cm} = \frac{m_1\vec{\mathbf{r}}_1 + m_2\vec{\mathbf{r}}_2}{m_1 + m_2} \tag{14.7a}$$

The (x, y, z) **coordinates** of the position vector of the centre of mass in the three-dimensional Cartesian coordinate system are given by:

$$X_{cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} \tag{14.7b}$$

$$Y_{cm} = \frac{m_1 y_1 + m_2 y_2}{m_1 + m_2} \tag{14.7c}$$

$$Z_{cm} = \frac{m_1 z_1 + m_2 z_2}{m_1 + m_2} \tag{14.7d}$$

where (x_1, y_1, z_1) and (x_2, y_2, z_2) are the coordinates of the particles with position vectors $\vec{\mathbf{r}}_1$ and $\vec{\mathbf{r}}_2$, respectively.

For the two-body system, **the centre of mass is situated on the line joining the two masses** (see Fig. 14.4).

What have we learnt so far? **We have found a point which moves as though the system's total mass existed at that point and the <u>net external force</u> was applied at that point**. This point whose position is defined by Eqs. (14.7a to d) is called the **<u>centre of mass</u>**. Note that we are referring to the **net external force and not considering the internal forces of interaction between the particles.** We shall discuss this point in greater detail in Sec. 14.3 when we set up the equation of motion.

So, using Eq. (14.7a) **we can study the motion of complicated systems in a simple way by focusing on the motion of the centre of mass**.

You have learnt that using the concept of the centre of mass, we can describe the complicated motion of two-particle systems in a simpler way. We can make the mathematics even simpler by introducing the **centre of mass** and **relative coordinates**. Let us define these two coordinates.

14.2.3 Centre of Mass and Relative Coordinates

The **centre of mass coordinate of a two-particle system** is defined by $\vec{\mathbf{R}}_{cm}$ as given in Eq. (14.7a). In terms of the Cartesian coordinates, its magnitude is

$$R_{cm} = \sqrt{X_{cm}^2 + Y_{cm}^2 + Z_{cm}^2} \qquad (14.8)$$

Centre of mass coordinate

where X_{cm}, Y_{cm} and Z_{cm} are given by Eqs. (14.7b to d). Henceforth, we shall refer to the **centre of mass coordinate as the c.m. coordinate**. The **relative coordinate** $\vec{\mathbf{r}}$ for the two-body system is defined as:

$$\vec{\mathbf{r}} = \vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2 \qquad (14.9a)$$

Relative coordinate

Note that the relative coordinate is **just the position of the mass** m_1 **relative to the mass** m_2. Fig. 14.5 shows the relative coordinate of a two-particle system. Let us state the formal definition of the relative coordinate for a two-particle system.

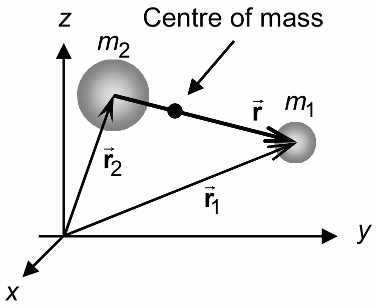


Fig. 14.5: **Relative coordinate for a two-particle system.**

RELATIVE COORDINATE

The relative coordinate of a two-body system is defined as the position of the mass m_1 **relative to the mass** m_2 :

$$\vec{\mathbf{r}} = \vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2 \qquad (14.9a)$$

In the three-dimensional Cartesian coordinate system, it is given by:

$$x = x_1 - x_2 \qquad (14.9b)$$

$$y = y_1 - y_2 \qquad (14.9c)$$

$$z = z_1 - z_2 \qquad (14.9d)$$

where (x_1, y_1, z_1) and (x_2, y_2, z_2) are the coordinates of the particles with position vectors $\vec{\mathbf{r}}_1$ and $\vec{\mathbf{r}}_2$, respectively. The magnitude of $\vec{\mathbf{r}}$ is

$$r = \sqrt{x^2 + y^2 + z^2} \qquad (14.9e)$$

Relative coordinate

Let us determine the centre of mass and relative coordinates of a couple of simple systems so that you get a better idea of this concept.

EXAMPLE 14.1: C. M. AND RELATIVE COORDINATES

a) What are the centre of mass and relative coordinates of a two-particle system in which the particles of masses M_1 and M_2, respectively, are located along the *x*-axis (Fig. 14.6)?

SOLUTION ■ The **KEY IDEA** here is to use Eq. (14.7b) and Eq. (14.9b) in one-dimension since the two particles are located along the *x*-axis and only the *x*-coordinate will be non-zero for them.

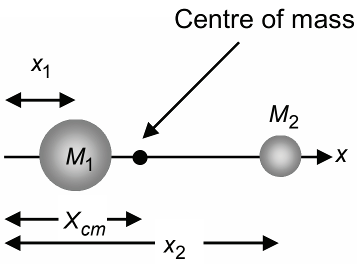


Fig. 14.6: Centre of mass of a two-particle system with the two particles lying along a straight line parallel to the *x*-axis. Note that it lies closer to the particle with greater mass.

Refer to Fig. 14.6. The *x*-coordinates of the particles are given by x_1 and x_2, respectively. Then from Eqs. (14.7b and 14.9b), the coordinate of the centre of mass and the relative coordinate are, respectively, given by

$$X_{cm} = \frac{M_1 x_1 + M_2 x_2}{M_1 + M_2} \quad \text{and} \quad x = x_1 - x_2$$

Let us apply this result to locate the position of the centre of mass of a **system of two particles** of mass 2.0 kg and 3.0 kg attached to a light rod *AB* of length 1.0 m at the points *A* and *B* (Fig. 14.7).

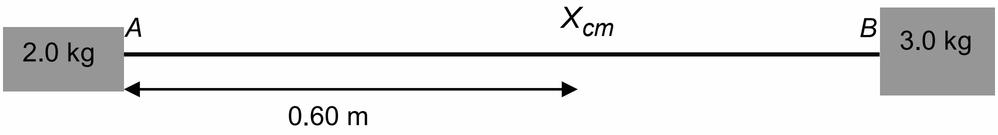


Fig. 14.7

Taking the origin to be at the point *A*, we have

$$x_1 = 0 \ \text{ for } M_1 = 2.0 \text{ kg and } x_2 = 1.0 \text{ m for } M_2 = 3.0 \text{ kg}$$

Hence, the centre of mass and relative coordinates for this system are:

$$X_{cm} = \frac{M_1 x_1 + M_2 x_2}{M_1 + M_2} = \frac{(2.0\,\text{kg})(0\,\text{m}) + (3.0\,\text{kg})(1.0\,\text{m})}{2.0\,\text{kg} + 3.0\,\text{kg}} = \frac{3.0}{5.0}\,\text{m} = 0.60\,\text{m}$$

and $\quad x = x_1 - x_2 = -1.0\,\text{m} \quad$ (coordinate of 1 with respect to 2)

b) Determine the centre of mass and relative coordinates of the two-particle system (both of mass 1.0 kg) kept in the *xy* plane? It is given that their coordinates in m are (0, 0) and (3.0, 2.0) (Fig. 14.8).

SOLUTION ■ The **KEY IDEA** here is to use Eqs. (14.7b and c) and Eqs. (14.9b and c) since the two particles are located in the *xy* plane and only the *xy* coordinates will be non-zero for them. From Eqs. (14.7b and c), we get

$$X_{cm} = \frac{(1.0 \times 0 + 1.0 \times 3.0)\,\text{kg.m}}{(1.0 + 1.0)\,\text{kg}} = 1.5\,\text{m}$$

$$Y_{cm} = \frac{(1.0 \times 0 + 1.0 \times 2.0)\,\text{kg.m}}{(1.0 + 1.0)\,\text{kg}} = 1.0\,\text{m}$$

$$\therefore \quad R_{cm} = \sqrt{(1.5)^2 + (1.0)^2} = 1.8\,\text{m}$$

From Eqs. (14.9b and c): $\quad x = x_1 - x_2 = (0 - 3.0)\,\text{m} = -3.0\,\text{m}$

$$y = y_1 - y_2 = (0 - 2.0)\,\text{m} = -2.0\,\text{m}$$

$$\therefore \quad r = \sqrt{(-3.0)^2 + (-2.0)^2} = 3.6\,\text{m}$$

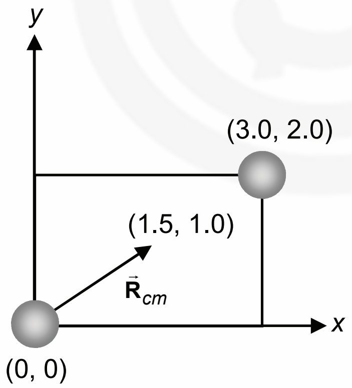


Fig. 14.8: $\vec{\mathbf{R}}_{cm}$ gives the position of the centre of mass. Its coordinates in m **are** (1.5, 1.0).

You may like to calculate the centre of mass and relative coordinates of some two-body systems.

SAQ 1 – Centre of mass and relative coordinates

Determine the centre of mass and relative coordinates for each of the following two-body systems:

a) An object of mass 2.5 kg at a distance of 1.0 m from another object of mass 25 kg.

b) The Earth-Moon system, where $M_e = 5.97 \times 10^{24}$ kg, $M_m = 7.35 \times 10^{22}$ kg and the distance between the Earth and the Moon is 3.84×10^8 m.

c) Two objects of equal mass (5.0 kg) having the coordinates (0,0) and (0, 2.0), respectively, in m.

What have we done in this section? We started with the two coordinates $\vec{\mathbf{r}}_1$ and $\vec{\mathbf{r}}_2$ (position vectors of the two bodies with mass m_1 and m_2, respectively) to describe the two-particle system. Now we have defined two **new coordinates** $\vec{\mathbf{R}}_{cm}$ and $\vec{\mathbf{r}}$. You can choose to describe the binary system either in the set of coordinates $(\vec{\mathbf{r}}_1, \vec{\mathbf{r}}_2)$ or in the new set of coordinates $(\vec{\mathbf{r}}, \vec{\mathbf{R}}_{cm})$. Using the coordinates $(\vec{\mathbf{r}}, \vec{\mathbf{R}}_{cm})$ makes it easier to solve the equation of motion for the two-body system. In the next section, you will see how this is possible.

14.3 EQUATION OF MOTION FOR TWO-BODY SYSTEMS IN C.M. AND RELATIVE COORDINATES

Let us consider the motion of a system of two particles 1 and 2 of masses m_1 and m_2, respectively. Let their position vectors be $\vec{\mathbf{r}}_1$ and $\vec{\mathbf{r}}_2$, respectively, at the instant *t* with respect to an origin *O* in an inertial frame of reference (Fig. 14.9). Let the **net force** on particle 1 be $\vec{\mathbf{F}}_1$ and the **net force** on particle 2 be $\vec{\mathbf{F}}_2$. Then, from Newton's second law, the equations of motion for the two-particle system are given by Eqs. (14.1c and d) repeated here as Eqs. (14.10a and b):

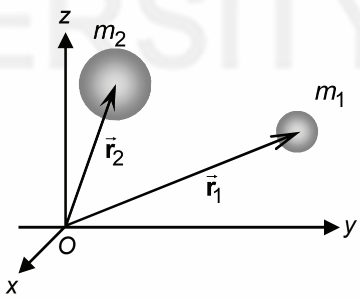


Fig. 14.9: Position vectors in a two-particle system.

$$m_1 \vec{\mathbf{a}}_1 = m_1 \frac{d^2 \vec{\mathbf{r}}_1}{dt^2} = \vec{\mathbf{F}}_1 \qquad (14.10a)$$

$$m_2 \vec{\mathbf{a}}_2 = m_2 \frac{d^2 \vec{\mathbf{r}}_2}{dt^2} = \vec{\mathbf{F}}_2 \qquad (14.10b)$$

Note that these equations are written in terms of the coordinates $(\vec{\mathbf{r}}_1, \vec{\mathbf{r}}_2)$. We have to solve these equations to determine the path of the two particles. Since it is difficult to solve these coupled equations, we shall now express them in terms of the relative and c.m. coordinates $(\vec{\mathbf{r}}, \vec{\mathbf{R}}_{cm})$. Note that $\vec{\mathbf{F}}_1$ and $\vec{\mathbf{F}}_2$ are the **net forces** on the particles 1 and 2. These are the resultants of the net external force on each particle and the internal forces of mutual interaction between the particles (Fig. 14.10). So we can write them as follows:

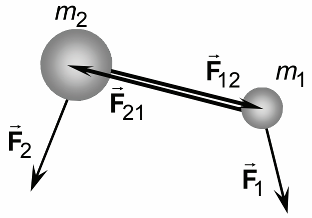


Fig. 14.10: **Forces on particle 1 and particle 2. Note that the forces of mutual interaction shown here are attractive.**

$$\vec{\mathbf{F}}_1 = \vec{\mathbf{F}}_{e1} + \vec{\mathbf{F}}_{21} \tag{14.11a}$$

and
$$\vec{\mathbf{F}}_2 = \vec{\mathbf{F}}_{e2} + \vec{\mathbf{F}}_{12} \tag{14.11b}$$

where $\vec{\mathbf{F}}_{e1}$ – net external force on particle 1,

$\vec{\mathbf{F}}_{21}$ – internal force on the particle 1 due to particle 2,

$\vec{\mathbf{F}}_{e2}$ – net external force on particle 2, and

$\vec{\mathbf{F}}_{12}$ – internal force on the particle 2 due to particle 1.

Note that from Newton's third law of motion, the force on particle 2 due to particle 1 is equal and opposite to the force on particle 1 due to particle 2:

$$\vec{\mathbf{F}}_{12} = -\vec{\mathbf{F}}_{21} \tag{14.11c}$$

Now there can be two situations:

1. No net external force is exerted on the system.
2. A non-zero net external force is exerted on the system.

For keeping the discussion simple, we start with the first case when no net external force is exerted on the system.

14.3.1 Two-body Problem for No External Force

In this case, the particles interact only with each other. Thus, the only forces being exerted on the two particles are the mutual forces of action and reaction: the force on particle 1 is only due to particle 2 and the force on particle 2 is only due to particle 1. From Eq. (14.11c), you know that these are equal and opposite. There are many examples of such systems in nature. For example, in the **Earth-Moon system**, the force on the Earth is just the force of gravitation due to the Moon and the force on the Moon is just the force of gravitation due to the Earth. The same is true for the **Sun-Planet system**. In the same way, in a **system of two charges**, the Coulomb force is attractive for unlike charges and repulsive for like charges. In all these systems, no external force is exerted on the system (unless in the latter case we place the charges in an external electric field).

Let us now write the equations of motion for the two-particle system on which the **net external force is zero** in terms of the coordinates $(\vec{\mathbf{r}}_1, \vec{\mathbf{r}}_2)$. We substitute $\vec{\mathbf{F}}_1$ and $\vec{\mathbf{F}}_2$ from Eqs. (14.11a and b) in Eqs. (14.10a and b) and put $\vec{\mathbf{F}}_{e1} = \vec{\mathbf{0}}$ and $\vec{\mathbf{F}}_{e2} = \vec{\mathbf{0}}$ in them to get:

$$m_1 \vec{\mathbf{a}}_1 = m_1 \frac{d^2 \vec{\mathbf{r}}_1}{dt^2} = \vec{\mathbf{F}}_{21} \tag{14.12a}$$

$$m_2\vec{\mathbf{a}}_2 = m_2\frac{d^2\vec{\mathbf{r}}_2}{dt^2} = \vec{\mathbf{F}}_{12} = -\vec{\mathbf{F}}_{21} \qquad (14.12b)$$

Note that in writing Eq. (14.12b), we have also used Eq. (14.11c).

We now do some simple algebra to express Eqs. (14.12a and b) in terms of the relative and c.m. coordinates $(\vec{\mathbf{r}}, \vec{\mathbf{R}}_{cm})$. We first add Eqs. (14.12a and b) and get:

$$m_1\vec{\mathbf{a}}_1 + m_2\vec{\mathbf{a}}_2 = m_1\frac{d^2\vec{\mathbf{r}}_1}{dt^2} + m_2\frac{d^2\vec{\mathbf{r}}_2}{dt^2} = \vec{\mathbf{0}}$$

or $$\frac{d^2}{dt^2}\left(m_1\vec{\mathbf{r}}_1 + m_2\vec{\mathbf{r}}_2\right) = \vec{\mathbf{0}} \quad \text{since } m_1 \text{ and } m_2 \text{ are constant} \qquad (14.13)$$

Using Eq. (14.7a), we can write Eq. (14.13) as

$$\frac{d^2}{dt^2}\left(m_1 + m_2\right)\vec{\mathbf{R}}_{cm} = \vec{\mathbf{0}}$$

or $$M\frac{d^2\vec{\mathbf{R}}_{cm}}{dt^2} = \vec{\mathbf{0}} \qquad (14.14)$$

Equation of motion of the centre of mass for zero net external force

since $M = m_1 + m_2$. We can also write Eqs. (14.12a and b) as follows:

$$\frac{d^2\vec{\mathbf{r}}_1}{dt^2} = \frac{\vec{\mathbf{F}}_{21}}{m_1} \qquad (14.15a)$$

and $$\frac{d^2\vec{\mathbf{r}}_2}{dt^2} = \frac{\vec{\mathbf{F}}_{12}}{m_2} = -\frac{\vec{\mathbf{F}}_{21}}{m_2} \qquad (14.15b)$$

Subtracting Eq. (14.15b) from Eq. (14.15a), we get

$$\frac{d^2}{dt^2}\left(\vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2\right) = \left(\frac{1}{m_1} + \frac{1}{m_2}\right)\vec{\mathbf{F}}_{21} \qquad (14.16)$$

Since $\vec{\mathbf{r}} = \vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2$, we can write Eq. (14.16) as

$$\frac{d^2\vec{\mathbf{r}}}{dt^2} = \left(\frac{1}{m_1} + \frac{1}{m_2}\right)\vec{\mathbf{F}}_{21} \qquad (14.17)$$

Note that in writing Eq. (14.18), we have used Eq. (14.19a) in Eq. (14.17) as follows:

$$\frac{d^2\vec{\mathbf{r}}}{dt^2} = \frac{1}{\mu}\vec{\mathbf{F}}_{21}$$

or $$\mu\frac{d^2\vec{\mathbf{r}}}{dt^2} = \vec{\mathbf{F}}_{21}$$

We can write Eq. (14.17) in the standard form of Newton' second law as

$$\mu\frac{d^2\vec{\mathbf{r}}}{dt^2} = \vec{\mathbf{F}}_{21} \qquad (14.18)$$

Equation of motion in relative coordinate

Here we have introduced a new quantity μ given by

$$\frac{1}{\mu} = \left(\frac{1}{m_1} + \frac{1}{m_2}\right) \qquad (14.19a)$$

or $$\mu = \frac{m_1 m_2}{(m_1 + m_2)} \qquad (14.19b)$$

Reduced mass

The quantity μ given by Eqs. (14.19a and b) is called the **reduced mass** of the two-particle system. You will learn why it is called so in the next section. You can write $\vec{\mathbf{F}}_{21} = -\vec{\mathbf{F}}_{12}$ and show that

$$\text{for} \quad \vec{\mathbf{r}} = \vec{\mathbf{r}}_2 - \vec{\mathbf{r}}_1, \qquad \mu \frac{d^2\vec{\mathbf{r}}}{dt^2} = \vec{\mathbf{F}}_{12} \tag{14.20}$$

NOTE

Note that we can use either Eq. (14.18) or Eq. (14.20) to obtain the solution of the two-body problem.

Work out the steps for Eqs. (14.18 and 14.20) before you study further. In fact Eq. (14.18) or Eq. (14.20) is the equation of motion of a particle of mass μ situated at a distance r from the origin. It is of course a <u>**fictitious particle**</u> and there is no such particle at the position $\vec{\mathbf{r}}$. Yet it does serve a purpose. The equation of motion for this fictitious particle is not a coupled equation if $\vec{\mathbf{F}}_{21}$ and $\vec{\mathbf{F}}_{12}$ depend only on $\vec{\mathbf{r}}$ and not on $\vec{\mathbf{r}}_1, \vec{\mathbf{r}}_2$. You can now use standard methods to solve this equation. Did you notice that **Eqs. (14.14)** and **(14. 18)** / **(14.20)** are the **equations of motion in the centre of mass and relative coordinates**? So we have reduced the coupled equations of motion in the coordinates $(\vec{\mathbf{r}}_1, \vec{\mathbf{r}}_2)$ to two separate uncoupled equations in terms of the relative and c.m. coordinates $(\vec{\mathbf{r}}, \vec{\mathbf{R}}_{cm})$. Let us put these equations together.

The word "fictitious" means "unreal" or "imaginary".

EQUATION OF MOTION IN C. M. AND RELATIVE COORDINATES FOR ZERO NET EXTERNAL FORCE

If the net external force on a two-particle system with masses m_1 and m_2 is zero, the **coupled equations of motion for the system in the coordinates** $(\vec{\mathbf{r}}_1, \vec{\mathbf{r}}_2)$ are given by

$$m_1 \frac{d^2\vec{\mathbf{r}}_1}{dt^2} = \vec{\mathbf{F}}_{21} \quad \text{and} \quad m_2 \frac{d^2\vec{\mathbf{r}}_2}{dt^2} = \vec{\mathbf{F}}_{12} = -\vec{\mathbf{F}}_{21}$$

If $\vec{\mathbf{F}}_{21}$ and $\vec{\mathbf{F}}_{12}$ depend only on $\vec{\mathbf{r}}$, these equations are reduced to the following equations in terms of the relative and c.m. coordinates $(\vec{\mathbf{r}}, \vec{\mathbf{R}}_{cm})$:

Equations of motion for two-particle system in c.m. and relative coordinates

$$M \frac{d^2\vec{\mathbf{R}}_{cm}}{dt^2} = \vec{\mathbf{0}} \qquad \text{where } M = m_1 + m_2 \tag{14.21a}$$

$$\text{and} \quad \mu \frac{d^2\vec{\mathbf{r}}}{dt^2} = \vec{\mathbf{F}}_{21} \quad \text{or} \quad \mu \frac{d^2\vec{\mathbf{r}}}{dt^2} = -\vec{\mathbf{F}}_{12} \tag{14.21b}$$

$$\text{where} \quad \frac{1}{\mu} = \left(\frac{1}{m_1} + \frac{1}{m_2}\right) \quad \text{or} \quad \mu = \frac{m_1 m_2}{(m_1 + m_2)} \tag{14.21c}$$

Can you say why this new quantity μ given by Eq. (14.21c) is called the reduced mass of the system? Let us find out.

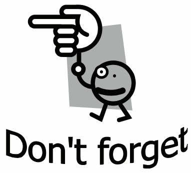
14.3.2 Reduced Mass

From Eq. (14.21c), you can see that μ **has the dimensions of mass**. But μ is neither the mass of particle 1 nor the mass of particle 2. It is the mass of the fictitious particle at a distance r from the origin at any instant t.

We call μ the **reduced mass** of the two-particle system. **Eq. (14.18) or Eq. (14.21b) is the equation of motion of a particle of mass μ moving under the influence of the mutual force of interaction $\vec{F}_{21}$.**

We say that

> **Eq. (14.21b) is the equation of motion of a "fictitious" particle of mass μ, called the reduced mass.**



You may be wondering: **Why is μ called the reduced mass**? We now answer this question with the help of some examples.

Let us find the value of the reduced mass for some two-particle systems (see Fig. 14.11) using Eq. (14.21c).

1. The value of μ for the **Sun-Earth system** is

$$\mu = \frac{m_s m_e}{(m_s + m_e)}$$

where m_s is the mass of the Sun and m_e is the mass of the Earth. We know that $m_s >> m_e$ since $m_s = 1.99 \times 10^{30}$ kg and $m_e = 5.97 \times 10^{24}$ kg. Hence, in the denominator, we can neglect m_e and write

$$m_s + m_e \approx m_s$$

Thus, we get
$$\mu \approx \frac{m_s m_e}{m_s} = m_e \qquad (14.22a)$$

So the mass of the fictitious particle of Eq. (14.18) is equal to the mass of the Earth, which is much less (or reduced) than the total mass of the Sun-Earth system.

We can generalise this result to any two-particle system in which one particle's mass is far greater than the other's.

The value of μ for a two-particle system for which $m_1 >> m_2$ is given by

$$\mu \approx m_2 \qquad \text{for} \quad m_1 >> m_2 \qquad (14.22b)$$

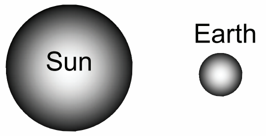
The mass of the fictitious particle is reduced and is approximately equal to m_2, the mass of the less massive particle.

2. The value of μ of a binary system of particles of equal masses is

$$\mu = \frac{m \times m}{(m+m)} = \frac{m}{2} \qquad (14.22c)$$

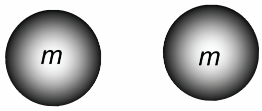
which is half of the mass of each particle.

Thus, in all the cases we have considered above, the mass of the "fictitious" particle is less than the total mass of the two-particle system. That is why it is called the **reduced mass**.



Reduced mass $= m_e$, mass of the Earth

(a)



Reduced mass $= \frac{m}{2}$, where m is the mass of each particle.

(b)

Fig. 14.11: **Reduced mass of a) the Sun-Earth system (not to scale); b) a system of two particles of equal mass.**

You may like to determine the value of the reduced mass of some two-particle systems to understand this concept.

SAQ 2 – **Reduced mass**

Determine the reduced mass of each of the following two-body systems:

a) The Earth-Moon system.

b) The system of two marbles, each of mass 0.2 kg.

c) The electron-proton system in the hydrogen atom.

d) The two-particle system of a star orbiting a black hole that is 100 times more massive than the star.

We hope that with these exercises, you have understood the concept of reduced mass Let us revise the formal definition of reduced mass.

Recap

REDUCED MASS

The **reduced mass** of a system of two particles having masses m_1 and m_2, respectively, is given by

$$\frac{1}{\mu} = \left(\frac{1}{m_1} + \frac{1}{m_2}\right)$$

Reduced mass

or $$\mu = \frac{m_1 m_2}{(m_1 + m_2)}$$

It is the mass of a **fictitious particle**, whose equation of motion (Eq. 14.21b) is given in terms of the relative coordinate of the two-particle system on which no net external force is exerted.

So far, you have learnt how to write the equations of motion for a two-particle system when the net external force is zero. We have also introduced the concept of reduced mass. You may now like to know: **How do these equations change when there is a net external force being exerted on the two-particle system**? This is the situation that we take up now.

14.3.3 Two-body Problem for Non-zero External Force

It is not possible to obtain the general analytical solutions for the paths of the individual particles of a two-particle system on which a net external force is being exerted except for special cases. We shall deduce the general equation of motion and apply it to the case when the only external force is that of gravity. Examples are the dumbbell and the baton of the leader of a music band. We start from Eqs. (14.10a and b) for a system of two particles 1 and 2 of masses m_1 and m_2. Let their position vectors at the instant t be $\vec{r}_1$ and $\vec{r}_2$, respectively, with respect to an origin O in an inertial frame of reference. The forces exerted on them are shown in Fig. 14.12. We repeat the equations of motion [Eqs. (14.10a and b)] here for convenience:

$$m_1\vec{\mathbf{a}}_1 = m_1\frac{d^2\vec{\mathbf{r}}_1}{dt^2} = \vec{\mathbf{F}}_1 \quad \text{and} \quad m_2\vec{\mathbf{a}}_2 = m_2\frac{d^2\vec{\mathbf{r}}_2}{dt^2} = \vec{\mathbf{F}}_2$$

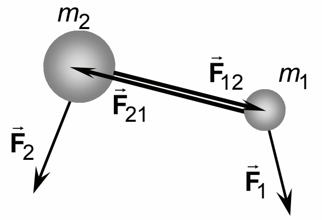


Fig. 14.12: **Forces on particle 1 and particle 2. Note that the forces of mutual interaction shown here are attractive.**

You know that in these equations, $\vec{\mathbf{F}}_1$ is the **net force** on particle 1 and $\vec{\mathbf{F}}_2$ is the **net force** on particle 2. We also write Eqs. (14.11a and 14.11b) for ready reference:

$$\vec{\mathbf{F}}_1 = \vec{\mathbf{F}}_{e1} + \vec{\mathbf{F}}_{21} \quad \text{and} \quad \vec{\mathbf{F}}_2 = \vec{\mathbf{F}}_{e2} + \vec{\mathbf{F}}_{12}$$

where $\vec{\mathbf{F}}_{e1}$ is the net external force on particle 1, $\vec{\mathbf{F}}_{21}$ is the internal force on the particle 1 due to particle 2, $\vec{\mathbf{F}}_{e2}$ is the net external force on particle 2, and $\vec{\mathbf{F}}_{12}$ is the internal force on the particle 2 due to particle 1. From Newton's third law of motion, the force on particle 1 due to particle 2 is equal and opposite to the force on particle 2 due to particle 1:

$$\vec{\mathbf{F}}_{12} = -\vec{\mathbf{F}}_{21}$$

We have to now write the equations of motion for the two-particle system on which the **net external force is non-zero** in terms of the coordinates $(\vec{\mathbf{r}}, \vec{\mathbf{R}}_{cm})$. Let us substitute $\vec{\mathbf{F}}_1$ and $\vec{\mathbf{F}}_2$ from Eqs. (14.11 a and b) in Eqs. (14.10a and b) and add them. We get

$$m_1\frac{d^2\vec{\mathbf{r}}_1}{dt^2} + m_2\frac{d^2\vec{\mathbf{r}}_2}{dt^2} = \vec{\mathbf{F}}_1 + \vec{\mathbf{F}}_2 \tag{14.23a}$$

On using Eq. (14.11a, b and c), we can write:

$$m_1\frac{d^2\vec{\mathbf{r}}_1}{dt^2} + m_2\frac{d^2\vec{\mathbf{r}}_2}{dt^2} = \vec{\mathbf{F}}_1 + \vec{\mathbf{F}}_2 = \vec{\mathbf{F}}_{e1} + \vec{\mathbf{F}}_{e2}$$

or $$m_1\frac{d^2\vec{\mathbf{r}}_1}{dt^2} + m_2\frac{d^2\vec{\mathbf{r}}_2}{dt^2} = \vec{\mathbf{F}}_e \quad \text{where} \quad \vec{\mathbf{F}}_e = \vec{\mathbf{F}}_{e1} + \vec{\mathbf{F}}_{e2} \tag{14.23b}$$

Thus, $\vec{\mathbf{F}}_e$ is the **net external force** on the two-particle system. Now recall the definition of the centre of mass coordinate of this system:

$$\vec{\mathbf{R}}_{cm} = \frac{(m_1\vec{\mathbf{r}}_1 + m_2\vec{\mathbf{r}}_2)}{m_1 + m_2} = \frac{(m_1\vec{\mathbf{r}}_1 + m_2\vec{\mathbf{r}}_2)}{M} \tag{14.24a}$$

Differentiating this equation twice with respect to time, we get (for constant masses of the particles):

$$M\frac{d^2}{dt^2}\left(\vec{\mathbf{R}}_{cm}\right) = \frac{d^2}{dt^2}(m_1\vec{\mathbf{r}}_1 + m_2\vec{\mathbf{r}}_2) = m_1\frac{d^2\vec{\mathbf{r}}_1}{dt^2} + m_2\frac{d^2\vec{\mathbf{r}}_2}{dt^2} \tag{14.24b}$$

Substituting Eq. (14.23b) in Eq. (14.24b), we can write

$$\boxed{M\frac{d^2\vec{\mathbf{R}}_{cm}}{dt^2} = \vec{\mathbf{F}}_e} \tag{14.25}$$

Equation of motion of c.m. for non-zero net external force

Eq. (14.25) is the equation of motion of a particle of mass M having the position vector $\vec{\mathbf{R}}_{cm}$, which moves under a net external force $\vec{\mathbf{F}}_e$. Using the methods you have learnt in Units 5 and 6, you can solve the equation of motion for the centre of mass if you know the force. Note that Eq. (14.25) is the equation for the motion of the centre of mass and its solution gives us the **path of the centre of mass of the system.**

The centre of mass of a system moves as if a particle of mass M (the total mass of the system) is situated at the point and acted upon by a force which is the resultant of all the external forces acting on the particles of the system.

Thus, if a two-body system were falling freely with a finite horizontal component of velocity, its centre of mass would move along a parabola. You can test this result by performing a simple activity.

Activity

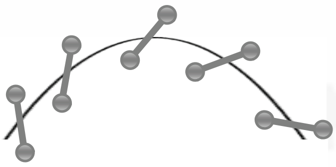


Fig. 14.13: Motion of c.m. of a dumb-bell falling under gravity.

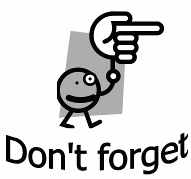
CENTRE OF MASS MOTION OF A TWO-PARTICLE SYSTEM

Take a dumb-bell with equal masses. You could make one yourself by making holes in two rubber balls of the same kind and inserting a light rod to connect the balls (see Fig. 14.13). Where does the centre of mass of this system lie?

Put a mark or a strip of paper at that point (which is at the middle of the rod). Now hold the dumb-bell at its centre of mass and throw it in the air. What is the path along which the centre of mass moves?

You will see that the dumb-bell moves without rotating if you apply the force at its centre of mass while throwing it. Moreover, its c.m. follows a parabolic path.

In the next section, you will learn that the same result applies to a freely-falling many-particle system. We end this discussion with a few important points that you should keep in mind.



In arriving at Eq. (14.25), we have assumed the following:

1. **The second law of Newton in writing Eq. (14.23a);**
2. **The third law of Newton in going from Eq. (14.23a) to Eq. (14.23b);**
3. **The masses of the particles and hence M are constant in arriving at Eq. (14.25) from Eqs. (14.24a) and (14.24b). This assumption is not correct when we take into account the relativistic variation of mass with velocity.**
4. $\left(m_1 + m_2\right) \neq 0$ **in writing Eq. (14.24a). This assumption is justified as the masses are all positive.**

Let us now find out whether we can reduce the problem to an equivalent one-body problem in this case. From Eqs. (14.10a, b and 14.11a, b), we can write

$$\frac{d^2\vec{\mathbf{r}}_1}{dt^2} = \frac{\vec{\mathbf{F}}_1}{m_1} = \frac{\vec{\mathbf{F}}_{e1} + \vec{\mathbf{F}}_{21}}{m_1} \tag{14.26a}$$

and
$$\frac{d^2\vec{\mathbf{r}}_2}{dt^2} = \frac{\vec{\mathbf{F}}_2}{m_2} = \frac{\vec{\mathbf{F}}_{e2} + \vec{\mathbf{F}}_{12}}{m_2} \tag{14.26b}$$

Subtracting Eq. (14.26b) from Eq. (14.26a), we get

$$\frac{d^2}{dt^2}(\vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2) = \left(\frac{1}{m_1} + \frac{1}{m_2}\right)\vec{\mathbf{F}}_{21} + \left(\frac{\vec{\mathbf{F}}_{e1}}{m_1} - \frac{\vec{\mathbf{F}}_{e2}}{m_2}\right) \quad \left(\because \vec{\mathbf{F}}_{21} = -\vec{\mathbf{F}}_{12}\right)$$

or
$$\frac{d^2\vec{\mathbf{r}}}{dt^2} = \frac{1}{\mu}\vec{\mathbf{F}}_{21} + \left(\frac{\vec{\mathbf{F}}_{e1}}{m_1} - \frac{\vec{\mathbf{F}}_{e2}}{m_2}\right) \tag{14.27}$$

Recall that

$$\frac{1}{\mu} = \left(\frac{1}{m_1} + \frac{1}{m_2}\right)$$

or $\mu = \dfrac{m_1 m_2}{(m_1 + m_2)}$

where μ is the reduced mass of the system.

Let us put these results together.

Recap

EQUATIONS OF MOTION IN C. M. AND RELATIVE COORDINATES FOR NON-ZERO EXTERNAL FORCE

If a non-zero net external force $\vec{\mathbf{F}}_e = \vec{\mathbf{F}}_{e1} + \vec{\mathbf{F}}_{e2}$ is exerted on a two-particle system with masses m_1 and m_2, the **coupled equations of motion for the system in the coordinates** $(\vec{\mathbf{r}}_1, \vec{\mathbf{r}}_2)$ are given by

$$m_1\frac{d^2\vec{\mathbf{r}}_1}{dt^2} = \vec{\mathbf{F}}_1 \quad \text{and} \quad m_2\frac{d^2\vec{\mathbf{r}}_2}{dt^2} = \vec{\mathbf{F}}_2$$

These are transformed to the following equations in terms of the c.m. and relative coordinates $(\vec{\mathbf{r}}, \vec{\mathbf{R}}_{cm})$:

$$M\frac{d^2\vec{\mathbf{R}}_{cm}}{dt^2} = \vec{\mathbf{F}}_e \quad \text{where } M = m_1 + m_2 \tag{14.25}$$

and
$$\frac{d^2\vec{\mathbf{r}}}{dt^2} = \frac{1}{\mu}\vec{\mathbf{F}}_{21} + \left(\frac{\vec{\mathbf{F}}_{e1}}{m_1} - \frac{\vec{\mathbf{F}}_{e2}}{m_2}\right) \tag{14.27}$$

Equations of motion in c.m. and relative coordinates for non-zero net external force

So we have obtained **two distinct equations**: Eq. (14.25) and Eq. (14.27). Eq. (14.25) is the equation of motion of the centre of mass of the system. We can solve it for the motion of the system as if its entire mass were concentrated at its centre of mass.

But we cannot obtain a general analytical solution of Eq. (14.27) for the motion of individual particles in the system except for special cases such as the following:

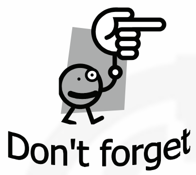
1. $\vec{\mathbf{F}}_{e1} = \vec{\mathbf{F}}_{e2} = \vec{\mathbf{0}}$

2. $\dfrac{\vec{\mathbf{F}}_{e1}}{m_1} = \dfrac{\vec{\mathbf{F}}_{e2}}{m_2}$

The force of gravity is an example of the second kind.

Thus, we need to solve two equations to determine the motion of the particles in the system and cannot reduce the problem to an equivalent one-body problem. Hence, we cannot obtain the general solution for the individual motion of the two particles except for very special cases mentioned above. However, **we can still determine how the c.m. of the system moves, that is, how the system moves if we take its entire mass to be concentrated at its c.m.**

In this section, you have learnt that



It is not possible to obtain the general analytical solutions for the paths of the individual particles of a two-particle system on which a net external force is being exerted except for special cases. We can obtain the path followed by its centre of mass and hence study the motion of the system as if its entire mass were concentrated at its centre of mass.

As an exercise, you may like to derive the equation of the motion of the centre of mass of a two-particle system falling under gravity.

SAQ 3 – **Two-body motion under gravity**

A drummer's band is led by a drum major whose baton is made up of two masses m_1 and m_2 separated by a thin, light rod of length L (Fig. 14.14). Determine the positions of the masses with respect to the baton's c.m. Write the equation of motion of the c.m. when it is thrown in air. Neglect air resistance and determine the path of the c.m.

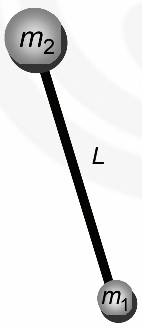


Fig. 14.14: **Drum major's baton**.

Let us now extend these concepts to briefly study the dynamics of many-particle systems.

14.4 DYNAMICS OF MANY-PARTICLE SYSTEMS

Let us begin with a simpler example of a three-body system of masses m_1, m_2 and m_3 with position vectors $\vec{\mathbf{r}}_1$, $\vec{\mathbf{r}}_2$ and $\vec{\mathbf{r}}_3$ with respect to an origin O. Fig. 14.15 shows an example of a three-body system: a weapon called the bola used for entangling animals. It is made up of three balls of iron or stone connected by leather thongs. What can we say about the motion of the bola when it is whirled and thrown in the air?

Extending what you have learnt so far, you can say that the c. m. of the system would move along a parabola. This is useful because when one throws the bola, one can aim it like a single body at the animal and forget about the complicated motion of the three balls. What are the equations of motion for this system? These are:

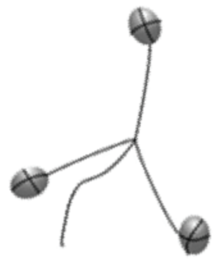


Fig. 14.15: The bola is an example of a three-particle system.

$$m_1 \frac{d^2 \vec{\mathbf{r}}_1}{dt^2} = \vec{\mathbf{F}}_1, \qquad m_2 \frac{d^2 \vec{\mathbf{r}}_2}{dt^2} = \vec{\mathbf{F}}_2 \qquad \text{and} \qquad m_3 \frac{d^2 \vec{\mathbf{r}}_3}{dt^2} = \vec{\mathbf{F}}_3 \tag{14.28}$$

where

$$\vec{\mathbf{F}}_1 = \vec{\mathbf{F}}_{e1} + \vec{\mathbf{F}}_{21} + \vec{\mathbf{F}}_{31}$$

$$\vec{\mathbf{F}}_2 = \vec{\mathbf{F}}_{e2} + \vec{\mathbf{F}}_{12} + \vec{\mathbf{F}}_{32} \tag{14.29a}$$

$$\vec{\mathbf{F}}_3 = \vec{\mathbf{F}}_{e3} + \vec{\mathbf{F}}_{13} + \vec{\mathbf{F}}_{23}$$

and

$$\vec{\mathbf{F}}_{12} = -\vec{\mathbf{F}}_{21}, \quad \vec{\mathbf{F}}_{23} = -\vec{\mathbf{F}}_{32} \quad \text{and} \quad \vec{\mathbf{F}}_{13} = -\vec{\mathbf{F}}_{31} \tag{14.29b}$$

We now add the three equations of motion given in Eq. (14.28) and use Eqs. (14.29a and b) to obtain

$$m_1 \frac{d^2 \vec{\mathbf{r}}_1}{dt^2} + m_2 \frac{d^2 \vec{\mathbf{r}}_2}{dt^2} + m_3 \frac{d^2 \vec{\mathbf{r}}_3}{dt^2} = \vec{\mathbf{F}}_1 + \vec{\mathbf{F}}_2 + \vec{\mathbf{F}}_3 = \vec{\mathbf{F}}_{e1} + \vec{\mathbf{F}}_{e2} + \vec{\mathbf{F}}_{e3} \tag{14.30a}$$

Using

$$\vec{\mathbf{R}}_{cm} = \frac{(m_1 \vec{\mathbf{r}}_1 + m_2 \vec{\mathbf{r}}_2 + m_3 \vec{\mathbf{r}}_3)}{(m_1 + m_2 + m_3)} \tag{14.30b}$$

we can write

$$M \frac{d^2 \vec{\mathbf{R}}_{cm}}{dt^2} = \vec{\mathbf{F}}_e \tag{14.30c}$$

where

$$M = m_1 + m_2 + m_3 \quad \text{and} \quad \vec{\mathbf{F}}_e = \vec{\mathbf{F}}_{e1} + \vec{\mathbf{F}}_{e2} + \vec{\mathbf{F}}_{e3} \tag{14.30d}$$

Now the external force on the three masses is just the force of gravity. Hence,

$$\vec{\mathbf{F}}_{e1} = m_1 \vec{\mathbf{g}}, \qquad \vec{\mathbf{F}}_{e2} = m_2 \vec{\mathbf{g}} \qquad \text{and} \qquad \vec{\mathbf{F}}_{e3} = m_3 \vec{\mathbf{g}} \tag{14.31a}$$

We now substitute Eq. (14.31a) in Eq. (14.30c) and use Eq. (14.30d) to write it as

$$M \frac{d^2 \vec{\mathbf{R}}_{cm}}{dt^2} = \vec{\mathbf{F}}_e = (m_1 + m_2 + m_3)\vec{\mathbf{g}} = M\vec{\mathbf{g}} \tag{14.31b}$$

or

$$\frac{d^2 \vec{\mathbf{R}}_{cm}}{dt^2} = \vec{\mathbf{g}} \tag{14.32}$$

Eq. (14.32) tells us that the c.m. of the bola falls freely under gravity. Thus, if thrown with a non-zero horizontal velocity component, it will move along a parabola just like a projectile whatever may be the motion of the individual particles in it. We extend this result to an *N*-particle system without going into its derivation.

Let us consider a system of N particles with masses $m_1, m_2, m_3, m_4, \ldots\ldots, m_N$ (see Fig. 14.16). Let the position vectors of these particles be $\vec{\mathbf{r}}_1, \vec{\mathbf{r}}_2, \vec{\mathbf{r}}_3, \vec{\mathbf{r}}_4, \ldots, \vec{\mathbf{r}}_N$ with respect to an origin O.

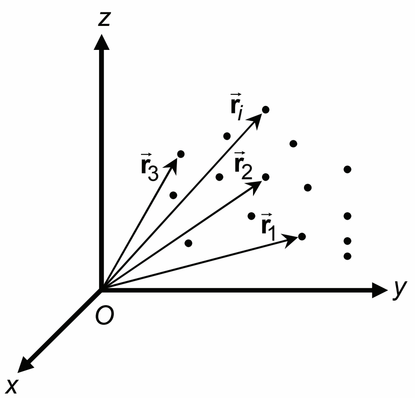


Fig. 14.16: Position vectors in a many-particle system.

Extending the definition of the c.m. for a three-particle system to this system, we get the definition of the centre of mass of an N-particle system:

Centre of mass of an *N*-particle system

$$\vec{\mathbf{R}}_{cm} = \frac{(m_1\vec{\mathbf{r}}_1 + m_2\vec{\mathbf{r}}_2 + m_3\vec{\mathbf{r}}_3 + m_4\vec{\mathbf{r}}_4 + \ldots + m_N\vec{\mathbf{r}}_N)}{(m_1 + m_2 + m_3 + m_4 + \ldots + m_N)} \tag{14.33a}$$

You will agree that this is a long expression to write all the time. So we introduce a special **summation notation for the sum**. We denote the sum

$$(m_1 + m_2 + m_3 + m_4 + \ldots + m_N) \quad \text{by} \quad \sum_{i=1}^{N} m_i$$

and $(m_1\vec{\mathbf{r}}_1 + m_2\vec{\mathbf{r}}_2 + m_3\vec{\mathbf{r}}_3 + m_4\vec{\mathbf{r}}_4 + \ldots + m_N\vec{\mathbf{r}}_N)$ by $\displaystyle\sum_{i=1}^{N} m_i\vec{\mathbf{r}}_i$

The symbol $\displaystyle\sum_{i=1}^{N}$ denotes the sum of the quantity written after it over the values taken by the **index** i. The symbol $\sum$ is pronounced as sigma. It is the capital sigma (the small sigma is denoted by σ). The index can be denoted by any letter instead of i, for example, j, n, p, m, etc.

So, whenever you see $\displaystyle\sum_{i=1}^{N}$ followed by a quantity with the subscript i or $\displaystyle\sum_{m=1}^{N}$ followed by a quantity with the subscript m, you should always think of it as the sum of N quantities. For example,

$$\sum_{i=1}^{N} c_i = (c_1 + c_2 + c_3 + c_4 + \ldots\ldots + c_N)$$

or $$\sum_{m=1}^{N} \vec{\mathbf{r}}_m = (\vec{\mathbf{r}}_1 + \vec{\mathbf{r}}_2 + \vec{\mathbf{r}}_3 + \vec{\mathbf{r}}_4 + \ldots\ldots + \vec{\mathbf{r}}_N)$$

You must practice using this notation by expressing the sums of some more physical quantities for this system. For example, write the total linear momentum, the total kinetic energy and the total angular momentum of the N-particle system and express it in this notation. Thus, using the summation notation, we can write the expression for the centre of mass of the system given by Eq. (14.33a) in a compact form as follows:

$$\vec{\mathbf{R}}_{cm} = \frac{\sum_{i=1}^{N} m_i \vec{\mathbf{r}}_i}{\sum_{i=1}^{N} m_i}$$

or

$$\vec{\mathbf{R}}_{cm} = \frac{\sum_{i=1}^{N} m_i \vec{\mathbf{r}}_i}{M} \qquad \text{where } M = \sum_{i=1}^{N} m_i \tag{14.33b}$$

Centre of mass of an N-particle system

Let us now write the equations of motion for the N-particle system using this notation. In terms of the x, y, z coordinates, we can write the centre of mass coordinates as:

$$X_{cm} = \frac{\sum_{i=1}^{N} m_i\, x_i}{\sum_{i=1}^{N} m_i}, \qquad Y_{cm} = \frac{\sum_{i=1}^{N} m_i\, y_i}{\sum_{i=1}^{N} m_i} \qquad Z_{cm} = \frac{\sum_{i=1}^{N} m_i\, z_i}{\sum_{i=1}^{N} m_i} \tag{14.33c}$$

Centre of mass coordinates for an N-particle system

A rigid body is a familiar example of a many-particle system. It is defined as a continuous aggregate of point masses such that the relative separation between any two point masses in the body always remains constant. Rigid bodies can be symmetrical or asymmetrical in shape. For example, sphere, bar, lamina, cylinder, cube and disc are symmetrical rigid bodies. The problem of finding the c.m. of a rigid body is complicated when the body is asymmetrical. Here we will not discuss the method of finding the c.m. of a rigid body. However, in Fig. 14.17 we show the c.m of some symmetrical bodies.

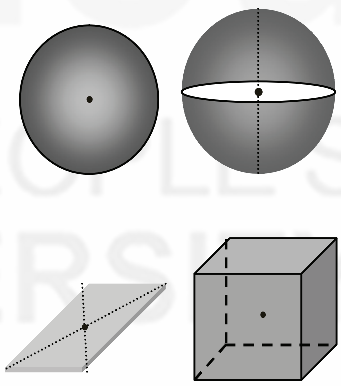


Fig. 14.17: **Location of the centre of mass of some symmetrical rigid bodies.**

Let us now write the equation of motion for the c.m. coordinate of an N-particle system. It is simply

$$M\ddot{\vec{\mathbf{R}}}_{cm} = \sum_{i=1}^{N} \vec{\mathbf{F}}_{ei} = \vec{\mathbf{F}}_e \tag{14.34}$$

where $\vec{\mathbf{R}}_{cm}$ is given by Eq. (14.33b) and $\vec{\mathbf{F}}_e$ is the **net external force being exerted on the system**.

Do you recognise that Eq. (14.34) is the equation of motion of a particle of mass M located at the centre of mass of the system?

So, what have we been able to do in this section?

We have **replaced the set of N equations for the system of N particles by a single equation for a particle of mass $M = \sum_{i=1}^{N} m_i$ located at the centre of mass of the system**. **Physically, the system of N particles then behaves**

as if the entire mass were located at the centre of mass of the system. This is the advantage of introducing the idea of a centre of mass. Let us state it once more.

Don't forget

> **As long as we are interested in the motion of the system as a whole, we may replace the system by a particle of mass *M* (equal to its total mass) located at its centre of mass.**

In the example of the bola, you have seen this result in action for the external force of gravity. The solution of the equation tells us that the centre of mass of a complex system or object follows a simple parabolic path under gravity even though its individual parts may follow complex motions. With this we end this section and summarise it.

Recap

EQUATION OF MOTION FOR THE C. M. OF AN *N*-PARTICLE SYSTEM

The equation of motion for a system of N particles with masses $m_1, m_2, m_3, m_4, \ldots, m_N$ having position vectors $\vec{\mathbf{r}}_1, \vec{\mathbf{r}}_2, \vec{\mathbf{r}}_3, \ldots, \vec{\mathbf{r}}_N$, respectively, with respect to an origin O, and acted upon by a net external force $\vec{\mathbf{F}}_e$ is,

$$M\ddot{\vec{\mathbf{R}}}_{cm} = \sum_{i=1}^{N} \vec{\mathbf{F}}_{ei} = \vec{\mathbf{F}}_e \qquad (14.34)$$

where $\vec{\mathbf{R}}_{cm}$ defines the position of the centre of mass of the N-particle system and is given by

$$\vec{\mathbf{R}}_{cm} = \frac{\sum_{i=1}^{N} m_i \vec{\mathbf{r}}_i}{\sum_{i=1}^{N} m_i} = \frac{\sum_{i=1}^{N} m_i \vec{\mathbf{r}}_i}{M} \quad \text{where } M = \sum_{i=1}^{N} m_i \qquad (14.33b)$$

We now end this unit and summarise what you have studied in it.

14.5 SUMMARY

Concept	Description
Centre of mass of a two-particle system	■ The **centre of mass** of a material body or a system of particles is the point, which moves as though the system's total mass existed at that point and all external forces were applied at that point. Its **position vector** for a **two-particle system** with masses m_1 and m_2 having position vectors $\vec{\mathbf{r}}_1$ and $\vec{\mathbf{r}}_2$, respectively, with respect to an origin O, is defined as $$\vec{\mathbf{R}}_{cm} = \frac{m_1\vec{\mathbf{r}}_1 + m_2\vec{\mathbf{r}}_2}{m_1 + m_2}$$

Relative coordinate

■ The **relative coordinate of a two-particle system** is defined as the position of the mass m_1 relative to the mass m_2 and is given by:

$$\vec{\mathbf{r}} = \vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2$$

Equation of motion in c.m. and relative coordinates for zero net external force

■ If the net external force on a two-particle system with masses m_1 and m_2 is zero, the **coupled equations of motion for the system in the coordinates** $(\vec{\mathbf{r}}_1, \vec{\mathbf{r}}_2)$ are given by

$$m_1 \frac{d^2 \vec{\mathbf{r}}_1}{dt^2} = \vec{\mathbf{F}}_{21} \quad \text{and} \quad m_2 \frac{d^2 \vec{\mathbf{r}}_2}{dt^2} = \vec{\mathbf{F}}_{12} = -\vec{\mathbf{F}}_{21}$$

These equations are reduced to the following equations in terms of the relative and c.m. coordinates $(\vec{\mathbf{r}}, \vec{\mathbf{R}}_{cm})$:

$$M \frac{d^2 \vec{\mathbf{R}}_{cm}}{dt^2} = \vec{\mathbf{0}} \quad \text{where } M = m_1 + m_2$$

$$\text{and} \quad \mu \frac{d^2 \vec{\mathbf{r}}}{dt^2} = \vec{\mathbf{F}}_{21} \quad \text{or} \quad \mu \frac{d^2 \vec{\mathbf{r}}}{dt^2} = -\vec{\mathbf{F}}_{12}$$

$$\text{where} \quad \frac{1}{\mu} = \left(\frac{1}{m_1} + \frac{1}{m_2}\right) \quad \text{or} \quad \mu = \frac{m_1 m_2}{(m_1 + m_2)}$$

Reduced mass

■ The **reduced mass of a two-particle system** with masses m_1 and m_2 is given by

$$\frac{1}{\mu} = \left(\frac{1}{m_1} + \frac{1}{m_2}\right) \quad \text{or} \quad \mu = \frac{m_1 m_2}{m_1 + m_2}$$

Equations of motion in c.m. and relative coordinates for non-zero external force

■ If non-zero net forces $\vec{\mathbf{F}}_1$ and $\vec{\mathbf{F}}_2$ are exerted on the two particles having masses m_1 and m_2 in a two-particle system, respectively, the **coupled equations of motion for the system in the coordinates** $(\vec{\mathbf{r}}_1, \vec{\mathbf{r}}_2)$ are given by

$$m_1 \frac{d^2 \vec{\mathbf{r}}_1}{dt^2} = \vec{\mathbf{F}}_1 \quad \text{and} \quad m_2 \frac{d^2 \vec{\mathbf{r}}_2}{dt^2} = \vec{\mathbf{F}}_2$$

These are reduced to the following equations in terms of the c.m. and relative coordinates $(\vec{\mathbf{r}}, \vec{\mathbf{R}}_{cm})$:

$$M \frac{d^2 \vec{\mathbf{R}}}{dt^2} = \vec{\mathbf{F}}_e \quad \text{where } M = m_1 + m_2$$

$$\text{and} \quad \frac{d^2 \vec{\mathbf{r}}}{dt^2} = \frac{1}{\mu} \vec{\mathbf{F}}_{21} + \left(\frac{\vec{\mathbf{F}}_{e1}}{m_1} - \frac{\vec{\mathbf{F}}_{e2}}{m_2}\right)$$

where $\vec{\mathbf{F}}_e = \vec{\mathbf{F}}_{e1} + \vec{\mathbf{F}}_{e2}$ is the net external force on the system and $\vec{\mathbf{F}}_{e1}$ and $\vec{\mathbf{F}}_{e2}$ are the net external forces on the respective particles.

Centre of mass of an N-particle system

- The **position vector of the centre of mass** for a **system of *N*-particles** with masses $m_1, m_2, m_3, m_4, \ldots, m_N$ having position vectors $\vec{\mathbf{r}}_1, \vec{\mathbf{r}}_2, \vec{\mathbf{r}}_3, \ldots, \vec{\mathbf{r}}_N$, respectively, with respect to an origin *O*, is defined as

$$\vec{\mathbf{R}}_{cm} = \frac{\sum_{i=1}^{N} m_i \vec{\mathbf{r}}_i}{M} \qquad \text{where } M = \sum_{i=1}^{N} m_i$$

In terms of the *x*, *y* and *z* coordinates, it can be written as

$$X_{cm} = \frac{\sum_{i=1}^{N} m_i\, x_i}{\sum_{i=1}^{N} m_i}, \quad Y_{cm} = \frac{\sum_{i=1}^{N} m_i\, y_i}{\sum_{i=1}^{N} m_i}, \quad Z_{cm} = \frac{\sum_{i=1}^{N} m_i\, z_i}{\sum_{i=1}^{N} m_i}$$

Equation of motion of the centre of mass for an N-particle system

- The **equation of motion of the centre of mass of a system of *N* particles** with masses $m_1, m_2, m_3, m_4, \ldots, m_N$ having position vectors $\vec{\mathbf{r}}_1, \vec{\mathbf{r}}_2, \vec{\mathbf{r}}_3, \ldots, \vec{\mathbf{r}}_N$, respectively, with respect to an origin *O*, and acted upon by a net external force $\vec{\mathbf{F}}_e$ is,

$$M\ddot{\vec{\mathbf{R}}}_{cm} = \sum_{i=1}^{N} \vec{\mathbf{F}}_{ei} = \vec{\mathbf{F}}_e$$

where $\vec{\mathbf{R}}_{cm}$ is the position vector of the centre of mass of the *N*-particle system.

14.6 TERMINAL QUESTIONS

1. Two particles of masses 2 kg and 8 kg are separated by a distance of 1 m. The distance of their centre of mass from the more massive particle is

   (a) 0.5 cm

   (b) 0.2 m

   (c) 0.8 m

   (d) 0.1 m

2. A proton and an electron, initially at rest, are allowed to move under their mutual attractive force. Their centre of mass will

   (a) move towards the proton.

   (b) move towards the electron.

   (c) remain stationary.

   (d) move in an unpredictable manner.

3. Three homogeneous solid spheres of masses 2.0 kg, 3.0 kg and 5.0 kg are arranged with their centres at $(2.0\,\text{m}\,\hat{\mathbf{i}} + 2.0\,\text{m}\,\hat{\mathbf{j}} + 5.0\,\text{m}\,\hat{\mathbf{k}})$, $(3.0\,\text{m}\,\hat{\mathbf{i}} - 4.0\,\text{m}\,\hat{\mathbf{j}} - 1.0\,\text{m}\,\hat{\mathbf{k}})$ and $(4.0\,\text{m}\,\hat{\mathbf{i}} - 2.0\,\text{m}\,\hat{\mathbf{j}} - 2.0\,\text{m}\,\hat{\mathbf{k}})$,

respectively. The y-coordinate of the centre of mass of the system of spheres is

(a) 3.3 m

(b) 0.3 m

(c) 1.8 m

(d) – 1.8 m

4. Locate the centre of mass and relative coordinates of a two-particle system consisting of two masses of 1.5 kg and 2.5 kg placed 3.0 m apart.

5. Particles of mass 1.0 kg, 2.0 kg, 3.0 kg and 4.0 kg are placed at four corners of a rectangle $ABCD$ as shown in Fig. 14.18. Given that $AB = 8.0$ cm and $AD = 4.0$ cm, locate the centre of mass of the system.

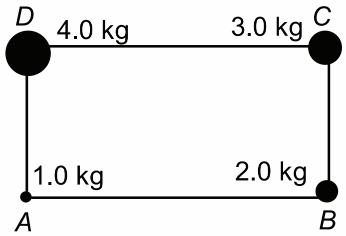


Fig. 14.18: Diagram for Terminal Question 5.

6. Three particles of masses 1.0 kg, 2.0 kg and 3.0 kg, respectively, are placed at the vertices A, B and C of a right-angled triangle (Fig. 14.19). Determine the centre of mass of the system.

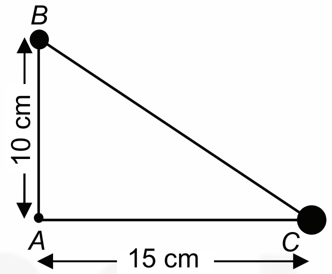


Fig. 14.19: Diagram for Terminal Question 6.

7. Determine the reduced mass (in au) of the carbon monoxide (CO) molecule given that the mass of the carbon and oxygen atoms are 12 au and 16 au, respectively.

8. Obtain the reduced mass (in units of solar mass) of a binary star system in which the masses of the two stars are 1.1 and 0.90 times the solar mass, respectively.

9. Determine the reduced mass of the system of planet Pluto and its moon. It is given that Pluto's mass is 1.31×10^{22} kg and that of its moon is 1.52×10^{21} kg.

10. The position vectors of three particle of masses $m_1 = 2.0\,\text{kg}$, $m_2 = 3.0\,\text{kg}$ and $m_3 = 5.0\,\text{kg}$ are given by $\vec{r}_1 = (2t+5t^2)\,\text{m}\,\hat{\mathbf{i}} + 3t\,\text{m}\,\hat{\mathbf{j}}$, $\vec{r}_2 = 4\,\text{m}\,\hat{\mathbf{i}} + (7t^2)\,\text{m}\,\hat{\mathbf{j}}$ and $\vec{r}_3 = 6t\,\text{m}\,\hat{\mathbf{i}} + 2\,\text{m}\,\hat{\mathbf{j}}$, respectively. Determine the velocity and acceleration of the centre of mass of the system.

14.7 SOLUTIONS AND ANSWERS

Self-Assessment Questions

1. a) Let the two objects be along the x-axis with the object of mass 2.5 kg at the origin. Then the coordinates of the two objects of masses $m_1 = 2.5\,\text{kg}$ and $m_2 = 25\,\text{kg}$ are $x_1 = 0$ and $x_2 = 1.0\,\text{m}$, respectively. From Eq. (14.7b) and Eq. (14.9b), respectively, the coordinate of the centre of mass and the relative coordinate are

$$X_{cm} = \frac{2.5\,\text{kg}\times 0 + 25\,\text{kg}\times 1.0\,\text{m}}{2.5\,\text{kg} + 25\,\text{kg}} = 0.91\,\text{m} \quad \text{and} \quad x = x_1 - x_2 = -1.0\,\text{m}$$

b) Let the centres of the Earth and the Moon be along the x-axis with the Earth at the origin. We use Eqs. (14.7b and 14.9b) with

$m_1 = M_e = 5.97\times10^{24}\,\text{kg}$ and $m_2 = M_m = 7.35\times10^{22}\,\text{kg}$

Let $x_1 = x$ coordinate of the Earth $= 0$

and $x_2 =$ distance between the Earth and the Moon $= 3.84 \times 10^8$ m

The centre of mass coordinate is

$$X_{cm} = \frac{5.97 \times 10^{24}\,\text{kg} \times 0 + 7.35 \times 10^{22}\,\text{kg} \times 3.84 \times 10^8\,\text{m}}{5.97 \times 10^{24}\,\text{kg} + 7.35 \times 10^{22}\,\text{kg}} = 4.67 \times 10^6\,\text{m}$$

So the centre of mass in this case is very close to the Earth, at a distance of 4.67×10^3 km from its centre. The relative coordinate is

$$x = x_1 - x_2 = -3.84 \times 10^8\,\text{m}$$

c) We use Eqs. (14.7b and c) and Eqs. (14.9b and c) with $m_1 = m_2 = 5.0\,\text{kg},\ x_1 = 0, y_1 = 0,\ x_2 = 0$ and $y_2 = 2.0\,\text{m}$

So the centre of mass coordinates are: $X_{cm} = 0$ and

$$Y_{cm} = \frac{5.0\,\text{kg} \times 0 + 5.0\,\text{kg} \times 2.0\,\text{m}}{5.0\,\text{kg} + 5.0\,\text{kg}} = 1.0\,\text{m}$$

The coordinates of the centre of mass in m are (0, 1.0). The relative coordinates are: $x = x_1 - x_2 = 0$ and $y = y_1 - y_2 = -2.0\,\text{m}$

2. a) We use Eq. (14.19b) with $m_1 = M_e = 5.97 \times 10^{24}$ kg and $m_2 = M_m = 7.35 \times 10^{22}$ kg. The reduced mass is

$$\mu = \frac{5.97 \times 10^{24}\,\text{kg} \times 7.35 \times 10^{22}\,\text{kg}}{5.97 \times 10^{24}\,\text{kg} + 7.35 \times 10^{22}\,\text{kg}} = 7.26 \times 10^{22}\,\text{kg} \cong M_m$$

b) For a binary system of particles of equal masses, the reduced mass is half the mass of each particle. In this case the reduced mass is half the mass of each marble, so

$$\mu = \frac{1}{2} \times (0.2\,\text{kg}) = 0.1\,\text{kg}$$

c) The mass of the proton is $m_p = 1.67 \times 10^{-27}$ kg and the mass of the electron is $m_e = 9.1 \times 10^{-31}$ kg. Since the mass of the proton is much greater than the mass of the electron $(m_p \approx 1836\, m_e)$, the reduced mass of the system is approximately equal to the mass of the electron.

d) We use Eq. (14.19b) with $m_1 =$ mass of the star, m

and $m_2 =$ mass of the black hole $(= 100m)$. Thus,

$$\mu = \frac{m \times 100\,m}{100\,m + m} = 0.99\,m \approx m \cong \text{mass of the star}$$

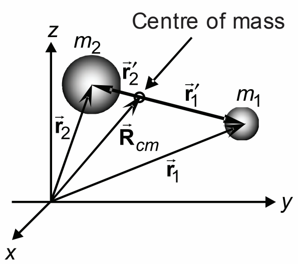


Fig. 14.20: Diagram for answer to SAQ 3.

3. Let the position vectors of the two masses m_1 and m_2 in the baton be $\vec{r}_1'$ and $\vec{r}_2'$, respectively, with respect to the c.m. (Fig. 14.20). Neglecting the mass of the thin rod, from Fig. 14.20, we can write the position vectors of the masses with respect to their c.m. as

$$\vec{r}_1' = \vec{r}_1 - \vec{\mathbf{R}}_{cm} \quad \text{and} \quad \vec{r}_2' = \vec{r}_2 - \vec{\mathbf{R}}_{cm}$$

Substituting the expression for the c.m. coordinate, we get

$$\vec{\mathbf{r}}_1' = \frac{m_2}{m_1 + m_2}\vec{\mathbf{r}} \quad \text{and} \quad \vec{\mathbf{r}}_2' = -\frac{m_1}{m_1 + m_2}\vec{\mathbf{r}}, \ \text{where } \vec{\mathbf{r}} = \vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2$$

The magnitudes of these vectors are given as

$$\left|\vec{\mathbf{r}}_1'\right| = \frac{m_2}{m_1 + m_2}\left|\vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2\right| = \frac{m_2 L}{m_1 + m_2} \text{ and } \left|\vec{\mathbf{r}}_2'\right| = \frac{m_1}{m_1 + m_2}\left|\vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2\right| = \frac{m_1 L}{m_1 + m_2}$$

The equation of motion of the c.m. under gravity is given by Eq. (14.25) with $M = m_1 + m_2$ and $\vec{\mathbf{F}}_e = \vec{\mathbf{F}}_{e1} + \vec{\mathbf{F}}_{e2} = m_1\vec{\mathbf{g}} + m_2\vec{\mathbf{g}} = (m_1 + m_2)\vec{\mathbf{g}}$

$$\therefore \quad \frac{d^2\vec{\mathbf{R}}_{cm}}{dt^2} = \vec{\mathbf{g}}$$

This is the equation of an object falling freely under the Earth's gravity. Hence, the path of the c.m. is a parabola.

Terminal Questions

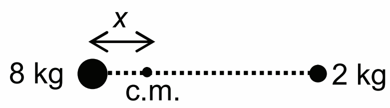


Fig. 14.21: **Not to scale.**

1. The correct option is (b). Let the particles be positioned on the x-axis, with the more massive particle (of mass 8 kg) at the origin (Fig. 14.21). Let the centre of mass be at a distance x from the origin. From Eq. (14.7b), the position of the centre of mass with respect to the origin is given by

$$x_{cm} = \frac{(8\,\text{kg} \times 0 + 2\,\text{kg} \times 1\,\text{m})}{10\,\text{kg}} = 0.2\,\text{m}$$

2. The correct option is (c) because an external force is needed to move the centre of mass of a system of particles. The mutual force of electrostatic attraction is an internal force which cannot change the position of the centre of mass of the system.

3. The correct option is (d). From Eq. (14.33c), the y-coordinate of the centre of mass is

$$Y_{cm} = \frac{[2.0\,\text{kg} \times 2.0\,\text{m} + 3.0\,\text{kg} \times (-4.0\,\text{m}) + 5.0\,\text{kg} \times (-2.0\,\text{m})]}{(2.0\,\text{kg} + 3.0\,\text{kg} + 5.0\,\text{kg})} = -1.8\,\text{m}$$

4. We consider both masses to be located on the x-axis with the mass of 1.5 kg at the origin. Using Eq. (14.7b), with $m_1 = 1.5\,\text{kg}$, $m_2 = 2.5\,\text{kg}$, $x_1 = 0$ and $x_2 = 3.0\,\text{m}$, we get the position of the centre of mass to be:

$$X_{cm} = \frac{1.5\,\text{kg} \times 0 + 2.5\,\text{kg} \times 3.0\,\text{m}}{1.5\,\text{kg} + 2.5\,\text{kg}} = 1.9\,\text{m}$$

The relative coordinate $x = x_1 - x_2 = -3.0\,\text{m}$

5. In Fig. 14.18, we choose the x and y-axis to lie along AB and AD, respectively, and A to be the origin of the coordinate system. We use Eq. (14.33c) with

$$\begin{array}{llll} m_1 = 1.0\,\text{kg} & m_2 = 2.0\,\text{kg} & m_3 = 3.0\,\text{kg} \text{ and} & m_4 = 4.0\,\text{kg} \\ x_1 = 0 & x_2 = 8.0\,\text{cm} & x_3 = 8.0\,\text{cm} \text{ and} & x_4 = 0\,\text{cm} \\ y_1 = 0 & y_2 = 0 & y_3 = 4.0\,\text{cm} \text{ and} & y_4 = 4.0\,\text{cm} \end{array}$$

So the coordinates of the centre of mass are

$$X_{cm} = \frac{1.0\,\text{kg} \times 0 + 2.0\,\text{kg} \times 8.0\,\text{cm} + 3.0\,\text{kg} \times 8.0\,\text{cm} + 4.0\,\text{kg} \times 0}{1.0\,\text{kg} + 2.0\,\text{kg} + 3.0\,\text{kg} + 4.0\,\text{kg}}$$

$$= 4.0\,\text{cm}$$

$$Y_{cm} = \frac{1.0\,\text{kg} \times 0 + 2.0\,\text{kg} \times 0 + 3.0\,\text{kg} \times 4.0\,\text{cm} + 4.0\,\text{kg} \times 4.0\,\text{cm}}{10\,\text{kg}}$$

$$= 2.8\,\text{cm}$$

6. In Fig. 14.19, let us take the x and y-axes to be along AC and AB, respectively. Then we can use Eq. (14.33c) to find the x and y coordinates of the centre of mass of the system. The coordinates of A, B and C (in m) are (0, 0), (0, 0.10) and (0.15, 0), respectively. Then

$$X_{cm} = \frac{1.0\,\text{kg} \times 0 + 2.0\,\text{kg} \times 0\,\text{m} + 3.0\,\text{kg} \times 0.15\,\text{m}}{6.0\,\text{kg}} = 7.5 \times 10^{-2}\,\text{m}$$

$$Y_{cm} = \frac{1.0\,\text{kg} \times 0 + 2.0\,\text{kg} \times 0.1\,\text{m} + 3.0\,\text{kg} \times 0}{6.0\,\text{kg}} = 3.3 \times 10^{-2}\,\text{m}$$

So the coordinates of the centre of mass are $(7.5 \times 10^{-2}\,\text{m}, 3.3 \times 10^{-2}\,\text{m})$

7. From Eq. (14.19b) with $m_1 = 12$ au and $m_2 = 16$ au, we get

$$\mu = \frac{12\,\text{au} \times 16\,\text{au}}{12\,\text{au} + 16\,\text{au}} = 6.9\,\text{au}$$

8. We use Eq. (14.19b) with $m_1 = 1.1 M_s$ and $m_2 = 0.9 M_s$, where M_s is the solar mass. The reduced mass is

$$\mu = \frac{1.1 M_s \times 0.90 M_s}{1.1 M_s + 0.90 M_s} = 0.495 M_s \approx 0.5 M_s$$

9. From Eq. (14.19b) with $m_1 = 1.31 \times 10^{22}$ kg and $m_2 = 1.52 \times 10^{21}$ kg,

$$\mu = \frac{1.31 \times 10^{22}\,\text{kg} \times 1.52 \times 10^{21}\,\text{kg}}{1.31 \times 10^{22}\,\text{kg} + 1.52 \times 10^{21}\,\text{kg}} = 1.36 \times 10^{21}\,\text{kg}$$

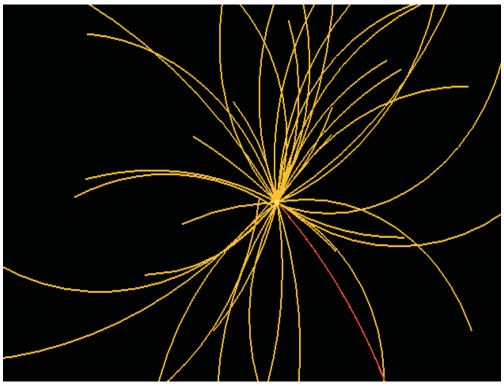
10. Using Eq. (14.33a), the position vector of the centre of mass of the system is

$$\vec{\mathbf{R}}_{cm} = (1.2 + 3.4t + t^2)\,\text{m}\,\hat{\mathbf{i}} + (1.0 + 0.6t + 2.1t^2)\,\text{m}\,\hat{\mathbf{j}}$$

The velocity and acceleration of the centre of mass are, respectively,

$$\vec{\mathbf{v}}_{cm} = \frac{d\vec{\mathbf{R}}_{cm}}{dt} = (3.4 + 2t)\,\text{ms}^{-1}\,\hat{\mathbf{i}} + (0.6 + 4.2t)\,\text{ms}^{-1}\,\hat{\mathbf{j}}$$

$$\vec{\mathbf{a}}_{cm} = \frac{d\vec{\mathbf{v}}_{cm}}{dt} = 2.0\,\text{ms}^{-2}\,\hat{\mathbf{i}} + 4.2\,\text{ms}^{-2}\,\hat{\mathbf{j}}$$



How do conservation laws applied to the collisions of elementary particles reveal the secrets of the microscopic world? This is what you will learn in this unit!

UNIT 15

CONSERVATION LAWS FOR MANY-PARTICLE SYSTEMS

Structure

STUDY GUIDE

In this unit, we determine the linear momentum, angular momentum and energy of two-particle and many-particle systems and deduce the respective conservation laws.

For understanding the concepts in this unit well, you should know the concepts of linear momentum, energy and angular momentum of single particles and the respective conservation laws. You may need to revise these concepts from Block 2. You should also revise the mathematical concepts of **vector algebra** given in Block 1. Some parts of this unit are quite mathematical and abstract. While studying them, **always keep a paper and pen with you and work out all the intermediate steps yourself**. This will help you learn the concepts of this unit well. You must also solve all examples, SAQs and Terminal Questions on your own!

"Nature never breaks her own laws."

Leonardo da Vinci

15.1 INTRODUCTION

In Unit 14, you have studied the dynamics of two-body systems and learnt the concept of centre of mass. You have learnt how to express the coupled equations of motion for two-particle systems in terms of decoupled equations in the **centre of mass** and **relative coordinates**. You have also learnt how to extend these concepts to many-particle systems. In this unit, we obtain the expressions of linear momentum, angular momentum and energy for two-particle and many-particle systems. We also revisit the respective conservation laws. You may recall from Units 8, 10 and 12 that these laws make the study of motion of objects quite simple. That is why we would like to apply them to many-particle systems.

In Sec. 15.2, we revisit the **concept of linear momentum for two-particle systems** and **the law of conservation of linear momentum** about which you have studied in Unit 8. We extend it to many-particle systems. In Sec. 15.3, we discuss the conservation of mechanical energy for two-particle systems and many-particle systems. In Sec. 15.4, we apply the laws of conservation of linear momentum and energy to study elastic collisions of two particles. In Sec. 15.5, we discuss the conservation of angular momentum for two-particle systems and many-particle systems.

With this unit, we complete our study of mechanics as applied to the translational and rotational motion of objects. In the next block you will study about oscillatory motion.

Expected Learning Outcomes

After studying this unit, you should be able to:

- determine the linear momentum, angular momentum and kinetic energy of two-particle and many-particle systems;
- derive the expressions of linear momentum, kinetic energy, and angular momentum of two-particle systems in terms of the c.m. and relative coordinates;
- write the expressions of linear momentum, kinetic energy, and angular momentum of many-particle systems in terms of the c.m. and relative coordinates;
- state the laws of conservation of linear momentum, mechanical energy and angular momentum for two-particle and many-particle systems;
- apply the laws of conservation of linear momentum, mechanical energy and angular momentum for simple two-particle systems; and
- solve problems on elastic collisions of two particles.

15.2 CONSERVATION OF LINEAR MOMENTUM

In this section, we first determine the linear momentum of a two-particle system in terms of the c.m. and the relative coordinates. Then we extend the concept to many-particle systems. We also discuss the law of conservation of linear momentum for two-particle and many-particle systems.

15.2.1 Two-particle System

The **total linear momentum** of the two-particle system is the sum of the linear momenta of the two particles in the system:

$$\vec{\mathbf{p}} = \vec{\mathbf{p}}_1 + \vec{\mathbf{p}}_2 \tag{15.1a}$$

where $\vec{\mathbf{p}}_1 = m_1\vec{\mathbf{v}}_1$ and $\vec{\mathbf{p}}_2 = m_2\vec{\mathbf{v}}_2$. Using the definition of the c.m. coordinate given in Eq. (14.2) of Unit 14, we can recast Eq. (15.1a) in terms of the c.m. coordinate. We write Eq. (15.1a) as

$$\vec{\mathbf{p}} = m_1\vec{\mathbf{v}}_1 + m_2\vec{\mathbf{v}}_2 = m_1\frac{d\vec{\mathbf{r}}_1}{dt} + m_2\frac{d\vec{\mathbf{r}}_2}{dt} \tag{15.1b}$$

We now divide and multiply the RHS of Eq. (15.1b) by $(m_1 + m_2)$.

Thus, we have
$$\vec{\mathbf{p}} = (m_1 + m_2)\left(\frac{m_1\frac{d\vec{\mathbf{r}}_1}{dt} + m_2\frac{d\vec{\mathbf{r}}_2}{dt}}{m_1 + m_2}\right) \tag{15.1c}$$

From Eq. (14.2), $\vec{\mathbf{R}}_{cm} = \dfrac{m_1\vec{\mathbf{r}}_1 + m_2\vec{\mathbf{r}}_2}{(m_1 + m_2)}$

Differentiating Eq. (14.2) given above with respect to time, we get the velocity of the c.m. of the two-particle system:

$$\vec{\mathbf{V}}_{cm} = \frac{d\vec{\mathbf{R}}_{cm}}{dt} = \frac{m_1\frac{d\vec{\mathbf{r}}_1}{dt} + m_2\frac{d\vec{\mathbf{r}}_2}{dt}}{m_1 + m_2} \tag{15.1d}$$

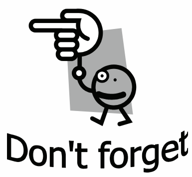
Substituting Eq. (15.1d) in Eq. (15.1c) we get

$$\vec{\mathbf{p}} = M\vec{\mathbf{V}}_{cm} \qquad \text{where} \qquad M = m_1 + m_2 \tag{15.1e}$$

Linear momentum of a two-particle system

Thus,

> **The total linear momentum of the two-particle system is just the linear momentum of its centre of mass.**



Let us apply this result.

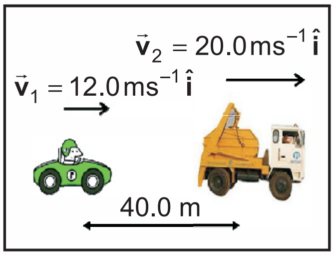


Fig. 15.1: **Linear momentum, position and velocity of c.m. of a two-particle system.**

EXAMPLE 15.1: MOTION OF TWO VEHICLES

A car of mass 1200 kg moves along a straight road with a speed of $12.0\ ms^{-1}$. A truck of mass 1800 kg and speed $20.0\ ms^{-1}$ moves in the same direction as the car. It is observed at some instant that the truck is 40.0 m ahead of the car. Determine the position of the centre of mass of the system consisting of the car and the truck at that instant. Calculate the total linear momentum of the system and the velocity of the centre of mass of the system at that instant.

SOLUTION ■ The **KEY IDEA** here is to treat the car and truck as particles and the system as a two-particle system. We can determine the coordinates of the car and the truck and hence the c.m. coordinates. The linear momentum of the system is given by Eq. (15.1a). The velocity of the c.m. can be determined using Eq. (15.1e).

Since both vehicles are moving in a straight line (Fig. 15.1), we choose the direction of motion to be along the *x*-axis. Thus, both objects are situated on the *x*-axis at any given instant. If the car is at the origin $x = 0$ m at the given instant, the truck's coordinate is $x = 40.0$ m. The position vectors of the car and the truck at that instant are given by $\vec{r}_1 = \vec{0}$ and $\vec{r}_2 = (40.0\ m)\hat{i}$

The position of the centre of mass of the system is

$$\vec{R}_{cm} = \frac{(1200\,kg)\,\vec{r}_1 + (1800\,kg)\,\vec{r}_2}{(1200 + 1800)\ kg} = \frac{(1800\,kg)\,(40.0\,m)\,\hat{i}}{3000\ kg} = 24.0\,m\ \hat{i}$$

The velocities of the car and the truck are $12.0\ ms^{-1}\hat{i}$ and $20.0\ ms^{-1}\hat{i}$, respectively. The total linear momentum of the system is

$$\vec{p} = m_1\vec{v}_1 + m_2\vec{v}_2 = (1200\ kg)\,\vec{v}_1 + (1800\,kg)\,\vec{v}_2$$
$$= \left(1200\ kg \times 12.0\ ms^{-1}\right)\hat{i} + \left(1800\ kg \times 20.0\ ms^{-1}\right)\hat{i} = 5.04 \times 10^4\ kgms^{-1}\ \hat{i}$$

The velocity of the centre of mass of the system is given by Eq. (15.1e) as

$$\vec{V}_{cm} = \frac{\vec{p}}{M} = \frac{5.04 \times 10^4\,kgms^{-1}}{3000\,kg}\ \hat{i} = 16.8\ ms^{-1}\ \hat{i}$$

Now recall Eq. (14.14) in Unit 14 for the case when there is no external force acting on the system; the only force is the force of mutual interaction between the two particles. When we integrate Eq. (14.14) with respect to time, we get

$$M\frac{d\vec{R}_{cm}}{dt} = \textbf{constant} \tag{15.2}$$

Eq. (15.2) together with Eq. (15.1e) tells us that **when no net external force is exerted on the system, the linear momentum of the centre of mass of the system is <u>constant</u>**. This brings us to the law of conservation of linear momentum for a two-particle system.

CONSERVATION OF LINEAR MOMENTUM FOR A TWO-PARTICLE SYSTEM

The total linear momentum of a two-particle system remains constant if the net external force exerted on the system is zero:

$$M \frac{d\vec{\mathbf{R}}_{cm}}{dt} = M\vec{\mathbf{V}}_{cm} = \textbf{constant} \qquad (15.2)$$

Conservation of linear momentum

Let us now take up a simple application of this law.

EXAMPLE 15.2: CONSERVATION OF LINEAR MOMENTUM

Two skaters, one of mass 40 kg and the other of mass 50 kg, stand in a smooth ice skating rink holding a pole of negligible mass. Starting from the ends of the pole, the skaters pull themselves along the pole with constant velocities until they meet. What is the velocity of the skater of mass 50 kg if the speed with which the skater of mass 40 kg moves is 10 ms^{-1}? Neglect friction.

SOLUTION ■ The **KEY IDEA** here is to treat the system as a two-particle system, determine whether the law of conservation of linear momentum can be applied to the system and then apply it.

Since there is no external force on the system, the total linear momentum of the system is conserved. Since the skaters are at rest initially, the total initial linear momentum of the system is zero. Let the velocities of the skaters of masses 40 kg and 50 kg be $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$, respectively. Since motion is in a straight line, we take the skater of mass 40 kg to move along the positive *x*-axis. Then from the law of conservation of linear momentum, we have

$$m_1\vec{\mathbf{v}}_1 + m_2\vec{\mathbf{v}}_2 = \vec{\mathbf{0}}$$

or

$$m_1 v_1 \hat{\mathbf{i}} - m_2 v_2 \hat{\mathbf{i}} = \vec{\mathbf{0}} \qquad \text{(i)}$$

since both skaters are moving opposite to each other. Substituting the values $m_1 = 40$ kg, $m_2 = 50$ kg and $v_1 = 10\text{ ms}^{-1}$ in Eq. (i), we get

$$v_2 = \frac{40\text{ kg} \times 10\text{ ms}^{-1}}{50\text{ kg}} = 8\text{ ms}^{-1}$$ along the negative *x*-axis.

SAQ 1 – Linear momentum of a two-particle system

a) A 2 kg ball and a 3 kg ball are moving towards each other with a speed of 5ms^{-1} each. What is the velocity of the centre of mass of the system?

b) A bullet of mass 10 g is fired into a block of wood of mass 10 kg and lodges into it. The speed of the block and the bullet is 0.2ms^{-1}. What is the initial speed of the bullet?

15.2.2 Many-particle System

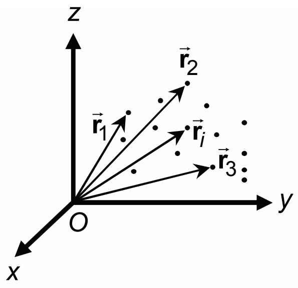


Fig. 15.2: **Many-particle system.**

Let us consider a system of N particles with masses $m_1, m_2, m_3, m_4,...,m_N$ such as the one we studied in Unit 14. Let the position vectors of these particles with respect to an origin O be $\vec{\mathbf{r}}_1, \vec{\mathbf{r}}_2, \vec{\mathbf{r}}_3, \vec{\mathbf{r}}_4,..., \vec{\mathbf{r}}_N$, respectively (see Fig. 15.2). What is the total linear momentum of this system? It is given by

$$\vec{\mathbf{P}} = m_1\dot{\vec{\mathbf{r}}}_1 + m_2\dot{\vec{\mathbf{r}}}_2 + ... + m_N\dot{\vec{\mathbf{r}}}_N \qquad (15.3a)$$

where $\dot{\vec{\mathbf{r}}}_1, \dot{\vec{\mathbf{r}}}_2,...,\dot{\vec{\mathbf{r}}}_N$ represent the velocities of the particles in the dot notation. We can use the summation notation to write the total linear momentum as

Linear momentum of an N-particle system

$$\vec{\mathbf{P}} = \sum_{i=1}^{N} m_i\dot{\vec{\mathbf{r}}}_i = M\dot{\vec{\mathbf{R}}}_{cm} \qquad (15.3b)$$

where we have used Eq. (14.33b) and put $\sum_{i=1}^{N} m_i\dot{\vec{\mathbf{r}}}_i = M\dot{\vec{\mathbf{R}}}_{cm}$.

Upon differentiating Eq. (15.3b) with respect to time, we get the equation of motion for a many-particle system.

Equation of motion for N-particle system

$$\frac{d\vec{\mathbf{P}}}{dt} = M\ddot{\vec{\mathbf{R}}}_{cm} = \vec{\mathbf{F}}_e \qquad (15.3c)$$

If the net external force on the system is zero, we have

$$\frac{d\vec{\mathbf{P}}}{dt} = M\ddot{\vec{\mathbf{R}}}_{cm} = \vec{\mathbf{0}} \qquad (15.4a)$$

or
$$\vec{\mathbf{P}} = M\dot{\vec{\mathbf{R}}}_{cm} = M\vec{\mathbf{V}}_{cm} = \textbf{constant} \qquad (15.4b)$$

In the dot notation,

$$\dot{\vec{\mathbf{r}}}_1 = \frac{d\vec{\mathbf{r}}_1}{dt}, \; \dot{\vec{\mathbf{r}}}_2 = \frac{d\vec{\mathbf{r}}_2}{dt} ...$$

and

$$\ddot{\vec{\mathbf{r}}}_1 = \frac{d^2\vec{\mathbf{r}}_1}{dt^2}, \; \ddot{\vec{\mathbf{r}}}_2 = \frac{d^2\vec{\mathbf{r}}_2}{dt^2} ...$$

This is just the **law of conservation of linear momentum** for an N-particle system.

CONSERVATION OF LINEAR MOMENTUM FOR A MANY- PARTICLE SYSTEM

The total linear momentum of an N-particle system remains constant if no external force acts on the system:

$$M\vec{\mathbf{V}}_{cm} = \textbf{constant} \qquad (15.4b)$$

Thus, **when no net external force acts on the many-particle system, the velocity of the centre of mass of the system remains constant.**

The centre of mass of the system moves in a straight line.

EXAMPLE 15.3: LINEAR MOMENTUM

Consider the system of three particles of equal mass m whose position vectors (in m) at three different instants of time are given below:

t (in s)	$\vec{\mathbf{r}}_1$	$\vec{\mathbf{r}}_2$	$\vec{\mathbf{r}}_3$
0	$\hat{\mathbf{i}} + \hat{\mathbf{j}}$	$2\hat{\mathbf{i}} + 2\hat{\mathbf{j}}$	$3\hat{\mathbf{i}} + 3\hat{\mathbf{j}}$
1	$\hat{\mathbf{i}}$	$\hat{\mathbf{j}}$	$3\hat{\mathbf{i}} + 3\hat{\mathbf{j}}$
2	$\hat{\mathbf{j}}$	$\hat{\mathbf{i}} + 2\hat{\mathbf{j}}$	$2\hat{\mathbf{i}}$

Determine the position of the centre of mass of the system at the three instants of time. Determine the average velocity of the c.m. between the time intervals of 0 to 1 s and 1s to 2s. Is there a net force being exerted on the system? Is its linear momentum conserved?

SOLUTION ■ The **KEY IDEA** here is to determine the average velocity of the c.m. of the three-particle system and see whether it is constant or changes. This will tell us whether a net external force is exerted on the system and its linear momentum is conserved or not.

The position of the c.m. (in m) at the three instants is given by

$$\vec{\mathbf{R}}_{cm}\ (t=0) = \frac{m(\hat{\mathbf{i}} + \hat{\mathbf{j}}) + m(2\hat{\mathbf{i}} + 2\hat{\mathbf{j}}) + m(3\hat{\mathbf{i}} + 3\hat{\mathbf{j}})}{3m} = 2\hat{\mathbf{i}} + 2\hat{\mathbf{j}}$$

$$\vec{\mathbf{R}}_{cm}\ (t=1\text{s}) = \frac{m\,\hat{\mathbf{i}} + m\,\hat{\mathbf{j}} + m(3\hat{\mathbf{i}} + 3\hat{\mathbf{j}})}{3m} = \frac{4}{3}\hat{\mathbf{i}} + \frac{4}{3}\hat{\mathbf{j}}$$

$$\vec{\mathbf{R}}_{cm}\ (t=2\text{s}) = \frac{m\,\hat{\mathbf{j}} + m(\hat{\mathbf{i}} + 2\hat{\mathbf{j}}) + m(2\hat{\mathbf{i}})}{3m} = \hat{\mathbf{i}} + \hat{\mathbf{j}}$$

Let us now determine the average velocity of the c.m between 0 and 1 s and 1s and 2s. The average velocity of the c.m (in ms^{-1}) between 0 and 1s and 1s and 2s is, respectively:

$$\vec{\mathbf{V}}_{cm1} = \frac{\vec{\mathbf{R}}_{cm}\ (t=1\text{s}) - \vec{\mathbf{R}}_{cm}\ (t=0)}{1\text{s}} = \left(\frac{4}{3}\hat{\mathbf{i}} + \frac{4}{3}\hat{\mathbf{j}}\right) - (2\hat{\mathbf{i}} + 2\hat{\mathbf{j}})$$

$$= -\frac{2}{3}(\hat{\mathbf{i}} + \hat{\mathbf{j}})$$

$$\vec{\mathbf{V}}_{cm2} = \frac{\vec{\mathbf{R}}_{cm}\ (t=2\text{s}) - \vec{\mathbf{R}}_{cm}\ (t=1\text{s})}{1\text{s}} = (\hat{\mathbf{i}} + \hat{\mathbf{j}}) - \left(\frac{4}{3}\hat{\mathbf{i}} + \frac{4}{3}\hat{\mathbf{j}}\right)$$

$$= -\frac{1}{3}(\hat{\mathbf{i}} + \hat{\mathbf{j}})$$

Thus, we find that the average velocity of the c.m. is different over the two time intervals, that is, **it is not constant**. This means that a net external force is being exerted on the system and its linear momentum is not conserved.

Let us now determine the kinetic energy of a system of many particles and arrive at the law of conservation of energy for such systems.

15.3 CONSERVATION OF ENERGY

In this section, we first determine the kinetic energy of a two-particle system in terms of the c.m. and the relative coordinates. Then we extend the concept to many-particle systems. Finally, we arrive at the law of conservation of mechanical energy for two-particle and many-particle systems. This section is quite mathematical and you should solve all intermediate steps yourself to understand it better.

15.3.1 Kinetic Energy of a Two-particle System

The kinetic energy (K) of a two-particle system is given by:

$$K = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 \tag{15.5}$$

where $\vec{\mathbf{v}}_1 = \frac{d\vec{\mathbf{r}}_1}{dt}$ and $\vec{\mathbf{v}}_2 = \frac{d\vec{\mathbf{r}}_2}{dt}$. We have to **determine** K **in terms of the c.m. and relative coordinates**. We can rewrite Eq. (14.2) given by

$$\vec{\mathbf{R}}_{cm} = \frac{m_1\vec{\mathbf{r}}_1 + m_2\vec{\mathbf{r}}_2}{M} \quad \text{(where } M = m_1 + m_2\text{) as follows:}$$

$$(m_1 + m_2)\vec{\mathbf{R}}_{cm} = m_1\vec{\mathbf{r}}_1 + m_2\vec{\mathbf{r}}_2 \tag{15.6a}$$

or
$$m_1(\vec{\mathbf{r}}_1 - \vec{\mathbf{R}}_{cm}) + m_2(\vec{\mathbf{r}}_2 - \vec{\mathbf{R}}_{cm}) = \vec{\mathbf{0}} \tag{15.6b}$$

As you have seen while solving SAQ 3 of Unit 14, the position vectors of the particles with respect to the c.m. are given by

$$\vec{\mathbf{r}}'_1 = \vec{\mathbf{r}}_1 - \vec{\mathbf{R}}_{cm} \quad \text{and} \quad \vec{\mathbf{r}}'_2 = \vec{\mathbf{r}}_2 - \vec{\mathbf{R}}_{cm} \tag{15.6c}$$

Substituting Eq. (15.6c) in Eq. (15.6b), we can write

$$m_1\vec{\mathbf{r}}'_1 + m_2\vec{\mathbf{r}}'_2 = \vec{\mathbf{0}} \tag{15.6d}$$

Differentiating Eqs. (15.6d and c) with respect to time, we get:

$$m_1\vec{\mathbf{v}}'_1 + m_2\vec{\mathbf{v}}'_2 = \vec{\mathbf{0}} \tag{15.6e}$$

$$\vec{\mathbf{v}}'_1 = \vec{\mathbf{v}}_1 - \vec{\mathbf{V}}_{cm} \quad \text{and} \quad \vec{\mathbf{v}}'_2 = \vec{\mathbf{v}}_2 - \vec{\mathbf{V}}_{cm} \tag{15.6f}$$

Using the fact that $v_1^2 = \vec{\mathbf{v}}_1 \cdot \vec{\mathbf{v}}_1$ and $v_2^2 = \vec{\mathbf{v}}_2 \cdot \vec{\mathbf{v}}_2$, and substituting $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$ from Eq. (15.6f) in Eq. (15.5), we can rewrite it as

$$K = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} m_1 (\vec{\mathbf{v}}_1 \cdot \vec{\mathbf{v}}_1) + \frac{1}{2} m_2 (\vec{\mathbf{v}}_2 \cdot \vec{\mathbf{v}}_2)$$

$$= \frac{1}{2} m_1 [(\vec{\mathbf{v}}'_1 + \vec{\mathbf{V}}_{cm}) \cdot (\vec{\mathbf{v}}'_1 + \vec{\mathbf{V}}_{cm})] + \frac{1}{2} m_2 [(\vec{\mathbf{v}}'_2 + \vec{\mathbf{V}}_{cm}) \cdot (\vec{\mathbf{v}}'_2 + \vec{\mathbf{V}}_{cm})]$$

or $K = \frac{1}{2} m_1 v_1'^2 + \frac{1}{2} m_2 v_2'^2 + \frac{1}{2}(m_1 + m_2)V_{cm}^2 + (m_1 \vec{\mathbf{v}}_1' + m_2 \vec{\mathbf{v}}_2').\vec{\mathbf{V}}_{cm}$ (15.6g)

Now, from Eq. (15.6e), $m_1\vec{\mathbf{v}}_1' + m_2\vec{\mathbf{v}}_2' = \vec{\mathbf{0}}$

Also $m_1 + m_2 = M$. So we can write Eq. (15.6g) as

$$K = \frac{1}{2}MV_{cm}^2 + \frac{1}{2} m_1 v_1'^2 + \frac{1}{2}m_2 v_2'^2 \quad (15.7)$$

Kinetic energy of a two-particle system

This is the **total kinetic energy of a two-particle system expressed in terms of the velocity of its c.m. and the velocities of the particles with respect to the c.m.** What does Eq. (15.7) tell us? It tells us that the kinetic energy of the two-particle system has two parts:

1. The kinetic energy of the total mass M (the sum of the masses of the two particles in the system) moving with the speed of the centre of mass. Thus, a certain amount of the total kinetic energy is locked in the motion of the c.m. of the system. When the net external force on the system is zero, the speed of the c.m. remains constant and therefore this part of the total kinetic energy does not change.
2. The sum of the kinetic energy of the two particles relative to the centre of mass.

We can also express the kinetic energy of the two-particle system in terms of the K. E. of the total mass moving with the velocity of the c.m. and the K.E. of a particle of reduced mass μ moving with the relative velocity, $\vec{\mathbf{v}}$.

You have proved in SAQ 3 of Unit 14 that

$$\vec{\mathbf{r}}_1' = \vec{\mathbf{r}}_1 - \vec{\mathbf{R}}_{CM} = \vec{\mathbf{r}}_1 - \frac{m_1\vec{\mathbf{r}}_1 + m_2\vec{\mathbf{r}}_2}{m_1 + m_2} = \frac{m_2(\vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2)}{m_1 + m_2} = \frac{m_2}{M}\vec{\mathbf{r}} \quad (15.8a)$$

where $\vec{\mathbf{r}} = \vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2$ and

$$\vec{\mathbf{r}}_2' = -\frac{m_1}{M}\vec{\mathbf{r}} \quad (15.8b)$$

Hence, $\frac{d\vec{\mathbf{r}}_1'}{dt} = \frac{m_2}{M}\frac{d\vec{\mathbf{r}}}{dt}$ or $\vec{\mathbf{v}}_1' = \frac{m_2}{M}\vec{\mathbf{v}}$ (15.8c)

and $\frac{d\vec{\mathbf{r}}_2'}{dt} = -\frac{m_1}{M}\frac{d\vec{\mathbf{r}}}{dt}$ or $\vec{\mathbf{v}}_2' = -\frac{m_1}{M}\vec{\mathbf{v}}$ (15.8d)

Substituting $\vec{\mathbf{v}}_1'$ and $\vec{\mathbf{v}}_2'$ from Eqs. (15.8c and d) and using the relation $v^2 = \vec{\mathbf{v}} . \vec{\mathbf{v}}$ for both velocities in Eq. (15.7), we get

$$K = \frac{1}{2}MV_{cm}^2 + \frac{1}{2} m_1(\vec{\mathbf{v}}_1' . \vec{\mathbf{v}}_1') + \frac{1}{2}m_2(\vec{\mathbf{v}}_2' . \vec{\mathbf{v}}_2')$$

$$\text{or} \qquad K = \frac{1}{2} M V_{cm}^2 + \frac{1}{2} m_1 \left(\frac{m_2^2}{M^2} \right) v^2 + \frac{1}{2} m_2 \left(\frac{m_1^2}{M^2} \right) v^2$$

$$\text{or} \qquad K = \frac{1}{2} M V_{cm}^2 + \frac{1}{2} \frac{m_1 m_2}{M} \left(\frac{m_1 + m_2}{M} \right) v^2$$

$$\text{or} \qquad K = \frac{1}{2} M V_{cm}^2 + \frac{1}{2} \left(\frac{m_1 m_2}{M} \right) v^2 \qquad (\because M = m_1 + m_2)$$

Kinetic energy of a 2-particle system

Thus,
$$\boxed{K = \frac{1}{2} M V_{cm}^2 + \frac{1}{2} \mu v^2} \qquad (\because \ \mu = \frac{m_1 m_2}{M}) \qquad (15.9)$$

What does Eq. (15.9) tell us? It tells us that the kinetic energy of the two-particle system has two parts:

1. The kinetic energy of the total mass *M* (the sum of the masses of the two particles in the system) moving with the speed of the centre of mass.
2. The kinetic energy of a fictitious particle of mass μ moving with the relative velocity $\vec{\mathbf{v}}$.

We now state the expression for the kinetic energy for a many-particle system without detailed derivation. The result can be derived using the same method that we have followed for a two-particle system.

15.3.2 Kinetic Energy of a Many-particle System

The total kinetic energy (*K*) of an *N*-particle system is given as:

$$K = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 + \frac{1}{2} m_3 v_3^2 + ... + \frac{1}{2} m_N v_N^2 = \frac{1}{2} \sum_{i=1}^{N} m_i v_i^2 \qquad (15.10a)$$

where m_i is the mass of the i^{th} particle and $\vec{\mathbf{v}}_i$, its velocity. Generalising Eq. (15.6f) to all *N* particles, we can express the total kinetic energy of this system in terms of the c.m. and relative coordinates as follows:

$$K = \frac{1}{2} \sum_{i=1}^{N} m_i (\vec{\mathbf{v}}_i . \vec{\mathbf{v}}_i) = \frac{1}{2} \sum_{i=1}^{N} m_i [(\vec{\mathbf{v}}'_i + \vec{\mathbf{V}}_{cm}).(\vec{\mathbf{v}}'_i + \vec{\mathbf{V}}_{cm})]$$

$$\text{or} \qquad K = \frac{1}{2} \sum_{i=1}^{N} m_i v_i'^2 + \frac{1}{2} (\sum_{i=1}^{N} m_i) V_{cm}^2 + (\sum_{i=1}^{N} m_i \vec{\mathbf{v}}'_i).\vec{\mathbf{V}}_{cm} \qquad (15.10b)$$

since $\vec{\mathbf{V}}_{cm}$ does not depend on *i*. Further, generalizing Eq. (15.6e), we can write $\sum_{i=1}^{N} m_i \vec{\mathbf{v}}'_i = \vec{\mathbf{0}}$. Also $\sum_{i=1}^{N} m_i = M.$ So we can write Eq. (15.10b) as

Kinetic energy of an *N*-particle system

$$\boxed{K = \frac{1}{2} M V_{cm}^2 + \frac{1}{2} \sum_{i=1}^{N} m_i v_i'^2} \qquad (15.11)$$

This is the **total kinetic energy of an *N*-particle system expressed in terms of its c.m. and relative coordinates**.

The first term in Eq. (15.11) gives the kinetic energy of the total mass *M* (the sum of the masses of the particles in the system) moving with the speed of the centre of mass. Thus, a certain amount of the total kinetic energy is locked in the motion of the c.m. of the system. When the net external force on the system is zero, the speed of the c.m. remains constant and therefore this part of the total kinetic energy does not change.

The second term in Eq. (15.11) is the sum of the kinetic energies of the particles with respect to the c.m. of the system.

We now deduce the law of conservation of mechanical energy for a system of particles.

15.3.3 Conservation of Mechanical Energy

Recall from the discussion in Units 10 and 13 that the total mechanical energy of the system is conserved if the force being exerted on it is **conservative**. This means that the force depends only on the distance between the particles, i.e., the relative positions of the particles. Let us deduce the law of conservation of mechanical energy for a single particle again using a different procedure from the one followed in Unit 10.

You know that for a single particle: $m\dfrac{d\vec{\mathbf{v}}}{dt} = \vec{\mathbf{F}}$

Taking the scalar product of both sides with $\vec{\mathbf{v}}\,dt$ and integrating the result, we get (study the intermediate steps in the margin):

$$\frac{1}{2}mv^2 = \int \vec{\mathbf{F}}\,.\,d\vec{\mathbf{r}} \tag{15.12a}$$

Recall from Unit 10 that if $\int \vec{\mathbf{F}}.d\vec{\mathbf{r}}$ depends only on the limits of integration and not on the actual path linking the initial and final positions of the particle, we can associate a **scalar potential energy function** *U* of the coordinates such that

$$U = -\int \vec{\mathbf{F}}\,.\,d\vec{\mathbf{r}} \tag{15.12b}$$

We can then write Eq. (15.12a) as

$$\boxed{\frac{1}{2}mv^2 + U = \text{constant}} \tag{15.12c}$$

Eq. (15.12c) gives the **law of conservation of mechanical energy for a single particle and applies to conservative forces** as you have studied in Units 10 and 13. We can extend this law to a **system of two particles** acted upon by conservative forces. Refer to Sec. 14.3 of Unit 14. Let the masses of the particles be m_1 and m_2, respectively. Let the position vectors of the particles be $\vec{\mathbf{r}}_1$ and $\vec{\mathbf{r}}_2$, respectively, at the instant *t* with respect to an origin *O* in an inertial frame of reference.

The law of conservation of mechanical energy for the two-particle system is given as

$$\boxed{\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + U = \text{constant}} \tag{15.13}$$

Law of conservation of mechanical energy for a 2-particle system

Taking the scalar product of both sides with $\vec{\mathbf{v}}\,dt$ and integrating it with respect to *t*, we get:

$$\int m\frac{d\vec{\mathbf{v}}}{dt}.\vec{\mathbf{v}}\,dt = \int \vec{\mathbf{F}}\,.\,\vec{\mathbf{v}}\,dt \tag{i}$$

We can simplify (i) to obtain Eq. (15.12a) as follows: We first express the LHS of (i) in terms of the kinetic energy as follows. We know that

$$\frac{d}{dt}(v^2) = \frac{d}{dt}(\vec{\mathbf{v}}.\vec{\mathbf{v}}) = 2\frac{d\vec{\mathbf{v}}}{dt}.\vec{\mathbf{v}}$$

Using this result we can write

$$m\frac{d\vec{\mathbf{v}}}{dt}.\vec{\mathbf{v}}\,dt = \frac{1}{2}m\frac{d}{dt}(v^2)$$

Integrating this equation with respect to time, we get:

$$\int m\frac{d\vec{\mathbf{v}}}{dt}.\vec{\mathbf{v}}\,dt = \int\frac{1}{2}m\frac{d}{dt}(v^2)dt = \frac{1}{2}mv^2 \tag{ii}$$

Further,

$$\vec{\mathbf{v}}\,dt = d\vec{\mathbf{r}}$$

and

$$\vec{\mathbf{F}}.\vec{\mathbf{v}}\,dt = \vec{\mathbf{F}}.d\vec{\mathbf{r}}$$

so that

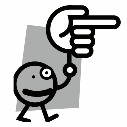
$$\int \vec{\mathbf{F}}.\vec{\mathbf{v}}\,dt = \int \vec{\mathbf{F}}.d\vec{\mathbf{r}} \tag{iii}$$

Substituting (ii) and (iii) in (i), we get Eq. (15.12a).

where U is the scalar potential energy function $U(\vec{\mathbf{r}}_1, \vec{\mathbf{r}}_2)$ defined by Eq. (15.12b). The derivation of Eq. (15.13) is beyond the scope of this syllabus.

CONSERVATION OF MECHANICAL ENERGY FOR A TWO-PARTICLE SYSTEM

If the net force exerted on each particle in a two-particle system depends only on its distance from the other particle (that is, it is a central conservative force), the total mechanical energy of the two-particle system remains constant [Eq. (15.13)].



Don't forget

The law of conservation of mechanical energy holds only for conservative forces.

In nature, forces like the force of gravitation and electrostatic force are conservative but forces like friction are not conservative. You have studied in Unit 10 that when non-conservative forces are present, the law of conservation of mechanical energy does not hold. However, by bringing in different forms of energy such as heat, electromagnetic energy etc. within this concept we can arrive at the **law of conservation of energy**.

We can extend the law of conservation of mechanical energy to many-particle systems on which conservative forces are exerted. Here we state the result without going into its derivation as it is beyond the scope of this course:

$$\sum_{i=1}^{N}\left(\frac{1}{2}m_i v_i^2\right) + U = \text{constant} \tag{15.14}$$

In this case U is the potential energy function for the many-particle system.

We now study the collisions of two particles as an application of the concepts described so far.

15.4 COLLISIONS OF TWO PARTICLES

We begin by explaining briefly: **What do we mean by collision between objects**? Study the description given in the box below.

COLLISION OF PARTICLES

A <u>collision</u> is said to take place between two or more objects when the objects come close enough so that there is some sort of interaction between them for a brief time interval.

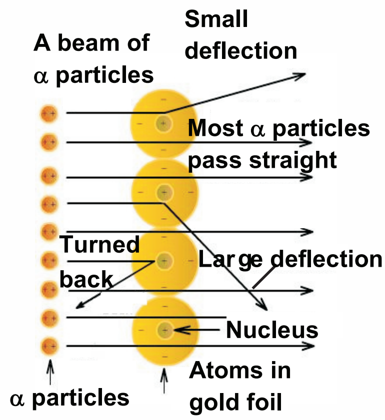
NOTE that in a collision process, there may or may not be any physical contact between the objects. Also external forces may or may not be exerted on them.

For example, the alpha particles bombarded on atomic nuclei (in the Rutherford scattering experiment) **do not actually hit the nuclei**. But there is an (electrostatic) interaction between them and we call this process as the **collision** of alpha particles with the nuclei. However, a billiards ball hit towards another ball **actually strikes** it. A car involved in a collision **actually hits** the other vehicle!

You may wonder: **Why should you learn about collisions**? You may be surprised to know that collision (also called scattering) of particles is a very important feature of the physical universe. Much of our knowledge of *atoms, molecules, nuclei and elementary particles has come from collision* (*also called scattering*) *experiments*. These experiments help us know the nature of the interaction (or type of force) between microscopic particles. In a collision between particles, the paths and energies of the interacting particles change. By measuring the energies and the angular distributions of the particles after collision, we get information about their structure and the nature of the forces involved.

Perhaps, the most dramatic of the scattering experiments was the experiment performed in 1911 by Ernest Rutherford (Fig. 15.3). In this experiment, a thin gold foil was bombarded by alpha particles (doubly ionized helium atoms). It was found that the number of scattered alpha particles varied with the angle of scattering and by studying their distribution, Rutherford proposed the **hypothesis of the atomic nucleus**. Today, high energy particle accelerators several kilometres in length help us study the forces of interaction between elementary particles by studying how they scatter from the targets.

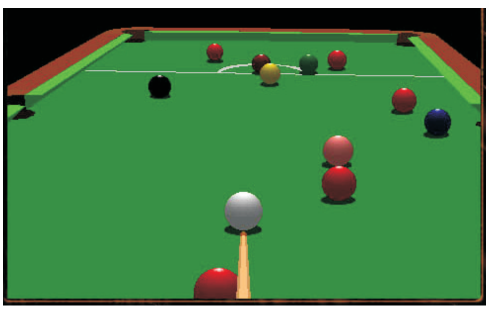
Collisions have applications in our everyday life as well. For example, we can find the velocities and angle of scattering of billiards balls after collision. This calculation actually helps the players decide how to hit a particular ball (see Fig. 15.4a). These calculations also help reconstruct an accident scene involving collisions of vehicles by helping determine their velocities before collision (Fig. 15.4b).



(a)

(b)

Fig. 15.3: **a) Rutherford scattering experiment; b) Ernest Rutherford (1871 –1937) was a British physicist and chemist born in New Zealand. He gave the nuclear model of the atom and is known as the father of nuclear physics. In 1908, he won the Nobel Prize in Chemistry for his contributions to the chemistry of radioactive substances.**

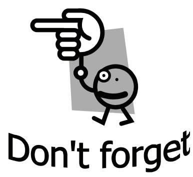


(a)

(b)

Fig. 15.4: **a) Collision of billiards balls; b) determination of velocities in vehicle collisions.**

<u>Do not</u> think of a collision process as one in which an object *<u>always strikes</u>* the other object. It can interact with the other object through some force of interaction.

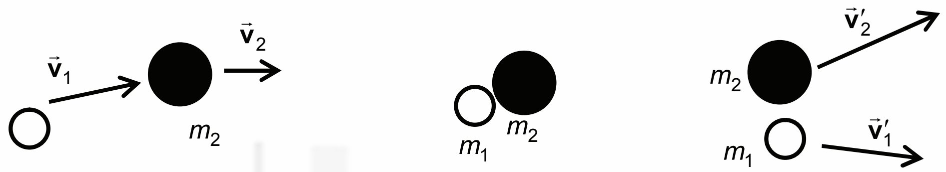


We can predict many details of a collision simply by applying conservation laws, even though we may not know the nature of the interaction or force. In this section you will study how to apply the conservation laws of momentum and energy to collisions of particles. To begin with, let us briefly describe the collision process. Study Fig. 15.5, which shows three distinct stages of the collision process in which the forces of interaction are important only at very small separations and for a very brief time interval:

Stage 1: At a time long before collision, each particle is effectively free, that is, the total energy is just the *KE* of the particle.

Stage 2: As the particles approach each other, the momentum and energy of each particle changes due to the forces of interaction between them.

Stage 3: At a time long after the collision, the particles are again free and move along straight lines with new velocities in new directions.



Before Collision **During Collision** **After Collision**

(a) **(b)** **(c)**

Fig. 15.5: **The collision process**.

The scattered particles may or may not be the same as the original particles. In scattering experiments we usually know the initial velocities. Often, one particle is initially at rest and is bombarded by particles of known energy. We measure the final velocities of the colliding particles with suitable particle detectors or using other methods.

Collisions are divided into **two** broad categories:

1. **Elastic collisions**, in which **both linear momentum and the total kinetic energy are conserved**, and
2. **Inelastic collisions**, in which **conservation of linear momentum holds good but the total kinetic energy is not conserved**. However, the **total energy is conserved**.

Thus, if $\vec{\mathbf{p}}_i$ and K_i are the initial linear momentum and kinetic energy of the system, respectively, before collision, and $\vec{\mathbf{p}}_f$ and K_f are the respective final linear momentum and kinetic energy after collision, then

Elastic Collisions

$$\vec{\mathbf{p}}_i = \vec{\mathbf{p}}_f \quad \text{and} \quad K_i = K_f \quad \textbf{for elastic collisions} \qquad (15.15a)$$

Inelastic Collisions

$$\vec{\mathbf{p}}_i = \vec{\mathbf{p}}_f \quad \text{and} \quad K_i \neq K_f \quad \textbf{for inelastic collisions} \qquad (15.15b)$$

Here we discuss only **elastic collisions in one and two-dimensions**.

15.4.1 Elastic Collisions in One-dimension

Let us consider an elastic collision between two particles moving in a straight line along the *x*-axis (see Fig. 15.6). This is a head-on collision. Note from Fig. 15.6 that a particle of mass m_1 (called the **incident particle** or **projectile**) moving with velocity $\vec{\mathbf{v}}_1$ collides with another particle of mass m_2 (called the **target particle**) moving with velocity $\vec{\mathbf{v}}_2$.

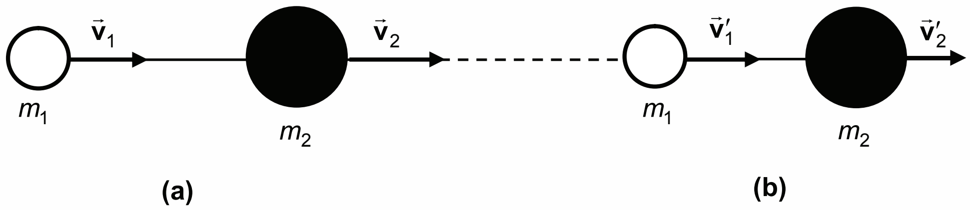


Fig. 15.6: Elastic collision between two particles in one-dimension – a) before collision; b) after collision, the velocities of the particles may change.

Then using Eq. (15.15a), we can determine the final velocities of the particles whatever the force of interaction between them may be. Let the velocity of mass m_1 after collision be $\vec{\mathbf{v}}'_1$ and that of mass m_2 be $\vec{\mathbf{v}}'_2$. Applying the laws of conservation of momentum and kinetic energy to the elastic collision of these two particles, we get:

$$m_1 \vec{\mathbf{v}}_1 + m_2 \vec{\mathbf{v}}_2 = m_1 \vec{\mathbf{v}}'_1 + m_2 \vec{\mathbf{v}}'_2 \qquad \textbf{Linear momentum} \qquad (15.16a)$$

$$\frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} m_1 v_1'^2 + \frac{1}{2} m_2 v_2'^2 \qquad \textbf{Kinetic energy} \qquad (15.16b)$$

Since the motion is along the *x*-axis, we can write Eq. (15.16a) as follows:

$$m_1 v_1 + m_2 v_2 = m_1 v'_1 + m_2 v'_2 \qquad (15.16c)$$

Eqs. (15.16a and b) are two equations for two unknowns and we can **solve for** v'_1 and v'_2 in terms of m_1, m_2, v_1 and v_2. We can eliminate v'_1 from these equations by using Eq. (15.16c) and solve for v'_2. The solution becomes easier if we assume that **the particle of mass m_2 is initially at rest**, that is, $v_2 = 0$. Then from Eq. (15.16c), we have

$$v'_1 = v_1 - \frac{m_2}{m_1} v'_2 \qquad (\because v_2 = 0) \qquad (15.17a)$$

Squaring Eq. (15.17a), we get

$$v_1'^2 = v_1^2 - 2\frac{m_2}{m_1} v_1 v'_2 + \frac{m_2^2}{m_1^2} v_2'^2 \qquad (15.17b)$$

We now substitute $v_1'^2$ from Eq. (15.17b) in Eq. (15.16b) and solve for v'_2:

$$m_1 v_1^2 = m_1 \left(v_1^2 - 2\frac{m_2}{m_1} v_1 v'_2 + \frac{m_2^2}{m_1^2} v_2'^2 \right) + m_2 v_2'^2 \qquad (15.17c)$$

or $$0 = \left(-2m_2 v_1 v_2' + \frac{m_2^2}{m_1} v_2'^2\right) + m_2 v_2'^2 \qquad (15.17\text{d})$$

or $$v_2'^2 \left(\frac{m_2}{m_1} + 1\right) - 2 v_1 v_2' = 0 \qquad (15.17\text{e})$$

or $$v_2' \left[v_2' \left(\frac{m_2}{m_1} + 1\right) - 2v_1\right] = 0 \qquad (15.17\text{f})$$

Eq. (15.17f) is a quadratic equation in v_2', which has two solutions:

1. $$v_2' = 0 \quad \Rightarrow \quad v_1' = v_1 \qquad (15.18)$$

   This solution simply restates the initial conditions. We always obtain such a solution in this type of problem because the initial velocities satisfy the conservation laws.

2. $$v_2' \left(\frac{m_2}{m_1} + 1\right) - 2v_1 = 0 \qquad (15.19\text{a})$$

   or $$\boxed{v_2' = \frac{2m_1}{(m_1 + m_2)} v_1} \qquad (15.19\text{b})$$

   and from Eq. (15.17a) $$\boxed{v_1' = \frac{(m_1 - m_2)}{(m_1 + m_2)} v_1} \qquad (15.19\text{c})$$

We now consider a few interesting **special cases** of Eqs. (15.19b and c) for head-on collisions.

1. Suppose the **masses of the colliding particles are equal**, that is, $m_1 = m_2$. Then Eqs. (15.19b and c) give

   $$v_1' = 0 \quad \text{and} \quad v_2' = v_1 \qquad (15.20\text{a})$$

   Thus, **when a particle collides with another particle of equal mass at rest, the incident particle comes to a stop, while the target particle starts moving with the velocity of the incident particle** (Fig. 15.7).

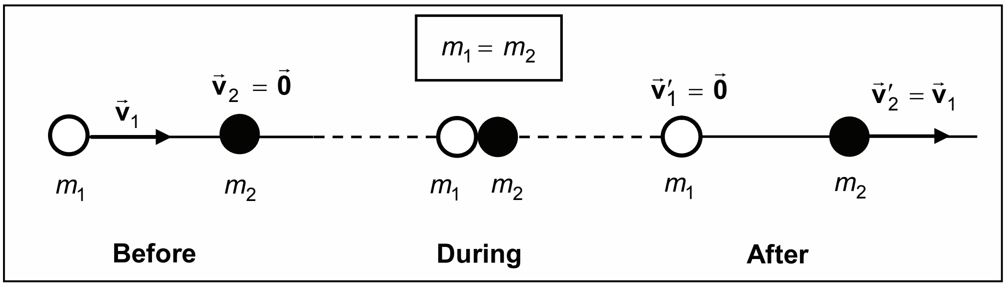


Fig. 15.7: **One-dimensional elastic collision between particles of equal mass**.

2. Suppose **a particle collides with a much more massive particle**, that is, $m_1 << m_2$. Then from Eqs. 15.19c and using the result for v_1' in 15.16b, we get

$$v_1' = -v_1 \quad \text{and} \quad v_2' \cong 0 \tag{15.20b}$$

Thus, **when a particle collides with another much more massive particle at rest, the incident particle is reflected back with the same speed, while the target particle hardly moves** (Fig.15.8).

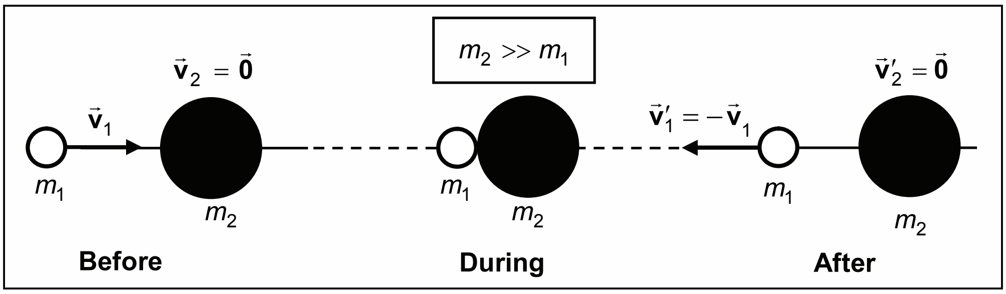


Fig. 15.8: **One-dimensional elastic collision between a less massive particle with a much more massive particle.**

3. Suppose **a particle collides with a particle of much less mass**, that is, $m_1 >> m_2$. Then from Eqs. (15.19b and c), we get

$$v_1' = v_1 \quad \text{and} \quad v_2' = 2v_1 \tag{15.20c}$$

Thus, **when a particle collides with another much less massive particle at rest, the incident particle keeps moving as if nothing happened, while the target particle takes off with twice the velocity of the incident particle** (Fig. 15.9).

You must try to picture the collision processes shown in Figs. 15.7, 15.8, 15.9, 15.10 and 15.11 in your mind or do similar activities with marbles, balls, etc. to understand them better.

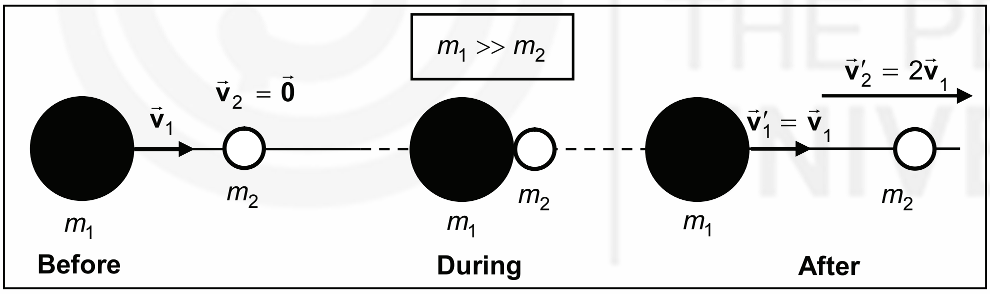


Fig. 15.9: **One-dimensional elastic collision between a more massive particle with a much less massive particle.**

What happens when the target particle is also moving? Let us find out.

EXAMPLE 15.4: HEAD-ON ELASTIC COLLISION

Two balls of masses m and $3m$ collide head-on in an elastic collision with equal and opposite velocities $\vec{v}$ (Fig. 15.10). Determine their final velocities.

SOLUTION ■ The **KEY IDEA** here is to use the fact that the linear momentum and kinetic energy of the particles are conserved. Since the collision is head-on, we can treat it as a one-dimensional problem and apply Eqs. (15.16b and c).

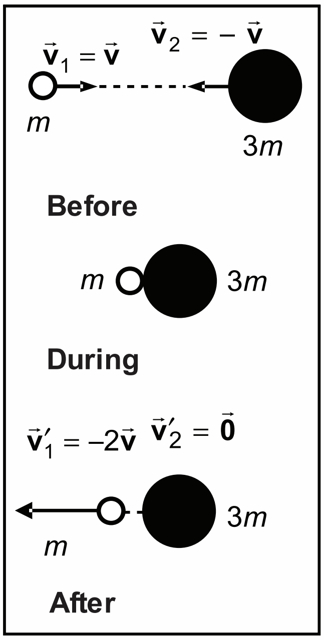


Fig. 15.10: **Head-on collision between particles moving in opposite directions**

Putting the values given in this problem in Eqs. (15.16b and c), we get

$$mv - 3mv = mv'_1 + 3mv'_2 \qquad \text{(i)}$$

$$\frac{1}{2}mv^2 + \frac{1}{2}3mv^2 = \frac{1}{2}mv'^2_1 + \frac{1}{2}3mv'^2_2 \qquad \text{(ii)}$$

We can eliminate v'_1 from (i) and (ii). From (i): $v'_1 = -2v - 3v'_2$ (iii)

Substituting (iii) in (ii) and solving for v'_2, we get

$$4v^2 = (-2v - 3v'_2)^2 + 3v'^2_2 \qquad \Rightarrow \qquad v'_2\,(v'_2 + v) = 0$$

This equation has two solutions, one of which represents the initial conditions: $v'_1 = v$ and $v'_2 = -v$

The interesting solution is: $v'_1 = -2v$ and $v'_2 = 0$

This solution shows that after the collision, the ball of mass m is reflected back and moves with twice its original speed and the ball of mass $3m$ comes to rest (Fig. 15.10).

SAQ 2 – Head-on elastic collision

a) A ball of mass 5.0 kg moves to the right with a velocity of $3.0\,\text{ms}^{-1}$ and collides head-on with a stationary ball of mass 8.0 kg. Calculate the velocity of each ball after collision given that the collision is elastic.

b) A 40 kg cart travelling at $6.0\,\text{ms}^{-1}$ collides head-on with a 50 kg cart. The final velocity of the first cart is $1.5\,\text{ms}^{-1}$ in the direction opposite to its initial direction of motion. What are the initial and final velocities of the second cart if the collision is elastic?

So far you have studied head-on elastic collisions (that is, collisions in one dimension). Very often we come across situations in which collisions are in two or three-dimensions. Let us study elastic collisions in two-dimensions. Examples are collisions of billiards balls, alpha particles with atoms, gas particles with each other.

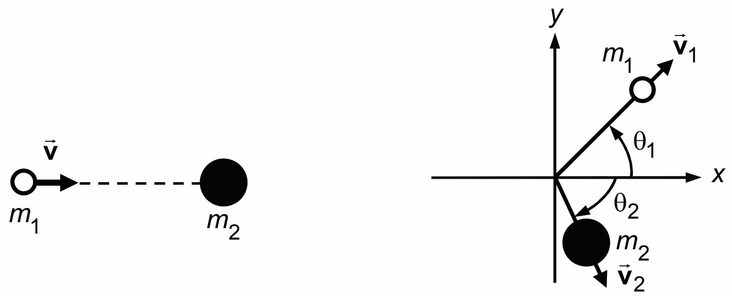
15.4.2 Elastic Collisions in Two-dimensions

Let us consider the collision of two particles of masses m_1 and m_2 and determine their velocities after collision given the initial conditions. For keeping the mathematics simple, we assume that the **target particle is initially at rest**. Suppose the projectile (particle 1) initially moves along the x-axis with velocity $\vec{\mathbf{v}}$ (Fig. 15.11a). Let the projectile and the target move in the xy plane after collision with velocities $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$, respectively (Fig. 15.11b). Since the linear momentum of the system is conserved, we have

$$m_1\vec{\mathbf{v}} = m_1\vec{\mathbf{v}}_1 + m_2\vec{\mathbf{v}}_2 \qquad (15.21)$$

We can express Eq. (15.21) in its component form along the x-axis:

$$m_1 v\hat{\mathbf{i}} = (m_1 v_1 \cos\theta_1 \hat{\mathbf{i}} + m_1 v_1 \sin\theta_1 \hat{\mathbf{j}}) + (m_2 v_2 \cos\theta_2 \hat{\mathbf{i}} - m_2 v_2 \sin\theta_2 \hat{\mathbf{j}}) \quad (15.22a)$$



Before collision **After collision**

(a) **(b)**

Fig. 15.11: **Elastic collision between two particles in two-dimensions**.

In Fig. 15.11b, θ_1 and θ_2 are the angles that $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$ make with the *x*-axis, respectively. Now equating the coefficients of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ in Eq. (15.22a), we get

$$m_1 v = m_1 v_1 \cos\theta_1 + m_2 v_2 \cos\theta_2 \quad (15.22b)$$

and

$$0 = m_1 v_1 \sin\theta_1 - m_2 v_2 \sin\theta_2 \quad (15.22c)$$

Since the kinetic energy of the system remains constant, we have

$$m_1 v^2 = m_1 v_1^2 + m_2 v_2^2 \quad (15.22d)$$

To eliminate θ_1 from Eqs. (15.22b and c), we can rewrite them as

$$m_1 v_1 \cos\theta_1 = m_1 v - m_2 v_2 \cos\theta_2 \quad (15.22e)$$

and

$$m_1 v_1 \sin\theta_1 = m_2 v_2 \sin\theta_2 \quad (15.22f)$$

Squaring Eqs. (15.22e and f) and adding the resulting equations, we get:

$$(m_1 v_1)^2 (\cos^2\theta_1 + \sin^2\theta_1) = (m_1 v - m_2 v_2 \cos\theta_2)^2 + (m_2 v_2 \sin\theta_2)^2$$

or

$$m_1^2 v_1^2 = m_2^2 v_2^2 + m_1^2 v^2 - 2 m_1 m_2 v v_2 \cos\theta_2 \quad (15.22g)$$

REMEMBER from elementary trigonometry that

$$\cos^2\theta + \sin^2\theta = 1$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$

and

$$\cos 2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$$

We now use Eq. (15.22d) to get an expression of v_2 in terms of v and θ_2. Multiplying Eq. (15.22d) by m_1, we get

$$m_1^2 v^2 = m_1^2 v_1^2 + m_1 m_2 v_2^2 \quad (15.22h)$$

Replacing Eq. (15.22h) in Eq.(15.22g) we get

$$0 = m_2^2 v_2^2 + m_1 m_2 v_2^2 - 2 m_1 m_2 v v_2 \cos\theta_2 \quad (15.22i)$$

Eq. (15.22i) is a quadratic equation in v_2 of which $v_2 = 0$ is a trivial solution. We disregard that and write only the acceptable solution, which is

$$v_2 = \frac{2 m_1 v \cos\theta_2}{m_1 + m_2} = \frac{2\alpha v \cos\theta_2}{1+\alpha} \quad \text{where } \alpha = \frac{m_1}{m_2} \quad (15.23)$$

We can determine the value of v_1 from Eq. (15.22f).

To determine θ_1, we divide Eq. (15.22f) by Eq. (15.22e) and write

$$\frac{m_1 v_1 \sin\theta_1}{m_1 v_1 \cos\theta_1} = \frac{m_2 v_2 \sin\theta_2}{m_1 v - m_2 v_2 \cos\theta_2} \tag{15.24a}$$

In order to determine the angles, we multiply the numerator and denominator of the right-hand side of Eq. (15.24a) by $2\cos\theta_2$.

Then we get $$\tan\theta_1 = \frac{m_2 v_2 \sin 2\theta_2}{2m_1 v \cos\theta_2 - 2m_2 v_2 \cos^2\theta_2} \tag{15.24b}$$

From Eq. (15.23), we replace $2m_1 v \cos\theta_2 = (m_1 + m_2)\, v_2$ in the denominator of Eq. (15.24b) and get

$$\tan\theta_1 = \frac{m_2 v_2 \sin 2\theta_2}{(m_1 + m_2) v_2 - 2m_2 v_2 \cos^2\theta_2} \tag{15.24c}$$

or $$\boxed{\tan\theta_1 = \frac{\sin 2\theta_2}{\alpha - \cos 2\theta_2}} \tag{15.25}$$

Eq. (15.25) tells us that the relation between θ_1 and θ_2 is strongly dependent on α, that is, the ratio of the mass of the projectile to that of the target particle.

Let us now apply this result to the problem of collision of billiards balls of equal mass.

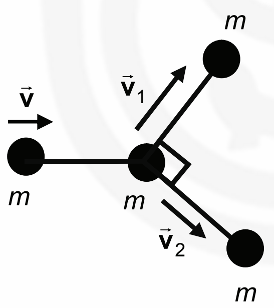


Fig. 15.12: **Elastic collision of two particles of equal mass in two-dimensions. Note that the particles move at right angles to each other after collision.**

EXAMPLE 15.5: ELASTIC COLLISION OF BILLIARDS BALLS

A billiards ball of mass *m* hits another billiards ball of equal mass **at rest** in an elastic collision and moves along a straight line at an angle of θ_1 from its original direction of motion. At what angle with each other do the target ball and the projectile move after collision?

SOLUTION ■ The **KEY IDEA** here is to treat the balls as particles and the system as a two-particle system. This is a collision in two-dimensions. We use the fact that linear momentum and kinetic energy are conserved for elastic collisions.

In this case, $m_1 = m_2$ and $\alpha = 1$. Therefore, from Eq. (15.25), we get

$$\tan\theta_1 = \frac{\sin 2\theta_2}{1 - \cos 2\theta_2} = \frac{2\sin\theta_2\cos\theta_2}{2\sin^2\theta_2} = \cot\theta_2 = \tan(90^\circ - \theta_2)$$

$$\therefore \quad \theta_1 = 90^\circ - \theta_2 \quad \text{or} \quad \theta_1 + \theta_2 = 90^\circ$$

Thus, **the billiards balls move perpendicular to each other after the elastic collision between them** (see Fig. 15.12).

What would happen if the projectile were much heavier than the target, for example, a proton colliding with an electron?

In that case $\alpha >> 1$, i.e., $m_1 >> m_2$. Since $\sin 2\theta_2$ and $\cos 2\theta_2$ lie between – 1 and + 1, in this case we have $\tan\theta_1 \to 0$ or $\theta_1 \to 0$. And as $\theta_1 \to 0$, we get from Eq. (15.25) that $\theta_2 \to 0$ also. **So when the projectile is much heavier than the target then both particles move along the same direction as that of the initial direction of the projectile** (Fig. 15.13).

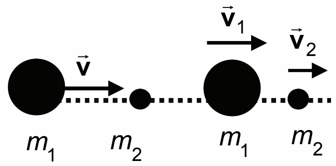


Fig. 15.13: **Elastic collision of two particles in two-dimensions when $m_1 >> m_2$. Both particles move along the same direction as that of the initial direction of the projectile.**

You may now like to stop and absorb this discussion. Try the following SAQ.

SAQ 3 – **Elastic collisions in two-dimensions**

Particle 1 of mass m is initially moving in the positive x-direction with speed v_1 (Fig. 15.14). It collides with particle 2 of mass $m/3$, which is initially moving in the opposite direction with an unknown speed v_2. Assume that the net external force acting on the particles is zero and the collision is elastic. After the collision, particle 1 moves with a speed $v_1' = v_1/2$ at an angle (– 90°) with respect to the positive x-direction. Particle 2 moves with an unknown speed v_2' at an angle 45° with respect to the positive x-direction as shown in the figure. Determine v_2 and v_2' in terms of v_1.

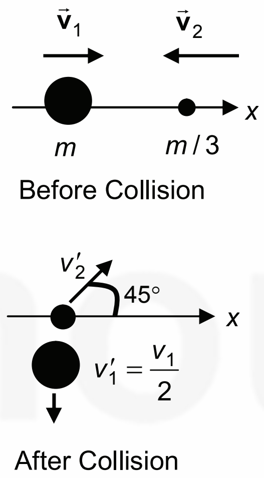


Fig. 15.14: **Diagram for SAQ 3.**

With this we end our discussion on collision of particles. In the last section of this unit, we shall discuss the conservation of angular momentum for two-particle and many-particle systems.

15.5 ANGULAR MOMENTUM

So far you have learnt about the linear momentum and mechanical energy of two-particle and many-particle systems and studied the respective conservation laws. We now determine the angular momentum for such systems and deduce the law of conservation of angular momentum.

15.5.1 Angular Momentum of a Two-particle System

The total angular momentum of the two-particle system is the vector sum of the angular momenta of the particles:

$$\vec{\mathbf{L}} = \vec{\mathbf{L}}_1 + \vec{\mathbf{L}}_2 \tag{15.26}$$

Recall the definition of angular momentum you have learnt in Unit 10. The angular momentum of each particle is given by

$$\vec{\mathbf{L}}_1 = \vec{\mathbf{r}}_1 \times \vec{\mathbf{p}}_1 = m_1\,\vec{\mathbf{r}}_1 \times \vec{\mathbf{v}}_1 \quad \text{and} \quad \vec{\mathbf{L}}_2 = \vec{\mathbf{r}}_2 \times \vec{\mathbf{p}}_2 = m_2\,\vec{\mathbf{r}}_2 \times \vec{\mathbf{v}}_2 \tag{15.27a}$$

Thus,
$$\vec{\mathbf{L}} = m_1\vec{\mathbf{r}}_1 \times \vec{\mathbf{v}}_1 + m_2\,\vec{\mathbf{r}}_2 \times \vec{\mathbf{v}}_2 \tag{15.27b}$$

Once again we use Eqs. (15.8a and b) to substitute the values of $\vec{\mathbf{r}}_1$ and $\vec{\mathbf{r}}_2$ and obtain the expression of the angular momentum of the system in terms of the c.m. and relative coordinates:

$$\vec{\mathbf{L}} = m_1\left(\vec{\mathbf{R}}_{cm} + \vec{\mathbf{r}}_1'\right) \times \vec{\mathbf{v}}_1 + m_2\left(\vec{\mathbf{R}}_{cm} + \vec{\mathbf{r}}_2'\right) \times \vec{\mathbf{v}}_2$$

On simplifying we get

$$\vec{\mathbf{L}} = m_1\left(\vec{\mathbf{R}}_{cm} \times \vec{\mathbf{v}}_1 + \vec{\mathbf{r}}_1' \times \vec{\mathbf{v}}_1\right) + m_2\left(\vec{\mathbf{R}}_{cm} \times \vec{\mathbf{v}}_2 + \vec{\mathbf{r}}_2' \times \vec{\mathbf{v}}_2\right)$$

$$= \vec{\mathbf{R}}_{cm} \times m_1\vec{\mathbf{v}}_1 + m_1(\vec{\mathbf{r}}_1' \times \vec{\mathbf{v}}_1) + \vec{\mathbf{R}}_{cm} \times m_2\vec{\mathbf{v}}_2 + m_2(\vec{\mathbf{r}}_2' \times \vec{\mathbf{v}}_2)$$

or $$\vec{\mathbf{L}} = \vec{\mathbf{R}}_{cm} \times (m_1\vec{\mathbf{v}}_1 + m_2\vec{\mathbf{v}}_2) + m_1(\vec{\mathbf{r}}_1' \times \vec{\mathbf{v}}_1) + m_2(\vec{\mathbf{r}}_2' \times \vec{\mathbf{v}}_2) \qquad (15.28a)$$

From Eq. (15.1e), $\vec{\mathbf{p}} = m_1\vec{\mathbf{v}}_1 + m_2\vec{\mathbf{v}}_2 = M\vec{\mathbf{V}}_{cm}$ and Eq. (15.28a) becomes

$$\vec{\mathbf{L}} = \vec{\mathbf{R}}_{cm} \times M\vec{\mathbf{V}}_{cm} + m_1(\vec{\mathbf{r}}_1' \times \vec{\mathbf{v}}_1) + m_2(\vec{\mathbf{r}}_2' \times \vec{\mathbf{v}}_2) \qquad (15.28b)$$

Note that $\vec{\mathbf{r}}_1'$ and $\vec{\mathbf{r}}_2'$ are the positions of the particles with respect to the c.m. Thus, the last two terms in Eq. (15.28b) are the respective angular momenta of the particles about the c.m. of the system. We denote their sum by $\vec{\mathbf{L}}_{cm}$:

Angular momentum of a 2-particle system about the c.m.

$$\boxed{\vec{\mathbf{L}}_{cm} = m_1(\vec{\mathbf{r}}_1' \times \vec{\mathbf{v}}_1) + m_2(\vec{\mathbf{r}}_2' \times \vec{\mathbf{v}}_2)} \qquad (15.28c)$$

We can also express $\vec{\mathbf{L}}_{cm}$ in terms of the reduced mass of the system and the relative velocity as follows:

Substituting $\vec{\mathbf{r}}_1' = \frac{m_2}{M}\vec{\mathbf{r}}$ and $\vec{\mathbf{r}}_2' = -\frac{m_1}{M}\vec{\mathbf{r}}$ from Eqs. (15.8a and b) in Eq. (15.28c), we can write

$$\vec{\mathbf{L}}_{cm} = m_1\left(\frac{m_2}{M}\vec{\mathbf{r}} \times \vec{\mathbf{v}}_1\right) + m_2\left(-\frac{m_1}{M}\vec{\mathbf{r}} \times \vec{\mathbf{v}}_2\right) = \frac{m_1m_2}{M}\left[\vec{\mathbf{r}} \times (\vec{\mathbf{v}}_1 - \vec{\mathbf{v}}_2)\right]$$

Angular momentum of a 2-particle system about the c.m.

or $$\boxed{\vec{\mathbf{L}}_{cm} = \mu(\vec{\mathbf{r}} \times \vec{\mathbf{v}}) \quad \text{where} \quad \vec{\mathbf{v}} = (\vec{\mathbf{v}}_1 - \vec{\mathbf{v}}_2)} \qquad (15.28d)$$

Using Eq. (15.28d), we can write Eq. (15.28b) as follows:

Angular momentum of a 2-particle system

$$\boxed{\vec{\mathbf{L}} = \vec{\mathbf{R}}_{cm} \times M\vec{\mathbf{V}}_{cm} + \vec{\mathbf{L}}_{cm} = \vec{\mathbf{R}}_{cm} \times \vec{\mathbf{P}}_{cm} + \vec{\mathbf{L}}_{cm}} \qquad (15.29)$$

Thus**, the angular momentum of the two-particle system is the sum of the angular momentum of the centre of mass and the total angular momenta of the particles about the centre of mass of the system**. **It is also the sum of the angular momentum of the c.m. and the angular momentum of a fictitious particle of mass μ moving with relative velocity $\vec{\mathbf{v}}$**.

We can extend this result to a system of *N* particles. We simply state the result here without deriving it.

The total angular momentum of the *N*-particle system about the origin is the vector sum of the angular momenta of the particles about the origin:

$$\vec{\mathbf{L}} = m_1\vec{\mathbf{r}}_1 \times \vec{\mathbf{v}}_1 + m_2\vec{\mathbf{r}}_2 \times \vec{\mathbf{v}}_2 + \ldots + m_N\vec{\mathbf{r}}_N \times \vec{\mathbf{v}}_N \qquad (15.30a)$$

We can write $\vec{\mathbf{L}}$ as

$$\vec{\mathbf{L}} = \vec{\mathbf{L}}_{cm} + \vec{\mathbf{R}}_{cm} \times \vec{\mathbf{P}}_{cm} \tag{15.30b}$$

where the **first term in Eq. (15.30b) is the sum of the angular momenta of all the particles of the system about the centre of mass of the system**. It is given by

$$\vec{\mathbf{L}}_{cm} = \sum_{i=1}^{N} m_i \vec{\mathbf{r}}_i' \times \vec{\mathbf{v}}_i \tag{15.30c}$$

In the last section of this unit, you will learn about the conservation of angular momentum. But first, we take up an example to determine the angular momentum of a two-particle system.

EXAMPLE 15.6: DETERMINING ANGULAR MOMENTUM

Determine the angular momentum of the two-particle system at the instant shown in Fig. 15.15 with respect to the origin and the c.m. of the system.

SOLUTION ■ The **KEY IDEA** here is to use Eqs. (15.27b and 15.28c) for the two-particle system.

The total angular momentum about the origin *O* is given by Eq. (15.27b). From Fig. 15.15, the coordinates (in m) of the 3.0 kg particle (say, particle 1) are (0, 2.5) and those of the 2.0 kg particle (particle 2) are (3.5, 0). The velocities of these particles are $-2.0\,\text{ms}^{-1}\,\hat{\mathbf{i}}$ and $2.0\,\text{ms}^{-1}\,\hat{\mathbf{j}}$, respectively.

From Eq. (15.27b), the angular momentum of the system about the origin *O* is

$$\vec{\mathbf{L}} = 3.0\,\text{kg}(2.5\,\hat{\mathbf{j}}) \times (-2.0\,\text{ms}^{-1}\,\hat{\mathbf{i}}) + 2.0\,\text{kg}(3.5\,\hat{\mathbf{i}}) \times (2.0\,\text{ms}^{-1}\,\hat{\mathbf{j}}) = 29\,\text{kgms}^{-1}\,\hat{\mathbf{k}}$$

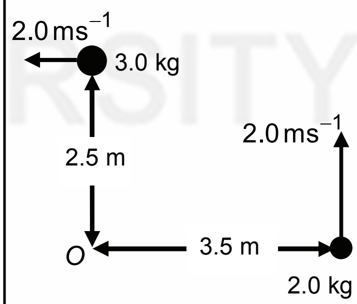


Fig. 15.15

The total angular momentum about the c.m. is given by Eq. 15.28c. The coordinates of the c.m. of the system are given (in m) by

$$\vec{\mathbf{R}}_{cm} = \frac{3.0\,\text{kg}(2.5\,\text{m}\,\hat{\mathbf{j}}) + 2.0\,\text{kg}(3.5\,\text{m}\,\hat{\mathbf{i}})}{5.0\,\text{kg}} = 1.4\,\text{m}\,\hat{\mathbf{i}} + 1.5\,\text{m}\,\hat{\mathbf{j}}$$

The coordinates of the 2 particles with respect to the c.m. are given by

$$\vec{\mathbf{r}}_1' = \vec{\mathbf{r}}_1 - \vec{\mathbf{R}}_{cm} = 2.5\,\hat{\mathbf{j}} - (1.4\,\text{m}\,\hat{\mathbf{i}} + 1.5\,\text{m}\,\hat{\mathbf{j}}) = -1.4\,\text{m}\,\hat{\mathbf{i}} + 1.0\,\text{m}\,\hat{\mathbf{j}} \text{ and}$$

$$\vec{\mathbf{r}}_2' = \vec{\mathbf{r}}_2 - \vec{\mathbf{R}}_{cm} = 3.5\,\hat{\mathbf{i}} - (1.4\,\text{m}\,\hat{\mathbf{i}} + 1.5\,\text{m}\,\hat{\mathbf{j}}) = 2.1\,\text{m}\,\hat{\mathbf{i}} - 1.5\,\text{m}\,\hat{\mathbf{j}}$$

Hence, from Eq. (15.28c),

$$\vec{\mathbf{L}}_{cm} = 3.0\,\text{kg}(-1.4\,\text{m}\,\hat{\mathbf{i}} + 1.0\,\text{m}\,\hat{\mathbf{j}}) \times (-2.0\,\text{ms}^{-1}\,\hat{\mathbf{i}}) + 2.0\,\text{kg}(2.1\,\text{m}\,\hat{\mathbf{i}} - 1.5\,\text{m}\,\hat{\mathbf{j}}) \times (2.0\,\text{ms}^{-1}\,\hat{\mathbf{j}})$$
$$= 14.4\,\text{kgms}^{-1}\,\hat{\mathbf{k}}$$

Alternatively, we could have used Eq. (15.28d) to arrive at this result for $\vec{\mathbf{L}}_{cm}$.

We now discuss the conservation of angular momentum for such systems.

15.5.2 Conservation of Angular Momentum

Once again we begin by considering a two-particle system and then extend the result to an N-particle system. From Eqs. (15.29) and (15.28d), we have

$$\vec{\mathbf{L}} = \vec{\mathbf{R}}_{cm} \times M\vec{\mathbf{V}}_{cm} + \mu\,\vec{\mathbf{r}} \times \vec{\mathbf{v}} \tag{15.31}$$

To arrive at the law of conservation of angular momentum, we differentiate Eq. (15.31) with respect to time and determine the net torque on the system. Using the dot notation for derivatives, we have:

$$\frac{d\vec{\mathbf{L}}}{dt} = \dot{\vec{\mathbf{R}}}_{cm} \times M\vec{\mathbf{V}}_{cm} + \vec{\mathbf{R}}_{cm} \times M\dot{\vec{\mathbf{V}}}_{cm} + \mu\,\dot{\vec{\mathbf{r}}} \times \vec{\mathbf{v}} + \mu\,\vec{\mathbf{r}} \times \dot{\vec{\mathbf{v}}} \tag{15.32a}$$

Since $\dot{\vec{\mathbf{R}}}_{cm} = \vec{\mathbf{V}}_{cm}$ and $\dot{\vec{\mathbf{r}}} = \vec{\mathbf{v}},$ the first and third terms in Eq. (15.32a) are equal to zero (because the vector product of a vector with itself is zero). Now if **no external force is exerted on the system**, the velocity of the c.m. is constant and $\dot{\vec{\mathbf{V}}}_{cm} = \vec{\mathbf{0}}$. Therefore, **when no external force is exerted on the system**, the second term in Eq. (15.32a) is also zero and we are left with

$$\frac{d\vec{\mathbf{L}}}{dt} = \mu\,\vec{\mathbf{r}} \times \dot{\vec{\mathbf{v}}} = \vec{\mathbf{r}} \times \mu\dot{\vec{\mathbf{v}}} = \vec{\mathbf{r}} \times \mu\frac{d\vec{\mathbf{v}}}{dt} \tag{15.32b}$$

or
$$\frac{d\vec{\mathbf{L}}}{dt} = \vec{\mathbf{r}} \times \vec{\mathbf{F}}_{21} \tag{15.32c}$$

where $\vec{\mathbf{F}}_{21}$ is the force of mutual interaction between the two particles.

Now **suppose that the force of mutual interaction between the two particles is central**, that is, **it is along the same direction as $\vec{\mathbf{r}}$, or in the opposite direction**. Then the definition of vector product gives

$$\vec{\mathbf{r}} \times \vec{\mathbf{F}}_{21} = \vec{\mathbf{0}} \tag{15.32d}$$

Thus,
$$\frac{d\vec{\mathbf{L}}}{dt} = \vec{\mathbf{0}} \tag{15.33}$$

This means that the net torque on the system is zero and the angular momentum $\vec{\mathbf{L}}$ of the system is conserved. Thus, we arrive at the law **of conservation of angular momentum for the two-particle system** on which the net external force is zero and the mutual force of interaction between particles is central.

CONSERVATION OF ANGULAR MOMENTUM FOR A TWO-PARTICLE SYSTEM

If the net external force on the two-particle system is zero and the force of mutual interaction is central, the net torque on the system is zero and the angular momentum of the two-particle system is conserved:

$$\vec{\mathbf{L}} = \textbf{constant} \tag{15.34}$$

This is also true for a many-particle system. We now end this unit and summarise what you have studied in it.

15.6 SUMMARY

Concept	Description
Linear momentum of a two-particle system	■ The **total linear momentum of the two-particle system** is just the linear momentum of its centre of mass: $$\vec{\mathbf{p}} = M\frac{d\vec{\mathbf{R}}_{cm}}{dt} = M\vec{\mathbf{V}}_{cm} \qquad \text{where } M = m_1 + m_2$$
Conservation of linear momentum for a two-particle system	■ The **total linear momentum of a two-particle system remains constant if no external force acts on the system**: $$\vec{\mathbf{p}} = M\vec{\mathbf{V}}_{cm} = \textbf{constant}$$
Conservation of linear momentum for an N-particle system	■ The **total linear momentum of an *N*-particle system remains constant if no external force acts on the system**: $$\vec{\mathbf{P}} = M\vec{\mathbf{V}}_{cm} = \textbf{constant} \quad \text{where } M = \sum_{i=1}^{N} m_i$$
Conservation of mechanical energy for a two-particle/N-particle system	■ If **the net force on each particle in a system depends only on its distance from the other particle** (that is, the **force is central conservative**), the **total mechanical energy of the system remains constant**.
Collision of particles	■ A **collision** is said to have taken place between two or more objects when the objects come close enough so that there is some sort of interaction between them for a brief time interval. In a collision process, there may or may not be any physical contact between the objects. Also there may or may not be any external forces being exerted on them.
Types of collisions	■ Collisions are divided into two broad categories: 1. **Elastic collisions**, in which both linear momentum and kinetic energy are conserved, $$\vec{\mathbf{p}}_i = \vec{\mathbf{p}}_f \quad \text{and} \quad K_i = K_f \quad \text{for elastic collisions}$$ 2. **Inelastic collisions**, in which conservation of linear momentum holds good, but kinetic energy is not conserved. However, the total energy is conserved. $$\vec{\mathbf{p}}_i = \vec{\mathbf{p}}_f \quad \text{and} \quad K_i \neq K_f \quad \text{for inelastic collisions}$$
Head-on elastic collisions – special cases	■ When a particle collides with another particle of **equal mass** at rest, the incident particle comes to a stop, while the target particle starts moving with the velocity of the incident particle.

When a particle collides with another **much more massive** particle at rest, the incident particle is reflected back with the same speed, while the target particle hardly moves.

When a particle collides with another **much less massive** particle at rest, the incident particle keeps moving as if nothing happened, while the target particle takes off with twice the velocity of the incident particle

Angular momentum of a two-particle system

- The **angular momentum of the two-particle system** can be expressed as the sum of the angular momentum of the centre of mass and the total angular momenta of the particles about the centre of mass of the system.

$$\vec{\mathbf{L}} = \vec{\mathbf{R}}_{cm} \times \vec{\mathbf{P}}_{cm} + \vec{\mathbf{L}}_{cm}$$

where $\vec{\mathbf{L}}_{cm} = m_1(\vec{\mathbf{r}}_1' \times \vec{\mathbf{v}}_1) + m_2(\vec{\mathbf{r}}_2' \times \vec{\mathbf{v}}_2)$

is the **angular momentum of the two particles about the c.m.**

Angular momentum of a N-particle system

- The angular momentum of an *N*-particle system is given by

$$\vec{\mathbf{L}} = \sum_{i=1}^{N} m_i \vec{\mathbf{r}}_i \times \vec{\mathbf{v}}_i$$

We can express it as the sum of the angular momentum of the c.m. and the angular momenta of the particles about the c.m.:

$$\vec{\mathbf{L}} = \vec{\mathbf{L}}_{cm} + \vec{\mathbf{R}}_{cm} \times \vec{\mathbf{P}}_{cm}$$

where $$\vec{\mathbf{L}}_{cm} = \sum_{i=1}^{N} m_i \vec{\mathbf{r}}_i' \times \vec{\mathbf{v}}_i$$

Conservation of angular momentum for a system of particles

- If the **net external force on system of particles is zero** and the **forces of mutual interaction between particles are central**, the **total angular momentum of the system is conserved**: $\vec{\mathbf{L}} =$ **constant**.

15.7 TERMINAL QUESTIONS

1. In a system of two particles of equal mass, one particle is initially at rest. The other particle approaches the particle at rest from the left at speed *v*. The speed of the centre of mass is

   a) 0
   b) 0.5*v*
   c) *v*
   d) 0.75*v*

2. Consider the same system as in Terminal Question 1. What is the velocity of the centre of mass after the two particles undergo an elastic collision?

   a) 0
   b) 0.5*v*

c) $-0.5v$
d) $0.75v$

3. In a car accident, a car *A* (mass 1000 kg) is initially at rest. It is hit at the back by a car *B* of mass 1500 kg. From the markings of the tyres on the road, the police are able to determine that after their collision, the speeds of car *A* and *B* are $12.0\,\text{ms}^{-1}$ and $8.0\,\text{ms}^{-1}$, respectively. Assuming that the collision is elastic, the speed of the car *B* before collision is
   a) $16\,\text{ms}^{-1}$
   b) $24\,\text{ms}^{-1}$
   b) $14\,\text{ms}^{-1}$
   c) $26\,\text{ms}^{-1}$

4. A ball of mass 5.0 kg moving with a speed of $3.5\,\text{ms}^{-1}$ collides with a ball of mass 2.5 kg at rest after which the balls move together. If the collision between the balls is elastic, the kinetic energy of the ball of mass 2.5 kg after collision is
   a) 1.7 J
   b) 3.4J
   c) 8.1 J
   d) 28 J

5. A box sliding on a smooth floor collides with an identical box initially at rest on the floor. After collision, the two boxes move together. Which of the following is true about the kinetic energy before and after the collision?
   a) $K_i = K_f$
   b) $K_i > K_f$
   c) $K_i < K_f$
   d) The information given is not enough.

6. A neutron moving with speed *v* collides elastically with a helium nucleus (at rest initially) and the helium nucleus is observed to move off at an angle θ with the initial direction of motion of the neutron. The mass of the helium nucleus is four times that of the neutron. Determine the direction in which the neutron moves after collision and the speeds of the two particles.

7. Four particles of equal mass are tied to a rigid mass-less rod as shown in Fig. 15.16. The rod is attached to a pivot at *O*. The system is rotated about *O* in a horizontal plane with a constant angular speed. Determine the angular momentum of the system of particles in terms of their mass, distance between the particles and their angular speed.

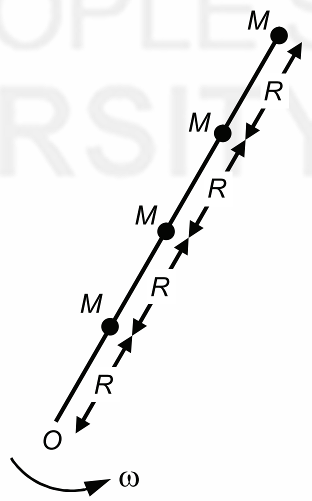


Fig. 15.16

8. A proton makes a head-on collision with an unknown particle at rest. The proton rebounds back with 4/9 of its initial kinetic energy. Obtain the ratio of the mass of the unknown particle to that of the proton.

9. An object is launched with a velocity of $20\,\text{ms}^{-1}$ at an angle of 45° with the vertical. At the top of its trajectory it breaks into two equal pieces. One piece falls vertically downwards. Where does the other piece fall? Take $g = 10\,\text{ms}^{-2}$.

10. Two astronauts (each of mass 80 kg) are tied by a light rope so that the distance between them is 8.0 m (Fig. 15.17). They are isolated in space and orbiting their c.m. at a speed of $5\,\text{ms}^{-1}$. Treat the astronauts as particles and calculate the angular momentum and kinetic energy of the system. After a while they pull the rope and move closer to each other so that the distance between them is reduced to 4.0 m. What is the new angular momentum of the system? What are their new speeds? Does the kinetic energy of the system change or remain the same?

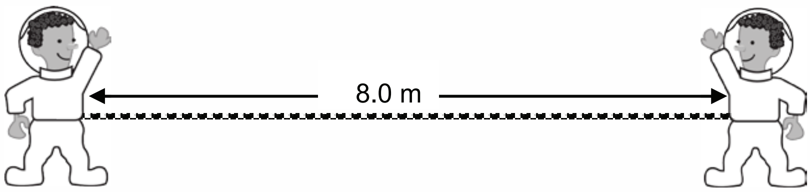


Fig. 15.17

15.8 SOLUTIONS AND ANSWERS

Self-Assessment Questions

1. a) From Eqs. (15.1d and e) with $m_1 = 2\,\text{kg}$, $m_2 = 3\,\text{kg}$, $\vec{\mathbf{v}}_1 = 5\,\text{ms}^{-1}\,\hat{\mathbf{i}}$ and $\vec{\mathbf{v}}_2 = -\,5\,\text{ms}^{-1}\,\hat{\mathbf{i}}$, we get

$$\vec{\mathbf{V}}_{cm} = \frac{2\,\text{kg} \times 5\,\text{ms}^{-1}\,\hat{\mathbf{i}} + 3\,\text{kg} \times (-5\,\text{ms}^{-1})\hat{\mathbf{i}}}{2\,\text{kg} + 3\,\text{kg}} = -\,1\text{ms}^{-1}\,\hat{\mathbf{i}}$$

b) Since there is no external force on the system, the linear momentum of the system is conserved:

$$m_i\vec{\mathbf{v}}_i = m_f\vec{\mathbf{v}}_f \qquad \text{(i)}$$

where m_i, m_f and $\vec{\mathbf{v}}_i$, $\vec{\mathbf{v}}_f$ are the initial and final masses and velocities of the system. Since the block is at rest initially, the linear momentum of the bullet equals the initial momentum of the system:

$$m_i\vec{\mathbf{v}}_i = 0.01\,\text{kg} \times \vec{\mathbf{v}}_i \qquad \text{(ii)}$$

Also since the bullet is lodged in the block, both move with the same velocity in the same direction after the impact and the final momentum of the system is given by:

$$m_f v_f = 10.01\,\text{kg} \times 0.2\,\text{ms}^{-1} = 2.002\,\text{kgms}^{-1} \qquad \text{(iii)}$$

Substituting Eqs. (ii) and (iii) in Eq. (i), we get

$$0.01\,\text{kg} \times v_i = 2.002\,\text{kgms}^{-1} \Rightarrow v_i = 200\,\text{ms}^{-1} = 2.0 \times 10^2\,\text{ms}^{-1}$$

2. a) For elastic collision, we use Eqs. (15.19c) and (15.19b) with $v_1 = 3.0\,\text{ms}^{-1}$, $m_1 = 5.0\,\text{kg}$ and $m_2 = 8.0\,\text{kg}$:

$$v_1' = \left(\frac{5.0\,\text{kg} - 8.0\,\text{kg}}{5.0\,\text{kg} + 8.0\,\text{kg}}\right) \times 3.0\,\text{ms}^{-1} = -0.69\,\text{ms}^{-1}$$

and $$v_2' = \frac{2 \times 5.0\,\text{kg}}{5.0\,\text{kg} + 8.0\,\text{kg}} \times 3.0\,\text{ms}^{-1} = 2.3\,\text{ms}^{-1}$$

b) We use Eqs. (15.16a and b) with $m_1 = 40\,\text{kg}$, $m_2 = 50\,\text{kg}$,
$\vec{\mathbf{v}}_1 = 6.0\,\text{ms}^{-1}\,\hat{\mathbf{i}}$ and $\vec{\mathbf{v}}'_1 = -1.5\,\text{ms}^{-1}\,\hat{\mathbf{i}}$. We need to determine $\vec{\mathbf{v}}_2$ and $\vec{\mathbf{v}}'_2$ which are the initial and final velocities of the second cart. Using Eq. (15.16a), we get

$$40\,\text{kg} \times 6.0\,\text{ms}^{-1} + 50\,\text{kg} \times v_2 = 40\,\text{kg} \times (-1.5\,\text{ms}^{-1}) + 50\,\text{kg} \times v'_2$$

$$\Rightarrow \qquad v'_2 - v_2 = 6.0 \qquad \text{(i)}$$

From Eq. (15.16b) we get

$$\frac{1}{2}(40\,\text{kg}) \times (6.0\,\text{ms}^{-1})^2 + \frac{1}{2}(50\,\text{kg}) \times v_2^2$$
$$= \frac{1}{2}(40\,\text{kg}) \times (-1.5\,\text{ms}^{-1})^2 + \frac{1}{2}(50\,\text{kg}) \times v_2'^2$$

$$\Rightarrow \qquad v_2'^2 - v_2^2 = 27 \qquad \text{(ii)}$$

Dividing Eq. (ii) by (i) we get $v'_2 + v_2 = 4.5$ (iii)

From (i) and (iii): $\vec{\mathbf{v}}_2 = -0.75\,\text{ms}^{-1}\,\hat{\mathbf{i}}$ and $\vec{\mathbf{v}}'_2 = 5.25\,\text{ms}^{-1}\,\hat{\mathbf{i}} \approx 5.3\,\text{ms}^{-1}\,\hat{\mathbf{i}}$

3. The initial and final velocities of particle 2 are $\vec{\mathbf{v}}_2$ and $\vec{\mathbf{v}}'_2$. Since there is no external force in the system, its linear momentum is conserved. So, the initial and final momenta of the system are, respectively,

$$\vec{\mathbf{p}}_i = mv_1\,\hat{\mathbf{i}} - \frac{m}{3}v_2\,\hat{\mathbf{i}} = \left(mv_1 - \frac{mv_2}{3}\right)\hat{\mathbf{i}}$$

$$\vec{\mathbf{p}}_f = -m\frac{v_1}{2}\,\hat{\mathbf{j}} + \frac{m}{3}(v'_2\cos 45^\circ\,\hat{\mathbf{i}} + v'_2\sin 45^\circ\,\hat{\mathbf{j}})$$

$$= \frac{mv'_2}{3\sqrt{2}}\,\hat{\mathbf{i}} + \left(\frac{mv'_2}{3\sqrt{2}} - \frac{mv_1}{2}\right)\hat{\mathbf{j}}$$

Since $\vec{\mathbf{p}}_i = \vec{\mathbf{p}}_f$, we equate the components of the momentum along the *x* and *y* directions, to get

$$mv_1 - \frac{mv_2}{3} = \frac{mv'_2}{3\sqrt{2}} \quad \text{and} \quad \frac{mv'_2}{3\sqrt{2}} - \frac{mv_1}{2} = 0 \qquad \text{(i)}$$

Solving for v'_2 and v_2 from (i), we get, $v'_2 = \dfrac{3\sqrt{2}}{2}v_1$ and $v_2 = \dfrac{3}{2}v_1$

From the second equation in (i), we have:

$$v'_2 = \frac{3\sqrt{2}}{2}v_1$$

Substituting this value in the first equation in (i), we get

$$mv_1 - \frac{mv_2}{3} = \frac{m}{3\sqrt{2}}\cdot\frac{3\sqrt{2}}{2}v_1$$

$$\Rightarrow v_2 = \frac{3}{2}v_1$$

Terminal Questions

1. The correct option is (b). The momentum of the system is equal to the momentum of the c.m. Let *m* be the mass of each particle and *v* be along the positive *x*-direction. Using Eq. (15.1e) we can write,

$$(m+m)\,\vec{\mathbf{V}}_{cm} = m\vec{\mathbf{v}} + m \times \vec{\mathbf{0}} \quad \text{or} \quad V_{cm} = \frac{mv}{2m} = 0.5v$$

2. The correct option is (b). Since there is no external force in the system, the total linear momentum remains constant and hence from Eq. (15.2), the velocity of the c.m. of the system is the same as it was before the collision, which is 0.5*v*.

3. The correct option is (a). Let the motion of the cars be in the positive x direction. We use Eq. (15.16a) for the conservation of linear momentum for an elastic collision in one direction with $m_1 = 1000\,\text{kg}$, $m_2 = 1500\,\text{kg}$, $v_1 = 0$, $v_1' = 12.0\,\text{ms}^{-1}$ and $v_2' = 8.0\,\text{ms}^{-1}$. We obtain v_2, the velocity of the car B before the collision, from Eq. (15.16a):

$$1000\,\text{kg}\times 0 + 1500\,\text{kg}\times v_2 = 1000\,\text{kg}\times 12.0\,\text{ms}^{-1} + 1500\,\text{kg}\times 8.0\,\text{ms}^{-1}$$

or $\quad v_2 = 16\,\text{ms}^{-1}$

4. The correct option is (d). We use Eq. (15.19b) with $m_1 = 5.0\,\text{kg}$, $m_2 = 2.5\,\text{kg}$ $v_1 = 3.5\,\text{ms}^{-1}$, $v_2 = 0$

   The velocity v_2' of the ball of mass 2.5 kg after the collision is then

   $v_2' = \dfrac{2\times 5.0\,\text{kg}}{(5.0\,\text{kg} + 2.5\,\text{kg})}\times 3.5\,\text{ms}^{-1} = 4.7\,\text{ms}^{-1}$ and the kinetic energy of the ball after collision is $K = \dfrac{1}{2}\times(2.5\,\text{kg})\times(4.7\,\text{ms}^{-1})^2 = 27.6\,\text{J} \approx 28\,\text{J}$

5. The correct option is (b). Let the mass of each box be m. Let the initial velocity of the box before collision be $v\,\hat{\mathbf{i}}$, and the velocity of the two boxes after collision be $\vec{\mathbf{v}}'$. Then, from conservation of linear momentum, we have

$$mv\,\hat{\mathbf{i}} + m(0) = (2m)\,\vec{\mathbf{v}}' \Rightarrow \vec{\mathbf{v}}' = \frac{v}{2}\,\hat{\mathbf{i}}$$

   The initial kinetic energy $K_i = \dfrac{1}{2}mv^2 + 0 = \dfrac{1}{2}mv^2$ and the final kinetic energy $K_f = \dfrac{1}{2}(2m)\left(\dfrac{v}{2}\right)^2 = \dfrac{mv^2}{4}$. Therefore, $K_i > K_f$

6. We use Eq. (15.23) with $m_1 = m$, $m_2 = 4m$, $\theta_2 = \theta$ and $\alpha = \dfrac{m_1}{m_2} = \dfrac{1}{4}$. In Eq. (15.23), v_2 is the velocity of the helium nucleus after collision. Let θ_1 be the angle that the neutron makes with the direction of motion after the collision. Thus,

$$v_2 = \frac{2v\dfrac{1}{4}\cos\theta}{1+\dfrac{1}{4}} = \frac{2}{5}v\cos\theta = 0.4v\cos\theta$$

   From Eq. (15.22d) $v_1^2 = v^2 - \dfrac{m_2}{m_1}v_2^2 = v^2 - 4(0.4v\cos\theta)^2$

$$\Rightarrow v_1 = v(1 - 0.64\cos^2\theta)^{1/2}$$

   From Eq. (15.25): $\tan\theta_1 = \dfrac{\sin 2\theta}{\dfrac{1}{4} - \cos 2\theta} \Rightarrow \theta_1 = \tan^{-1}\left(\dfrac{\sin 2\theta}{\dfrac{1}{4} - \cos 2\theta}\right)$

7. Since the particles are executing circular motion about O, the angular momentum of each particle is given by $mr^2\omega$, where m is the mass of the particle, r its distance from O and ω its angular speed. Therefore, the total angular momentum of the four-particle system is

$$L = mR^2\omega + m(2R)^2\omega + m(3R)^2\omega + m(4R)^2\omega = 30\,mR^2\omega$$

8. Let the motion of the proton (of mass m) be along the positive x-axis. Let m' be the mass of the unknown particle. Suppose the initial velocity of the proton is $v_1\,\hat{\mathbf{i}}$ and the final velocities of the proton and the unknown particle are $-v_1'\,\hat{\mathbf{i}}$ and $v_2'\,\hat{\mathbf{i}}$, respectively. For elastic collision, both linear momentum and kinetic energy are conserved. Therefore, we have

$$mv_1\,\hat{\mathbf{i}} + m' \times \vec{\mathbf{0}} = -mv_1'\,\hat{\mathbf{i}} + m'v_2'\,\hat{\mathbf{i}} \qquad \text{(i)}$$

and $$\frac{1}{2}mv_1^2 = \frac{1}{2}mv_1'^2 + \frac{1}{2}m'v_2'^2 \qquad \text{(ii)}$$

It is given that $$\frac{1}{2}mv_1'^2 = \frac{4}{9}\left(\frac{1}{2}mv_1^2\right) \Rightarrow v_1' = \frac{2}{3}v_1 \qquad \text{(iii)}$$

Substituting Eq. (iii) in Eq. (i), we get

$$m'v_2'\,\hat{\mathbf{i}} = m\,(v_1 + v_1')\,\hat{\mathbf{i}} = \frac{5}{3}mv_1\,\hat{\mathbf{i}} \quad \Rightarrow \quad m'v_2' = \frac{5}{3}mv_1 \qquad \text{(iv)}$$

Using Eq. (iv), we get

$$\frac{1}{2}m'v_2'^2 = \frac{1}{2}\left(\frac{(m'v_2')^2}{m'}\right) = \frac{1}{2}\frac{\left(\frac{5}{3}mv_1\right)^2}{m'} \Rightarrow \frac{1}{2}m'v_2'^2 = \frac{25}{18}\frac{m^2v_1^2}{m'} \qquad \text{(v)}$$

Substituting Eqs. (v) and (iii) in (ii), we get

$$\frac{1}{2}mv_1^2 = \frac{1}{2}m\left(\frac{2}{3}v_1\right)^2 + \frac{25}{18}\frac{m}{m'}(mv_1^2)$$

$$\Rightarrow \quad \frac{1}{2}mv_1^2 = \frac{4}{18}mv_1^2 + \frac{25}{18}\frac{m}{m'}(mv_1^2) \quad \Rightarrow \quad \frac{25}{18}\frac{m}{m'} = \frac{5}{18} \quad \Rightarrow m' = 5m$$

9. See Fig. 15.18. The highest point on the trajectory occurs when the object is a distance $\frac{R}{2}$ from the point of launch, where R is the range of the object given by

$$R = \frac{v_0^2 \sin 2\theta}{g} = \frac{v_0^2\, 2\sin\theta\cos\theta}{g}$$

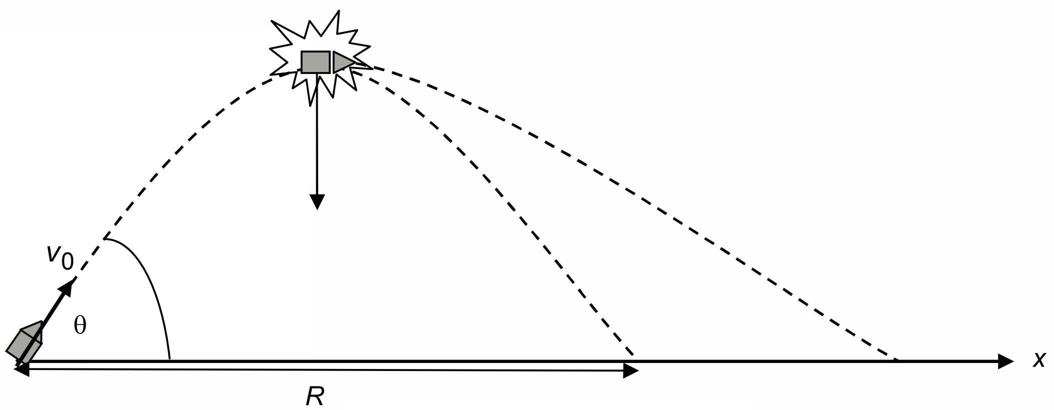


Fig. 15.18: Not to scale

The velocity of the object when it breaks into two parts is only along the horizontal direction since, by definition, the object has no vertical velocity at the highest point. Now let $2m$ be the mass of the object before it breaks up.

The system is made of two parts: Part 1 (of mass m) that falls vertically downwards and part 2 of mass m. Let v_2 be the velocity of part 2 in the horizontal direction. Since no forces are acting along the horizontal direction, linear momentum is conserved along the horizontal direction:

$$p_{ix} = p_{fx} \quad \text{or} \quad (m + m)v_0 \cos\theta = m \times 0 + mv_2 \Rightarrow v_2 = 2v_0 \cos\theta$$

Thus part 2 has a new horizontal velocity which is **twice** the horizontal velocity of the original object. But there is no change in its vertical velocity since it is falling under the force of gravity. Normally, the whole object would have traveled a horizontal distance of $\frac{R}{2}$ from the midpoint. Since part 2 takes the same amount of time to fall to the ground from the highest point as the object but travels at twice the horizontal velocity of the object, it will travel twice as far horizontally from the midpoint as the object. Therefore, the horizontal distance part 2 travels from the initial position of the object is

$$\frac{R}{2} + 2 \times \frac{R}{2} = \frac{3R}{2} = \frac{3 \times v_0^2 \sin\theta \cos\theta}{g} = \frac{3 \times (20\ \text{ms}^{-1})^2 \sin 45^\circ \cos 45^\circ}{10\ \text{ms}^{-2}}$$

$$= 60\,\text{m}$$

10. Refer to Fig. 15.17. We treat the system as a two-particle system. The angular momenta of the particles are parallel (perpendicular to the plane of the paper and pointing towards us) and equal in magnitude. Since the particles are executing circular motion, the magnitude of the initial total angular momentum of the system is given by

$$L = L_1 + L_2 = mvr + mvr = 2mvr$$

where $\quad m = 80\,\text{kg}, \quad v = 5.0\,\text{ms}^{-1} \quad \text{and} \quad r = \left(\frac{8.0}{2}\right)\text{m} = 4.0\,\text{m}$

$$\therefore \quad L = 2(80\,\text{kg})\ (5.0\,\text{ms}^{-1})\ (4.0\,\text{m}) = 3.2 \times 10^3\ \text{kg}\,\text{m}^2\text{s}^{-1}$$

The initial total K.E. of the system is the sum of the K.E. of the individual particles.

$$\therefore \ \text{K.E.} = 2\left(\frac{1}{2}\,mv^2\right) = (80\,\text{kg})\ (5\text{ms}^{-1})^2 = 2.0 \times 10^3\ \text{J}$$

The astronauts move close to each other due to equal and opposite internal forces that act along the line joining them. This means that the mutual force is central. And there is no external force. Hence, the angular momentum of the system remains conserved, i.e. its magnitude is $3.2 \times 10^3\ \text{kg}\,\text{m}^2\text{s}^{-1}$ and direction is the same as before.

Let V and R be the new speed and radius, respectively. Then from conservation of angular momentum, we have,

$$2mVR = 3.2 \times 10^3\ \text{kg}\,\text{m}^2\text{s}^{-1} \quad \text{where } m = 80\,\text{kg and } R = \frac{4.0\,\text{m}}{2} = 2.0\,\text{m}$$

$$\therefore \qquad V = \frac{3.2 \times 10^3\ \text{kg}\,\text{m}^2\text{s}^{-1}}{2(80\,\text{kg})\ (2.0\,\text{m})} = 10\,\text{ms}^{-1}$$

The new total $\text{K.E.} = 2\left(\frac{1}{2}mV^2\right) = (80\,\text{kg})\ (10\,\text{ms}^{-1})^2 = 8.0 \times 10^3\ \text{J}.$

So the new K.E. is greater.

FURTHER READINGS

1. **Principles of Physics**; D. Halliday, R. Resnick and J. Walker, Tenth Edition, Wiley India Ltd. (2015).

2. **Mechanics** (**Berkeley Physics Course**, **Volume I**); C. Kittel, W. D. Knight, M. A. Ruderman, A. C. Helmholz, B. J. Moyer; McGraw Hill International Book Company (2017).

TABLE OF PHYSICAL CONSTANTS

Symbol	**Quantity**	**Value**
c	Speed of light in vacuum	$3.00 \times 10^{8}\ \mathrm{ms^{-1}}$
μ_0	Permeability of free space	$1.26 \times 10^{-6}\ \mathrm{NA^{-2}}$
ε_0	Permittivity of free space	$8.85 \times 10^{-12}\ \mathrm{C^2 N^{-1} m^{-2}}$
$1/4\pi\varepsilon_0$		$8.99 \times 10^{9}\ \mathrm{Nm^2 C^{-2}}$
e	Charge of the proton	$1.60 \times 10^{-19}\ \mathrm{C}$
$-e$	Charge of the electron	$-1.60 \times 10^{-19}\ \mathrm{C}$
h	Planck's constant	$6.63 \times 10^{-34}\ \mathrm{Js}$
$\hbar$	$h/2\pi$	$1.05 \times 10^{-34}\ \mathrm{Js}$
m_e	Electron rest mass	$9.11 \times 10^{-31}\ \mathrm{kg}$
$-e/m_e$	Electron charge to mass ratio	$-1.76 \times 10^{11}\ \mathrm{Ckg^{-1}}$
m_p	Proton rest mass	$1.67 \times 10^{-27}\ \mathrm{kg}$ (1 amu)
m_n	Neutron rest mass	$1.68 \times 10^{-27}\ \mathrm{kg}$
a_0	Bohr radius	$5.29 \times 10^{-11}\ \mathrm{m}$
N_A	Avogadro constant	$6.02 \times 10^{23}\ \mathrm{mol^{-1}}$
R	Universal gas constant	$8.31\ \mathrm{Jmol^{-1} K^{-1}}$
k_B	Boltzmann constant	$1.38 \times 10^{-23}\ \mathrm{J\,K^{-1}}$
G	Universal gravitational constant	$6.67 \times 10^{-11}\ \mathrm{Nm^2 kg^{-2}}$

<u>Astrophysical Data</u>

Celestial Body	**Mass (kg)**	**Mean radius (m)**	**Mean distance from the centre of Earth (m)**
Sun	1.99×10^{30}	6.96×10^{8}	1.50×10^{11}
Moon	7.35×10^{22}	1.74×10^{6}	3.84×10^{8}
Earth	5.97×10^{24}	6.37×10^{6}	0

LIST OF BLOCKS AND UNITS: BPHCT-131

SYLLABUS: MECHANICS (BPHCT-131) 4 Credits

Vector Algebra: Geometrical and algebraic representation of vectors, Vector algebra; Scalar and vector products; Derivatives of a vector with respect to a scalar.

First Order Ordinary Differential Equations: First order homogeneous differential equations (separable and linear first order differential equations).

Second Order Ordinary Differential Equations: 2^{nd} order homogeneous differential equations with constant coefficients.

Laws of Motion: Frames of reference; Newton's Laws of motion; Straight line motion; Motion in a plane; Uniform circular motion; 3-d motion.

Applications of Newton's Laws of Motion: Friction; Tension; Gravitation; Spring-mass system – Hooke's law; Satellite in circular orbit and applications; Geosynchronous orbits; Basic idea of global positioning system (GPS); Weight and Weightlessness.

Linear Momentum and Impulse: Conservation of momentum; Impulse; impulse-momentum Theorem; Motion of rockets.

Work and Energy: Work and energy; Conservation of energy; Head-on and 2-d collisions.

Kinematics of Angular Motion: Kinematics of angular motion: Angular displacement, angular velocity and angular acceleration; General angular motion.

Dynamics of Rotational Motion: Torque; Rotational inertia; Kinetic energy of rotation; Angular momentum; Conservation of angular momentum and its applications.

Motion under Central Force: Motion of a particle in a central force field (motion in a plane, conservation of angular momentum, constancy of areal velocity; Kepler's Laws (statement only).

Dynamics of Many Particle Systems: Dynamics of a system of particles; Centre of Mass, determination of the centre of mass of discrete mass distributions, centre of mass of a rigid body (qualitative).

Conservation Laws: Linear momentum, angular momentum and energy conservation for many-particle systems.

Simple Harmonic Motion: Simple Harmonic Motion; Differential equation of SHM and its solutions; Kinetic Energy, Potential Energy, and Total Energy of SHM and their time averages.

Superposition of Harmonic Oscillations: Linearity and Superposition Principle; Superposition of Collinear Oscillations having equal frequencies and having different frequencies (beats); Superposition of Orthogonal Oscillations with equal and unequal frequency; Lissajous Figures and their uses.

Damped Oscillations: Equation of Motion of Damped Oscillations and its solution (without derivation); Qualitative description of the solution for Heavy, Critical and Weak Damping; Characterising Damped Oscillations – Logarithmic Decrement, Relaxation Time and Quality Factor.

Wave Motion: Qualitative Description (Wave formation and Propagation; Describing Wave Motion, Frequency, Wavelength and Velocity of Wave; Mathematical Description of Wave Motion).