Indira Gandhi National
Open University
School of Sciences

BPHCT-131
MECHANICS

Block

4

HARMONIC OSCILLATIONS

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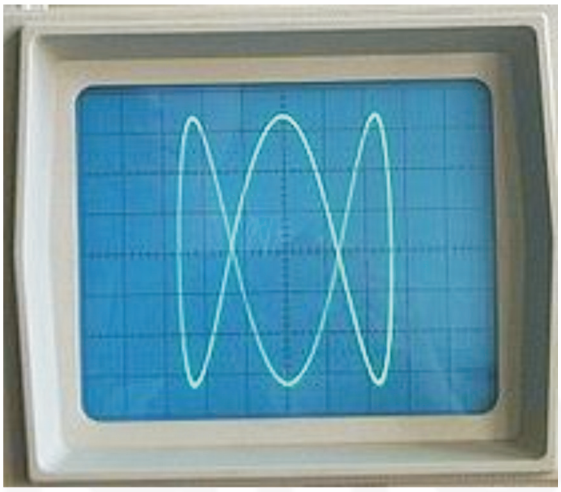
CONTENTS

BLOCK 4: HARMONIC OSCILLATIONS

In this block, you will study the oscillatory behaviour of an isolated system and wave motion. The discussion of oscillatory motion is mainly confined to mechanical oscillators. But, the mathematical techniques developed through these discussions apply equally well to other types of physical systems such as electrical systems. Further, we have adopted the familiar approach of going from easy to difficult in so far as mathematical treatment and conceptual difficulty level of the topics is concerned. Therefore, we have first considered the oscillations of an ideal isolated mechanical oscillator and then taken up the analysis of superposition of two oscillations and damped oscillations. In the end, we have discussed the formation and propagation of waves.



In **Unit 16**, you will study the oscillations of an idealised spring-mass system to appreciate the basic characteristics of the simplest kind of oscillatory motion called **simple harmonic motion (SHM)**. You will also learn how the mathematical techniques developed for SHM can be used to determine the energy associated with oscillatory systems.

In **Unit 17**, you will study the **superposition principle** and learn how this principle can be used to analyse the motion of an object on which two or more harmonic oscillations act simultaneously. You will discover that when the superposing oscillations are orthogonal to each other, the object traces very interesting paths known as **Lissajous figures**.

As you know, any motion – linear or oscillatory – in the real world is always resisted, generally by frictional forces. For example, the oscillations of a swing, left to itself, gradually die out due to air drag. So, to appreciate the behaviour of a real oscillator, you must know how a **drag force affects its oscillatory motion**. This is the subject matter of **Unit 18** wherein we have not only analysed the effect of damping on the motion of an oscillator but have also defined and obtained mathematical expressions for some parameters which are used to characterise the extent of damping in an oscillatory system.

In **Unit 19**, we discuss wave motion. **What is a wave?** For most of us, the first image which comes to mind when we think of waves is the wave moving across the surface of ocean, lake or pond. However, in science, wave is a generic term which refers to propagation of ‘disturbance’ created by something oscillating or vibrating. In case of water waves, the disturbance caused in the still water surface (say, by dropping a piece of stone) propagates as wave. Similarly, we hear each other because our vocal cord creates disturbance in the surrounding air and the disturbance propagates as sound waves. And, the electromagnetic waves are the variations or 'disturbance' in the electric and magnetic fields in the area surrounding an antenna in which electrons execute oscillatory motion. The phenomena of oscillations and waves are related and they have many common characteristics such as amplitude, frequency and phase. The **formation and propagation of waves** is the subject matter of Unit 19. You will learn how to represent waves graphically as well as mathematically. You will also study about the phase of waves and phase difference between them.

We hope that you enjoy studying oscillatory motion, and wish you success.

This is a picture of Foucault's pendulum kept at the National Museum of Science and Technology in Milan, Italy. Foucault's pendulum, named after the French physicist Léon Foucault, is a simple pendulum which demonstrates the rotation of the Earth. (Source of picture: commons.wikimedia.org)

UNIT 16

SIMPLE HARMONIC MOTION

Structure

STUDY GUIDE

In this unit, you will learn how to describe simple harmonic motion. For better understanding of the subject matter of this unit, you need to refresh some basic concepts of mechanics, differential calculus and differential equations from your school physics and mathematics courses. In particular, you should revise the concept of derivatives in calculus before studying this unit. You know that knowledge of the force being exerted on an object is essential to describe its motion. You should focus on the nature of the force responsible for oscillatory motion. The equation of motion of a simple harmonic oscillator is a second order linear ordinary differential equation and you have learnt how to solve it in Unit 4 of this course. Depending on the complexity of the oscillatory system, different mathematical techniques are used to solve differential equations describing its motion. However, we would like you to focus more on the physical inferences and conclusions we can derive on the basis of the solutions of these equations of motion.

"It doesn't matter how beautiful your theory is, it doesn't matter how smart you are. If it doesn't agree with experiments, it's wrong."

Richard P. Feynman

16.1 INTRODUCTION

In your school physics course, you have learnt about rectilinear motion, motion in a plane and periodic motion. You are also familiar with the motion of falling bodies, planets and satellites. A body dropped from rest falls freely (under the action of gravity) along a straight line. Food packet dropped from an aeroplane or a cricket ball thrown by the fielder at the stumps follows a curved path. The motion of the planets and satellites are periodic in nature.

You may have observed the motion of the pendulum of a wall clock, a swing and a vibrating string of a musical instrument such as a violin, a *sitar* or a guitar. These are examples of oscillatory motion. In Block 2 of this course, you have learnt how to analyse rectilinear motion as well as motion in a plane using the laws of mechanics. We can also analyse and understand the salient features of **oscillatory motion using the laws of mechanics**. Further, the study of oscillatory motion is necessary to understand wave phenomenon because waves are generated when energy is exchanged among a large number of interconnected oscillating systems.

In this unit, we have discussed **simple harmonic motion (SHM) – the simplest kind of oscillatory motion.** The study of SHM is very useful because (i) oscillatory motion of a variety of mechanical systems is indeed SHM for small displacements from the equilibrium position and (ii) it enables us to analyse even complex oscillatory motion in terms of SHM. **Further, the oscillatory motion of different types of physical systems can be visualised as SHM under certain approximations**.

In Sec. 16.2, we begin by discussing the physics of the oscillatory motion of a spring-mass system and obtain conditions under which it can be characterised as SHM. In Sec. 16.3 you will learn how to establish the equation of motion of a spring-mass system and solve it to obtain a relation between instantaneous displacement and time. You will also learn how to use this relation to obtain instantaneous velocity, acceleration and phase of an oscillator. The energy associated with the oscillatory motion of a spring-mass system executing SHM is discussed in Sec. 16.4.

Expected Learning Outcomes

After studying this unit, you should be able to:

- state the basic criteria for the motion of a system to be called simple harmonic motion;
- establish the equation of motion for a simple harmonic oscillator;
- define the terms amplitude, time period and phase of an oscillator;
- derive expressions for velocity and acceleration of an oscillator; and
- obtain expressions for potential energy, kinetic energy and total energy of a body executing SHM.

16.2 SIMPLE HARMONIC MOTION: BASIC CHARACTERISTICS

You know that periodic motion is very common in our everyday life. For example, the hands of a clock come back to a particular position after the passage of a fixed time. Similarly, the beating of our heart, our breathing, the motion of the Earth around the Sun, the motion of the Moon around the Earth etc., are familiar examples of ***periodic motion***. **When a body in periodic motion moves to-and-fro (or back and forth) about a fixed position, the motion is said to be *vibratory or oscillatory*.** So, we can say that the motion of the hands of a clock is periodic but not oscillatory.

Like periodic motion, oscillatory motion is also a common phenomenon. Well known examples of oscillatory motion are oscillating bob of a pendulum clock, swing, piston of an engine, motion of piston in a shock absorber, vibrating strings of a musical instrument, atoms in a solid, etc. Even large buildings and bridges may execute oscillatory motion. Fig. 16.1 shows some typical systems in which oscillatory motion takes place.

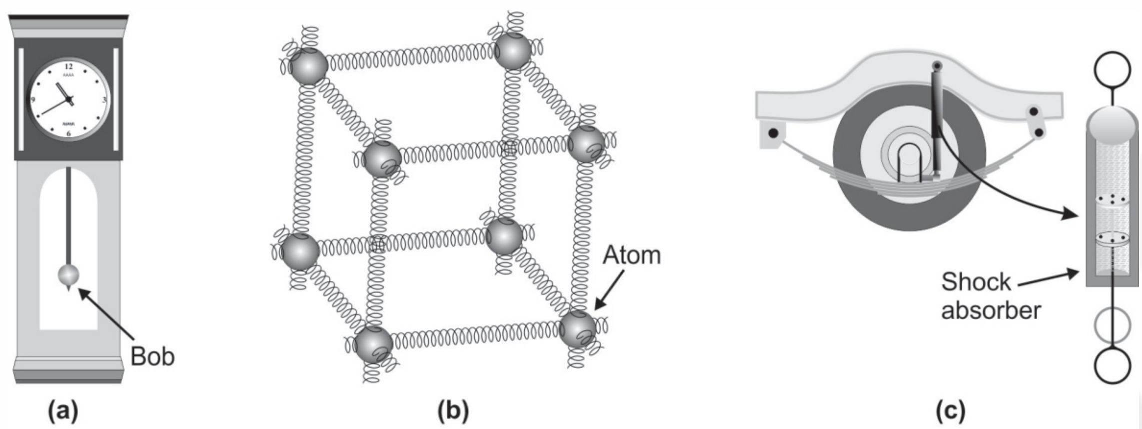


Fig. 16.1: A few typical systems in which oscillatory motion takes place: a) bob of a pendulum clock; b) atoms in a solid (bound to each other through inter-atomic forces represented here by imaginary springs); c) a shock absorber.

Left to itself, oscillatory motion dies down gradually due to resistance by the medium air, liquid, etc.. Further, the oscillatory motion around us is generally quite complex. The simplest kind of oscillatory motion is SHM. A system executing SHM is called **simple harmonic oscillator**. We believe that you know a lot about oscillatory motion from your +2 classes. However, for the sake of completeness, we discuss a simple oscillator system called spring-mass system.

> If a spring is stretched within its elastic limit, Hooke's law holds. It implies that the restoring force will be linearly proportional to the extension of spring. In other words, Hooke's law is valid only if the extension of the spring is 'small' so that the stretching does not cross the elastic limit of the spring. Eq. (16.1) will not be valid for large displacements. In fact, F will then be a rather complex function of x.

16.2.1 Oscillations of a Spring-Mass System

Refer to Fig. 16.2a. It shows a spring-mass system which consists of a spring of negligible mass, whose one end is fixed to a rigid support S and the other end has a block of mass m attached to it. (The mass of a spring is said to be negligible if it is much less than the mass of the block attached to it.) *We assume that the system lies flat on a horizontal frictionless surface.*

We choose a coordinate system whose x-axis is along the length of the spring to analyse the motion of this system. When the block is at rest, we mark a point O on it and define the origin of the coordinate system by this point. That is, at equilibrium, O lies at $x = 0$ as shown in Fig. 16.2a.

To appreciate the fact that an equal force will produce larger extension in a longer spring than in a shorter one, let us assume that a spring has been cut into two unequal pieces such that the shorter one has 10 coils (or turns) and the longer one has 20 coils. Suppose the piece having 10 coils is stretched by 20 cm. For this extension, each coil must be stretched by 2 cm. Further, suppose that the longer piece of the spring having 20 coils is also stretched by 20 cm. In this case, each coil will be stretched only by 1 cm. The force required to stretch a coil by 2 cm will be larger than the force required to stretch it by one cm. Thus, to produce the same extension, we need to apply greater force on the shorter spring. Conversely, if equal force is applied on two similar springs of unequal lengths, they will be stretched or compressed unequally.

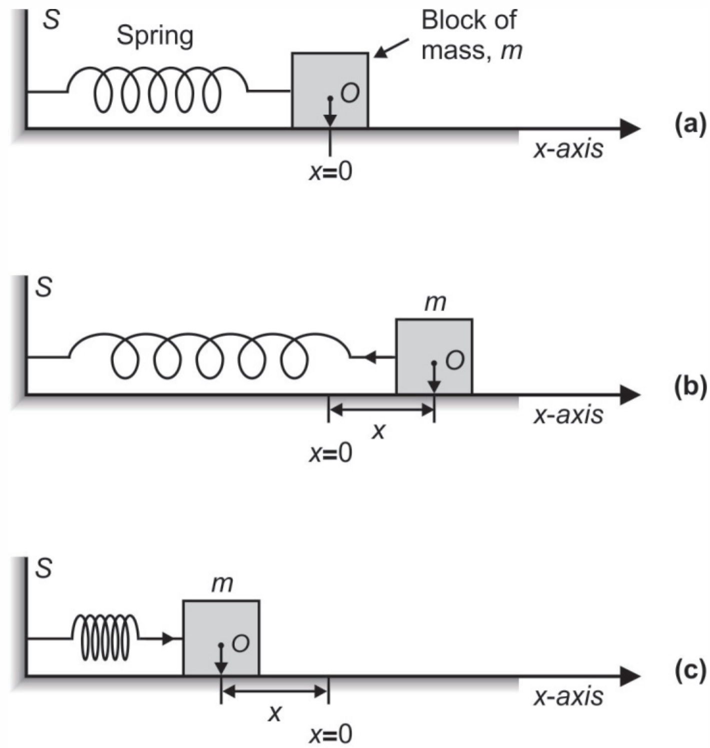


Fig. 16.2: A spring-mass system as an ideal oscillator: a) equilibrium configuration; b) extended configuration; c) compressed configuration.

We now pull the mass longitudinally (along the positive x-axis), so that the spring is stretched through a distance x, (Fig. 16.2b) and then release it. What will you observe? The spring-mass system moves back and forth about the equilibrium position $(x = 0)$. That is, the system executes oscillatory motion.

To understand the nature of oscillations, we note that **due to elasticity, a restoring force, say F is generated in the spring which tends to bring the mass back to the equilibrium position.** When the spring is stretched (Fig. 16.2b), the restoring force tends to compress the spring but when the spring is compressed (Fig. 16.2c), the restoring force tends to extend the spring. (The more you stretch/compress the spring, the greater is the restoring force.) That is, ***the direction of restoring force is always opposite to the displacement of the block***.

To determine the magnitude of the restoring force, we recall that for small extension, Hooke's law holds (see the margin remark on page 9). Then magnitude of the restoring force (F) will be linearly proportional to the extension, x of the spring and we can write:

$$F = -kx \tag{16.1}$$

The negative sign in Eq. (16.1) signifies that the restoring force opposes displacement of the mass attached to the spring. Note that the magnitude of the restoring force is a function of displacement, x, and is, therefore, variable. The quantity k is called the **spring constant** or **force constant** of the spring. The spring constant is a measure of the stiffness of the spring. For a given value of x, a stiffer spring will exert larger restoring force on the mass. The value of the spring constant depends on the physical properties such as density and bulk modulus of elasticity of the material the spring is made up of. However, if a spring is cut into pieces of unequal lengths and equal force is applied on each one of them, the longer spring will be stretched/compressed more than the shorter one (see the margin remark on page 10). The spring constant is numerically equal to the magnitude of restoring force exerted by the spring for unit extension. Its SI unit is Nm^{-1}.

For simplicity, here *we confine our discussion only to such oscillations for which Eq.* (16.1) *is valid.* This is known as **small oscillation approximation**. The magnitude of acceleration, a_c produced in mass, m due to the restoring force can be obtained using Newton's second law of motion:

$$F = ma_c$$

On combining the above equation with Eq. (16.1), we can write

$$-kx = ma_c$$

or

$$a_c = -\left(\frac{k}{m}\right)x \tag{16.2}$$

Eq. (16.2) shows that acceleration of the mass attached to the spring is (i) directly proportional to its displacement (because (k/m) is a constant), and (ii) directed opposite to the displacement (as indicated by the negative sign on RHS). When the oscillatory motion of a body is characterised by these two features, we say that the body is executing simple harmonic motion. We may, therefore, conclude as follows:

NOTE

Newton's second law of motion is written in the vector form as:

$$\vec{\mathbf{F}} = m\vec{\mathbf{a}}_c$$

where force, $\vec{\mathbf{F}}$ and acceleration, $\vec{\mathbf{a}}_c$ are vector quantities. Since the motion of the spring-mass system is one-dimensional, we have written Newton's second law in the scalar form.

Any oscillatory motion which has the following two characteristics is said to be **simple harmonic motion**:

- The acceleration of a body is directly proportional to its displacement from the equilibrium position.
- The acceleration is always directed opposite to the displacement, i.e. towards the equilibrium position.

At this stage, you may ask: **Why does the spring-mass system oscillate?** To know the answer to this question, we note that when we pull the mass of the spring-mass system from its equilibrium position, the spring is stretched. The restoring force in the spring tries to bring back the mass to its equilibrium position. In this process, the mass acquires kinetic energy. Due to this kinetic energy, the mass comes back to the equilibrium position and continues its motion. That is, the mass overshoots the equilibrium position; it happens **because of inertia (of motion)**. Once the mass overshoots and moves to the other side of the equilibrium position, the spring is compressed and the restoring force again comes into play but, in the opposite direction. This process repeats in time. Thus, we may conclude as follows:

The oscillatory motion of a spring-mass system arises due to two intrinsic properties of the system: elasticity (of the spring) and inertia (of motion) of the mass.

You should now answer the following SAQ to get a feel of the numerical values of different quantities associated with the SHM.

SAQ 1 – **Spring - constant and its determination**

a) Suppose that a spring is cut into two pieces *A* and *B*. The length of *A* is one-third and that of *B* is two-thirds of the length of the original spring. One end of *A* as well as *B* is fixed in a rigid wall.

   i) If equal force is applied to the free ends of *A* and *B* separately, will their extensions be equal?

   ii) If not, which spring will stretch more?

   iii) Which spring has a higher value of spring constant?

b) Suppose that the spring in Fig.16.2a is stretched by 5 cm when a force of 2 N is applied. Calculate the spring constant. If a force of 2.5 N is applied on this spring, determine the compression of the spring.

We have, so far, considered the oscillatory motion of a horizontally aligned spring-mass system and discovered that it executes SHM when it undergoes small displacement. You may now like to know: **Will the motion of the spring-mass system be different when we suspend it vertically from a rigid support? What forces will act on the system in this configuration?** We shall discuss it now.

Vertically aligned spring-mass system

Refer to Fig. 16.3, which shows a spring-mass system suspended vertically from a rigid support *S*. Unlike the earlier case of horizontally aligned spring-mass system, here we also need to take into account the force of gravity on the mass. (We ignore the mass of the spring in comparison with the suspended mass.) Note that **two forces – the restoring force due to spring and the force due to gravity – act on the mass simultaneously.**

You may now ask: **What will the nature of its oscillations be**?

To analyse the motion of the vertically suspended spring-mass system, let us choose the *x*-axis to be along the length of the spring. We take the bottom of the spring as our reference point, $x = 0$, when no mass is attached to it (Fig.16.3a).

When mass *m* is suspended from the spring, it stretches to the point, say $x = x_0$, and the system comes to rest. That is, the reference point shifts to $x = x_0$ (Fig. 16.3b). You will agree that in this situation, the weight, *mg* of mass *m* balances the restoring force kx_0 acting in the upward direction and we can write:

$$mg - kx_0 = 0$$

or

$$mg = kx_0 \tag{16.3}$$

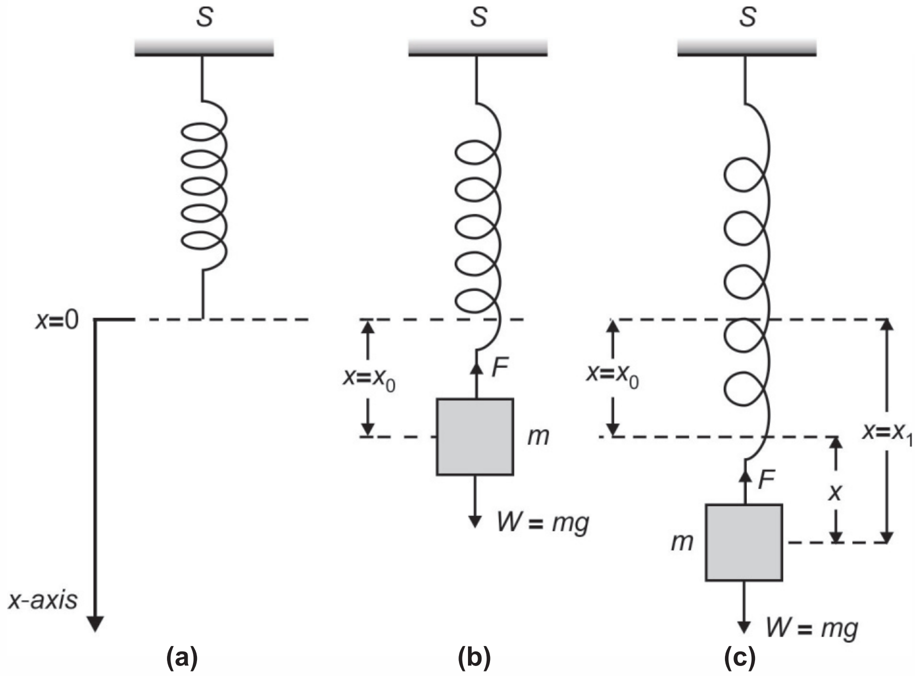


Fig. 16.3: A vertically hanging spring-mass system: a) spring with no mass suspended from it; b) spring in equilibrium when mass *m* has been attached to it; c) spring-mass system displaced from its equilibrium position.

Now, we pull the mass further downwards so that the spring stretches to $x = x_1$ (Fig.16.3c) and then release it. We observe that the system starts oscillating about the point $x = x_0$. You may ask: **How do we explain this motion**? We note that, at the instant we release the mass, the total restoring force acting on the mass is kx_1 and it points in the upward direction. It means that the total downward force acting on the mass is also equal to kx_1. But, the total downward force includes the externally applied force and the force due to gravity. Thus, the net downward force, F' acting on mass *m* at that instant is

$$F' = kx_1 - mg$$

Using Eq. (16.3), we can write

$$F' = kx_1 - kx_0 = k(x_1 - x_0) = kx$$

where $x = x_1 - x_0$. Recall that F' is equal to the force that the spring exerts on the mass after it is released and the force exerted by the spring is directed upward. Therefore, we can write the restoring force as

$$F = -kx$$

Note that the **net restoring force** on the mass executing vertical oscillations has the same form as Eq. (16.1). Therefore, we can say that **the oscillatory motion of a vertically suspended spring-mass system is simple harmonic and the force due to gravity does not affect its nature of oscillations**. The effect of gravity is that it shifts the equilibrium position.

Eq. (16.1) gives **the force law obeyed by the spring-mass system.** We now use Newton's second law of motion to establish the equation of motion of a simple harmonic oscillator and address questions like: How does the motion of a simple harmonic oscillator evolve with time? What is the velocity and acceleration of the oscillator at any given instant of time or at a given point in space? How much time does the oscillator take to complete one oscillation?

16.3 THE DIFFERENTIAL EQUATION OF SHM

> **NOTE**
>
> Velocity $\vec{\mathbf{v}}$ is defined as the rate of change of displacement $\vec{\mathbf{x}}$ with time. Mathematically, we can write
>
> $$\vec{\mathbf{v}} = d\vec{\mathbf{x}}/dt$$
>
> Similarly, acceleration is defined as rate of change of velocity with time. Thus, we can write:
>
> $$\vec{\mathbf{a}}_c = \frac{d\vec{\mathbf{v}}}{dt} = \frac{d}{dt}\left(\frac{d\vec{\mathbf{x}}}{dt}\right) = \frac{d^2\vec{\mathbf{x}}}{dt^2}$$
>
> Note that displacement, velocity and acceleration are vector quantities and they have been represented by their respective symbols in bold face and with arrows above their symbols to distinguish them from scalar quantities such as time, t. Since we are dealing with one-dimensional motion of the oscillator, we can confine to the magnitude of various vector quantities. Thus, we can write:
>
> $$v = dx/dt$$
>
> and $$a_c = d^2x/dt^2$$

We reconsider the motion of an ideal spring-mass system and refer again to Fig. 16.2b. Suppose that, at instant t, the displacement of mass m from its equilibrium position is x. The restoring force F, acting on the mass, is given by Eq. (16.1). Hence, the acceleration, a_c experienced by the mass in one dimensional motion is given by Newton's second law of motion:

$$F = ma_c$$

On combining this result with Eq. (16.1), we can write

$$ma_c = -kx \tag{16.4a}$$

Since for one-dimensional motion,

$$a_c = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2}$$

we can write Eq. (16.4a) as

$$m\frac{d^2x}{dt^2} = -kx \tag{16.4b}$$

or

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0 \tag{16.4c}$$

Eq. (16.4c) is the equation of motion of a simple harmonic oscillator. Note that the unit of (k/m) is $\text{Nm}^{-1}\,\text{kg}^{-1} = (\text{kg ms}^{-2})\,(\text{m}^{-1}\,\text{kg}^{-1}) = \text{s}^{-2}$. It means that we can replace (k/m) by ω_0^2, where ω_0 is **angular frequency** and has the unit s^{-1} and dimension T^{-1}, where T is time.

You may now ask: What is the physical significance of ω_0? Can we relate a physical quantity having the dimensions of reciprocal of time with SHM?

In terms of ω_0, we can write Eq. (16.4c) as

$$\frac{d^2x}{dt^2} + \omega_0^2\,x = 0 \tag{16.5}$$

where $\omega_0 = \sqrt{k/m}$. Eq. (16.5) is yet another form of the equation of motion of an oscillator executing SHM. Note that Eq. (16.5) is a linear, second order homogeneous ordinary differential equation and **describes simple harmonic motion in one dimension**. It is important to mention here that we have arrived at Eq. (16.5) for a spring-mass system, but it describes SHM in general. This result is reproduced below:

> The differential equation describing the SHM of mass m attached to a spring of force constant k is $\dfrac{d^2x}{dt^2} + \omega_0^2\,x = 0$.

You have learnt how to solve Eq. (16.5) in Unit 4 of this course. You may recall Eq. (4.4b) and see that the general solution of Eq. (16.5) is given as

$$x(t) = C_1 \sin\omega_0 t + C_2 \cos\omega_0 t \tag{16.6a}$$

where C_1 and C_2 are arbitrary constants. You have learnt in Unit 4 that with an appropriate choice of the constants C_1 and C_2, the general solution can be recast in any one of the following forms:

$$x(t) = a\sin(\omega_0 t + \phi) \tag{16.6b}$$

$$x(t) = a\cos(\omega_0 t + \phi) \tag{16.6c}$$

$$x(t) = a\sin(\omega_0 t - \phi) \tag{16.6d}$$

$$x(t) = a\cos(\omega_0 t - \phi) \tag{16.6e}$$

where a, ω_0 and ϕ are constants. Before studying further, you should convince yourself, by solving the following SAQ, that the above solutions satisfy the ordinary differential equation [Eq. (16.5)], which describes the oscillatory motion of a spring-mass system.

SAQ 2 **– Solutions of the equation of motion for SHM**

Show that the solutions given by Eqs. (16.6b, c, d, e) satisfy Eq. (16.5).

From Eqs. (16.6b to e), you may note that the variation of displacement with time is given by a sine or cosine function. Therefore, SHM is said to be **sinusoidal motion**. We will examine their physical significance soon. Note further that

- we can consider any one of these solutions [Eq. (16.6b), Eq. (16.6c), Eq. (16.6d) or Eq. (16.6e)] in our discussion;
- the solution of the equation of motion of a spring-mass system gives displacement of the oscillating mass as a function of time ; and
- the solution enables us to determine physical parameters such as velocity, acceleration, energy, etc. of the mass executing SHM and also helps us to know the physical significance of the parameter ω_0 appearing in Eq. (16.5). You will learn about it later in this unit.

The plots between displacement of the oscillator from its equilibrium position and time, based on Eqs. (16.6b) and (16.6c), are shown in Figs. 16.4a and 16.4b, respectively. For simplicity, we have taken $\phi = 0$ while plotting these graphs. Note that the qualitative features of variation of *x* with *t* in both the graphs are similar.

From Fig. 16.4a, we note that at $t = 0, x = 0,$ i.e., the oscillating mass is at the equilibrium position denoted by point *A* on the graph. As time elapses, the displacement increases and attains maximum positive value, equal to *a*. This is denoted by point *B* on the graph.

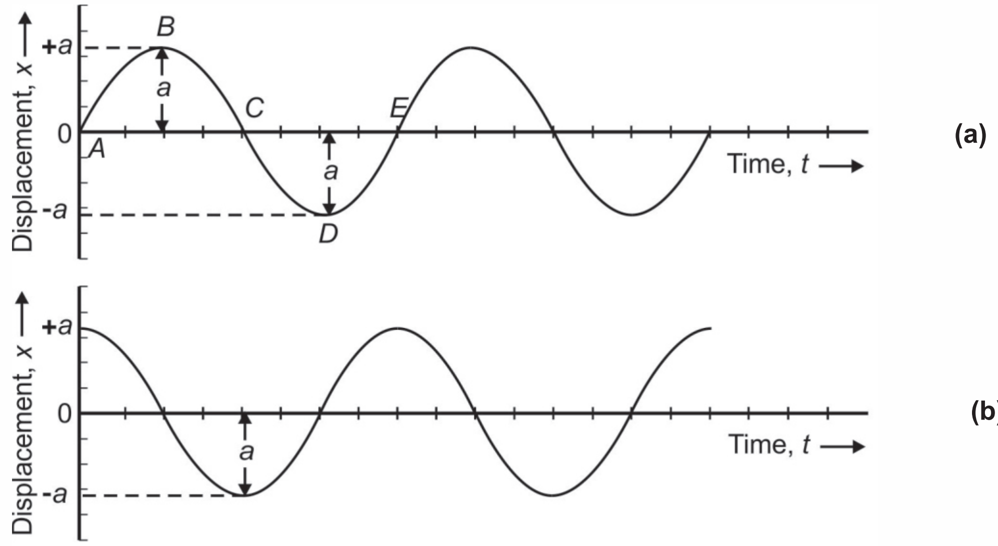


Fig. 16.4: The variation of displacement with time of a body executing SHM according to a) Eq. (16.6b); b) Eq. (16.6c).

With further passage of time, the displacement of the mass begins to decrease, becomes zero (point *C*), increases in the opposite direction, attains maximum (point *D*) and again becomes zero corresponding to (point *E*) the equilibrium position. Beyond *E*, the variation in displacement repeats itself. You will agree that the actual motion of the oscillating mass is truly represented by the graph in Fig. 16.4a. The nature of displacement-time graph in Fig. 16.4b is similar to that of Fig. 16.4a except for the fact that, in this case, at $t = 0,$ the mass is at maximum displacement in the positive *x*-direction.

Let us now discover the physical meaning of the parameters a, ω_0 and ϕ. The constant a is called the **amplitude** of motion; *it is the maximum value that the displacement* (x) *of an oscillator can attain*. Since the sine (or cosine) functions can have values only between $+1$ and –1, the motion takes place entirely between the limits $+a$ and $-a$ (Refer to Fig. 16.4). To understand the physical meaning of ω_0 and ϕ, we need to learn the meaning of phase of an oscillator. For this, we need to determine the velocity and acceleration of an oscillator. Let us do it now.

16.3.1 Velocity and Acceleration of an Oscillator

You may recall from your school physics that instantaneous velocity v is the first derivative of displacement x with time. Therefore, using Eq. (16.6b) we can write the instantaneous velocity of the oscillating mass as

$$v = \frac{dx}{dt} = a\,\omega_0 \cos(\omega_0 t + \phi) \qquad (16.7a)$$

Note that the product $a\omega_0$ will be the maximum velocity of the oscillating mass because $\cos(\omega t + \phi)$ can have maximum value of $+1.$ We can also express velocity of the oscillator in terms of its displacement. We square both sides of Eq. (16.7a) and get

$$v^2 = \omega_0^2 a^2 \cos^2(\omega_0 t + \phi)$$

$$= \omega_0^2 a^2 (1 - \sin^2(\omega_0 t + \phi)) = \omega_0^2 (a^2 - x^2)$$

so that

$$v = \omega_0 \sqrt{(a^2 - x^2)} \tag{16.7b}$$

Further, as $\sin\left(\frac{\pi}{2} + \theta\right) = \cos\theta,$ we can rewrite Eq. (16.7a) as

$$v = a\,\omega_0 \sin\left[\frac{\pi}{2} + (\omega_0 t + \phi)\right] \tag{16.7c}$$

Note that the arguments of the sine functions for displacement and velocity differ by ($\pi/2$).

Since acceleration a_c is the time rate of change of velocity, from Eq. (16.7a), we can write

$$a_c = \frac{dv}{dt} = -a\,\omega_0^2 \sin(\omega_0 t + \phi) \tag{16.8a}$$

$$= a\,\omega_0^2 \sin[\pi + (\omega_0 t + \phi)] \tag{16.8b}$$

because $\sin(\pi + \theta) = -\sin\theta$. Note that $a\omega_0^2$ is the maximum value of acceleration. On combining Eqs. (16.8a) and (16.6b), we get

$$a_c = -\,\omega_0^2\, x \tag{16.9}$$

Therefore, we note that, if $x(t)$ is known, the velocity and acceleration can be calculated easily. Before proceeding further, let us recapitulate the important results obtained in this sub-section.

Recap

- The velocity and acceleration of a harmonic oscillator are given in terms of the displacement as

  $$v = \omega_0 \sqrt{(a^2 - x^2)}$$

  and $\quad a_c = \omega_0^2 x$

- The maximum values of velocity and acceleration of a harmonic oscillator are $a\,\omega_0$ and $a\,\omega_0^2$, respectively.

16.3.2 Phase of an Oscillator

You now know that an oscillator executes to and fro motion about its equilibrium position and passes through the equilibrium position again and again. Now, refer to Fig. 16.5. Suppose $x = 0$ denotes the equilibrium position of the spring-mass system when mass m is hanging freely. If the mass m is pulled downwards and then released, it starts oscillating vertically. While oscillating, the mass passes through the point Q, say while moving from $x = 0$ to $x = a$ (maximum displacement) and again while coming back from $x = a$ to $x = 0$. Note that, at these two instants, when the mass is at point Q, the value of displacement is same but the direction of motion of the oscillating mass is different: it is moving down, away from the equilibrium position, while going from $x = 0$ to $x = a$ and it is moving up, towards the equilibrium position, while going from $x = a$ to $x = 0$. *This means that its velocities (at Q) at these two instances are different* (*because velocity is a vector quantity*). **We then say that the states of motion of mass m are different at these two**

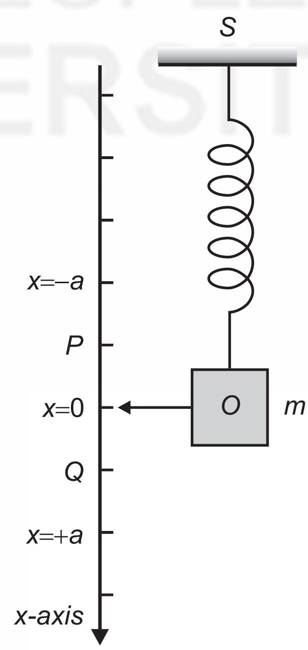


Fig. 16.5: A vertically oscillating spring-mass system.

instants. The state of motion of an oscillator is specified by its displacement and velocity.

If the oscillating mass passes through point Q at times t_1 and t_2 while moving down from $x = 0$ to $x = a$, its displacement as well as velocity are the same. **Then, we say that the oscillator is in the same state of motion. If the state of motion of an oscillator is identical at any two instants, it is said to be in the same phase at those instants.**

So, we specify the state of motion at a point, say Q, by saying that the **phase of the oscillator** is the same at those instants when it passes through Q while going from $x = 0$ to $x = a$ or from $x = a$ to $x = 0$. The same holds true for all other points between $x = a$ and $x = -a$, such as point P.

In other words, we can say that when an oscillator is in the same phase at two different times, its displacements and velocities are the same at these instants. **The time interval between two consecutive instants at which the phase of an oscillator is the same defines the time period of oscillation.** Thus, if we denote the time period of an oscillator by T, we can write

$$x(t) = x(t+T) \tag{16.10}$$

Note that Eq. (16.10) defines the **periodicity condition** for displacement: for an oscillating mass, displacement at time t is the same as its displacement at time $(t + T)$. Using this definition of the time period, we can say that the **periodicity condition** must also hold for the instantaneous velocity of the oscillator:

$$v(t) = v(t+T) \tag{16.11}$$

These two periodicity conditions (Eqs. (16.10) and (16.11)) help us understand the physical meaning of the constant ω_0. Using Eq. (16.10), we can rewrite Eq. (16.6b) as

$$a\sin(\omega_0 t + \phi) = a\sin(\omega_0 t + \phi + \omega_0 T)$$

If we put $\theta = \omega_0 t + \phi$, we can write

$$\sin\theta = \sin(\theta + \omega_0 T) \tag{16.12a}$$

Similarly, using Eq. (16.11) in Eq. (16.7a), we can write

$$a\omega_0\cos(\omega_0 t + \phi) = a\omega_0\cos(\omega_0 t + \phi + \omega_0 T)$$

In terms of θ, we can rewrite this equation as

$$\cos\theta = \cos(\theta + \omega_0 T) \tag{16.12b}$$

Note that Eqs. (16.10) and (16.11) or Eqs. (16.12a) and (16.12b) must hold simultaneously. Can you say why? It is because we are considering the two states of motion of the oscillator separated in time by the time period T. Now, can you use your knowledge of basic trigonometry to tell as to when the two periodicity conditions will hold simultaneously? We know that periodicity of sine and cosine functions is 2π, i.e. $\sin(2\pi+\theta) = \sin\theta$ and $\cos(2\pi+\theta) = \cos\theta$. Therefore, if we assume that

$$2\pi + \theta = \theta + \omega_0 T \tag{16.13}$$

then the two periodicity conditions will hold simultaneously.

Further, it readily follows from Eq. (16.13) that

$$\omega_0 = \frac{2\pi}{T} = 2\pi f \qquad (16.14)$$

where $f = (1/T)$ is the frequency of the oscillator. **The frequency of the oscillator is defined as the number of complete oscillations it makes in one second.** Eq. (16.14) gives us ω_0 in terms of the time period (or frequency) of the oscillator. ω_0 is called the **angular frequency** of the oscillator.

Let us pause for a moment and reflect on what we have achieved so far. Recall that using the concept of phase, we have been able to unfold the physical meaning of ω_0. Further, an important aspect of our discussion about the phase of an oscillator is that we have obtained a common condition [Eq. (16.13)] for periodicity.

Note that the factor $(\omega_0 t + \phi)$ appears as the argument of the sine and cosine functions specifying displacement and velocity. This factor also plays a vital role in relating T and ω_0. So, it is taken as the measure of phase. Thus,

$$\text{Phase of an oscillator executing SHM} = \omega_0 t + \phi \qquad (16.15)$$

where ϕ is called the **initial phase** or the **phase constant**. It is also known as the **epoch** of SHM.

Before proceeding further, let us recapitulate the definitions of the physical parameters characterising SHM.

Recap

- **Displacement** gives the instantaneous position of an oscillator with reference to its equilibrium position.
- **Amplitude** of oscillation is the maximum value of displacement on either side of the equilibrium position.
- **Phase** of an oscillator specifies its state of motion at a given instant.
- **Time period** is the interval of time between two consecutive instants at which phase of the oscillator is the same.
- **Frequency** of an oscillator is equal to the number of complete oscillations it makes in one second.

Now, to understand the graphical representation of the expressions for displacement, velocity and acceleration of an oscillator, refer to Fig. 16.6. It shows the variation of displacement, velocity and acceleration of an oscillator with time based on Eqs. (16.6b), (16.7a) and (16.8a), respectively. For simplicity, we have taken initial phase, ϕ of the oscillator to be zero.

Let us compare the time variations of displacement and velocity. You will note that v attains maximum and minimum values before x by a quarter of a period (i.e., $T/4$). Since one-fourth of a period corresponds to a phase change of $\pi/2$ rad or 90°, we say that **velocity leads displacement by** $\pi/2$. This is also evident from the comparison of Eqs. (16.7c) and (16.6b).

Similarly, by comparing the time variations of displacement and velocity with that of **acceleration, you will conclude that acceleration leads**

displacement by π and velocity by $\pi/2$. This is also evident from the comparison of Eq. (16.8b) with Eqs. (16.6b) and (16.7c), respectively.

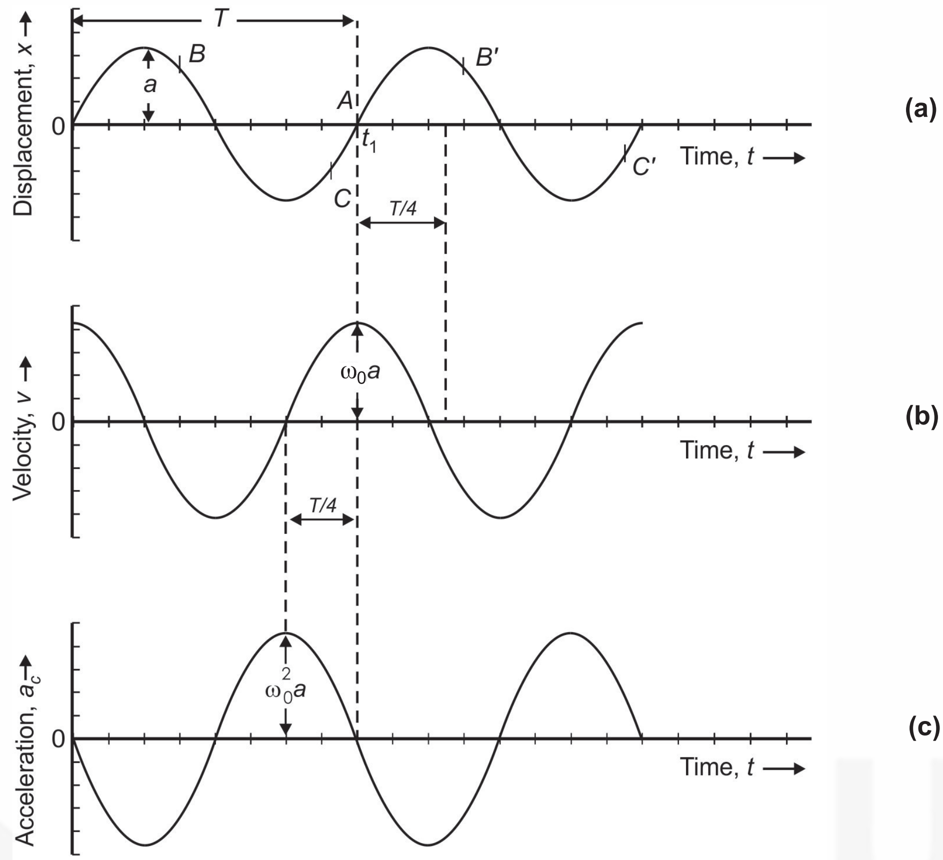


Fig. 16.6: Time variation of a) displacement; b) velocity; c) acceleration of a harmonic oscillator.

Before proceeding further, we work out a few examples to give you a feel for the numerical values of the parameters associated with SHM.

EXAMPLE 16.1 : PHASE OF AN OSCILLATOR

The block of mass m shown in Fig. 16.1 executes SHM with amplitude a. The time is measured from the instant when it is at (i) $x = a$, (ii) $x = -a$, and (iii) $x = a/\sqrt{2}$. Calculate its initial phase ϕ, if its displacement is given by

$$x = a \sin(\omega_0 t + \phi)$$

SOLUTION ■ (i) If time is measured from the instant $x = a$, we say that $x = a$ when $t = 0$. So, the expression for displacement reduces to

$$a = a\sin\phi \quad \Rightarrow \quad \sin\phi = 1 \quad \Rightarrow \quad \phi = \pi/2$$

(ii) In this case, we have $x = -a$ when $t = 0$. So, the expression for displacement simplifies to:

$$-a = a\sin\phi \quad \Rightarrow \quad \sin\phi = -1 \quad \Rightarrow \quad \phi = 3\pi/2$$

(iii) In this case, $x = \frac{a}{\sqrt{2}}$ when $t = 0$. So, the expression for displacement takes the form:

$$\frac{a}{\sqrt{2}} = a\sin\phi \quad \Rightarrow \quad \sin\phi = \frac{1}{\sqrt{2}} \quad \Rightarrow \quad \phi = \frac{\pi}{4}$$

EXAMPLE 16.2 : AMPLITUDE, TIME PERIOD AND DISPLACEMENT OF AN OSCILLATOR

The displacement of an object executing SHM is given by:

$$x = 0.01 \cos 4\pi\,(t + 0.0625)\ \text{m}$$

Determine (i) amplitude of the oscillatory motion, (ii) time period of oscillation, (iii) maximum velocity, (iv) maximum acceleration, and (v) initial displacement of the object.

SOLUTION ■ The standard expression for the displacement of an oscillator executing SHM is given by Eq. (16.6b):

$$x = a \sin(\omega_0 t + \phi)$$

On comparing this expression with the expression for displacement given in the problem, we obtain

i) Amplitude, $a = 0.01\ \text{m}$; Angular frequency, $\omega_0 = 4\pi$

ii) The time period of oscillation is defined as $T = 2\pi/\omega_0$. So, on substituting the value of $\omega_0\ (= 4\pi)$, we get

$$\text{Time period, } T = \frac{2\pi}{\omega_0} = \frac{2\pi}{4\pi} = 0.5\ \text{s}$$

iii) From Eq. (16.7a), we recall that
Maximum velocity $= \omega_0 a = (4\pi\,\text{s}^{-1}) \times (0.01\ \text{m}) = 0.13\ \text{ms}^{-1}$

iv) From Eq. (16.8a), we recall that

Maximum acceleration $= \omega_0^2 a = (4\pi)^2\ \text{s}^{-2} \times (0.01\ \text{m}) = 1.6\ \text{ms}^{-2}$

v) The initial displacement, x_0 is obtained by putting $t = 0$ in the given expression. So, we have

$$x_0 = (0.01\ \text{m}) \times \cos(4\pi \times 0.0625)$$

$$= 0.01 \times \frac{1}{\sqrt{2}}\ \text{m} = 7.1 \times 10^{-3}\ \text{m}$$

EXAMPLE 16.3 : AMPLITUDE AND TIME PERIOD OF AN OSCILLATOR

The velocities of an object executing SHM are $10\ \text{cms}^{-1}$ and $24\ \text{cms}^{-1}$ when its displacements are 12 cm and 5 cm, respectively. Calculate the amplitude and time period of oscillations.

SOLUTION ■ The displacement of the object executing SHM is given by Eq. (16.6b). If we take the initial phase ϕ to be zero, we can write

$$x(t) = a \sin \omega_0 t$$

Thus, instantaneous velocity of the body is given by

$$v = \frac{dx}{dt} = \omega_0 a \cos \omega_0 t$$

so that $v^2 = \omega_0^2 a^2 \cos^2 \omega_0 t = \omega_0^2 a^2 (1 - \sin^2 \omega_0 t) = \omega_0^2 (a^2 - x^2)$ (i)

Now, suppose the velocities of the object are v_1 and v_2 at displacements x_1 and x_2, respectively. Then from Eq. (i), we can write the following two equations:

$$v_1^2 = \omega_0^2 (a^2 - x_1^2) \quad \text{(ii)}$$

and $$v_2^2 = \omega_0^2 (a^2 - x_2^2) \quad \text{(iii)}$$

Substituting the value of ω_0^2 from Eq. (ii) in Eq. (iii), we can write

$$v_2^2 = \frac{v_1^2 (a^2 - x_2^2)}{(a^2 - x_1^2)}$$

or $$v_2^2 a^2 - v_2^2 x_1^2 = v_1^2 a^2 - v_1^2 x_2^2$$

On collecting the terms containing a^2, we can write

$$a^2 (v_2^2 - v_1^2) = v_2^2 x_1^2 - v_1^2 x_2^2$$

so that $$a^2 = \frac{v_2^2 x_1^2 - v_1^2 x_2^2}{v_2^2 - v_1^2} \quad \text{(iv)}$$

As per the problem, we have $v_1 = 10 \text{ cm s}^{-1}$ and $v_2 = 24 \text{ cm s}^{-1}$ at $x_1 = 12 \text{ cm}$ and $x_2 = 5 \text{ cm}$, respectively. Therefore, on substituting these values in Eq. (iv), we get

$$a^2 = \frac{(24 \text{ cm s}^{-1})^2 (12 \text{ cm})^2 - (10 \text{ cm s}^{-1})^2 (5 \text{ cm})^2}{(24 \text{ cm s}^{-1})^2 - (10 \text{ cm s}^{-1})^2} = 169 \text{ cm}^2$$

so that $a = 13$ cm.

On substituting this value of a in Eq. (ii) along with the values of v_1 and x_1, we get

$$\omega_0^2 = \frac{v_1^2}{(a^2 - x_1^2)} = \frac{(10 \text{ cm s}^{-1})^2}{(13 \text{ cm})^2 - (12 \text{ cm})^2} = 4 \text{ s}^{-2}$$

The time period of SHM is given by Eq. (16.14)

$$T = \frac{2\pi}{\omega_0} = \pi \text{ s} = 3.14 \text{ s}$$

So, we note that the amplitude of SHM is 13 cm and its time period is 3.14 s.

To check your understanding of the basic concepts associated with SHM, you should answer the following SAQs.

SAQ 3 – Parameters associated with SHM

a) Show that the frequency of oscillation of a spring-mass system can be expressed as

$$f = \frac{1}{2\pi}\sqrt{k/m}$$

b) The oscillation of a simple harmonic oscillator is described by the equation

$$x(t) = 0.4\,\sin(0.1t + 0.5)$$

where x and t are expressed in metre and second, respectively. Determine the amplitude, time period and frequency of oscillation, maximum velocity, maximum acceleration and initial displacement of the oscillator.

Now that you have learnt how to establish the equation of motion describing SHM and obtained expressions for velocity and acceleration of an oscillator, you should be ready for further analysis. One of the important quantities associated with SHM is energy. You will learn about it now.

16.4 ENERGY IN SHM

A mechanical system executing SHM possesses potential energy as well as kinetic energy. For qualitative understanding of the origin of these energies associated with SHM, refer to Fig. 16.7, which shows a spring-mass system. Let us first obtain the expression for potential energy of the spring-mass system. You will agree that as the mass oscillates, the spring is stretched and compressed alternately. At every instant during oscillation, except when the mass is at the equilibrium position $(x = 0)$, the spring exerts a restoring force on the mass. Due to this force, energy is stored in the spring as **elastic potential energy** (*P.E.*) of the spring.

NOTE

The mathematical analysis of SHM does not change whether we use Eq. (16.6b), (16.6c), (16.6d) or (16.6e) for the displacement of the oscillator. To show the equivalence of these equations, we have used cosine representation for displacement in Sec. 16.4. You can convince yourself that the final result is the same, irrespective of the sine or cosine representation for displacement of the oscillator.

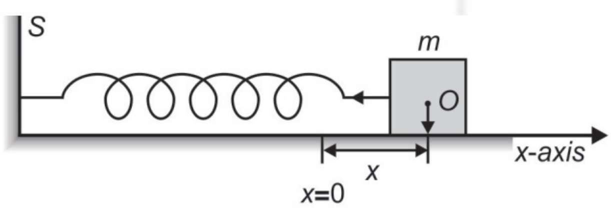


Fig. 16.7: The spring-mass system at the instant when the mass is displaced by a distance *x* from its equilibrium position.

To obtain an expression for the *P.E.* of the spring-mass system, suppose that the mass is displaced by a distance *x* from its equilibrium position (Fig. 16.7).

From your school physics, you may recall that the work done by the spring on the mass *m* in moving it by a distance *x* is given by

$$W = \int_0^x F\,dx$$

For the system under consideration, the force exerted by the spring stretched by a distance x is

$$F = -kx$$

Note that the equality between the change in *P.E.* and the negative of work done is valid only for a conservative force. Hooke's law for an ideal spring is an example of one-dimensional conservative force.

So, we can write

$$W = \int_0^x (-kx)dx = -\left[(1/2)kx^2\right]_0^x = -(1/2)kx^2$$

Now, the change in the elastic potential energy (of the spring) is equal to the negative of the work done by the spring on the mass. If $(P.E.)_i$ and $(P.E.)_f$ are, respectively, the initial and final potential energies of the spring as it is stretched from equilibrium $(x = 0)$ to x, we can write the change in potential energy, Δ (*P.E.*) of the spring as

$$(P.E.)_f - (P.E.)_i = -(-1/2)\,kx^2 = (1/2)\,kx^2$$

If we take $(P.E.)_i = 0$ when $x = 0$, and $(P.E.)_f = (P.E.)$, we can write

$$P.E. = (1/2)kx^2 \tag{16.16}$$

Now you may ask: **Does potential energy of the spring change with time?** To check this, we substitute the expression for displacement, x from Eq. (16.6c) in Eq. (16.16). This gives

$$P.E. = (1/2)\,ka^2\,\cos^2(\omega_0 t + \phi) \tag{16.17}$$

Eq. (16.17) shows that the *P.E.* of the spring-mass system exhibits sinusoidal behaviour in time. For $\phi = 0$, the initial potential energy $(t = 0)$ is $(1/2)\,ka^2$.

To obtain the expression for the kinetic energy of a spring-mass system, we consider the configuration when the mass is released from its displaced position at x. We know that due to the restoring force of the spring, it will move towards the equilibrium position. You may ask: As the mass moves towards the equilibrium position, what happens to x *and hence the P.E.* stored in the spring? The decrease in the magnitude of x suggests that the *P.E.* decreases.

You may again ask: **What happens to the lost *P.E.*? The *P.E.* changes into kinetic energy of the mass.**

To understand this, we note that at the instant mass is released, it is at rest which implies that its velocity and hence its kinetic energy (*K.E.*) is zero. As it moves towards the equilibrium position, its velocity increases gradually. That is, its *K.E.* increases. Therefore, we can say that once the mass is released, the *P.E.* of the spring gradually transforms to *K.E.* of the mass. If there is no loss of energy due to friction, this transformation will be cent-per-cent.

Further, recall that the kinetic energy of mass, m moving with speed v is given by: $K.E. = (1/2)\,mv^2$. You may also recall that for the spring-mass system under consideration, the displacement of the mass can be represented by Eq. (16.6c). Therefore, the velocity, $v\,(= dx/dt)$ of the mass can be written as

$$v = -a\,\omega_0 \sin(\omega_0 t + \phi)$$

Using this result in the expression for *K.E.*, the instantaneous kinetic energy, *K.E.* of the spring-mass system can be written as

$$K.E. = (1/2)m\omega_0^2\, a^2 \sin^2(\omega_0 t + \phi)$$

$$\boxed{K.E. = (1/2)ka^2 \sin^2(\omega_0 t + \phi)} \qquad (16.18)$$

since $k = \omega_0^2 m.$ Note that for $\phi = 0$ and $t = 0,$ *K.E.* is $(1/2)ka^2.$

Using Eq. (16.18), we can also express *K.E.* in terms of displacement, *x* by writing

$$K.E. = (1/2)ka^2\, [1 - \cos^2(\omega_0 t + \phi)]$$

since $\cos^2\theta + \sin^2\theta = 1.$ On simplification, we can rewrite it as

$$K.E. = (1/2)ka^2 - (1/2)ka^2 \cos^2(\omega_0 t + \phi) = (1/2)ka^2 - (1/2)kx^2$$

$$\boxed{K.E. = (1/2)\, k\, (a^2 - x^2)} \qquad (16.19)$$

Eq. (16.19) shows that the *K.E.* of an oscillator

- is maximum when it passes through the equilibrium position $x = 0$; and
- the maximum value of *K.E.* is equal to $(1/2)ka^2$.

From your school physics you may recall that the total mechanical energy of a system is the sum of its potential energy and kinetic energy. Hence, by combining Eqs. (16.17) and (16.18), we obtain the total mechanical energy, *E* of the oscillator at any instant *t*:

$$\begin{aligned} E &= P.E. + K.E. \\ &= (1/2)\, ka^2 \cos^2(\omega_0 t + \phi) + (1/2)\, ka^2 \sin^2(\omega_0 t + \phi) \\ &= (1/2)ka^2 \end{aligned} \qquad (16.20)$$

> Eq. (16.20) shows that **total energy of the spring-mass system**
>
> - does not change with time; and
> - is proportional to the square of amplitude.

The analysis of the oscillatory motion of the spring-mass system leads us to conclude that: *the potential energy as well as the kinetic energy of an oscillator varies with time but its total energy remains constant*.

To understand the transformation of potential energy into kinetic energy and *vice-versa*, study Fig. 16.8. Fig. 16.8a depicts the situation when the spring is stretched and the displacement of the mass is maximum: $x = a.$ From Eqs. (16.16) and (16.19), we note that the energy of the system in this configuration is entirely potential and stored in the spring and kinetic energy of the mass is zero. This is shown by the associated bar diagram for *P.E.* and *K.E.* Note that the height of *P.E.* bar is maximum whereas that of the *K.E.* bar is zero.

As the mass is released from $x = a,$ the *P.E.* stored in the spring begins to decrease (because $P.E. \propto x^2$) and is transferred to the mass, which acquires *K.E.*

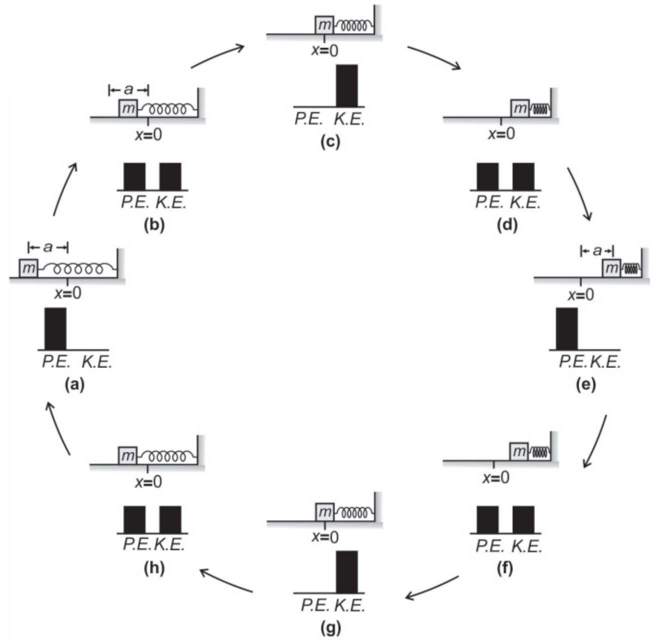


Fig. 16.8: Energy transformation in spring-mass system at various instants during one oscillation. The bar diagrams indicating the values of potential and kinetic energies are shown at intervals of $t = T/8$.

So, at the instant when the mass is between $x = a$ and $x = 0$, as shown in Fig. 16.8b, part of the *P.E.* of the spring has been transferred to the *K.E.* of the mass. That is, energy of the system is partly potential and partly kinetic. This is shown as shortened *P.E.* bar and finite *K.E.* bar in Fig. 16.8b. As the mass reaches the equilibrium position $(x = 0)$, entire energy of the system becomes kinetic and its potential energy drops down to zero in accordance with Eq. (16.16). This is illustrated in Fig. 16.8c. The corresponding energy bars indicate *P.E.* as zero and *K.E.* as maximum. At this stage, due to inertia, the mass, which has acquired kinetic energy, moves beyond $x = 0$ in the opposite direction and begins to compress the spring. As a result, its kinetic energy begins to decrease and transforms into potential energy of the spring. As the mass reaches maximum displacement $(x = -a)$ in the opposite direction, its *K.E.* becomes zero and *P.E.* of the spring becomes maximum again. This is depicted in Fig. 16.8e. This transformation between *P.E.* and *K.E.* continues interchangeably. Therefore, we may conclude that **as the mass oscillates, energy in the spring-mass system alternates between potential and kinetic forms, keeping the total energy constant**.

The graphs of *P.E.* and *K.E.* as a function of displacement, x and based on Eqs. (16.16) and (16.19), respectively, are shown in Fig. 16.9. You may note that:

- the shape of *P.E.* curve as well as *K.E.* curve is parabolic;
- the graphs are symmetric about the origin; and
- the *P.E.* versus x and *K.E.* versus x graphs are inverted with respect to one another.

Note that the total energy, E is represented by the horizontal line AA' in Fig.16.9. At any value of x, the total energy is the sum of kinetic and potential energies and is equal to $(1/2)\,ka^2$.

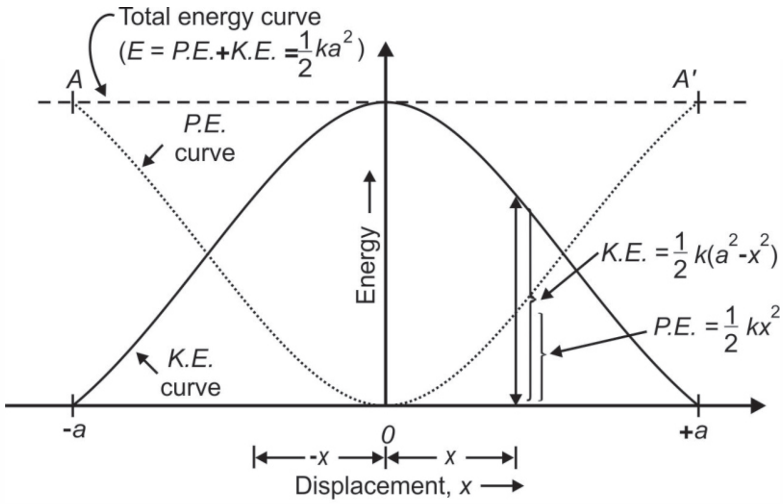


Fig. 16.9: Graphs of potential energy (*P.E.*), kinetic energy (*K.E.*) and total energy (*E*) versus displacement *x* based on Eqs. (16.16), (16.19) and (16.20), respectively.

The points *A* and *A′* where this horizontal line touches the *P.E.* curve define the **turning points**.

At the turning points,

- velocity of the mass is zero and its acceleration is maximum [Eq. (16.9)].
- displacement of the oscillating mass about the equilibrium position is maximum, (i.e., $x = \pm a$).
- the total energy of the oscillator is entirely potential, i.e., *K.E* is zero.

So far, we have discussed energy of SHM and transformation of *P.E.* into *K.E.* and *vice-versa* by referring to a spring-mass system as our model simple harmonic oscillator. **However, this result is valid in general for any mechanical system executing SHM.** Let us now work out an example to use these results for calculating the energy of a simple harmonic oscillator.

EXAMPLE 16.4 : ENERGY OF A HARMONIC OSCILLATOR

An object of mass 0.5 kg is executing simple harmonic motion. Its amplitude is 10 cm and its period is 0.1 s. Calculate the potential energy and the kinetic energy of the object (a) when it is 5 cm from the equilibrium position, and (b) at instants $t = T/8$ and $T/2$. Assume that the initial phase of the oscillations is zero.

SOLUTION ■ As per the problem, we have

$$m = 0.5 \text{ kg}; \; a = 10 \text{ cm} = 0.1 \text{ m}; \; \text{and} \; T = 0.1 \text{ s}$$

a) The potential energy of the particle executing SHM is given in terms of displacement *x* by Eq. (16.16):

$$P.E. = (1/2)\, kx^2 \qquad \text{(i)}$$

We have to calculate *P.E.* when displacement $x = 0.05$ m. Further, for calculating *k*, we use the relation

$$\omega_0^2 = k/m \qquad \Rightarrow \qquad k = \omega_0^2 m \qquad \text{(ii)}$$

We know that

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{0.1\text{ s}} \qquad \text{(iii)}$$

Using this result in Eq. (ii), we get

$$k = \left(\frac{2\pi}{0.1\text{ s}}\right)^2 \times (0.5\text{ kg}) \qquad \text{(iv)}$$

On substituting the value of k from Eq. (iv) in Eq. (i) and putting $x = 0.05$ m, we get

$$P.E. = (1/2)\left(\frac{2\pi}{0.1\text{s}}\right)^2 \times (0.5\text{ kg}) \times (0.05\text{ m})^2 = 2.47\text{ J}$$

The kinetic energy of the particle executing SHM is given by Eq. (16.19):

$$K.E. = (1/2)\, k\,(a^2 - x^2)$$

On substituting the values of k, a, and x, we get

$$K.E. = (1/2)\left(\frac{2\pi}{0.1\text{ s}}\right)^2 \times (0.5\text{ kg})\,[(0.1\text{ m})^2 - (0.05\text{ m})^2] = 7.41\text{J}$$

b) The potential energy of a particle executing SHM is given by Eq. (16.17):

$$P.E. = (1/2)\, ka^2 \cos^2(\omega_0 t + \phi)$$

Since the initial phase is zero, we have $\phi = 0$ and the expression for instantaneous $P.E.$ simplifies to

$$P.E. = (1/2)\, ka^2 \cos^2 \omega_0 t$$

For $t = T/8$, we have

$$P.E. = (1/2)ka^2 \cos^2\left(\frac{2\pi}{T} \times \frac{T}{8}\right)$$

$$= (1/2) \times \left(\frac{2\pi}{0.1\text{ s}}\right)^2 \times (0.5\text{ kg}) \times (0.1\text{ m})^2 \times \cos^2\left(\frac{\pi}{4}\right) = 24.2\text{ J}$$

The instantaneous kinetic energy is given by Eq. (16.18):

$$K.E. = (1/2)ka^2 \sin^2(\omega_0 t + \phi)$$

For $\phi = 0$, we have $K.E. = (1/2)ka^2 \sin^2(\omega_0 t)$

Substituting the value of k, a, ω_0 and $t\,(=T/8)$, we get

$$K.E. = (1/2) \times \left(\frac{2\pi}{0.1\text{ s}}\right)^2 \times (0.5\text{ kg}) \times (0.1\text{ m})^2 \times \sin^2\left(\frac{\pi}{4}\right) = 24.2\text{ J}$$

Similarly, the $P.E.$ and $K.E.$ at the instant $t = T/2$ are

$$P.E. = (1/2)ka^2 \cos^2\left(\frac{2\pi}{T} \times \frac{T}{2}\right) = (1/2)ka^2 \cos^2(\pi)$$

$$= (1/2) \times \left(\frac{2\pi}{0.1\text{ s}}\right)^2 \times (0.5\text{ kg}) \times (0.1\text{ m})^2 \times 1 = 4.39 \times 10^{-2}\text{ J}$$

and $$K.E. = (1/2)ka^2 \sin^2\left(\frac{2\pi}{T}\times\frac{T}{2}\right) = (1/2)ka^2\sin^2(\pi)$$

$$= (1/2)\times\left(\frac{2\pi}{0.1\text{ s}}\right)^2 \times (0.5\text{ kg})^2 \times (0.1\text{ m})^2 \times 0 = 0$$

Note that *P.E.* and *K.E.* of the oscillator are equal at $t = T/8$ and *K.E.* is zero at $t = T/2$. You should compare these values with the physical situations of a spring-mass system depicted in Fig. 16.8. You will note that $t = T/8$ corresponds to configuration depicted in Fig. 16.8b and $t = T/2$ corresponds to the one depicted in Fig. 16.8e.

Before proceeding further, you may like to work out following SAQ.

SAQ 4 – Energy of a harmonic oscillator

a) For what value of displacement the *K.E.* and *P.E.* of a simple harmonic oscillator become equal?

b) In a spring-mass system, a 0.55 kg mass is attached to a spring of force constant 25 Nm^{-1}. The mass is released from rest at $x = 40$ mm. Calculate the *P.E.* and *K.E.* of the system at (i) $x = 20$ mm and (ii) $t = T/4$. Take the initial phase to be zero.

c) Show that the *P.E.* and *K.E.* repeat in *T*/2.

16.4.1 Average Energy Associated with SHM

Refer to Fig. 16.6 again. You will note that in each case, the area under the curve for the first half cycle is exactly equal to the area under the curve in the second half cycle. But these are on opposite sides of the horizontal axis. It means that over one complete cycle of oscillation, the algebraic sum of these areas will be zero. So we can say that the average values of displacement, velocity and acceleration over one complete cycle are zero.

However, the situation in case of *P.E.* and *K.E.* of the oscillator is somewhat different. Refer to Fig. 16.9 and note that the curves representing variation of *P.E.* and *K.E* lie in the upper-half only. So, the total area under either curve is positive for one complete cycle. This means that unlike displacement, velocity and acceleration, we can talk about average values of kinetic and potential energies. You will now learn how to obtain expressions for these quantities.

NOTE

The positive values of *P.E.* and *K.E.* over one complete cycle are due to the fact that *P.E.* is proportional to square of displacement and *K.E.* is proportional to square of velocity.

The time average of kinetic energy over one complete cycle is defined as

$$< K.E. > = \frac{\int_0^T (K.E.)\,dt}{T}$$

On substituting for *K.E.* from Eq. (16.18), we get

$$< K.E. > = \frac{ka^2}{2T}\int_0^T \sin^2(\omega_0 t + \phi)\,dt$$

To solve the integral

$$I = \int_0^T \sin^2(\omega_0 t + \phi)\,dt$$

we assume that $\phi = 0$.

Then we can write

$$I = \int_0^T \sin^2(\omega_0 t)\,dt$$

Recall the trigonometric relation

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

Using this in the expression for I, we get

$$I = \int_0^T \left[\frac{1-\cos 2(\omega_0 t)}{2}\right] dt$$

$$= \int_0^T \frac{1}{2}\,dt - \int_0^T \left[\frac{\cos 2(\omega_0 t)}{2}\right] dt$$

$$= \frac{T}{2} - \frac{1}{2} \times \left[\frac{1}{2}\sin 2(\omega_0 t)\right]_0^T$$

$$= \frac{T}{2} - \frac{1}{2} \times \frac{1}{2}\left[\sin 2\left(\frac{2\pi}{T} \times T\right) - \sin 2(0)\right]$$

$$= T/2$$

The value of the integral in the expression for average kinetic energy is $T/2$ (read the margin remark). So the expression for average kinetic energy of a body executing SHM reduces to

$$\boxed{< K.E. > = \frac{ka^2}{4}} \qquad (16.21)$$

You can follow similar steps for *P.E.* and convince yourself that the average value of the potential energy over one complete cycle for a body executing SHM is

$$\boxed{< P.E. > = \frac{ka^2}{4}} \qquad (16.22)$$

Eqs. (16.21) and (16.22) show that, over one complete cycle of oscillation, the average kinetic energy of a harmonic oscillator is equal to its average potential energy. The sum of average kinetic and average potential energies is:

$$< K.E. > + < P.E. > = (1/4)ka^2 + (1/4)ka^2 = (1/2)ka^2 = E, \text{ total energy.}$$

This result shows that the sum of the average *K.E.* and average *P.E.* of a harmonic oscillator is equal to its total mechanical energy [Eq. (16.20)].

Before proceeding further, let us recall the important results of this section:

Recap

P.E. AND K.E. OF AN OSCILLATOR

- The potential energy (*P.E.*) and the kinetic energy (*K.E.*) of an oscillator vary with time but total energy remains constant.
- The *P.E.*, *K.E.* and total energy, *E* of an oscillator are given by

$$P.E. = \frac{1}{2}kx^2$$

$$K.E. = \frac{1}{2}(a^2 - x^2)$$

$$E = \frac{1}{2}ka^2$$

- The average *P.E.* and average *K.E.* of an oscillator are equal:

$$< P.E. > = \frac{1}{4}ka^2 = < K.E. >$$

- The sum of average *P.E.* and average *K.E.* is equal to total energy of an oscillator:

$$< K.E. > + < P.E. > = \frac{1}{4}ka^2 + \frac{1}{4}ka^2 = \frac{1}{2}ka^2 = E$$

Now, you should solve an SAQ.

SAQ 5 – Average energy of a simple harmonic oscillator

The amplitude of oscillation of a simple harmonic oscillator is 40 cm. Show that its instantaneous kinetic energy is more than its average kinetic energy when the displacement is 20 cm.

Let us now summarise what you have learnt in this unit.

16.5 SUMMARY

Concept	Description
Simple harmonic motion	■ An oscillatory motion is said to be **simple harmonic** when acceleration is (i) proportional to the displacement and (ii) along a direction opposite to the displacement.
Equation of motion	■ The **equation of motion** or the **differential equation** describing SHM is given by $$\frac{d^2x}{dt^2} + \omega_0^2\, x = 0$$
Solution	■ The **solution** of the differential equation describing SHM is given by $$x(t) = \begin{cases} a\sin(\omega_0 t + \phi) \\ a\cos(\omega_0 t + \phi) \\ a\sin(\omega_0 t - \phi) \\ a\cos(\omega_0 t - \phi) \end{cases}$$
Velocity	■ The **velocity** and **acceleration** of a body executing SHM is given by $$v = \omega_0\sqrt{a^2 - x^2} \quad \text{and} \quad a_c = -\,\omega_0^2 x$$
Phase	■ The **phase** of an oscillator refers to its state of motion. If the state of motion (that is, the value of its displacement and velocity) of an oscillator is identical at any two instants, the oscillator is said to be in the same phase at those instants.
Time period	■ The **time period** and **frequency** of oscillation of a body executing SHM are respectively given by the relations $$T = \frac{2\pi}{\omega_0} \quad \text{and} \quad f = \frac{\omega_0}{2\pi} = \frac{1}{T}$$
Potential and kinetic energy	■ A simple harmonic oscillator possesses **potential energy** and **kinetic energy**. The expressions for *P.E.* and *K.E.* are $$P.E. = \frac{1}{2}kx^2 = \frac{1}{2}ka^2\cos^2(\omega_0 t + \phi)$$ $$K.E. = \frac{1}{2}k(a^2 - x^2) = \frac{1}{2}ka^2\sin^2(\omega_0 t + \phi)$$

Total energy

- The **total energy** of a simple harmonic oscillator is given by

$$E = P.E. + K.E. \ = \frac{1}{2}ka^2$$

Time averaged kinetic energy and potential energy

- The **time averaged kinetic energy and potential energy of a simple harmonic oscillator** are same and its value is equal to $\frac{1}{4}ka^2$.

16.6 TERMINAL QUESTIONS

1. An object executes SHM with amplitude, angular frequency and initial phase equal to 65 mm, $4.0\,\text{s}^{-1}$ and zero, respectively. Write expressions for displacement, velocity and acceleration of the object. Also, determine the values of these parameters at $t = 1.5$ s.
2. In Fig. 16.10, three combinations of two springs of force constants k_1 and k_2 are given. Calculate the period of oscillation in each case.

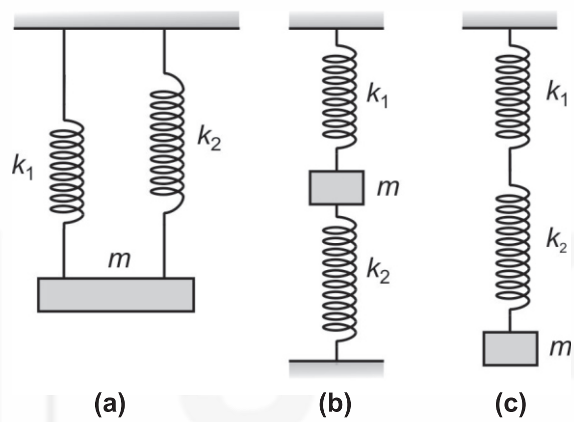


Fig. 16.10: Three different combinations of spring-mass system.

3. An object undergoes SHM with frequency $f = 0.45$ Hz. The initial displacement is 0.025 m and the initial velocity is $1.5\ \text{ms}^{-1}$. Calculate the amplitude, maximum velocity and maximum acceleration of the object.
4. For a horizontally placed simple harmonic oscillator, the mass, amplitude of oscillation, frequency and initial phase are 0.5 kg, 5 cm, 60 oscillations per minute and ($\pi/3$) rad, respectively. Write the expression for displacement of the oscillator at instant *t*. Also calculate the force constant and mechanical energy of the oscillator.
5. Determine the amplitude and time period of a harmonic oscillator if at distances x_1 and x_2 from the equilibrium position, its velocities are v_1 and v_2, respectively.
6. A spring-mass system executing SHM has $m = 0.5$ kg, $k = 25\ \text{Nm}^{-1}$ and its total energy is 25 mJ. Calculate (a) the amplitude of oscillation, (b) the maximum velocity of the mass, (c) the velocity of the mass when displacement is 15 mm and (d) the distance of the mass from the equilibrium position when its velocity is $0.2\ \text{ms}^{-1}$.
7. A rubber pad acts as an elastic spring. When a mass of 100 g is placed, it is compressed by 1 cm. Then the mass is gently tapped downwards. It begins to oscillate. Calculate the frequency of oscillation.

16.7 SOLUTIONS AND ANSWERS

Self-Assessment Questions

1. a) i) As per the problem, we know that springs *A* and *B* are the parts of the same original spring and spring *A* has smaller number of turns than *B*. It is so because the length of spring *A* is one-third of the original spring. Therefore, if equal force is applied on the free ends of *A* and *B*, their extensions will not be equal.

ii) Spring *B* will stretch more because its length (or number of turns) is larger than the length of spring *A*.

iii) From Eq. (16.1), we note that the spring (or force) constant *k* is inversely proportional to extension *x*. Therefore, same force will produce smaller extension in spring *A* as compared to spring *B*. Thus, spring *A* will have greater value of spring constant.

b) From Eq. (16.1), we can write the spring (or force) constant as

$$k = \frac{\text{Force}}{\text{Displacement}} = \frac{2.0\text{ N}}{5.0\times10^{-2}\text{ m}} = 40\text{ Nm}^{-1}$$

Let the compression of spring be x' when a force of magnitude 2.5 N is applied. So, from Eq. (16.1), we have

$$x' = \frac{\text{Force}}{\text{Spring constant}} = \frac{2.5\text{ N}}{40\text{ Nm}^{-1}} = 6.3\times10^{-2}\text{ m}$$

2. The differential equation for SHM is given by Eq. (16.5): $\dfrac{d^2x}{dt^2} + \omega_0^2\, x = 0$

Let us first take the solution given by Eq. (16.6b): $x(t) = a\sin(\omega_0 t + \phi)$

Differentiating it twice with respect to *t*, we get

$$\frac{dx}{dt} = \omega_0\, a\cos(\omega_0 t + \phi) \text{ and } \frac{d^2x}{dt^2} = -\,\omega_0^2\, a\sin(\omega_0 t + \phi) = -\,\omega_0^2\, x$$

So, $$\frac{d^2x}{dt^2} + \omega_0^2\, x = 0$$

Thus, Eq. (16.6b) satisfies Eq. (16.5). The solution given by Eq. (16.6c) is

$$x(t) = a\cos(\omega_0 t + \phi)$$

So, $$\frac{dx}{dt} = -\omega_0\, a\sin(\omega_0 t + \phi) \text{ and } \frac{d^2x}{dt^2} = -\,\omega_0^2\, a\cos(\omega_0 t + \phi) = = -\,\omega_0^2\, x$$

So, $$\frac{d^2x}{dt^2} + \omega_0^2\, x = 0$$

Eq. (16.6d) is $x(t) = a\sin(\omega_0 t - \phi)$

So, $$\frac{dx}{dt} = \omega_0\, a\cos(\omega_0 t - \phi) \text{ and } \frac{d^2x}{dt^2} = -\,\omega_0^2\, a\sin(\omega_0 t - \phi) = = -\,\omega_0^2\, x$$

So, $$\frac{d^2x}{dt^2} + \omega_0^2\, x = 0$$

Eq. (16.6e) is $x(t) = a\cos(\omega_0 t - \phi)$

So, $\dfrac{dx}{dt} = -\omega_0 a \sin(\omega_0 t - \phi)$ and $\dfrac{d^2x}{dt^2} = -\omega_0^2 a \cos(\omega_0 t - \phi) = = -\omega_0^2 x$

So, $$\frac{d^2x}{dt^2} + \omega_0^2 x = 0$$

3. a) From Eq. (16.14), we recall that frequency $f = \omega_0 / 2\pi$. But, we have defined the angular frequency ω_0 [see Eq. (16.5)] as

$$\omega_0 = \sqrt{k/m} \qquad \Rightarrow \qquad f = (1/2\pi)\sqrt{k/m}$$

b) As per the problem, the displacement of a simple harmonic oscillator is given by

$$x(t) = 0.4 \sin(0.1t + 0.5)$$

On comparing this equation with the standard equation of SHM

$$x(t) = a \sin(\omega_0 t + \phi)$$

we can write

i) Amplitude of oscillation $a = 0.4$ m

ii) Time period $T = \dfrac{2\pi}{\omega_0} = \dfrac{2\pi}{0.1\text{ s}^{-1}} = 62.8$ s

iii) Frequency $f = \dfrac{1}{T} = \dfrac{1}{62.8\text{ s}} = 0.01\text{ s}^{-1}$

iv) Maximum velocity $= \omega_0 a = (0.1\text{ s}^{-1}) \times (0.4\text{ m}) = 0.04\text{ ms}^{-1}$

v) Maximum acceleration $= \omega_0^2 a = (0.1\text{ s}^{-1})^2 \times (0.4\text{ m}) = 0.004\text{ ms}^{-2}$

vi) Initial displacement, x_0 is the displacement at $t = 0$. So, on substituting $t = 0$ in the given equation for $x(t)$, we get

$$x(t=0) = 0.4 \sin(0.5) = 0.2\text{ m}$$

4. a) Let the *K.E.* and *P.E.* of a simple harmonic oscillator be equal when its displacement is *x*. Then the expressions for *P.E.* and *K.E.* given by Eqs. (16.16) and (16.19), can be written as

$$P.E. = (1/2)\,kx^2 \qquad \text{(i)}$$

and $$K.E. = (1/2)\,k\,(a^2 - x^2) \qquad \text{(ii)}$$

Since *P.E.* and *K.E.* are equal, from Eqs. (i) and (ii), we have

$$a^2 - x^2 = x^2 \qquad \Rightarrow \qquad x = \pm\frac{a}{\sqrt{2}}$$

b) As per the problem, $m = 0.55$ kg; $k = 25\text{ Nm}^{-1}$

Since mass is released from rest at $x = 40$ mm, the amplitude of oscillation, $a = 40\text{ mm} = 0.04$ m

i) We need to determine *P.E.* and *K.E.* when the mass is at $x = 20\text{ mm} = 0.02$ m. From Eq. (16.16), we have

$$P.E. = (1/2)kx^2 = (1/2) \times (25\text{ Nm}^{-1}) \times (0.02\text{ m})^2 = 5 \times 10^{-3}\text{ J}$$

For *K.E.*, we note from Eq. (16.19) that $K.E. = (1/2)k\,(a^2 - x^2)$

$$\therefore \quad K.E. = (1/2) \times (25\text{ Nm}^{-1}) \times [(0.04\text{ m})^2 - (0.02\text{ m})^2] = 1.5 \times 10^{-2}\text{ J}$$

ii) We need to determine $P.E.$ and $K.E.$ at $t = T/4$ when initial phase, $\phi = 0$. From Eq. (16.17), we have $P.E. = (1/2)ka^2\cos^2(\omega_0 t)$

$$\therefore \quad P.E. = (1/2)\times(25\text{ Nm}^{-1})\times(0.04\text{ m})^2\times\cos^2\left(\frac{2\pi}{T}\times\frac{T}{4}\right) = 0$$

since $\cos(\pi/2) = 0$. From Eq. (16.18), we have

$$K.E. = (1/2)ka^2\sin^2(\omega_0 t)$$
$$= (1/2)\times(25\text{ Nm}^{-1})\times(0.04\text{ m})^2\times\sin^2\left(\frac{2\pi}{T}\times\frac{T}{4}\right) = 2\times10^{-2}\text{ J}$$

c) From Eqs. (16.17) and (16.18), we note that $P.E.$ and $K.E.$ changes sinusoidally. Since the time period of the oscillator is $(2\pi/\omega_0)$, we have to show that the time period of $P.E.$ will be $(T/2)$ or (π/ω_0). Thus, using Eq. (16.17), we can write P.E. at $t = (t + (\pi/\omega_0))$ as

$$P.E. = (t = t + \pi/\omega_0)$$
$$= (1/2)\, ka^2\cos^2[\omega_0(t + (\pi/\omega_0)) + \phi]$$
$$= (1/2)\, ka^2\cos^2(\omega_0 t + \phi)$$
$$= P.E.(t)$$

Similarly, you can prove that the time period of K.E. is (π/ω_0).

5. The $K.E.$ of a simple harmonic oscillator is given by Eq. (16.19):

$$K.E. = (1/2)k(a^2 - x^2)$$

We are given that the amplitude of oscillation, $a = 40\text{ cm} = 0.4\text{ m}$. At $x = 20\text{ cm} = 0.2\text{ m}$, we can write

$$K.E. = (1/2)k\,[(0.4\text{ m})^2 - (0.2\text{ m})^2] = 0.06\,k\,\text{m}^2 \qquad \text{(i)}$$

And, the average $K.E.$ of an oscillator is given by Eq. (16.21):

$$\langle K.E.\rangle = (1/4)ka^2 = (1/4)k(0.4\text{ m})^2 = 0.04\,k\,\text{m}^2 \qquad \text{(ii)}$$

On comparing Eqs. (i) and (ii), we discover that the $K.E.$ of the system at $x = 20\text{ cm}$ is more than its average value.

Terminal Questions

1. The expressions for displacement, velocity and acceleration of an object executing SHM are given by Eqs. (16.6b), (16.7a) and (16.8a), respectively:

$$x(t) = a\sin(\omega_0 t + \phi) \qquad \text{(i)}$$
$$v(t) = \omega_0 a\cos(\omega_0 t + \phi) \qquad \text{(ii)}$$

and
$$a_c(t) = -\omega_0^2 a\sin(\omega_0 t + \phi) \qquad \text{(iii)}$$

As given in the problem, amplitude $a = 0.065\text{ m}$, angular frequency $\omega_0 = 4.0\text{ s}^{-1}$ and initial phase $\phi = 0$. Substituting these values in Eqs. (i), (ii) and (iii), we get

$$x(t) = (0.065\text{ m})\sin[(4\text{ s}^{-1})t] \qquad \text{(iv)}$$
$$v(t) = (4\text{ s}^{-1})\times(0.065\text{ m})\cos[(4\text{ s}^{-1})t] \qquad \text{(v)}$$

and
$$a_c(t) = -(4\text{ s}^{-1})^2\times(0.065\text{ m})\sin[(4\text{ s}^{-1})t] \qquad \text{(vi)}$$

Note that ω_0 is actually in rad s^{-1} and we have to express the argument of the sine and cosine functions in degrees using the relation $\pi(\text{rad}) = 180°$. On substituting $t = 1.5\text{ s}$ in Eqs. (iv), (v) and (vi), we obtain

$$x(t = 1.5 \text{ s}) = (0.065 \text{ m}) \sin[(4 \text{rads}^{-1}) \times (1.5 \text{ s})]$$

$$= (0.065 \text{m}) \sin(343.6°) = -1.84 \times 10^{-2} \text{ m}$$

$$v(t = 1.5 \text{ s}) = (4 \text{s}^{-1}) \times (0.065 \text{ m}) \cos[(4 \text{rads}^{-1}) \times (1.5 \text{s})]$$

$$= (4 \text{s}^{-1}) \times (0.065 \text{ m}) \cos(343.6°) = 0.25 \text{ ms}^{-1}$$

$$a_c(t = 1.5 \text{ s}) = -(4 \text{s}^{-1})^2 \times (0.065 \text{ m}) \sin[(4 \text{rads}^{-1}) \times (1.5 \text{s})] = 0.29 \text{ ms}^{-2}$$

2. a) In this arrangement, both springs will be extended by the same length x and the restoring force is given by

$$F = -k_1 x - k_2 x \qquad \text{(i)}$$

Thus, the equation of motion (Eq. (16.4b)) is modified to

$$m\frac{d^2x}{dt^2} + (k_1 + k_2)x = 0 \qquad \text{(ii)}$$

The angular frequency characterising this arrangement is

$$\omega_0 = [(k_1 + k_2)/m]^{1/2}$$

Hence, the time period of oscillation for the system will be given by

$$T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{m/(k_1 + k_2)} \qquad \text{(iii)}$$

b) In this arrangement, if the mass is displaced up or down through a distance x, the restoring forces are

$$F_1 = -k_1 x \quad \text{and} \quad F_2 = -k_2 x$$

Hence, the net restoring force acting on the mass is

$$F = -k_1 x - k_2 x \qquad \text{(iv)}$$

Note that Eq. (iv) is identical to Eq. (i). Therefore, the equation of motion and time period of oscillation will be given by Eqs. (ii) and (iii), respectively.

c) In this case, the two springs are connected in series. When the mass is displaced by x, the same restoring force will be exerted by each spring. But the extensions of the springs will be different, say x_1 and x_2, because their spring constants, k_1 and k_2 are different. Since the restoring force, F is the same, we can write

$$F = -k_1 x_1 = -k_2 x_2$$

The total extension of the spring is $x = x_1 + x_2$

which can be written as

$$x = -(F/k_1) - (F/k_2) = -[(1/k_1) + (1/k_2)]F$$

so that $\quad F = -[x/\{(1/k_1) + (1/k_2)\}] = -k'x$

where $\quad k' = [1/\{(1/k_1) + (1/k_2)\}]$

is the effective spring constant of the system. Therefore, time period of oscillation for the system is given by

$$T = 2\pi\sqrt{m/k'} = 2\pi\sqrt{m[(1/k_1) + (1/k_2)]}$$

3. The velocity of an object executing SHM is given by

$$v = \omega_0 \sqrt{a^2 - x^2}$$

so that $\quad v^2 = \omega_0^2 (a^2 - x^2)$

or $\quad a = \sqrt{(v^2/\omega_0^2) + x^2} \qquad \text{(i)}$

In the given problem, $f = 0.45$ Hz; $x = 0.025$ m and $v = 1.5\ \text{ms}^{-1}$. So,

$$\omega_0 = 2\pi f = (2 \times 3.14 \times 0.45\ \text{s}^{-1}) = 2.83\,\text{s}^{-1}$$

On substituting the values of v, ω_0 and x in Eq. (i), we obtain the amplitude of oscillations:

$$a = \sqrt{\frac{(1.5\ \text{ms}^{-1})^2}{(2.83\ \text{s}^{-1})^2} + (0.025\ \text{m})^2} = 0.53\ \text{m}$$

We know that the maximum velocity v_{max} is given by

$$v_{max} = \omega_0 a = (2.83\ \text{s}^{-1}) \times (0.53\ \text{m}) = 1.5\ \text{ms}^{-1}$$

The maximum acceleration $(a_c)_{max}$ is

$$(a_c)_{max} = \omega_0^2 a = (2.83\ \text{s}^{-1})^2 \times (0.53\ \text{m}) = 4.24\ \text{ms}^{-2}$$

4. The expression for displacement of a simple harmonic oscillator at any instant t is given by Eq. (16.6b): $x(t) = a\sin(\omega_0 t + \phi)$ (i)

As per the problem, $m = 0.5$ kg, $\phi = \pi/3$ and $a = 5\,\text{cm} = 0.05$ m. Also, we have frequency $f = 60$ oscillations per minute. So, $f = \dfrac{60}{60\,\text{s}} = 1\ \text{s}^{-1}$

This gives angular frequency $\omega_0 = 2\pi f = 2\pi\ \text{s}^{-1}$. On substituting the values of a, ω_0 and ϕ in Eq. (i), we get

$$x(t) = (0.05\ \text{m})\sin[2\pi t + (\pi/3)]$$

The force constant can be obtained using the relation

$$k = \omega_0^2 m = (2\pi\ \text{s}^{-1})^2 \times (0.5\ \text{kg}) = 19.7\ \text{Nm}^{-1}$$

Total mechanical energy of the oscillator is given by Eq. (16.20):

$$E = (1/2)ka^2 = \frac{1}{2}(19.7\ \text{Nm}^{-2}) \times (0.05\text{m})^2 = 2.46 \times 10^{-2}\ \text{J}$$

5. The expression for instantaneous displacement of an oscillator executing SHM is given by $x = a\cos(\omega_0 t + \phi)$

$$\therefore \qquad v = \frac{dx}{dt} = -a\omega_0 \sin(\omega_0 t + \phi) = -a\omega_0\sqrt{1-(x^2/a^2)}$$

Hence, when displacement is x_1 and velocity is v_1, we can write

$$v_1/a\omega_0 = -\sqrt{1-(x_1^2/a^2)} \qquad \text{(i)}$$

Similarly, for displacement x_2 and velocity v_2, we can write

$$v_2/a\omega_0 = -\sqrt{1-(x_2^2/a^2)} \qquad \text{(ii)}$$

On squaring Eqs. (i) and (ii), we obtain

$$(v_1/a\omega_0)^2 = 1-(x_1^2/a^2) \qquad \text{(iii)}$$

and $$(v_2/a\omega_0)^2 = 1-(x_2^2/a^2) \qquad \text{(iv)}$$

On subtracting Eq. (iv) from Eq. (iii), we get

$$[v_1^2 - v_2^2]/(a\omega_0)^2 = (x_2^2 - x_1^2)/a^2$$

This expression may be simplified to obtain $\omega_0 = \sqrt{(v_1^2 - v_2^2)/(x_2^2 - x_1^2)}$

On using this value of ω_0 in Eq. (i), you can convince yourself that amplitude of oscillations is given by

$$a = \sqrt{(v_1^2 x_2^2 - v_2^2 x_1^2)/(v_1^2 - v_2^2)}$$

6. We are given that $m = 0.5$ kg, $k = 25$ Nm^{-1} and $E = 25 \times 10^{-3}$ J.

   a) To determine the amplitude of oscillation, we use the expression for total energy of the oscillator (Eq. (16.20)):

   $$E = (1/2)ka^2 \Rightarrow a = \sqrt{2E/k} = \sqrt{2 \times (25 \times 10^{-3}\text{ J})/25\text{ Nm}^{-1}} = 0.044\text{ m}$$

   b) From Eq. (16.7a), we note that the maximum velocity of the oscillator is

   $$v_{max} = \omega_0 a = (\sqrt{k/m})\, a = \sqrt{\frac{(25\text{Nm}^{-1})}{0.5\text{ kg}}} \times (0.044\text{ m}) = 0.014\text{ ms}^{-1}$$

   c) The velocity and displacement of an oscillator is related by Eq. (16.7b):

   $$v = \omega_0 \sqrt{a^2 - x^2} = \sqrt{k/m}\,.\,\sqrt{a^2 - x^2}$$

   So, at $x = 15$ mm $= 0.015$ m, the velocity of the mass is given by

   $$v = \sqrt{\frac{25\text{ Nm}^{-1}}{0.5\text{ kg}}} \times \sqrt{(0.044\text{ m})^2 - (0.015\text{ m})^2} = 0.29\text{ ms}^{-1}$$

   d) We use Eq. (16.7b) to obtain displacement when velocity is 0.2 ms^{-1}:

   $$v^2 = \omega_0^2\,(a^2 - x^2) \qquad \Rightarrow \qquad x^2 = a^2 - \frac{v^2}{\omega_0^2}$$

   $$\text{or } x^2 = (0.044\text{ m})^2 - \frac{(0.2\text{ ms}^{-1})^2 \times 0.5\text{ kg}}{25\text{ Nm}^{-1}} = 0.001\text{m}^2 \Rightarrow x = 0.031\text{ m}$$

7. Since $m = 100$ g, the deforming force acting on the rubber pad is given by

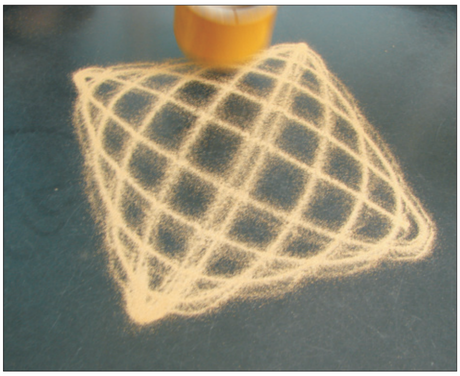
   $$F = mg = (0.1\text{ kg}) \times (9.8\text{ ms}^{-2}) = 0.98\text{ N}$$

   From Newton's 3rd law of motion, we can say that the deforming force is equal in magnitude to the restoring force exerted by the rubber pad. Since compression is $1\text{ cm} = 0.01\text{ m}$, the force constant of the rubber pad is

   $$k = (0.98\text{ N})/(0.01\text{ m}) = 98\text{ Nm}^{-1}$$

   The frequency of oscillation is given by

   $$f = \omega_0/2\pi = (1/2\pi)\sqrt{k/m} = (1/2\pi)\sqrt{(98\text{ Nm}^{-1})/(0.1\text{kg})} = 4.98\text{ Hz}$$



This is a photograph of Lissajous figures formed when a box full of sand with a hole in its bottom is hung from a string and swung freely. Sand flowing out from the hole forms the above pattern on a surface below the box. This family of curves was investigated in detail by Jules Antoine Lissajous (1822–1880), French mathematician, in 1857 and hence they are known as Lissajous figures.
(Source of picture: commons.wikimedia.org)

UNIT 17

SUPERPOSITION OF HARMONIC OSCILLATIONS

Structure

STUDY GUIDE

The central concept of this unit is the principle of superposition which helps us analyse a variety of physical phenomena. The major portion of this unit deals with the applications of the superposition principle to determine the nature of the resultant motion of a body when two harmonic oscillations act on it simultaneously. To do so, we use algebra involving trigonometric identities. You should, therefore, refresh your knowledge of school level trigonometry and coordinate geometry. Moreover, instead of getting overwhelmed by the algebra and losing track of the underlying physics, you should try to make sense of the expression for the resultant displacement in each case. We hope you will be able to appreciate the physical situation described by these expressions better if you devote enough time in examining the figures containing graphs more carefully.

"Study without desire spoils the memory, and it retains nothing that it takes in."

Leonardo da Vinci

17.1 INTRODUCTION

In Unit 16, you have studied the basic concepts of simple harmonic motion (SHM). You have learnt that a body is said to execute SHM if its acceleration is proportional to displacement and is directed opposite to the displacement. For a spring-mass system, we obtained the equation of motion of the mass by considering the forces acting on it. Further, the solutions of the equation of motion led to the expression for displacement of the mass as a function of time.

As you now know, SHM is an idealised model of oscillatory motion. The oscillations or vibrations in actual physical systems, such as the vibrations of the strings of musical instruments and the diaphragm in our eardrums, are rather complex when they are subjected to more than one oscillation at the same time. **So, you may logically ask: How do we determine the resultant motion of a body when it is subjected to a number of oscillations simultaneously?** To do so, we use the principle of superposition. The analysis of the resultant motion in such situations is simplified considerably if we assume that the simultaneously acting oscillations are simple harmonic. As mentioned in the previous unit, one of the advantages of the idealised model of SHM is that we can represent a complex oscillation by combining two or more simple harmonic oscillations of appropriate amplitudes and frequencies. Conversely, it also means that when two or more harmonic oscillations act on a body at the same time, the motion is likely to be quite complex. The study of such motions forms the subject matter of this unit.

In Sec. 17.2, you will learn the principle of superposition. You would agree that the physical parameters such as amplitude, frequency and phase of the superposing oscillations may be same or different. These factors determine the nature of the resultant motion of the body. In Sec. 17.3, you will learn how to apply the superposition principle to analyse the resultant motion when two **collinear** (that is, along the same line) oscillations of the same frequency and different frequencies are superposed. In Sec. 17.4, we have discussed the resultant motion of a body when two **mutually perpendicular** oscillations act on it simultaneously. You will learn how to obtain expressions for the trajectories of the resultant motion, known as Lissajous figures.

Expected Learning Outcomes

After studying this unit, you should be able to:

- State the principle of superposition;
- Explain the conditions under which the principle of superposition can be used for two or more oscillations;
- Apply the principle of superposition to analyse the motion of a body on which two collinear harmonic oscillations of the same frequency and different frequencies act simultaneously;
- Apply the principle of superposition to determine the resultant motion of a body on which two mutually perpendicular harmonic oscillations act simultaneously; and
- Explain the formation of Lissajous figures.

17.2 PRINCIPLE OF SUPERPOSITION

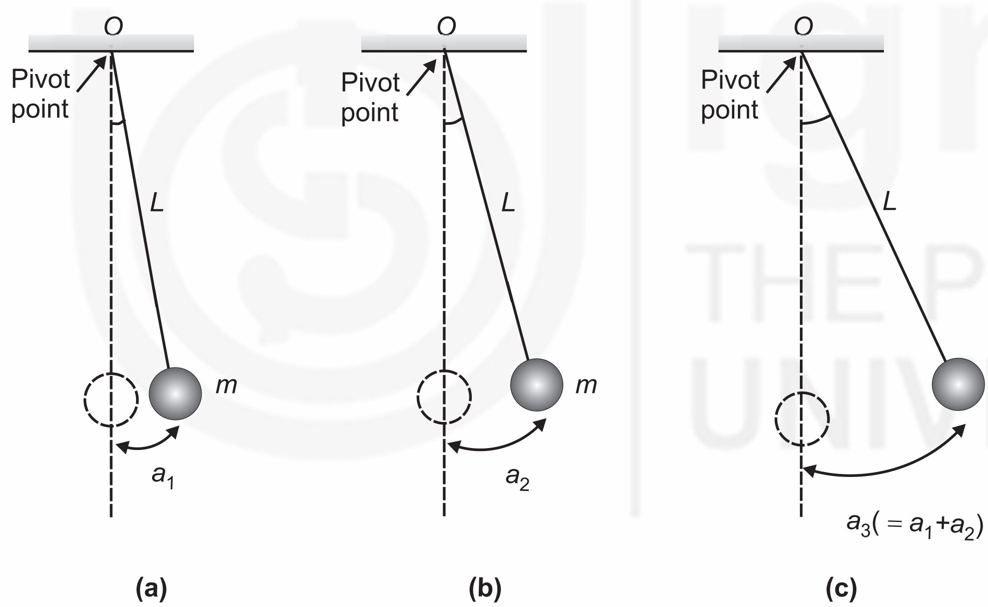
In Unit 16, you have learnt that an oscillatory motion is simple harmonic, if the force acting on it is directly proportional to its displacement and is directed opposite to it. For example, the restoring force, F $(= -kx)$ of the spring acting on the mass in a spring-mass system gives rise to SHM. From your school physics classes, you may recall that in the case of a simple pendulum, the restoring force giving rise to the oscillatory motion of the pendulum is provided by the tangential component of the weight of the bob.

NOTE

The principal of superposition is a general principle valid for a variety of physical phenomena studied in different branches of physics. These include mechanics, electricity and magnetism, optics and quantum mechanics. We will, however confine ourselves to the use of this principle here for mechanical systems to analyse superposition of oscillations.

We come across many physical situations where a body is simultaneously subjected to two or more harmonic forces (or harmonic oscillations). For example, in a market place, sound waves of different amplitudes and frequencies are incident on our ear diaphragm simultaneously. These tend to displace our ear diaphragm individually in different directions. In such a situation, the resultant motion of the diaphragm can be determined by using the superposition principle. For the given situation, the superposition principle implies that *at any given time, the displacement of the diaphragm will be equal to the sum of the displacements due to individual harmonic oscillations*.

To elaborate the meaning of this statement, let us consider a few experiments with a simple pendulum executing SHM under small angle approximation.



It is only when the value of the angle of oscillation (θ) of a simple pendulum is such that $\sin\theta \sim \theta$, the pendulum is said to execute simple harmonic oscillation. This condition is called small angle approximation.

Fig. 17.1: Simple pendulum with different initial conditions: a) initial displacement = a_1**, initial velocity** = 0**; b) initial displacement** = a_2**, initial velocity** = 0; **c) initial displacement** = a_3 (= a_1 + a_2)**, initial velocity =** 0**.**

Suppose that the bob of a simple pendulum is constrained to oscillate in a plane and its initial displacement is a_1 (Fig. 17.1a). So, the **initial conditions** on the displacement and velocity of the bob (at $t = 0$) are: displacement $x = a_1$ and velocity $v = 0$. Let the displacement of the bob measured at a subsequent time t_1 be x_1.

Let us repeat the experiment with **another set of initial conditions**: initial displacement $x = a_2$, and initial velocity $v = 0$, as shown in Fig. 17.1b. When the bob oscillates under these conditions, let the displacement of the bob measured after the same interval of time, t_1 be x_2.

Now, let us do this experiment once again with initial conditions which are the sum (or the superposition) of the initial conditions of the above mentioned two experiments. That is, the initial displacement, a_3 of the bob is the sum of the initial displacements, a_1 and a_2, that is, $a_3 = (a_1 + a_2)$ and initial velocity, $v = 0$ as shown in Fig. 17.1c. **It is assumed here that, with this initial condition, the pendulum oscillates under the combined influence of two oscillations – represented by the oscillations of the pendulums in the above two experiments – simultaneously.** According to the principle of superposition, the displacement, x_3 of the bob after the same interval of time t_1 should be equal to $(x_1 + x_2)$. In other words, the resultant displacement x_3 is the sum of the individual displacements x_1 and x_2.

We can, therefore, state the principle of superposition as follows:

Don't forget

The resultant displacement due to superposition of two (or more) harmonic oscillations is the algebraic sum of individual displacements at all subsequent times.

As such, the principle of superposition has universal validity. We can generalise it as: **The net effect is the sum of individual effects**. Before proceeding further, let us recall the important points discussed in this section.

Recap

- When two or more harmonic oscillations act on a body simultaneously, the resultant motion of the body can be determined using the principle of superposition.
- According to the principle of superposition, the resultant displacement is the algebraic sum of the individual displacements. If $x_1(t)$ and $x_2(t)$ are the individual displacements of the superposing oscillations, the resultant displacement, $x(t)$ is given by

$$x(t) = x_1(t) + x_2(t)$$

Linearity and Superposition Principle

As mentioned above, the principle of superposition is a general principle in the sense that it is valid for a variety of phenomena observed in different branches of physics. **But, it is important to note that it is valid only for those phenomena which can be described by linear homogeneous ordinary differential equations.** To elaborate the above statement, let us revisit the equation of the motion of a spring-mass system given by Eq. (16.5):

$$\frac{d^2x}{dt^2} + \omega_0^2 x = 0 \tag{17.1}$$

Recall from Unit 4 that Eq. (17.1) is a linear homogeneous ODE. It does not contain any non-zero term which is higher than the first power of the variable x, or of its derivatives $\frac{dx}{dt}, \frac{d^2x}{dt^2}$, etc. You know that such a differential equation which contains only first power terms of the variable or its derivatives (such as $\frac{dx}{dt}, \frac{d^2x}{dt^2}$ etc.) is called a **linear differential equation**. (If a differential equation contain term(s) with higher powers of the variable or their derivatives, the equation is said to be nonlinear.)

Further, this differential equation does not contain any non-zero term independent of the variable (which, in the present case is displacement, x). So, it is a **homogeneous differential** equation. Thus, we find that the equation of motion of a spring-mass system - a simple harmonic oscillator - is a linear homogeneous ODE.

At this stage, you may ask: What is the theoretical basis of the fact that the superposition principle can be used only when the motion of a system can be described by a linear homogeneous ordinary differential equation? The theoretical basis is provided by a property of linear homogeneous ordinary differential equations, according to which the sum of any two solutions of a linear homogeneous ODE is also a solution.

To understand this property, let us consider Eq. (17.1) which gives the equation of motion of a spring-mass system. Note that it is a second order, linear, homogeneous ODE. Now, recall from Sec. 4.2 of Unit 4 that such a differential equation will have two linearly independent solutions. **According to one of the properties of linear ODEs, the sum of its two linearly independent solutions is also a solution.** To elaborate the meaning of this property, let $x_1(t)$ and $x_2(t)$ be two different solutions of Eq. (17.1). Therefore, $x_1(t)$ and $x_2(t)$ will satisfy Eq. (17.1) and we can write

$$\frac{d^2x_1}{dt^2} = -\omega_0^2\, x_1 \tag{17.2}$$

and

$$\frac{d^2x_2}{dt^2} = -\omega_0^2\, x_2 \tag{17.3}$$

On adding Eqs. (17.2) and (17.3), we get

$$\frac{d^2(x_1 + x_2)}{dt^2} = -\omega_0^2 (x_1 + x_2) \tag{17.4}$$

Eq. (17.4) shows that $x_1(t) + x_2(t)$ is also a solution of Eq. (17.1). So, we find that if $x_1(t)$ and $x_2(t)$ are two different solutions of Eq. (17.1), their sum, $\{x_1(t) + x_2(t)\}$ is also its solution. So, we can say that, according to the superposition principle, if two harmonic oscillations, described by displacements $x_1(t)$ and $x_2(t)$, act on a body simultaneously, the resultant motion described by $x(t)$ is given by:

$$x(t) = x_1(t) + x_2(t) \tag{17.5}$$

You should understand the relation between linearity of the equation of motion of harmonic oscillators and the principle of superposition. We can now discuss the use of superposition principle to determine the resultant motion of an object on which more than one harmonic oscillations act at the same time.

When two harmonic oscillations act on a body simultaneously, we expect its resultant motion to depend on the amplitudes, frequencies, and phases of the superposing oscillations. In addition, it also depends on whether the superposing oscillations are collinear or perpendicular to each other. For simplicity, we begin our discussion by first considering the superposition of two collinear oscillations.

17.3 SUPERPOSITION OF TWO COLLINEAR HARMONIC OSCILLATIONS

When two collinear oscillations act on a body, each of them will displace it along the same line. Therefore, you should expect that the resultant motion will also be along the same line. But, the amplitudes, frequencies and phases of these oscillations can influence the resultant motion differently. Here we consider a few typical cases, in increasing order of complexity: oscillations having same frequency but different amplitudes, and oscillations having unequal frequencies and unequal amplitudes.

17.3.1 Collinear Oscillations of Same Frequency

You may recall from Section 16.3.1 that, if a body executes SHM, its displacement, x (t) at time t can be expressed as

$$x(t) = a\cos(\omega_0 t + \phi) \tag{17.6}$$

where a, ω_0 and ϕ respectively denote the amplitude, frequency and initial phase of the oscillation. We wish to consider superposition of two such harmonic oscillations whose frequencies are same but their amplitudes are unequal. **Let us further assume that the initial phase difference between these two oscillations is π.** Such oscillations are said to be in opposite phase. Under these assumptions, we can write the expressions for the displacements, $x_1(t)$ and $x_2(t)$ for the two collinear superposing oscillations at time t as

$$x_1(t) = a_1 \cos \omega_0 t \tag{17.7}$$

and

$$x_2(t) = a_2 \cos(\omega_0 t + \pi) = -a_2 \cos \omega_0 t \tag{17.8}$$

Now, according to the principle of superposition, we can write the resultant displacement of the body at a given time t as

$$x(t) = x_1(t) + x_2(t)$$

On substituting the values of $x_1(t)$ and $x_2(t)$ from Eqs. (17.7) and (17.8), we get

$$x(t) = a_1 \cos \omega_0 t - a_2 \cos \omega_0 t = (a_1 - a_2) \cos \omega_0 t \tag{17.9}$$

Eq. (17.9) gives the displacement-time relation for the resultant motion of the body. On comparing Eqs. (17.9) with Eq. (17.6), we can say that when two collinear oscillations with initial phase difference π act on a body simultaneously, it will execute SHM with amplitude $(a_1 - a_2)$ and zero initial phase. **Eq. (17.9) also shows that if the amplitudes of superposing oscillations were equal, the resultant displacement will be zero at all times.**

Displacement-time graphs for these oscillations are shown in Fig. 17.2. While Figs. 17.2a and b depict the displacement-time graphs for individual superposing oscillations $x_1(t)$ and $x_2(t)$, respectively, Fig. 17.2c depicts the displacement-time graph for the resultant oscillation (Eq. (17.9)). You may note that displacement of the resultant oscillation at any instant of time is obtained by algebraic addition of the displacements of individual superposing oscillations. You should convince yourself about the validity of this statement by closely examining Figs. 17.2a, b and c at different values of time, *t*.

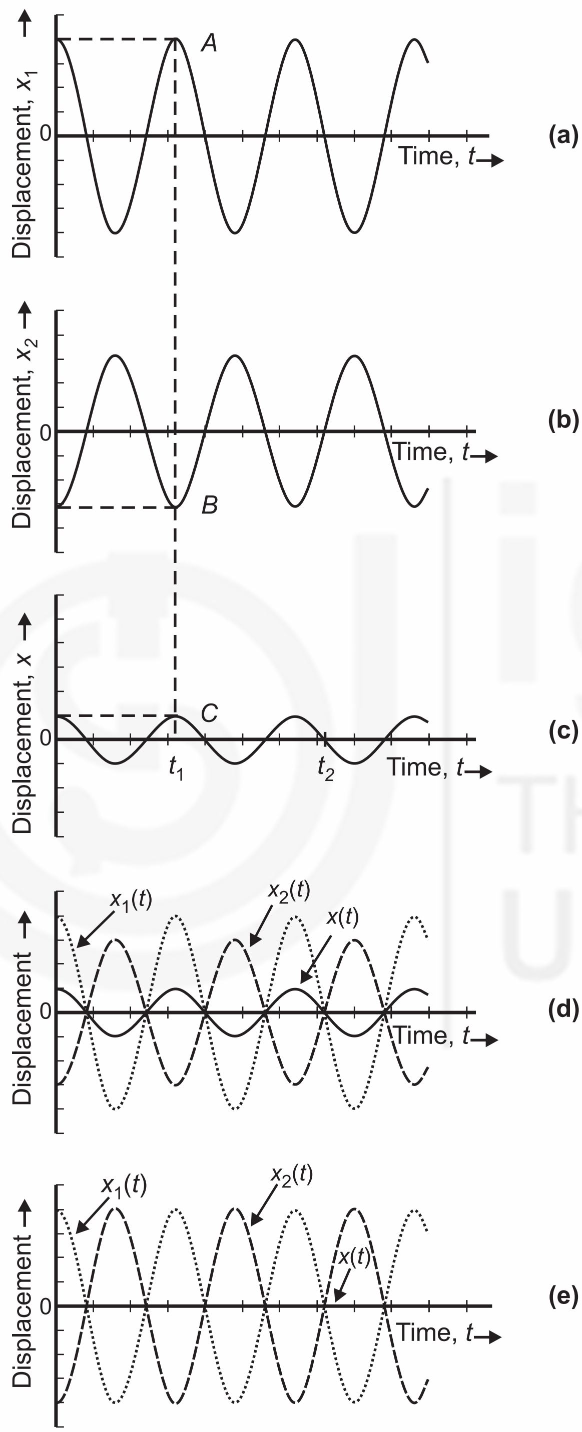


Fig. 17.2: Plots of displacement-time graph for two collinear oscillations and their superposition under different conditions: a) $x_1(t)$; b) $x_2(t)$; c) the resultant displacement, $x(t)$ due to their superposition when $a_1 \neq a_2$ and initial phase difference is π; d) $x_1(t)$, $x_2(t)$ and $x(t)$ together for the case $a_1 \neq a_2$ and initial phase difference π; e) $x_1(t)$, $x_2(t)$ and $x(t)$ together for the case $a_1 = a_2$ and initial phase difference π.

Fig. 17.2d represents all the three displacement-time graphs viz. $x_1(t)$, $x_2(t)$ and $x(t)$ on the same graph. Further, Fig. 17.2e depicts the superposing oscillations $x_1(t)$ and $x_2(t)$ as well as the resultant oscillation, $x(t)$ for the case $a_1 = a_2$. Note that, in this case, the resultant displacement is along the x-axis (time-axis) which means that the resultant displacement of the oscillator is zero at all times.

In the above discussion, we considered superposition of two collinear oscillations in opposite phases. Let us now consider superposition of two collinear in-phase oscillations of different amplitudes. (You may recall that for in-phase oscillations, the initial phase difference is defined as $\phi = 2n\pi;\ n = 0, 1, 2, ...$). In the following example, we have discussed this case.

EXAMPLE 17.1 : SUPERPOSITION OF TWO COLLINEAR IN-PHASE OSCILLATIONS

Two collinear in-phase harmonic oscillations of amplitudes a_1 and a_2 have the same frequency, ω_0. Show that their superposition gives rise to a harmonic oscillation of amplitude $|a_1 + a_2|$.

SOLUTION ■ We can represent two in-phase collinear harmonic oscillations having different amplitudes but same frequency as

$$x_1(t) = a_1 \cos\omega_0 t$$

and

$$x_2(t) = a_2 \cos\omega_0 t$$

According to the principle of superposition, the displacement of the resultant oscillation is given by

$$x(t) = x_1(t) + x_2(t) = (a_1 + a_2)\cos\omega_0 t$$

This result shows that the resultant displacement is sinusoidal in time. It means that superposition of two-in-phase harmonic oscillations gives rise to a harmonic oscillation. Further, we know that the cosine function varies between + 1 and – 1. So, the amplitude of the resultant oscillation can be expressed as $|a_1 + a_2|$.

Let us now consider superposition of two collinear harmonic oscillations **having same frequency but different amplitudes and different initial phases**. *Note that this is the most general case of superposition of two collinear harmonic oscillations of same frequency because we have taken arbitrary values of their amplitudes and initial phases.* Suppose that the first oscillation is characterised by amplitude a_1 and initial phase ϕ_1 and the second oscillation is characterised by amplitude a_2 and initial phase ϕ_2. Let the angular frequency of these collinear oscillations be ω_0. Then, we can write the displacements, $x_1(t)$ and $x_2(t)$ as

$$x_1(t) = a_1 \cos(\omega_0 t + \phi_1) \tag{17.10}$$

and

$$x_2(t) = a_2 \cos(\omega_0 t + \phi_2) \tag{17.11}$$

From the principle of superposition, we can write

$$\begin{aligned} x(t) &= x_1(t) + x_2(t) \\ &= a_1 \cos(\omega_0 t + \phi_1) + a_2 \cos(\omega_0 t + \phi_2) \end{aligned}$$

Using the cosine formula for the sum of two angles (see margin remark), we get

$$x(t) = a_1 \cos\omega_0 t \cos\phi_1 - a_1 \sin\omega_0 t \sin\phi_1 + a_2 \cos\omega_0 t \cos\phi_2 - a_2 \sin\omega_0 t \sin\phi_2$$

$\cos(A + B) = \cos A \cos B - \sin A \sin B$

$\cos(A - B) = \cos A \cos B + \sin A \sin B$

On collecting the coefficients of $\cos\omega_0 t$ and $\sin\omega_0 t$, we obtain

$$x(t) = (a_1 \cos\phi_1 + a_2 \cos\phi_2)\cos\omega_0 t - (a_1 \sin\phi_1 + a_2 \sin\phi_2)\sin\omega_0 t \quad (17.12)$$

Since a_1, a_2, ϕ_1 and ϕ_2 are constant, we express the terms containing these constants by introducing two new constants, say a and δ, by defining

$$a_1 \cos\phi_1 + a_2 \cos\phi_2 = a\cos\delta \quad (17.13)$$

and

$$a_1 \sin\phi_1 + a_2 \sin\phi_2 = a\sin\delta \quad (17.14)$$

We combine Eqs. (17.12), (17.13) and (17.14) to obtain the desired expression for resultant oscillation:

$$x(t) = a\cos\delta\cos\omega_0 t - a\sin\delta\sin\omega_0 t = a\cos(\omega_0 t + \delta) \quad (17.15)$$

We can express constants a and δ in terms of the amplitudes and initial phases of superposing oscillations. You will learn about it in a moment.

Note that Eq. (17.15) is similar to Eq. (17.10) or Eq. (17.11). Moreover, the resultant oscillation has the same frequency but different amplitude and initial phase compared to the superposing oscillations. **Thus, we conclude that the resultant motion is simple harmonic but its amplitude and initial phase are different from those of the superposing oscillations.** The displacement-time graphs of the individual superposing oscillations, $x_1(t)$ and $x_2(t)$ and the resultant oscillation, $x(t)$ are shown in Fig. 17.3.

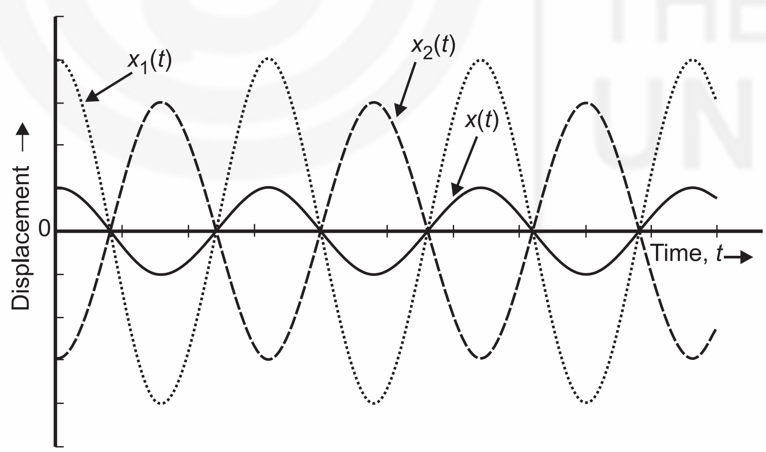


Fig. 17.3: Graphical depiction of superposition of two collinear harmonic oscillations of same frequency but different amplitudes and initial phases. (For ease in comprehension, we have taken the phase difference as π.)

We now express the amplitude and phase of the resultant oscillation in terms of amplitudes, a_1 and a_2, and initial phases ϕ_1 and ϕ_2 of the superposing oscillations. To do so, we take squares of Eqs. (17.13) and (17.14) and add the resultant expressions. On simplification, we get

$$a = \sqrt{a_1^2 + a_2^2 + 2a_1 a_2 \cos(\phi_1 - \phi_2)} \quad (17.16)$$

For $a_1 = a_2$, we can write Eq. (17.16) as

$a^2 = 2a_1^2 + 2a_1^2 \cos\phi$
$= 2a_1^2 (1 + \cos\phi)$

(Contd.)

where $\phi = \phi_1 - \phi_2$ denotes the phase difference between the superposing oscillations. Using the relation

$\cos 2\phi = 2\cos^2\phi - 1$

the expression for a^2 simplifies to

$a^2 = 2a_1^2 \times 2\cos^2\left(\frac{\phi}{2}\right)$

$= 4a_1^2 \cos^2\left(\frac{\phi}{2}\right)$

Hence $a = 2a_1 \cos\left(\frac{\phi}{2}\right)$

Further, to obtain expression for phase, δ of the resultant oscillation, we divide Eq. (17.14) by Eq. (17.13):

$$\delta = \tan^{-1}\left[\frac{a_1 \sin\phi_1 + a_2 \sin\phi_2}{a_1 \cos\phi_1 + a_2 \cos\phi_2}\right] \tag{17.17}$$

For the special case when $a_1 = a_2$, that is, when the amplitudes of superposed oscillations are equal ($a_1 = a_2$), Eq. (17.16) simplifies to (see the margin remark for the derivation):

$$a = 2a_1 \cos\left(\frac{\phi}{2}\right) \tag{17.18}$$

where $\phi = \phi_1 - \phi_2$. Here ϕ denotes the difference between the initial phases of superposing oscillations.

We now summarise the important results obtained in this section.

Recap

- If superposing collinear harmonic oscillations are of the same frequency, the resultant motion is always simple harmonic.
- When two collinear harmonic oscillations of same frequency, ω_0 but different amplitudes, a_1 and a_2 and initial phases, ϕ_1 and ϕ_2 are superposed, the resultant oscillation is simple harmonic along the same line and its amplitude, a and phase, δ are given by

$$a = \sqrt{a_1^2 + a_2^2 + 2a_1 a_2 \cos(\phi_1 - \phi_2)}$$

and $$\delta = \tan^{-1}\left(\frac{a_1 \sin\phi_1 + a_2 \sin\phi_2}{a_1 \cos\phi_1 + a_2 \cos\phi_2}\right)$$

We hope that now you can derive expressions for the displacement and phase of the resultant oscillation obtained by superposition of two collinear harmonic oscillations of same frequency but having different amplitudes and initial phases. You must have noted that the nature of the resultant motion is determined by the interplay of the amplitudes and initial phases of the superposing oscillations.

In the following example, we show how the phase difference of superposing oscillations influences the amplitude of the resultant oscillation.

EXAMPLE 17.2 : SUPERPOSITION OF TWO HARMONIC OSCILLATIONS OF SAME FREQUENCY BUT ARBITRARY AMPLITUDES AND INITIAL PHASES

Two collinear harmonic oscillations, each of frequency ω_0, have amplitudes, a_1 and a_2 and initial phases, ϕ_1 and ϕ_2. Show that, when these oscillations are superposed, the amplitude of the resultant motion is equal to $(a_1 + a_2)$ if their phase difference $(\phi_1 - \phi_2)$ is $2n\pi$, where n is an integer. What will be the value of the resultant amplitude when $(\phi_1 - \phi_2) = (2n + 1)\pi$?

SOLUTION ■ From Eq. (17.16), we recall that amplitude of the resultant motion arising due to superposition of two harmonic oscillations having amplitudes, a_1 and a_2 and initial phases, ϕ_1 and ϕ_2 is given by

$$a^2 = a_1^2 + a_2^2 + 2a_1 a_2 \cos(\phi_1 - \phi_2) \quad \text{(i)}$$

When $(\phi_1 - \phi_2) = 2n\pi$, the value of cosine function is unity, i.e., $\cos(\phi_1 - \phi_2) = 1$. Therefore, Eq. (i) reduces to

$$a^2 = a_1^2 + a_2^2 + 2a_1 a_2 = (a_1 + a_2)^2$$

so that $$a = \pm(a_1 + a_2) = (a_1 + a_2) \quad \text{(ii)}$$

The negative sign has been dropped as it will lead to physically unrealistic result.

For $(\phi_1 - \phi_2) = (2n+1)\pi$, we have $\cos(\phi_1 - \phi_2) = -1$ and Eq. (i) reduces to

$$a^2 = a_1^2 + a_2^2 - 2a_1 a_2 = (a_1 - a_2)^2$$

so that $$a = |a_1 - a_2| \quad \text{(iii)}$$

From the results contained in Eqs. (ii) and (iii), we may conclude that the phase difference between superposing oscillations plays an important role in determining amplitude of the resultant motion.

To check your understandingof the concepts discussed above, you should solve an SAQ.

SAQ 1 **– Superposition of harmonic oscillations**

Two harmonic oscillations, each of frequency ω_0 but amplitudes 5 cm and 3 cm act on a body simultaneously along the same direction. If the initial phase difference between these oscillations is $(\pi/2)$, calculate the amplitude and the phase of the resultant oscillation.

In practice, we have to deal with harmonic oscillations of different frequencies. For instance, sounds of different frequencies incident simultaneously on our eardrums in a gathering make it to vibrate in a complex manner. So, you may like to know: Will the conclusions drawn above hold even if the frequencies of the two superposing collinear oscillations are not equal? Let us discover answer to this question now.

17.3.2 Collinear Oscillations of Different Frequencies

Let us consider superposition of two harmonic oscillations having amplitudes a_1 and a_2 and angular frequencies ω_1 and ω_2 respectively such that $\omega_1 > \omega_2$. Let us represent these oscillations as

$$x_1(t) = a_1 \cos(\omega_1 t + \phi_1) \quad \text{(17.19a)}$$

and $$x_2(t) = a_2 \cos(\omega_2 t + \phi_2) \quad \text{(17.19b)}$$

The phase difference between these two oscillations is given by

$$\phi = (\omega_1 - \omega_2)t + (\phi_1 - \phi_2) \quad \text{(17.19c)}$$

Note that the first term on right hand side of Eq. (17.19c) changes continuously with time whereas the second term is constant in time. Since we are interested to know the evolution of resultant oscillation with time, initial phase difference, $(\phi_1 = \phi_2)$ will not play any role. Therefore, for convenience, we may take the initial phases of these two oscillations to be zero, i.e. $\phi_1 = \phi_2 = 0.$ Thus, Eqs. (17.19a) and (17.19b) reduces to

$$x_1(t) = a_1 \cos \omega_1 t \tag{17.20a}$$

and

$$x_2(t) = a_2 \cos \omega_2 t \tag{17.20b}$$

If these two collinear oscillations are superposed, we can write the expression for the displacement of the resultant oscillation, $x(t)$ using the principle of superposition:

$$x(t) = x_1(t) + x_2(t)$$
$$= a_1 \cos\omega_1 t + a_2 \cos\omega_2 t \tag{17.21}$$

To express Eq. (17.21) in a physically more meaningful form, we introduce two new terms, namely, **average angular frequency,** ω_a, defined as

$$\omega_a = \frac{\omega_1 + \omega_2}{2} \tag{17.22}$$

and **angular frequency of modulation**, ω_m as

$$\omega_m = \frac{\omega_1 - \omega_2}{2} \tag{17.23}$$

So, with the help of Eqs. (17.22) and (17.23), we can express ω_1 and ω_2 in terms of ω_a and ω_m as

$$\omega_1 = \omega_a + \omega_m \tag{17.24}$$

and

$$\omega_2 = \omega_a - \omega_m \tag{17.25}$$

On substituting Eqs. (17.24) and (17.25) in Eq. (17.21), we get

$$x(t) = a_1 \cos(\omega_a + \omega_m)t + a_2 \cos(\omega_a - \omega_m)t$$
$$= a_1 [\cos \omega_a t \cos\omega_m t - \sin\omega_a t \sin\omega_m t]$$
$$+ a_2 [\cos\omega_a t \cos\omega_m t + \sin\omega_a t \sin\omega_m t]$$

On collecting coefficients of $\cos \omega_a t \cos\omega_m t$ and $\sin \omega_a t \sin\omega_m t$, we get

$$x(t) = (a_1 + a_2)\cos\omega_a t \cos\omega_m t - (a_1 - a_2)\sin\omega_a t \sin\omega_m t \tag{17.26}$$

Let us now make the substitutions:

$$(a_1 + a_2) \cos\omega_m t = a_m \cos\theta_m \tag{17.27a}$$

and

$$(a_1 - a_2) \sin\omega_m t = a_m \sin\theta_m \tag{17.27b}$$

where a_m and θ_m are the constants to be determined.

On making these substitutions, we can write Eq. (17.26) as

$$x(t) = a_m \cos\omega_a t \cos\theta_m - a_m \sin\omega_a t \sin\theta_m$$
$$= a_m \cos(\omega_a t + \theta_m) \tag{17.28}$$

Eq. (17.28) is the expression for the resultant oscillation in terms of modulated amplitude a_m, and phase constant θ_m. To obtain the expressions for a_m in terms of the known quantities a_1, a_2, ω_1 and ω_2, we use Eqs. (17.27a) and (17.27b) and write:

$$a_m^2 (\cos^2 \theta_m + \sin^2 \theta_m) = (a_1 + a_2)^2 \cos^2 \omega_m t + (a_1 - a_2)^2 \sin^2 \omega_m t$$

$$= (a_1^2 + a_2^2 + 2a_1 a_2)\cos^2 \omega_m t + (a_1^2 + a_2^2 - 2a_1 a_2)\sin^2 \omega_m t$$

$$= (a_1^2 + a_2^2) + 2a_1\, a_2 (\cos^2 \omega_m t - \sin^2 \omega_m t)$$

$$= a_1^2 + a_2^2 + 2a_1\, a_2\, \cos 2\, \omega_m t$$

Hence

$$a_m = [a_1^2 + a_2^2 + 2a_1\, a_2\, \cos 2\omega_m t]^{1/2} \qquad (17.29)$$

Again, using Eqs. (17.27a) and (17.27b) we can write:

$$\tan\theta_m = \frac{(a_1 - a_2)\sin\omega_m t}{(a_1 + a_2)\cos\omega_m t}$$

or
$$\theta_m = \tan^{-1}\left[\frac{(a_1 - a_2)\sin\omega_m t}{(a_1 + a_2)\cos\omega_m t}\right] \qquad (17.30)$$

We now discuss some important conclusions that we can draw from these results:

- If the amplitudes of the superposing oscillations are equal, that is, for $a_1 = a_2$ $(= a, \text{say})$ the numerator in Eq. (17.30) vanishes giving $\theta_m = 0$.

  Further, on using the relation $\cos 2\theta = 2\cos^2\theta - 1$, Eq. (17.29) reduces to

  $$a_m(t) = [a^2 + a^2 + 2a^2(2\cos^2 \omega_m t - 1)]^{1/2}$$
  $$= 2a\cos\omega_m t \qquad (17.31)$$

  Substituting the value of $\theta_m (= 0)$ in Eq. (17.28), we get

  $$x(t) = a_m \cos\omega_a t \qquad (17.32)$$

  where

  $$a_m = 2a\cos\omega_m t$$

  So, for a simpler case of the superposition of collinear oscillations of **equal amplitudes** and different frequencies, the resultant oscillation is given by Eq. (17.32) with modulated amplitude given by Eq. (17.31).

- You may note that Eq. (17.28) resembles the expression for displacement of a body executing SHM. But, this resemblance is misleading. The resultant motion represented by Eq. (17.28) is not harmonic because the amplitude of motion varies with time according to Eq. (17.29). However, the resultant oscillation will be periodic, that is, $x(t) = x(t+T)$, for some specific values of time period, T for which $\omega_1 T = 2\pi n_1$ and $\omega_2 T = 2\pi n_2$ where n_1 and n_2 are integers. This means that the resultant oscillation will be periodic when

  $$T = \frac{2\pi}{\omega_1} n_1 = n_1 T_1$$

  and
  $$T = \frac{2\pi}{\omega_2} n_2 = n_2 T_2$$

  Thus, the condition for resultant oscillation to be periodic is

  $$n_1 T_1 = n_2 T_2$$

  or
  $$n_1\omega_2 = n_2\omega_1 \qquad (17.33)$$

- When the frequencies of the superposing oscillations are nearly equal, that is, $\omega_1 \approx \omega_2$, Eqs. (17.22) and (17.23) show that $\omega_m << \omega_a$. This means that modulation (or change) in amplitude of the resultant motion is very slow as compared to the average frequency ω_a of oscillation. Thus, the resultant amplitude may be considered as constant over the time period, $(2\pi/\omega_a)$ of the resultant oscillation. Under this condition, the resultant oscillation can be considered to be harmonic oscillation of frequency ω_a.

Further, when two harmonic oscillations of nearly equal frequencies (i.e. $\omega_1 \approx \omega_2$) are superposed, we observe a periodic variation in the amplitude of the resultant oscillation. This periodic variation in the amplitude is known as the phenomenon of **beats**. The phenomenon of beats is easily observed/heard in case of sound. When two sources of sound, such as tuning forks, of nearly same frequencies are vibrating simultaneously, a listener hears that the intensity of the resulting sound increases and decreases periodically. This fluctuation in the intensity of sound, which is caused due to change in amplitude, is called beats.

For graphical representation of beats, let us consider the case when the amplitudes of the two superposing oscillations of nearly same frequencies are same. Two such oscillations are depicted in Fig. 17.4a and 17.4b. When these oscillations are superposed, their resultant will be as depicted in Fig. 17.4c. In Fig. 17.4c, you may note that the amplitude varies periodically with time. This periodic variation of amplitude gives rise to beats.

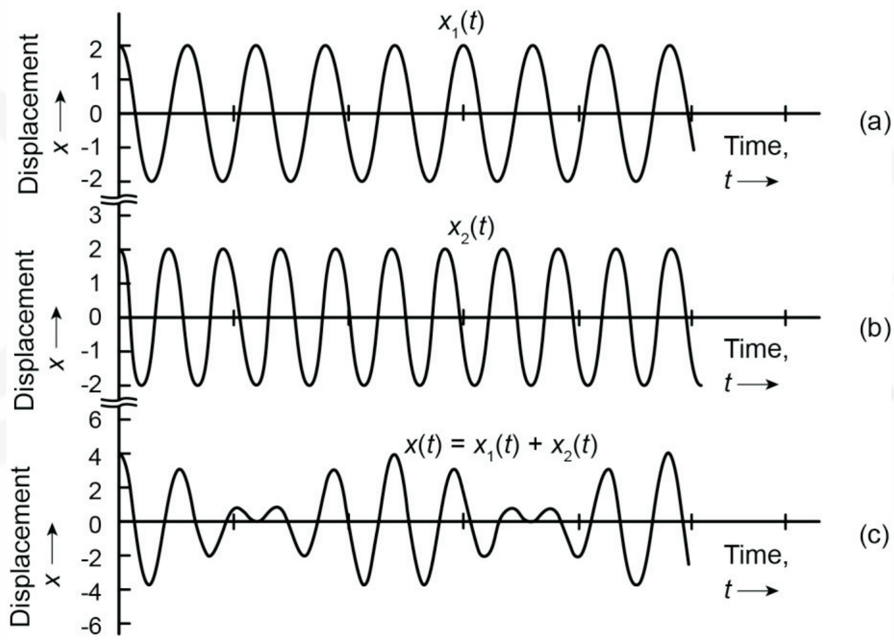


Fig. 17.4: Superposition of two oscillations of equal amplitude and nearly equal frequencies giving rise to beats; a) and b) depict the component oscillations and c) depicts the resultant oscillation having time varying amplitude.

Now, you may like to know: How is the periodicity of beats or the beat frequency related to the frequencies of the superposing oscillations? To discover the answer to this question, we need to obtain expressions for the time intervals between two consecutive amplitude maxima and amplitude minima (Fig. 17.4).

From the expression (Eq. (17.29)) for modulated amplitude, we note that it will attain the maximum values $(= a_1 + a_2)$ when

$$\cos 2\omega_m t = 1$$

This means that ω_m should satisfy the condition

$$2\omega_m t = 2n\pi; \qquad n = 0, 1, 2, ...$$

On using Eq. (17.23), we can write the above condition as

$$(\omega_1 - \omega_2)t = 2n\pi; \qquad n = 0, 1, 2, ...$$

Since the angular frequency, $\omega = 2\pi f$, where *f* is the frequency of oscillation, this condition can be rewritten in terms of *f* as

$$(f_1 - f_2)t = n; \qquad n = 0, 1, 2, ...$$

Therefore, we can conclude that the resultant amplitude will be maximum at

$$t = 0, \frac{1}{(f_1 - f_2)}, \frac{2}{(f_1 - f_2)}, ..., \frac{n}{(f_1 - f_2)} \qquad (17.34)$$

where $f_1 = (\omega_1 / 2\pi)$ and $f_2 = (\omega_2 / 2\pi)$ are the frequencies of superposing harmonic oscillations.

Similarly, you can convince yourself that amplitude of the resultant oscillation attains the minimum value $(= |a_1 - a_2|)$ when

$$\cos 2\omega_m t = -1$$

This requires that ω_m should satisfy the condition

$$2\omega_m t = (2n + 1)\pi; \qquad n = 0, 1, 2, ...$$

Thus, the resultant amplitude will be minimum for

$$t = \frac{1}{2(f_1 - f_2)}, \frac{3}{2(f_1 - f_2)}, \frac{5}{2(f_1 - f_2)}, ..., \frac{(2n+1)}{2(f_1 - f_2)} \qquad (17.35)$$

Eqs. (17.34) and (17.35) show that the time interval between two consecutive maxima or minima of the amplitudes of the resultant oscillation are equal. As such, an amplitude maxima followed by a minima is called a beat. However, the time period between two consecutive beats is called **beat period**:

$$t_b = \frac{1}{(f_1 - f_2)}$$

Hence, the beat frequency is given as

$$f_b = |f_1 - f_2| \qquad (17.36)$$

Eq. (17.36) shows that the **beat frequency** is equal to the difference in the frequencies of the superposing oscillations.

Now, before proceeding further, you should answer an SAQ.

SAQ 2 – Superposition of collinear oscillations of different frequencies

a) A particle is simultaneously acted upon by the following two collinear harmonic oscillations:

$$x_1(t) = 0.05 \cos(5\pi t)\text{m}; \quad x_2(t) = 0.03 \cos(3\pi t)\text{m}$$

Calculate the amplitude and phase of the resultant motion at time $t = 5\,\text{s}$.

b) Two tuning forks of frequencies 385 Hz and 389 Hz are sounded simultaneously. Calculate the beat frequency.

Before we proceed further, let us recall some important points of this section.

Recap

- When two collinear oscillations of unequal frequencies are superposed, the resultant motion is not simple harmonic because its amplitude is modulated (i.e., varies with time).
- However, the resultant motion is periodic if the angular frequencies ω_1 and ω_2 of the superposing oscillations satisfy the following condition

$$n_1 \omega_2 = n_2 \omega_1$$

- Further, when $\omega_1 \approx \omega_2$, $\omega_m << \omega_a$. Under this condition, the resultant motion is almost harmonic with angular frequency ω_a.
- When the frequencies are nearly equal, $\omega_1 \approx \omega_2$, the periodic variation in the amplitude of resultant oscillation gives rise to the phenomenon of beats. The beat frequency is given as

$$f_b = |f_1 - f_2|$$

So far we have confined our discussion to harmonic oscillations in one dimension. But oscillatory motion in two dimensions (2-D) is also possible.To get a feel of such a motion, you may like to do the following activity.

Activity

Superposition in 2-D

Take a funnel with a narrow bore and fill it with sand. Hang it from a rectangular piece of card board as shown in Fig. 17.5. The funnel can now oscillate independently in two mutually perpendicular directions.

Displace the funnel in the *x*-direction, and then release it with an impulse in the *y*-direction. As the funnel oscillates, you will note that the sand drops on the floor and generates a typical figure. Watch the shape of the figure. Is it curved?

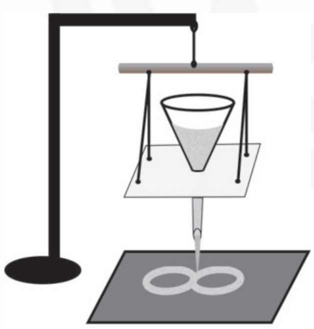


Fig. 17.5: Superposition in two-dimensions (2-D): trace of resultant motion.

17.4 SUPERPOSITION OF TWO MUTUALLY PERPENDICULAR HARMONIC OSCILLATIONS: LISSAJOUS FIGURES

From the discussions in this unit so far, you know that when two harmonic oscillations are superposed, the nature and path of the resultant motion depend on frequencies, amplitudes and initial phases of the superposing oscillations. We now learn to apply the principle of superposition to explain superposition in two-dimensions (2-D). Let us first consider the case when the superposing orthogonal oscillations are of the same frequency.

17.4.1 Orthogonal Oscillations of Equal Frequency

Consider two mutually perpendicular (orthogonal) oscillations: one along the *x*-axis and the other along the *y*-axis. Let us suppose that each oscillation has same angular frequency (ω_0) but their amplitudes, a_1 and a_2 are different such that $a_1 > a_2$. These oscillations can be represented as

$$x(t) = a_1 \cos \omega_0 t \tag{17.37}$$

and $$y(t) = a_2 \cos(\omega_0 t + \phi) \tag{17.38}$$

Note that the initial phases of the oscillations along the *x*- and the *y*-axes are zero and ϕ, respectively.

We are interested in knowing the nature of and path traced by the resultant oscillation when these two mutually perpendicular oscillations act on a body simultaneously. We can do so using the criterion valid for superposition of collinear oscillations: $n_1\ \omega_2 = n_2\ \omega_1$ (Sec. 17.3.2). In the instant case, the superposing oscillations are of the same frequency: $n_1 = 1 = n_2$. Thus, we can say that the resultant oscillation will be periodic.

To ascertain the path of the resultant oscillation, we have to determine how instantaneous displacements along the *x*- and *y*-axes are related. We can obtain the required relation by eliminating the terms containing *t* in Eqs. (17.37) and (17.38). For mathematical ease, we consider a few typical values of initial phase ϕ and gradually move from simple to complex situations.

Case I: $\phi = 0$ or π

For $\phi = 0$, Eqs. (17.37) and (17.38) take the form

$$x(t) = a_1 \cos \omega_0 t \quad \text{and} \quad y(t) = a_2 \cos \omega_0 t$$

To eliminate terms involving *t* from the above equations, we note that $\cos \omega_0 t = x / a_1$ and substitute this value in the expression for *y*. This gives

$$y(t) = a_2 \left(\frac{x}{a_1} \right)$$

or $$y = \frac{a_2}{a_1} x \tag{17.39}$$

The equation of a straight line is:

$y = mx + c$

where *m* is slope of the line and *c* is its intercept on the *y*-axis. In Eq. (17.39), $c = 0$.

Do you recognise this equation? It describes a **straight line** passing through the origin and having positive slope $(= a_2 / a_1)$. Therefore, we can conclude that **superposition of two orthogonal oscillations of same frequency and zero initial phases results in motion along a straight line (Fig. 17.6a)**. The arrows on the straight line in Fig. 17.6 indicate the direction of motion of the body along the path.

To ascertain whether or not the resultant motion is oscillatory, we determine the direction of motion using Eqs. (17.37) and (17.38) for $\phi = 0$. From these equations, we note that, for $\omega_0 t = 0$, we have $x = a_1$ and $y = a_2$ which specifies point *A* in Fig. 17.6a. When $\omega_0 t = \pi / 2$, we have $x = 0$ and $y = 0$, which specify the origin of the coordinate axes. For $\omega_0 t = \pi$, we have $x = -a_1$ and $y = -a_2$ which specify point *B*. Similarly, you can convince yourself that for values of $\omega_0 t$ between π and 2π, the body traces the path *BA* and when $\omega_0 t = 2\pi$ it reaches point *A* after completing one oscillation.

Further, for $\phi = \pi$, Eqs. (17.37) and (17.38) take the form

$$x(t) = a_1 \cos \omega_0 t$$

and $$y(t) = -a_2 \cos \omega_0 t$$

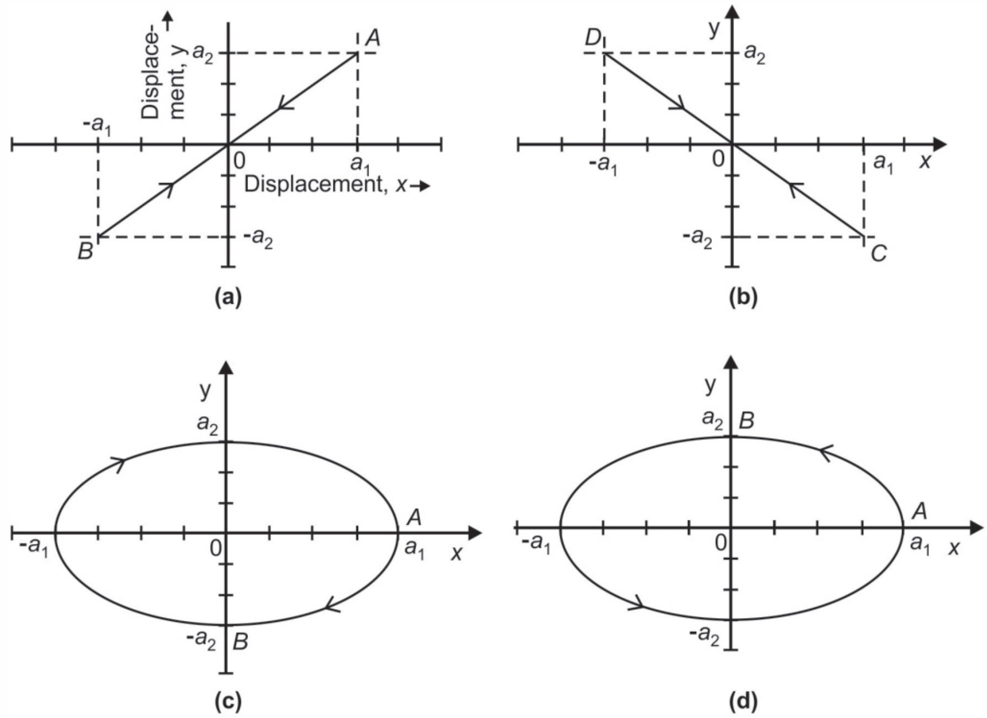


Fig. 17.6: Path traced by a body due to superposition of two orthogonal harmonic oscillations of unequal amplitudes, equal frequencies but different initial phases: a) $\phi = 0$; **b)** $\phi = \pi$; **c)** $\phi = (\pi / 2)$; **and d)** $\phi = (3\pi / 2)$.

Again, by substituting $\cos \omega_0 t = (x / a_1)$ in the expression for y, we get

$$y = -\frac{a_2}{a_1} x \tag{17.40}$$

This result (Eq. (17.40))shows that the path traced by the resultant motion of the body is along a **straight line** but the slope is negative. This corresponds to the motion along the diagonal *CD*, as shown in Fig. 17.6b.

<u>Case II: $\phi = \pi / 2$</u>

In this case, Eqs. (17.37) and (17.38) take the form

$$x(t) = a_1 \cos \omega_0 t \tag{17.41}$$

and

$$y(t) = a_2 \cos\left(\omega_0 t + \frac{\pi}{2}\right) = -a_2 \sin \omega_0 t \tag{17.42}$$

On squaring Eqs. (17.41) and (17.42) on both sides and adding, we get

$$\frac{x^2(t)}{a_1^2} + \frac{y^2(t)}{a_2^2} = \cos^2 \omega_0 t + \sin^2 \omega_0 t$$

or

$$\frac{x^2}{a_1^2} + \frac{y^2}{a_2^2} = 1 \tag{17.43}$$

Eq. (17.43) represents an **ellipse** whose semi-major and semi-minor axes are respectively a_1 and a_2. Thus, we can conclude: **when two orthogonal oscillations of equal frequency, unequal amplitudes but initial phases differing by** $\pi/2$ **are superposed, the resultant motion is along an ellipse whose principal axes lie along the** *x*- **and the** *y*-**axes** (Fig. 17.6c).

To know the direction of motion of the body in this case, we note from Eqs. (17.41) and (17.42) that at $t = 0$, $x = a_1$ and $y = 0$. This means that, initially, the body is at point *A* in Fig. 17.6c. But as *t* increases, *x* decreases from its maximum positive value and *y* becomes more and more negative. **This means that the elliptical path is traced from point *A* towards point *B*. Thus, the ellipse is described in the clockwise direction.**

Similarly, you can obtain the equation for the path traced by the resultant motion for $\phi = 3\pi/2$. You should convince yourself that the resultant motion is along an ellipse traced in the anticlockwise direction (Fig. 17.6d).

These traces for the path of the resultant motion are called **Lissajous figures**.

So far we have analysed superposition of two mutually perpendicular oscillations of same frequency but unequal amplitudes and different initial phases. You may now like to know: **What happens when their amplitudes are same?** For $a_1 = a_2 = a$, Eq. (17.43) reduces to

$$x^2(t) + y^2(t) = a^2 \tag{17.44}$$

You may recall from your school mathematics course that Eq. (17.44) represents a **circle** of radius *a*. **Thus, when two mutually perpendicular oscillations of equal frequency, equal amplitude and initial phase difference, $\phi = \pi/2$ are superposed, the resultant motion is along a circle traversed in clockwise direction (Fig. 17.7a).** Similarly, if $\phi = 3\pi/2$ and $a_1 = a_2$, the circular path will be traversed in anticlockwise direction (Fig. 17.7b).

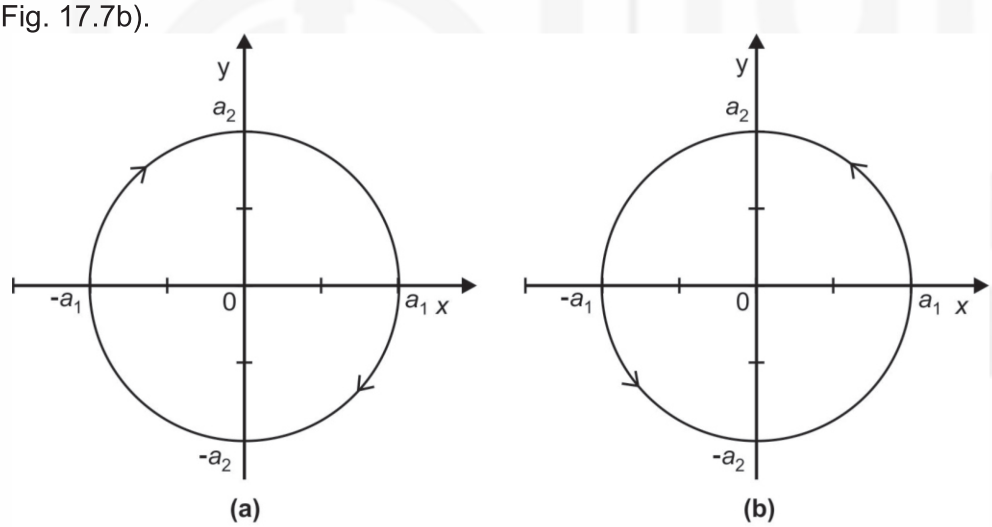


Fig. 17.7: Path traced by resultant motion of a body due to superposition of two orthogonal oscillation of equal amplitude, equal frequency and different initial phase difference: a) $\phi = \pi/2$; b) $\phi = 3\pi/2$.

Now, before proceeding further, you should answer an SAQ.

SAQ 3 – **Superposition of orthogonal oscillations of equal frequency**

The following two orthogonal harmonic oscillations act on a body simultaneously:

$$x(t) = 0.02 \cos(5\pi t) \text{ m} \quad \text{and} \quad y(t) = 0.02 \cos\left(5\pi t + \frac{\pi}{2}\right) \text{ m}$$

Determine the path of the resultant motion of the body.

So far in this section, you have learnt about the nature of resultant oscillations of a body for some particularly simple values of initial phase difference, ϕ. You may now ask: What will the resultant motion be if initial phases take arbitrary value? To seek answer to this question, we consider superposition of two orthogonal oscillations of unequal amplitudes but same frequency and arbitrary phase difference.

Case III: General Case

Let the initial phase difference between the superposing oscillations be ϕ. To obtain the expression for the path traced by the resultant oscillation, we need to eliminate *t* from Eqs. (17.37) and (17.38). So, we rewrite Eq. (17.38) as

$$\frac{y(t)}{a_2} = \cos(\omega_0 t + \phi) = \cos\omega_0 t \cos\phi - \sin\omega_0 t \sin\phi \qquad (17.45)$$

From Eq. (17.37), we note that

$$\cos\omega_0 t = \frac{x(t)}{a_1} \Rightarrow \sin\omega_0 t = \sqrt{1 - \frac{x^2(t)}{a_1^2}}$$

To obtain time independent relation between the displacements of the body along the *x*- and *y*-directions, we substitute these values of $\cos\omega_0 t$ and $\sin\omega_0 t$ in Eq. (17.45). This gives

$$\frac{y(t)}{a_2} = \frac{x(t)\cos\phi}{a_1} - \sqrt{1 - \frac{x^2(t)}{a_1^2}}\sin\phi$$

or
$$\frac{x(t)}{a_1}\cos\phi - \frac{y(t)}{a_2} = \sqrt{1 - \frac{x^2(t)}{a_1^2}}\sin\phi$$

On squaring both sides and rearranging terms, we get the following expression for the resultant path:

$$\boxed{\frac{x^2(t)}{a_1^2} + \frac{y^2(t)}{a_2^2} - 2\frac{x(t)\,y(t)}{a_1 a_2}\cos\phi = \sin^2\phi} \qquad (17.46)$$

Eq. (17.46) is the general equation of an ellipse whose axes are inclined to the coordinate axes. **So, we can conclude: when two mutually perpendicular harmonic oscillations of same frequency but different amplitudes and initial phases are superposed, the resultant motion is along an elliptical path.** For some typical values of ϕ lying between 0 and 2π, the paths traced out by the resultant oscillation are shown in Fig. 17.8. Note that the resultant path for $\phi = 0, \pi/2, \pi$ and $3\pi/2$ have been shown in Fig. 17.6 and 17.7. Also note that the paths for $\phi = 0$ and $\phi = \pi$ are not elliptical; they are straight lines.

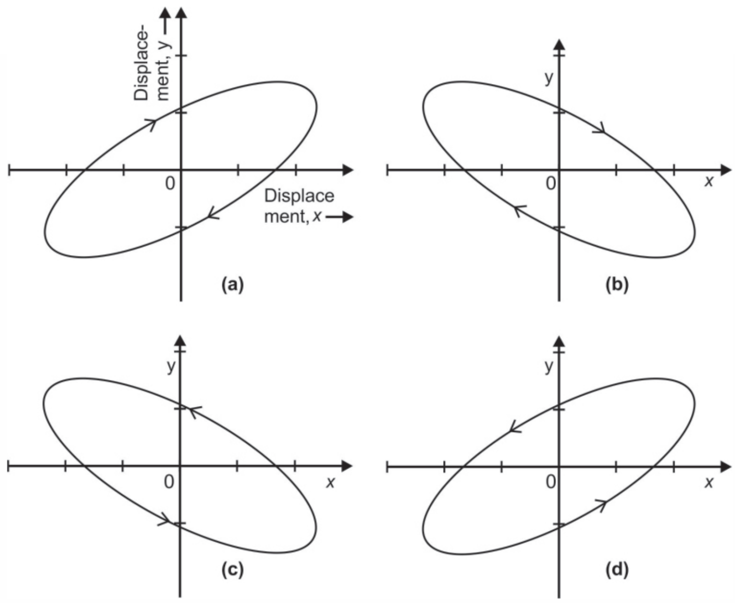


Fig. 17.8: Resultant paths traced by a body when two mutually perpendicular harmonic oscillations of equal frequency but different amplitudes are superposed for different values of initial phase difference a) $\pi / 4$; **b)** $3\pi /4$; **c)** $5\pi /4$; **d)** $7\pi /4$.

In your physics laboratory, you will get an opportunity to obtain some of the paths depicted in Fig. 17.8 using a cathode ray oscilloscope (CRO). To this end, you will have to apply different alternating sinusoidal voltages at horizontal plates (*XX*) and vertical plates (*YY*) of the CRO. An electron beam is made to move under the simultaneous influence of the two AC voltages (which are equivalent to harmonic forces). Thus, the path traced by the electron beam will be analogous to the path of the resultant motion arising due to superposition of two orthogonal harmonic oscillations. When AC voltages of same frequency are applied, you will obtain various curves on the CRO screen as shown in Fig. 17.8 by adjusting the initial phases and amplitudes of the signals. Further, note that the elliptical paths traced by the electron, as shown in Fig. 17.8, under the influence of two orthogonal superposing signals correspond to some typical values (integral multiples of $\pi/4$) of phase difference between the signals. However, using a CRO in a physical laboratory, you can also determine any random value of the phase difference between the two signals by analysing the shape of elliptical path. You will get an opportunity to do it in the laboratory course on mechanics.

We now work out an example to illustrate these conclusions.

EXAMPLE 17.3 : SUPERPOSITION OF ORTHOGONAL OSCILLATIONS OF SAME FREQUENCY

In a cathode ray oscilloscope, the displacements of electrons due to two mutually perpendicular and sinusoidally varying voltages applied between the *XX* and *YY* plates are given by

$$x(t) = 4\cos 2\pi f\,t\,; \quad \text{and} \quad y(t) = 4\cos\left(2\pi f\,t + \frac{\pi}{6}\right)$$

Determine the resultant path of the electron beam on the CRO screen.

SOLUTION ■ From the given expressions for x and y, we have:

$$a_1 = 4 \text{ units}; \quad a_2 = 4 \text{ units and } \phi = \pi/6$$

To determine the path traced by an electron on the CRO screen, under the influence of these two orthogonal voltages, we use Eq. (17.46). On substituting the values of a_1, a_2 and ϕ, we get

$$\frac{x^2}{4^2} + \frac{y^2}{4^2} - \frac{2xy}{4\times 4}\cos\frac{\pi}{6} = \sin^2\frac{\pi}{6}$$

or $$\frac{x^2}{16} + \frac{y^2}{16} - \frac{2xy}{16}\frac{\sqrt{3}}{2} = \frac{1}{4}$$

or $$x^2 + y^2 - \sqrt{3}\,xy - 4 = 0$$

This equation represents an ellipse. So, the resultant path of an electron on the CRO screen is along an ellipse.

You may now like to solve an SAQ.

SAQ 4 **– Superposition of orthogonal oscillations of equal frequency**

An object is acted upon simultaneously by the following two orthogonal oscillations:

$$x(t) = 0.03\,\sin(4\pi t)\,\text{m}; \quad y(t) = 0.04\,\sin(4\pi t + 1.5\pi)\,\text{m}$$

Obtain the equation of the path traced by the resultant motion of the object.

In the following, we summarise important results of this section.

Recap

- When two mutually perpendicular oscillations of equal frequency but different amplitudes and initial phases are superposed, the resultant displacement is given by

  $$\frac{x^2(t)}{a_1^2} + \frac{y^2(t)}{a_2^2} - 2\frac{x(t)\,y(t)}{a_1\,a_2}\cos\phi = \sin^2\phi$$

  where a_1 and a_2 are amplitudes of the two oscillations and ϕ is the phase difference between them. The path traced by the resultant motion is elliptical.

- For some typical values of the phase difference between the orthogonal superposing oscillations, the elliptical path reduces to a straight line (for $\phi = 0$ or π) and circular path (for $\phi = \pi/2$ and $a_1 = a_2$).

17.4.2 Orthogonal Oscillations of Unequal Frequencies

You now know that when two superposing orthogonal oscillations have the same frequency, the shape of the curve traced out by the resultant oscillation primarily depends on their initial phases. A slightly more complex situation arises when the frequencies of two superposing orthogonal oscillations are also unequal. However, if the frequencies of the superposing orthogonal oscillations are in the ratio 2:1 (that is, if $\omega_1 = 2\omega_0$, then $\omega_2 = \omega_0$), the path traced by the resultant motion for some typical values of the phase difference, ϕ is rather simple. Two such orthogonal oscillations are represented by

$$x(t) = a_1 \cos(2\omega_0 t + \phi)$$

and $$y(t) = a_2 \cos \omega_0 t$$

The expressions for the path traced by the resultant motion due to superposition of these two orthogonal oscillations for different values of the phase difference, ϕ between them are given below (for the derivation of these results, refer to Terminal Question 4):

i) $\phi = 0$

For this value of ϕ, the expression for the path of the resultant motion is given by

$$y^2(t) = \frac{a_2^2}{2a_1}[(x(t) + a_1)] \qquad (17.47)$$

Eq. (17.47) represents a parabola. Thus, a body subjected to two orthogonal oscillations having frequencies in the ratio 2:1 will trace a parabolic path. This is shown in Fig. 17.9a.

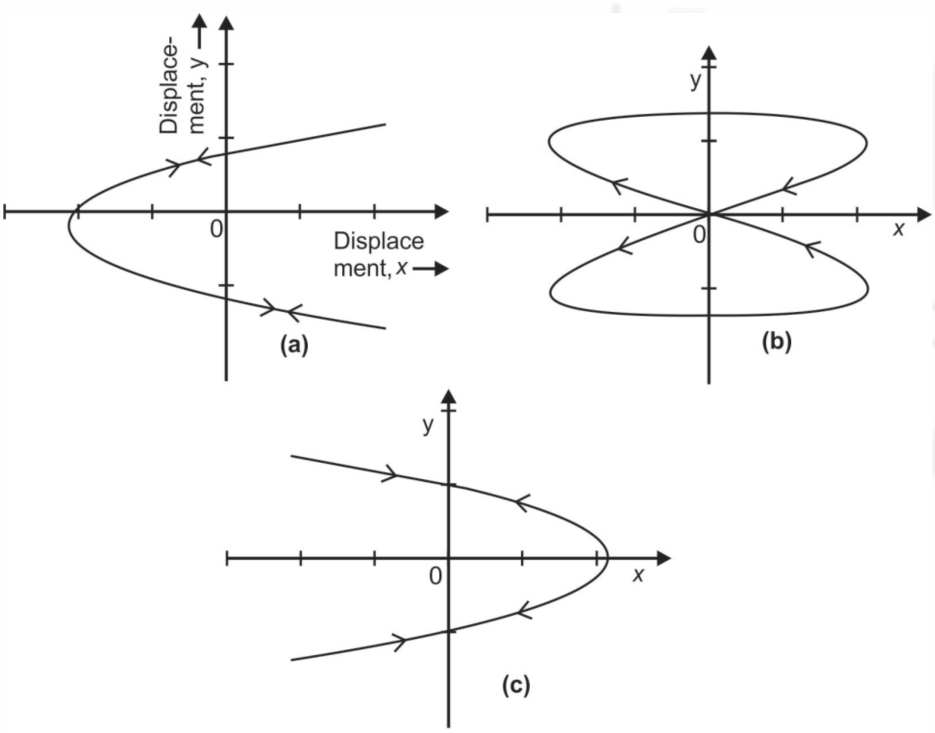


Fig. 17.9: Lissajous figures generated due to the superposition of two orthogonal harmonic oscillations of different amplitudes and having frequencies in the ratio 2:1. The initial phase difference between the two superposing orthogonal oscillations is a) $\phi = 0$; **b)** $\pi/2$; **c)** π.

ii) $\phi = \dfrac{\pi}{2}$

For this value of ϕ, the expression for the path of the resultant motion is given by

$$\frac{4y^2(t)}{a_2^2}\left(\frac{y^2(t)}{a_2^2} - 1\right) + \frac{x^2(t)}{a_1^2} = 0 \qquad (17.48)$$

Eq. (17.48) signifies a path which is like number '8' in shape. This is shown in Fig. 17.9b.

iii) $\phi = \pi$

For this value of ϕ, the expression for the path traced by the resultant motion is given by

$$y^2(t) = -\frac{a_2^2}{2a_1}(x(t) - a_1) \tag{17.49}$$

Eq. (17.49) also represents a parabola. The parabolic path represented by Eq. (17.49) is shown in Fig. 17.9c. We note that the direction of the parabola for $\phi = \pi$ is opposite to that for $\phi = 0$.

We hope you now understand the genesis of Lissajous figures, which we discussed in this section for some representative values of ϕ when the frequencies of the superposing orthogonal oscillations are in the ratio 2:1.

It is interesting to mention here that before the days of digital frequency meters, Lissajous figures were used to determine the frequency of sound as well as that of radio signals. This was done by applying a signal of known frequency to the horizontal plates of a cathode ray oscilloscope and the signal whose frequency was to be measured was applied to the vertical plates. By observing the resulting Lissajous figure, the unknown frequency was easily estimated because the shape of the Lissajous figure is a function of the ratio of the frequencies of the superposing signals.

Let us now sum up what you have learnt in this unit.

17.5 SUMMARY

Concept	Description
Principle of superposition	■ The **principle of superposition** states that when two or more harmonic oscillations are superposed, the displacement of the resultant oscillation at any given time *t* is the algebraic sum of individual displacements. If $x_1(t)$ and $x_2(t)$ are the displacements of two superposing harmonic oscillations at time *t* then the displacement $x(t)$ of the resultant oscillation at time *t* is given by $$x(t) = x_1(t) + x_2(t)$$
Linear differential equation	■ The principle of superposition is **valid only for those phenomena which can be described by linear differential equations**.
Superposition of collinear oscillations with same frequency	■ When two collinear harmonic oscillations of the same frequency, given by $$x_1(t) = a_1 \cos(\omega_0 t + \phi_1)$$ and $$x_2(t) = a_2 \cos(\omega_0 t + \phi_2)$$ are superposed, the resultant motion is described by $$x(t) = a \cos(\omega_0 t + \delta),$$

where $a = [a_1^2 + a_2^2 + 2a_1 a_2 \cos(\phi_1 - \phi_2)]^{1/2}$

and $\delta = \tan^{-1}\left(\dfrac{a_1 \sin\phi_1 + a_2 \sin\phi_2}{a_1 \cos\phi_1 + a_2 \cos\phi_2}\right)$

Superposition of collinear oscillations with different frequencies

- When two collinear harmonic oscillations of different frequencies are superposed, the **resultant motion is modulated** and is represented as $x(t) = a_m(t)\cos\omega_a t$, where **modulated amplitude**, $a_m(t) = 2a\cos\omega_m t$, **angular frequency of modulation**, $\omega_m = \dfrac{\omega_1 - \omega_2}{2}$ and **average angular frequency**, $\omega_a = \dfrac{\omega_1 + \omega_2}{2}$.

Superposition of mutually orthogonal harmonic oscillations

- When two **mutually orthogonal harmonic oscillations** act on a body simultaneously, the resultant motion traces out a variety of paths. If the oscillations have equal frequencies, the shape of the path depends on their initial phase difference. In general, the path traced by the resultant motion is elliptical but for certain values of initial phases of the superposing oscillations, it reduces to a straight line.

Superposition of mutually orthogonal oscillations of unequal frequencies

- When two **mutually orthogonal oscillations of unequal frequencies** and different initial phases act on a body simultaneously, the resultant paths are complex curves. These paths are collectively known as **Lissajous figures**.

17.6 TERMINAL QUESTIONS

1. The motion of a simple pendulum is described by the differential equation

$$\frac{d^2x}{dt^2} + 4x = 0$$

Write the solutions of this differential equation for the following two sets of initial conditions:

a) at $t = 0$, $x = 3\,\text{cm}$ and $\dfrac{dx}{dt} = 0$

b) at $t = 0$, $x = 2\,\text{cm}$ and $\dfrac{dx}{dt} = 4\,\text{cm}\,\text{s}^{-1}$

Denote these two solutions by $x_1(t)$ and $x_2(t)$ and show that for a new displacement $x_3(t) = x_1(t) + x_2(t)$, the initial conditions of the bob are essentially the superposition of the initial conditions of $x_1(t)$ and $x_2(t)$.

2. Two collinear harmonic oscillations are represented by

$$x_1(t) = 3\sin\left(20\pi t + \frac{\pi}{6}\right)\,\text{cm}$$

and $$x_2(t) = 4\sin\left(20\pi t + \frac{\pi}{3}\right)\,\text{cm}$$

Calculate the amplitude, phase constant and the period of resultant oscillation obtained on superposing these two collinear oscillations.

3. A particle is simultaneously subject to two mutually perpendicular oscillations given by

a) $x = 3 \sin \omega t$ cm, $y = 3 \cos \omega t$ cm

b) $x = \sin \omega t$ cm, $y = 4 \sin (\omega t + \pi)$ cm

Determine the trajectories of its motion.

4. The frequencies of two orthogonal harmonic oscillations of unequal amplitudes are in the ratio 2:1. When these two oscillations are applied simultaneously on a body, obtain the expressions for the path traced by the resultant motion of the body for phase difference, ϕ between the superposing oscillations equal to (i) zero, (ii) $\pi/2$, and (iii) π.

17.7 SOLUTIONS AND ANSWERS

Self-Assessment Questions

1. From Eq. (17.16), we recall that the amplitude of the resultant oscillation due to superposition of two collinear oscillations is given by

$$a = \sqrt{a_1^2 + a_2^2 + 2a_1 a_2 \cos(\phi_1 - \phi_2)} \qquad \text{(i)}$$

As per the problem, we have $a_1 = 5\,\text{cm} = 0.05\,\text{m}$; $a_2 = 3\,\text{cm} = 0.03\,\text{m}$; $\phi_1 = (\pi/2)$ and $\phi_2 = 0$ so that $(\phi_1 - \phi_2) = \pi/2$

On substituting these values in Eq. (i), we get

$$a = \sqrt{(0.05\,\text{m})^2 + (0.03\,\text{m})^2 + 2(0.05\,\text{m})\times(0.03\,\text{m})\cos(\pi/2)} = 0.06\,\text{m}$$

Further, from Eq. (17.17), we note that the phase of resultant oscillation is given by

$$\delta = \tan^{-1}\left[\frac{a_1 \sin\phi_1 + a_2 \sin\phi_2}{a_1 \cos\phi_1 + a_2 \cos\phi_2}\right]$$

$$= \tan^{-1}\left[\frac{(0.05\,\text{m})\sin(\pi/2) + (0.03\,\text{m})\sin(0)}{(0.05\,\text{m})\cos(\pi/2) + (0.03\,\text{m})\cos(0)}\right] = 59.1° = 1.03\,\text{rad}$$

2. a) This is the case of superposition of two collinear harmonic oscillations of unequal amplitudes and unequal frequencies. The amplitude of resultant motion due to such superposition is given by Eq. (17.29):

$$a_m(t) = [a_1^2 + a_2^2 + 2a_1 a_2 \cos(2\omega_m t)]^{1/2} \qquad \text{(i)}$$

As per the problem

$$a_1 = 0.05 \text{ m}; \quad a_2 = 0.03 \text{ m}; \quad \omega_1 = 5\pi; \; \omega_2 = 3\pi \text{ and } t = 5\text{s}.$$

So, $$\omega_m = \frac{\omega_1 - \omega_2}{2} = \frac{2\pi}{2} = \pi\,\text{s}^{-1}$$

On substituting these values in Eq. (i), we get

$$a_m(t = 5\,\text{s}) = [(0.05 \text{ m})^2 + (0.03 \text{ m})^2 + 2(0.05 \text{ m}) \times (0.03 \text{ m}) \times (\cos(2\times(\pi\,\text{s}^{-1})\times(5\,\text{s}))]^{1/2} = 0.08 \text{ m}$$

The phase of the resultant motion is given by Eq. (17.30):

$$\theta_m = \tan^{-1}\left[\frac{(a_1 - a_2)\sin(\omega_m t)}{(a_1 + a_2)\cos(\omega_m t)}\right] = \tan^{-1}\left[\frac{(0.05 - 0.03)\text{m} \times \sin(5\pi)}{(0.05 + 0.03)\text{m} \times \cos(5\pi)}\right] = 0$$

b) From Eq. (17.36), we know that the beat frequency f_b is given by

$$f_b = |f_1 - f_2| = |385 - 389| = 4\text{Hz}$$

3. From the given expressions, we note that the orthogonal oscillations have equal amplitude $(= 0.02\text{ m})$ equal frequency (= 5π) and differ in phase by $\pi/2$. Hence from Eq. (17.38), we can write the equation for the trajectory of the resultant motion as

$$\frac{x^2}{(0.02\text{ m})^2} + \frac{y^2}{(0.02\text{ m})^2} = 1 \quad \text{or} \quad x^2 + y^2 = (0.02\text{ m})^2$$

This is the equation of a circle of radius 0.02 m. So, the body moves along a circular path of radius 0.02 m.

4. We note from the expressions for $x(t)$ and $y(t)$ that the two orthogonal oscillations have same frequency but different amplitudes $(a_1 = 0.03\text{ m}$ and $a_2 = 0.04\text{ m})$ and the initial phase difference between them is $1.5\pi\ (= 3\pi/2)$. The general expression for the resultant path of a body on which two orthogonal oscillations act simultaneous is given by Eq. (17.41):

$$\frac{x^2}{a_1^2} + \frac{y^2}{a_2^2} - 2\frac{xy}{a_1 a_2}\cos\phi = \sin^2\phi$$

On substituting the values of a_1, a_2 and ϕ, we get

$$\frac{x^2}{(0.03\text{ m})^2} + \frac{y^2}{(0.04\text{ m})^2} = 1$$

This is the equation of an ellipse. It means that the resultant motion will be along an elliptical path.

Terminal Questions

1. The given differential equation for a simple pendulum is

$$\frac{d^2x}{dt^2} + 4x = 0 \qquad \text{(i)}$$

On comparing it with the standard equation for SHM, $\frac{d^2x}{dt^2} + \omega_0^2 x = 0$, we note that $\omega_0 = 2\,\text{s}^{-1}$. Therefore, the solution of this equation can be written as

$$x(t) = a\cos(2t + \phi) \qquad \text{(ii)}$$

Differentiating this with respect to t, we get

$$\frac{dx}{dt} = -2a\sin(2t + \phi) \qquad \text{(iii)}$$

a) As per the given initial conditions: at $t = 0$, $x = 3\text{cm}$ and $dx/dt = 0$. Therefore, from Eqs. (ii) and (iii), we obtain

$$3\text{cm} = a\cos\phi \qquad \text{(iv)}$$

and $$0 = -2a \sin\phi \tag{v}$$

Eq. (v) implies that $\phi = 0$ since a is finite. Using this result in Eq. (iv), we get

$$a = 3\,\text{cm}$$

Hence, the complete solution of the given differential equation can be expressed as

$$x_1(t) = 3 \cos 2t \text{ cm} \tag{vi}$$

b) The second set of initial conditions are: at $t = 0$, $x = 2\,\text{cm}$ and $\frac{dx}{dt} = 4\,\text{cm s}^{-1}$. On substituting these values in Eqs. (ii) and (iii), we get

$$2\,\text{cm} = a \cos\phi \tag{vii}$$

and $$4\,\text{cms}^{-1} = -2a \sin\phi \quad \Rightarrow \quad 2\,\text{cms}^{-1} = -a \sin\phi \tag{viii}$$

On dividing Eq. (viii) by Eq. (vii), we get

$$\tan\phi = -1 \quad \Rightarrow \quad \phi = -\frac{\pi}{4}$$

On substituting $\phi = -\pi/4$ in Eq. (vii), we get

$$a = 2\sqrt{2} \text{ cm}$$

Therefore, we can write the solution for the given initial condition as

$$x_2(t) = 2\sqrt{2} \cos\left(2t - \frac{\pi}{4}\right)\text{cm} \tag{ix}$$

Since superposition of x_1 and x_2 yields x_3, we get can write

$$x_3(t) = x_1(t) + x_2(t)$$

On substituting the values of $x_1(t)$ and $x_2(t)$ from Eqs. (vi) and (ix) respectively, we get

$$x_3(t) = 3 \cos 2t + 2\sqrt{2} \cos\left(2t - \frac{\pi}{4}\right)$$

$$= 3 \cos 2t + 2\sqrt{2}\left[\cos 2t \cos\left(\frac{\pi}{4}\right) + \sin 2t \sin\left(\frac{\pi}{4}\right)\right]$$

$$= 3 \cos 2t + 2\sqrt{2}\left(\frac{1}{\sqrt{2}}\cos 2t + \frac{1}{\sqrt{2}} \sin 2t\right)$$

$$= (5 \cos 2t + 2 \sin 2t) \text{ cm} \tag{x}$$

Now if we superpose the initial conditions corresponding to x_1 and x_2, we have at

$t = 0$, $x = 5\,\text{cm}$ and $\frac{dx}{dt} = 4\,\text{cm s}^{-1}$. On substituting these values of x and dx/dt in Eqs. (ii) and (iii), we get

$$5\,\text{cm} = a \cos\phi \tag{xi}$$

and $$4\,\text{cms}^{-1} = -2a \sin\phi \tag{xii}$$

On dividing Eq. (xii) by Eq. (xi), we get $\tan\phi = -2/5$

From this we can write $\sin\phi = -2/\sqrt{29}$ and $\cos\phi = 5/\sqrt{29}$

and $a = \sqrt{29}$ cm

Therefore, the solution obtained on superposing the two initial conditions is

$$x_3(t) = \sqrt{29}\cos(2t + \phi)$$

$$= \sqrt{29}\,[\cos 2t\, \cos\phi - \sin 2t\, \sin\phi]\text{ cm}$$

On substituting for cos ϕ and sin ϕ, we get

$$x_3(t) = (5\cos 2t + 2\sin 2t)\text{ cm} \qquad \text{(xiii)}$$

So, you may note that the value of resultant displacement, $x_3(t)$ given by Eq. (xiii) is the same as the one obtained by the superposition of individual displacements, $x_1(t)$ and $x_2(t)$ (Eq. (x)).

2. We can write the displacements $x_1(t)$ and $x_2(t)$ of the harmonic oscillations in terms of cosine functions as:

$$x_1(t) = 3\cos\left(20\pi t + \frac{\pi}{6} - \frac{\pi}{2}\right)\text{cm} = 3\cos\left(20\pi t - \frac{\pi}{3}\right)\text{cm} \qquad \text{(i)}$$

and $$x_2 = 4\cos\left(20\pi t + \frac{\pi}{3} - \frac{\pi}{2}\right)\text{cm} = 4\cos\left(20\pi t - \frac{\pi}{6}\right)\text{cm} \qquad \text{(ii)}$$

When these two collinear oscillations of equal frequencies are superposed, the expression for the displacement of resultant oscillation is given as (Eq. (17.15)):

$$x(t) = a\cos(\omega_0 t + \delta) \qquad \text{(iii)}$$

From Eqs. (i) or (ii), we get that $\omega_0 = 20\pi$. To determine the value of the resultant amplitude a, we recall from Eq. (17.16) that

$$a = [(a_1^2 + a_2^2 + 2a_1 a_2 \cos(\phi_1 - \phi_2)]^{1/2}$$

On substituting the values of a_1, a_2, ϕ_1 and ϕ_2 from Eqs. (i) and (ii), we get

$$a = \left(3^2 + 4^2 + \left(2\times 3\times 4\times\cos\frac{\pi}{6}\right)\right)^{1/2}\text{cm} = 6.77\text{ cm}$$

The phase constant δ of the resultant oscillation is given by Eq. (17.17):

$$\delta = \tan^{-1}\left(\frac{3\sin(-\pi/3) + 4\sin(-\pi/6)}{3\cos(-\pi/3) + 4\cos(-\pi/6)}\right) = \tan^{-1}\left(\frac{3\sqrt{3}+4}{3+4\sqrt{3}}\right) = -0.24\pi$$

So, the expression for the resultant oscillation is obtained by substituting the values of ω_0, a and δ in Eq. (iii):

$$x(t) = 6.77\cos(20\pi t - 0.24\pi)\text{ cm}$$

3. a) We are given

$$x(t) = 3\sin\omega t\text{ cm} \quad \text{and} \quad y(t) = 3\cos\omega t = 3\sin\left(\omega t + \frac{\pi}{2}\right)\text{cm}$$

These orthogonal oscillations are of equal amplitude and same frequency but differ in phase by $\pi/2$. Hence, using Eq. (17.44), we may conclude that the resultant motion will be along a circle defined by

$$x^2 + y^2 = 9\text{ cm}^2 \quad\Rightarrow\quad x^2 + y^2 = (3\text{ cm})^2$$

b) We are given that

$$x(t) = \sin \omega t \text{ cm} \quad \text{and} \quad y(t) = 4 \sin(\omega t + \pi) \text{ cm}$$

These orthogonal oscillations have different amplitudes $(a_1 = 1 \text{ cm and } a_2 = 4 \text{ cm})$, same frequency ω and differ in phase by π. Hence using this data in Eq. (17.46), we get

$$x^2 + \frac{y^2}{16} + \frac{2xy}{4} = 0$$

or $$16x^2 + y^2 + 8xy = 0 \quad \Rightarrow \quad (4x + y)^2 = 0$$

So the trajectory of the resultant motion will be along a straight line defined by

$$y = -4x$$

4. As per the problem, if the frequencies of two orthogonal superposing oscillations are ω_1 and ω_2, then

$$\omega_1 = 2\omega_0 \quad \text{and} \quad \omega_2 = \omega_0$$

Thus, the expression for such orthogonal harmonic oscillations can be written as

$$x(t) = a_1 \cos(2\omega_0 t + \phi) \quad \text{and} \quad y(t) = a_2 \cos \omega_0 t$$

where, ϕ is the phase difference between the two oscillations.

i) For $\phi = 0$, the expression reduces to

$$x(t) = a_1 \cos 2\omega_0 t \quad \text{and} \quad y(t) = a_2 \cos \omega_0 t$$

So,

$$x(t) = a_1 \cos 2\omega_0 t = a_1 (2\cos^2 \omega_0 t - 1)$$

Substituting $\cos \omega_0 t = y / a_2$, we can write

$$x(t) = a_1 [(2y^2 / a_2^2) - 1]$$

or $$(x / a_1) = (2y^2 / a_2^2) - 1$$

or $$y^2(t) = (a_2^2 / 2a_1)\,[x(t) + a_1] \qquad \text{(i)}$$

Eq. (i) is the equation of parabola and the resultant motion will be along a parabola.

ii) For $\phi = \pi/2$, the expressions for $x(t)$ and $y(t)$ reduces to

$$x(t) = -a_1 \sin 2\omega_0 t \quad \text{and} \quad y(t) = a_2 \cos \omega_0 t \qquad \text{(ii)}$$

We can write $x(t)$ as

$$-(x / a_1) = 2 \sin \omega_0 t \, \cos \omega_0 t$$

Now, substituting $\cos \omega_0 t = (y / a_2)$ and $\sin \omega_0 t = \sqrt{1 - \cos^2 \omega_0 t}$

$$= \sqrt{1 - (y / a_2)^2}$$

We can write Eq. (ii) as

$$-(x/a_1) = (2y/a_2)\left(\sqrt{1-(y/a_2)^2}\right)$$

Squaring both sides and rearranging the terms, we get

$$\left(4y^2/a_2^2\right)\left[\left(y^2/a_2^2\right)-1\right]+\left[x^2/a_1^2\right]=0 \qquad \text{(iii)}$$

Eq. (iii) represents a curve which is like number "8".

iii) For $\phi = \pi$, the expressions for $x(t)$ and $y(t)$ reduces to

$$x(t) = -a_1 \cos 2\omega_0 t \quad \text{and} \quad y(t) = a_2 \cos \omega_0 t$$

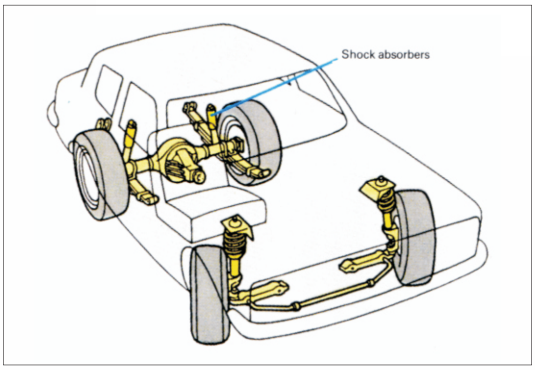
or $$(x/a_1) = 2\cos^2 \omega_0 t - 1$$

Again, substituting $\cos\omega_0 t = (y/a_2)$, we can write,

$$(2y^2/a_2^2) = -(x/a_1) + 1$$

or $$y^2(t) = (a_2^2/2a_1)\,[x(t) - a_1] \qquad \text{(iv)}$$

Eq. (iv) represents a parabola but it is directed opposite to that represented by Eq. (i) for $\phi = 0$. So, in this case also, the resultant motion will be along a parabolic path.



This is a picture showing four shock absorbers fitted near each wheel of a car. The shock absorber is a very good example of damped oscillation. It contains spring-loaded check valves and orifices to control the flow of oil through an internal piston. When the piston moves up and down in the hydraulic liquid, the vibratory motion of the automobile is reduced considerably. Shock absorbers are an important part of automobile and motorcycle suspensions, aircraft landing gear, and the supports for many industrial machines.

UNIT 18

DAMPED OSCILLATIONS

Structure

STUDY GUIDE

So far, our discussion in the previous two units of this block has been confined to ideal physical systems which execute simple harmonic motion. In this unit, you will learn to appreciate that real physical systems are not ideal. In fact, the oscillatory motion of these systems are rather difficult to analyze mathematically. The study of damped oscillations is the first step towards dealing with real oscillatory systems. As such, we can describe the motion of a damped oscillator by a second order linear differential equation. You have learnt how to solve the equation of motion of a damped oscillator in Unit 4 of this course. Here, our emphasis will be on the physical implications of damping on the motion of an oscillator. You may recall that the solution of equation of motion of a damped system involves exponential and hyperbolic functions. So, you will do better if you refresh the basic algebra and calculus involving such functions. Further, you should work out the problems involving logarithmic decrement, relaxation time and quality factor to get a feel for their numerical values.

> ***"Science is the attempt to make the chaotic diversity of our sense-experience correspond to a logically uniform system of thought."***
>
> ***Albert Einstein***

18.1 INTRODUCTION

In Unit 16, you learnt about simple harmonic motion (SHM) and its characteristics. You may recall that the displacement-time graph of a mechanical system executing SHM is sinusoidal and the total energy of such a system remains constant in time. This implies that once such a system is set into motion, it should continue to oscillate indefinitely. Such an oscillation is said to be ***free* oscillation**. Do you know of any *real physical system* that keeps oscillating indefinitely? Probably, we can think of none.

In the real world, the amplitude (and hence energy) of oscillation of an oscillatory system reduces gradually due to the presence of frictional forces in the medium. Recall the oscillations of a swing, a simple pendulum or a vertical spring-mass system when left to itself. In all these systems, oscillations die down gradually because air exerts a **drag force**. This implies that every oscillating system loses energy as time passes. The motion of such a system is said to be **damped motion**. You may now ask: Where does this energy go? The oscillating system has to work against the damping force, which causes dissipation of its energy. *That is, the energy of an oscillating body is used up in overcoming damping*.

NOTE

The **drag force** is the frictional force experienced by an object moving in a medium such as air or liquid.

In general, damping causes wasteful loss of energy. Therefore, we invariably try to minimise it. But in some engineering systems, we knowingly introduce damping. A familiar example is automobile shock absorber. When an automobile goes over a bump, spring of the shock absorber is set into motion. The shock absorber is so designed that it dampens (that is, minimises) the oscillations and we enjoy almost jerk free ride even on a bumpy road.

In this unit, you will learn the salient features of damped motion. In Sec. 18.2, you will learn to establish the equation of motion of a damped oscillator and study physical implications of its solutions for over-damped, critically damped and weakly damped systems. (You have learnt how to solve the equation of motion for a damped system in Unit 4 of this course). Therefore, we will just quote the results here. You will discover that weak damping leads to oscillatory motion of gradually diminishing amplitude. In Sec. 18.3, you will learn to characterise weak damping in terms of logarithmic decrement, relaxation time and quality factor. You will learn to obtain expressions for these parameters which help us obtain numerical measures of damping in an oscillator.

Expected Learning Outcomes

After studying this unit, you should be able to:

- ❖ Establish the equation of motion of a damped harmonic oscillator;
- ❖ Differentiate between weakly damped, critically damped and over-damped systems;
- ❖ Discuss the effect of damping on amplitude, period of oscillation and energy of an oscillator; and
- ❖ Obtain expressions for logarithmic decrement, relaxation time and quality factor.

18.2 EQUATION OF MOTION OF A DAMPED OSCILLATOR

In Sec. 16.2, you have learnt that the force law for a spring-mass system executing SHM is given by

$$F = -k\,x$$

Here F is magnitude of the restoring force and x denotes instantaneous displacement of the system from the equilibrium position. You may recall that while discussing the motion of a spring-mass system (Unit 16), we completely ignored the effect of friction or air drag on its motion. **But in practice, every oscillating system experiences some frictional force, which slows down its motion.** Such a force is called **damping force.** And to predict the behaviour of an actual oscillator more realistically, we must study the motion of a damped harmonic oscillator.

According to Stokes' law, the drag force experienced by a spherical body of radius r falling freely in a viscous medium (like water or oil) is given by

$$F_d = 6\pi\,\eta\,r\,v$$

Here η is the coefficient of viscosity of the medium and v is the velocity of the body.

To discuss the motion of a damped oscillator, we consider a spring-mass system in which the mass is made to oscillate horizontally in a **viscous medium**, say in a oil container, as shown in Fig. 18.1. As the mass oscillates, it experiences a damping force. Let us denote it by F_d. *Usually, it is difficult to quantify exact magnitude of this force*. However, **for oscillations of sufficiently small amplitude, it is fairly reasonable to model the damping force on Stokes' law.** That is, we assume F_d to be proportional to velocity of the mass and write

$$F_d = -\gamma v \qquad (18.1)$$

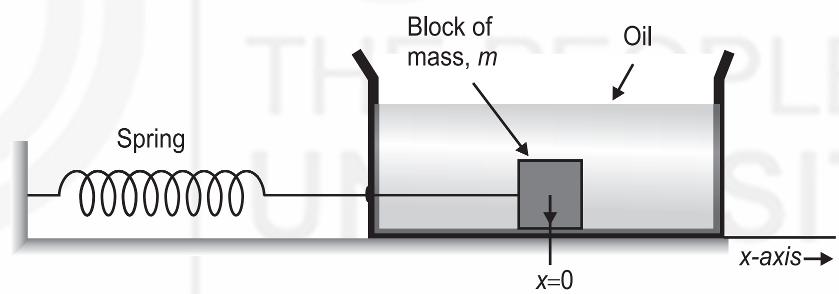


Fig. 18.1: **A damped spring-mass system. Note that damping is introduced in the system because the oscillating mass is immersed in oil – a viscous medium.**

The negative sign on the RHS of Eq. (18.1) signifies that damping opposes motion. The constant of proportionality, γ is called the **damping coefficient** or **damping constant**. *Numerically, it is equal to force per unit velocity* and is measured in $\dfrac{\text{N}}{\text{ms}^{-1}} = \dfrac{\text{kg ms}^{-2}}{\text{ms}^{-1}} = \text{kg s}^{-1}$. Eq. (18.1) implies that the faster a body moves, greater will be the opposition by the viscous medium to its motion. Note that, unlike the restoring force which is proportional to displacement, the damping force is proportional to velocity.

Equation of Motion

To establish the equation of motion of a damped oscillator, we choose x-axis to be along the length of the spring (Fig. 18.1). We take the equilibrium position of mass as the origin $(x = 0)$. Suppose that the mass (in the spring-mass system) is pulled longitudinally and then released. Since the mass gets displaced from its equilibrium position, a restoring force and a damping force will act on it simultaneously. We express these forces as follows:

- *Restoring force*: $-kx$, where k is force constant, and
- *Damping force*: $-\gamma v$, where $v\ (= dx/dt)$ is the instantaneous velocity of the oscillator and γ is the damping constant.

Note that **for a damped oscillator, the force law must include the restoring force as well as the damping force**. Therefore, we can write

$$F = -kx - \gamma v = -kx - \gamma \frac{dx}{dt}$$

Using this expression for force in Newton's second law of motion $(F = ma_c)$, we can write the equation of motion for a damped oscillator as

$$m\frac{d^2x}{dt^2} = -kx - \gamma\frac{dx}{dt} \tag{18.2}$$

On rearranging terms and dividing throughout by m, Eq. (18.2) takes the form

$$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + \omega_0^2 x = 0 \tag{18.3}$$

where $\omega_0^2 = \dfrac{k}{m}$ and $2b = \dfrac{\gamma}{m}$

Note the factor of 2 that has been deliberately introduced in the damping term. You will soon learn that it helps us in expressing the solutions of Eq. (18.3) in a simpler form. Also note that the constant b, called **damping factor**, has the dimensions of T^{-1}:

$$b = \frac{\gamma}{2m} = \frac{\text{force}}{\text{velocity} \times \text{mass}} = \frac{\mathrm{MLT}^{-2}}{\mathrm{LT}^{-1}\mathrm{M}} = \mathrm{T}^{-1}$$

This result shows that the unit of b is s^{-1}, which is the same as that of ω_0.

Do you notice any similarity between the differential equations represented by Eqs. (16.5) and (18.3)? Both the differential equations are linear, of second order and homogeneous with constant coefficients. This similarity will help us analyse the motion of a damped oscillator.

Solution of Differential Equation

From Eq. (4.34) of Unit 4, you may recall that the solution of the differential equation given by Eq. (18.3) can be written as:

$$x(t) = \exp(-bt)\left[a_1 \exp\left\{(b^2 - \omega_0^2)^{1/2} t\right\} + a_2 \exp\left\{-(b^2 - \omega_0^2)^{1/2} t\right\}\right] \tag{18.4}$$

This expression shows that the exact solution is governed by the **relative magnitudes of the damping force (represented by the damping factor, b) and the restoring force (represented by the force constant k and hence the parameter, ω_0^2).** The quantity $b^2 - \omega_0^2$ can be negative, zero or positive, depending on whether the damping factor is less than, equal to or greater than the angular frequency, respectively. These conditions give rise to following three distinct kinds of damped motion:

- If $b > \omega_0$, the system takes *a very long time* to return back to its equilibrium position. Then we say that damping is heavy and the system is **heavily damped** or **over-damped**.
- If $b = \omega_0$, the system takes *minimal time* to return to its initial position. Then we say that the system is **critically damped**. Such a system never overshoots the equilibrium position.
- If $b < \omega_0$, the system executes *oscillatory motion* with gradually decreasing amplitude. Then we say that the system is **weakly damped**. A weakly damped system is *of maximum interest in physics.*

Now go through the following example.

EXAMPLE 18.1 : A DAMPED OSCILLATOR

The equation of motion of an oscillating body of mass 0.5 kg is

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 9x = 0$$

a) Calculate (i) force constant, k, (ii) angular frequency, ω_0, (iii) damping constant, γ and (iv) damping factor, b.

b) What is the nature of damping?

SOLUTION ■ a) The equation of motion of damped harmonic oscillator is given by Eq. (18.3):

$$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + \omega_0^2 x = 0$$

On comparing this equation with the given equation of motion, we get:

$$2b = 4\,\mathrm{s}^{-1} \quad \text{and} \quad \omega_0^2 = 9\,\mathrm{s}^{-2}$$

i) To calculate the force constant k, we recall that

$$\omega_0^2 = \frac{k}{m} \quad \Rightarrow \quad k = \omega_0^2\, m = (9\,\mathrm{s}^{-2}) \times (0.5\,\mathrm{kg}) = 4.5\ \mathrm{kg\,s}^{-2}$$

ii) For calculating angular frequency ω_0, we have:

$$\omega_0^2 = 9\,\mathrm{s}^{-2} \qquad \Rightarrow \qquad \omega_0 = 3\,\mathrm{s}^{-1}$$

iii) To calculate damping constant γ, we use the relation

$$2b = \frac{\gamma}{m} \qquad \Rightarrow \qquad \gamma = 2bm = (4\,\mathrm{s}^{-1}) \times (0.5\ \mathrm{kg}) = 2\ \mathrm{kg\,s}^{-1}$$

iv) For calculating the damping factor b, we have

$$2b = 4\,\mathrm{s}^{-1} \qquad \Rightarrow \qquad b = 2\,\mathrm{s}^{-1}$$

b) From the calculated values of $b\ (= 2\,\mathrm{s}^{-1})$ and $\omega_0\ (= 3\,\mathrm{s}^{-1})$, we find that $b < \omega_0$. This signifies weak damping. Therefore, the motion of the oscillator is weakly damped.

To fix the ideas discussed above, you should answer the following SAQ.

SAQ 1 **– Determining the nature of damping in a oscillatory system**

The differential equation of an oscillator is given by

$$\frac{d^2x}{dt^2} + 20\frac{dx}{dt} + 25x = 0$$

Calculate the force constant, damping constant, angular frequency and damping factor. What is the nature of damping?

18.2.1 Heavy Damping

On solving SAQ 1, you must have seen that the given equation represents non-oscillatory motion corresponding to a heavily damped system. We can write the solution for an over-damped system as

$$x(t) = \frac{v_0}{2\beta}\exp(-bt)\,[\exp(\beta t) - \exp(-\beta t)] = \frac{v_0}{\beta}\exp(-bt)\sinh\beta t \quad (18.5)$$

where $\beta = \sqrt{b^2 - \omega_0^2}$

> $\sinh\beta t = \frac{\exp(\beta t) - \exp(-\beta t)}{2}$ is hyperbolic sine function.

From Eq. (18.5), we note that $x(t)$ is a product of two terms: an increasing hyperbolic function, say $\phi(t) = \sinh\beta t$ and a decaying exponential function, say $\psi(t) = \exp(-bt)$. Interplay of these two terms determines the time-variation of displacement of a heavily damped oscillator. The exponential term will tend to reduce the displacement as time passes while the hyperbolic term will tend to increase it. The plots of these two terms are indicated in separately in Fig. 18.2. Note that β and b are constants and they will not affect in any way the time variations of functions $\phi(t)$ and $\psi(t)$, respectively. When these two terms (Eq. (18.5)) are plotted together, as shown by the continuous curve in Fig. 18.2, we get the displacement-time graph for a heavily damped system. You may note that initially the displacement increases with time. But soon the exponential term begins to dominate and displacement begins to decrease gradually. However, **the motion is non-oscillatory** because $x(t)$ does not become negative at any time. Such a motion is called **dead-beat**.

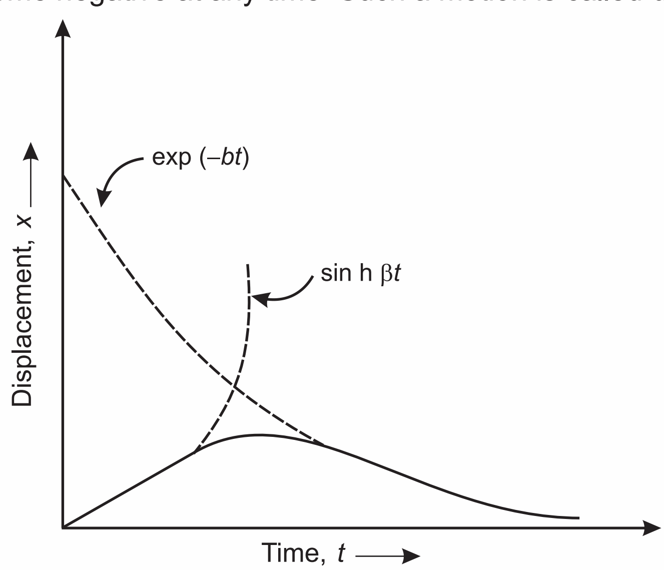


Fig. 18.2: **Time-variation of displacement of an over-damped system.**

We may now summarise the result obtained for an over-damped system.

Recap

The motion of an over-damped $(b > \omega_0)$ system is non-oscillatory. The instantaneous displacement, $x(t)$ of such a system is determined by an interplay of a decaying exponential function and an increasing hyperbolic function. Mathematically, we write it as

$$x(t) = \frac{v_0}{\beta} \exp(-bt) \sinh \beta t$$

18.2.2 Critical Damping

The critically damped oscillation is characterised by the condition $b = \omega_0$. This means that $(b^2 - \omega_0^2) = 0$, and Eq. (18.4), the solution of Eq. (18.3) reduces to:

$$\begin{aligned} x(t) &= (a_1 + a_2) \exp(-bt) \\ &= a \exp(-bt) \end{aligned} \tag{18.6}$$

where $a = a_1 + a_2$

You can verify (SAQ 2) that for critical damping, the complete solution of Eq. (18.3) is given by

$$x(t) = (p + qt) \exp(-bt) \tag{18.7}$$

where p and q are constants. Note that p has the dimension of length and q has the dimensions of velocity. We can determine the values of these constants using the initial conditions.

SAQ 2 – **Solution of the differential equation of a critically damped oscillator**

Show that Eq. (18.7) represents the complete solution of Eq. (18.3) for a critically damped oscillator.

On the basis of the general solution of a critically damped oscillator given by Eq. (18.7), we can draw the following conclusions about its behaviour:

- The time variation of displacement of a critically damped oscillator is governed by a decaying exponential function. Therefore, **the time taken by the oscillator to come back to equilibrium position will be minimum.**
- The displacement remains positive at all times. This means that a critically damped system does not overshoot equilibrium position or oscillate about it.

Fig. 18.3 depicts time variation of displacement of a critically damped system.

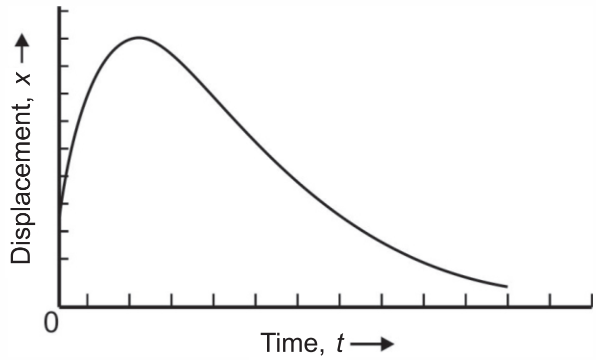


Fig. 18.3: **Displacement-time graph for a critically damped system.**

The door closing mechanism in a door closer and shock absorbers in an automobile are the most common examples of critically damped systems. Due to critical damping of the spring used in a door closer, the door closes quickly but gently. Similarly, shock absorbers quickly damp the bounce that a car may experience immediately after hitting a road bump or a pit. Some other examples of critically damped systems are the indicator needles in electrical and electronic instruments.

We may now conclude as follows:

Recap

A critically damped system attains the equilibrium condition in minimum time. The instantaneous displacement of a critically damped oscillator is given by

$$x(t) = (p + qt)\exp(-bt)$$

where p and q are constants.

You now know that an over-damped as well as a critically damped system returns to the equilibrium position without any oscillation.

You may, therefore, ask**: What is the essential difference between a critically damped and a heavily damped system? A critically damped system returns back to the equilibrium position in the quickest possible time, whereas a heavily damped system attains equilibrium position very slowly.**

In physics, over-damped and critically damped oscillations have limited use. The case of more general interest is weak damping. You will now learn about it.

18.2.3 Weak Damping

When $b < \omega_0$, the system is said to be weakly damped and **this gives rise to damped oscillatory motion**.

For a weakly damped system, the solution of Eq. (18.3) is given by:

$$x(t) = \frac{v_0}{\omega_d}\exp(-bt)\cos\left(\omega_d t - \frac{\pi}{2}\right) = a(t)\sin\omega_d t \quad (18.8)$$

where $$a(t) = \frac{v_0}{\omega_d}\exp(-bt) = a_0\exp(-bt) \quad (18.9)$$

and angular frequency of the damped oscillator is less than angular frequency of free oscillations:

$$\omega_d = (\omega_0^2 - b^2)^{1/2} = \left[\frac{k}{m} - \frac{\gamma^2}{4m^2}\right]^{1/2} \tag{18.10}$$

Eq. (18.8) gives the general solution of Eq. (18.3) for weak damping. Note that it represents sinusoidal motion of frequency $\omega_d\ (< \omega_0)$. However, the amplitude, $a(t)\ (= a_0 \exp(-bt))$ of oscillation *decreases exponentially with time.* **So, we can say that the motion of a weakly damped system is oscillatory but not simple harmonic**. The displacement-time graph for such an oscillator is shown in Fig. 18.4. Since the sine function varies between +1 and –1, the displacement-time curve lies between $a_0 \exp(-bt)$ and $-a_0 \exp(-bt)$.

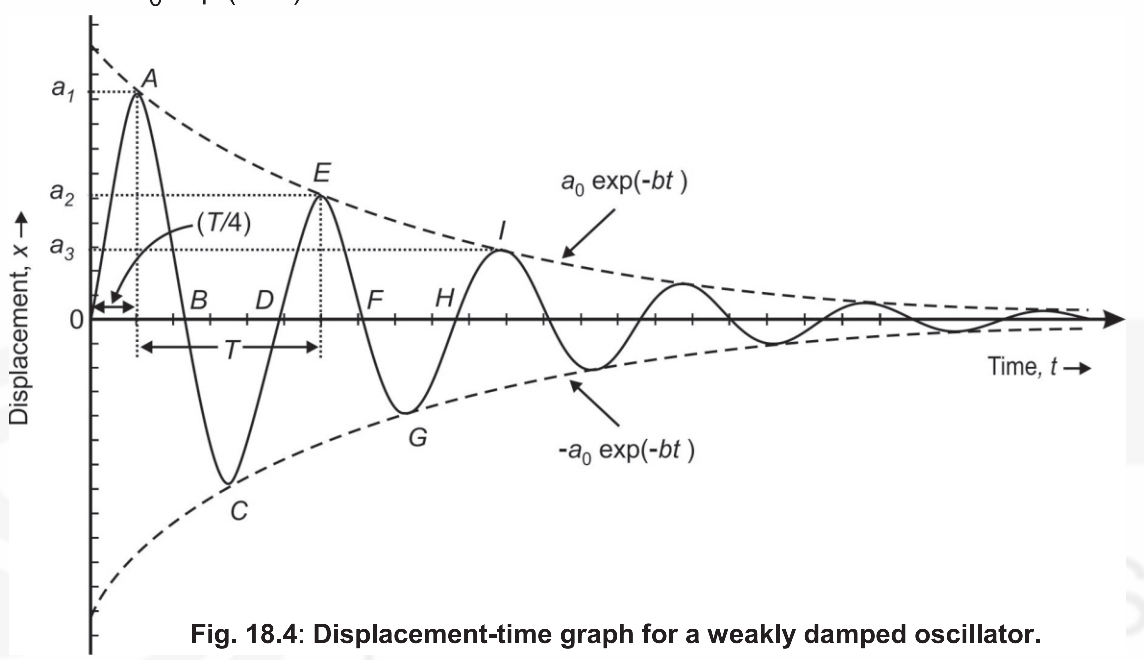


Fig. 18.4: **Displacement-time graph for a weakly damped oscillator.**

In addition to amplitude of oscillation, **weak damping also affects the frequency of oscillation**. You may ask: What is the period of the damped motion depicted in Fig. 18.4? Strictly speaking, it is difficult to define time period for weakly damped motion because the motion does not repeat itself. That is, the value of amplitude, say A at time t_1, is never reached again at any subsequent time. But, using the analogy with simple harmonic motion loosely, we define the time period as the time elapsed between three consecutive values when displacement becomes zero. That is, time taken by the oscillator between points B and F in Fig. 18.4 Further, since angular frequency, ω_d of the damped oscillator is less than its natural frequency, ω_0, **damping increases the period of oscillation of an oscillator**. Mathematically, we express the period of oscillation of a weakly damped oscillator as

$$T_d = \frac{2\pi}{\omega_d} = \frac{2\pi}{(\omega_0^2 - b^2)^{1/2}} = \frac{2\pi}{\sqrt{\dfrac{k}{m} - \dfrac{\gamma^2}{4m^2}}} \tag{18.11}$$

Before proceeding further, let us recapitulate the results obtained for a weakly damped oscillator.

Recap

- For a weakly damped oscillator, the instantaneous displacement is given by

$$x(t) = a(t)\sin\omega_d t$$

where amplitude of oscillation decreases exponentially with time:

$$a(t) = a_0 \exp(-bt) \text{ and } a_0 = v_0/\omega_d$$

- The motion of a weakly damped system is oscillatory but not simple harmonic. The period of oscillation is given by

$$T_d = \frac{2\pi}{\sqrt{\frac{k}{m} - \frac{\gamma^2}{4m^2}}}$$

You should now go through the following example carefully to learn calculate the magnitude of various quantities associated with a weakly damped system.

EXAMPLE 18.2 : OSCILLATIONS OF A WEAKLY DAMPED SYSTEM

Restoring and frictional forces of magnitudes kx and $\gamma\frac{dx}{dt}$, respectively act simultaneously on an object of mass m attached to a spring. Under the influence of these forces, the mass oscillates with a frequency 0.5 Hz and its amplitude reduces to half in 2 s. Calculate the force constant k, the damping constant γ and the damping factor b. Also write the differential equation for the system.

SOLUTION ■ Since the object is subjected to frictional (or damping) force, it constitutes a damped system which oscillates with frequency 0.5 Hz. Thus, we can write the angular frequency, ω_d of this damped oscillator as

$$\omega_d = 2\pi f = \pi\ \text{s}^{-1}$$

Since the system is oscillatory despite being damped, we can say that it is a weakly damped system. The amplitude of a weakly damped system at any given time t is given by Eq. (18.9):

$$a(t) = a_0 \exp(-bt)$$

Further, we are told that the amplitude of oscillation reduces to half in 2 s. Therefore, we can write

$$\frac{1}{2}a_0 = a_0 \exp(-2b)$$

or $\qquad \exp(-2b) = 1/2 \quad \Rightarrow \quad \exp(2b) = 2$

On taking natural logarithm of both sides, we get

$$b = \frac{1}{2}\ln 2$$

We know that $b = \gamma/2m$. So, we can write

$$\frac{\gamma}{2m} = \frac{2.303}{2}\times\log_{10} 2 = \frac{2.303\times 0.3010}{2}$$

or $\qquad \gamma = 0.6932\,m \qquad$ (i)

To calculate k, you may recall from Eq. (18.10) that

$$\omega_d^2 = \frac{k}{m} - \frac{\gamma^2}{4m^2}$$

It can be rearranged to write

$$\frac{k}{m} = \omega_d^2 + \left(\frac{\gamma}{2m}\right)^2 = \pi^2 + (0.3466)^2 = 9.98$$

Hence $\quad k = 9.9\,m \qquad$ (ii)

The differential equation of a damped harmonic oscillator is given by Eq. (18.3):

$$\frac{d^2x}{dt^2} + \frac{\gamma}{m}\frac{dx}{dt} + \frac{k}{m}x = 0$$

Substituting the values of (γ/m) and (k/m) from Eqs. (i) and (ii) respectively, we obtain the differential equation representing the motion of the mass, m as:

$$\frac{d^2x}{dt^2} + 0.693\frac{dx}{dt} + 9.98\,x = 0$$

You may now like to answer an SAQ.

***SAQ 3* – Comparison of time periods of a damped and an undamped oscillator**

The amplitude of vibration of a damped spring-mass system decreases from 10 cm to 2.5 cm in 200 s. If this oscillator completes 100 oscillations in this time, compare the periods with and without damping.

You now know that the motion of an over-damped system and a critically damped system is non-oscillatory. However, the motion of a weakly damped system is oscillatory and its frequency is lower than that of a free (or undamped) system. You must also have noted that the amplitude of a weakly damped oscillator decreases continuously and the rate of decay is characterised by damping factor, b. Let us now learn about the energy of a weakly damped oscillator.

Energy of a weakly damped oscillator

From Sec. 16.4.1, you may recall that the average energy, E_0 of a harmonic oscillator is given by

$$E_0 = (1/2)\,ka^2$$

where a is the amplitude of oscillation. This expression indicates that larger the amplitude of oscillation, greater will be the average energy of the oscillator. When such a system is damped, its energy is spent on overcoming the effect of damping. The loss of energy of the system manifests as gradual decrease in the value of amplitude of oscillation of the system.

Since amplitude is a measure of energy of an oscillator, you can get a qualitative idea about the energy dissipation in a weakly damped system from Fig. 18.4. Note that oscillations dies down gradually with time. It means that a damped oscillating system loses energy at a rate determined by the magnitude of damping factor, b.

To obtain an expression for average energy of a weakly damped system, we recall that average energy of a free oscillator is proportional to the square of amplitude. We extend this argument to a weakly damped oscillator and write

$$< E > \propto a^2 = Ca^2$$

where C is the constant of proportionality. Using Eq. (18.9), we can write the expression for average energy as

$$< E > = Ca_0^2 \exp(-2bt) = E_0 \exp(-2bt) \qquad (18.12)$$

where $E_0 = Ca_0^2$ is energy of an undamped oscillator.

Eq. (18.12) shows that the average energy of a weakly damped oscillator decreases at a **rate faster** [exp (– 2*bt*)] **than the rate of decrease of amplitude** [exp (– *bt*)]**.** In Fig. 18.5, we have plotted the time variation of average energy. If you compare this with the envelope of plot shown by the dotted lines in Fig. 18.4, you will note that the energy curve is steeper.

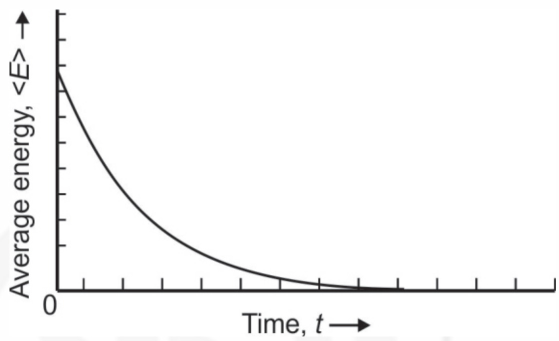


Fig. 18.5: **Time variation of average energy for a weakly damped system.**

So, we can conclude as follows:

Recap

- The average energy of a weakly damped oscillator decreases exponentially with time:

$$< E > = E_0 \exp(-2bt)$$

- The average energy of a weakly damped oscillator decreases faster than its amplitude.

You may now like to answer an SAQ.

SAQ 4 – Motion of a weakly damped system

An object of one kg mass executes one-dimensional motion. It experiences a restoring force characterised by force constant $4\ \text{Nm}^{-1}$ and a resistive force with damping constant $0.6\ \text{Nm}^{-1}\text{s}$.

i) Will its motion be oscillatory?

ii) Calculate the damping constant for which the motion will be critically damped.

iii) For what mass will the motion be critically damped for the given forces?

You must have noted that a weakly damped oscillator exhibits oscillatory behaviour, with continuously decaying amplitude with time. However, we need to know the magnitude (or extent) of damping operative in a system, even if it is weak. This is because a quantitative measure of the damping in a weakly damped oscillator is useful for investigating the motion of a variety of physical systems. Therefore, it is important to learn to characterize weak damping and obtain its quantitative measure. You will learn about it now.

18.3 CHARACTERISING WEAK DAMPING

To obtain a quantitative estimate of damping in the system, we define three parameters, namely, logarithmic decrement (λ), relaxation time (τ) and quality factor (Q). These parameters are defined in terms of angular frequency ω_0 and damping factor b. Depending on the system, one or more of these parameters may have to be calculated to determine the extent of damping. We now discuss these parameters one by one.

18.3.1 Logarithmic Decrement

In this method of characterising weak damping, *the damping present in a system is measured in terms of the rate at which the amplitude of oscillation dies down with time.* Suppose a weakly damped system, initially at rest at its equilibrium position, is given an impulse, i.e. at $t = 0$, $x = 0$ and $v = v_0$. The instantaneous displacement of this oscillator can be expressed in terms of the time period of oscillations as (see margin remark):

The expression for the instantaneous displacement of a weakly damped system is given by Eq. (18.8):

$x(t) = a(t) \sin \omega_d t$

For small values of b, we can take $\omega_d \approx \omega_0$ and write

$x(t) = a(t) \sin \omega_0 t$

$= a(t) \sin (2\pi t / T)$

$$x(t) = a(t) \sin\left(\frac{2\pi t}{T}\right)$$

where $a(t) = a_0 \exp(-bt)$ and a_0 is amplitude of free oscillations $(b = 0)$.

To obtain an expression for the rate of decrease of amplitude of a weakly damped oscillator, refer to Fig. 18.4 again. Note that at $t = T/4$, the displacement rises to its first maximum value (point A). Let the value of amplitude corresponding to point A be a_1. So, using above expression for displacement, we can write

$$x\left(\frac{T}{4}\right) = a_0 \exp\left(-\frac{bT}{4}\right) \sin\left(\frac{2\pi}{T} \times \frac{T}{4}\right)$$

or
$$a_1 = a_0 \exp\left(-\frac{bT}{4}\right)$$

since $\sin(\pi/2) = 1$. After one complete oscillation, that is, after time $\left(\frac{T}{4} + T\right)$, the amplitude becomes maximum again (point E in Fig. 18.4). Let us denote the value of amplitude corresponding to point E by a_2. *Note that both these amplitudes, a_1 and a_2 will lie in the same direction/quadrant.*

Thus, using $t = (5T/4)$ in the expression for $a(t)$, we can write the amplitude a_2 as

$$a_2 = a_0 \exp\left(-\frac{5bT}{4}\right)$$

The next maximum of displacement, a_3 will occur at $\left(\frac{5T}{4}+T\right)=\frac{9T}{4}$. The corresponding amplitude (point *I* in Fig. 18.4), a_3 will be given by

$$a_3 = a_0 \exp\left(-\frac{9bT}{4}\right)$$

Similarly, we can write

$$a_4 = a_0 \exp\left(-\frac{13bT}{4}\right)$$

Note that the ratio of successive amplitudes separated by one time period, T is constant:

$$\frac{a_1}{a_2} = \frac{a_2}{a_3} = \frac{a_3}{a_4} = \ldots \qquad \frac{a_{n-1}}{a_n} = \exp(bT) = d \tag{18.13}$$

The constant *d* which denotes the ratio of two successive amplitudes separated by one period, is called the decrement of the motion. The logarithm of decrement is called **logarithmic decrement of the motion**. We denote logarithmic decrement by λ and write

$$\lambda = \ln d = \ln(\exp(bT)) = bT = \frac{\gamma T}{2m} \tag{18.14}$$

Note that λ depends on damping factor (b) as well as the time period (T) of the oscillating system. From Eqs. (18.13) and (18.14), it readily follows that

$$\frac{a_1}{a_2} = d = \exp(\lambda)$$

$$\frac{a_1}{a_3} = \frac{a_1}{a_2}\times\frac{a_2}{a_3} = d^2 = \exp(2\lambda)$$

and so on. This result shows that we can measure λ by knowing two successive amplitudes separated by one period. However, from an experimental point of view, it is more convenient to compare amplitudes of oscillations separated by n periods. That is, by measuring (a_1/a_n). We, therefore, write

$$\frac{a_1}{a_n} = \frac{a_1}{a_2}\cdot\frac{a_2}{a_3}\cdot\frac{a_3}{a_4}\ldots\frac{a_{n-1}}{a_n} = \exp[(n-1)\lambda]$$

Taking logarithm on both sides and simplifying, we get:

$$\lambda = \frac{1}{(n-1)}\ln\left(\frac{a_1}{a_n}\right) \tag{18.15}$$

If you plot $\ln(a_1/a_n)$ along y-axis and $(n-1)$ along x-axis for different values of n, you will obtain a straight line. The slope of this line gives the value of logarithmic decrement, λ.

Let us now work out an example to illustrate the utility of the above results.

EXAMPLE 18.3 : LOGARITHMIC DECREMENT OF A DAMPED HARMONIC OSCILLATOR

A damped harmonic oscillator has the first amplitude of 20 cm. The value of the amplitude reduces to 2 cm after 100 oscillations. If period of oscillations is 4.6 s, calculate the logarithmic decrement and damping factor. Also determine the number of oscillations in which the amplitude drops by 50 percent.

SOLUTION ■ The logarithmic decrement is given by Eq. (18.15):

$$\lambda = \frac{1}{(n-1)} \ln\left(\frac{a_1}{a_n}\right)$$

We are given that $a_1 = 20$ cm; $a_n = 2$ cm; and $n = 100$ .So, we can write

$$\lambda = \frac{1}{99} \ln\left(\frac{20\text{ cm}}{2\text{ cm}}\right) = \frac{1}{99}\ln(10) = 0.023$$

The logarithmic decrement is given in terms of damping factor, b by Eq. (18.14):

$$\lambda = bT$$

So the damping factor b is

$$b = \frac{\lambda}{T} = \frac{0.023}{4.6\text{ s}} = 0.005\text{ s}^{-1}$$

We know that the amplitude of a damped oscillator is given by Eq. (18.9):

$$a(t) = a_0 \exp(-bt)$$

Let the initial amplitude be a_0 and after time t_1, it drops by 50 percent. Then, we can write

$$\frac{a(t_1)}{a_0} = \exp(-bt_1)$$

$$\frac{1}{2} = \exp(-bt_1)$$

or, $$\exp(bt_1) = 2$$

Taking logarithm on both sides, we get

$$bt_1 = \ln(2)$$

or $$t_1 = \frac{\ln(2)}{b} = \frac{0.693}{0.005\text{ s}^{-1}} \cong 139\text{ s}$$

Since the period of oscillation is 4.6 s, total number of oscillations completed in time t_1 is

$$= \frac{139\text{ s}}{4.6\text{ s}} \cong 30$$

So, in 30 oscillations, the amplitude of the damped oscillator will drop by 50 percent.

Before proceeding further, you should solve an SAQ.

SAQ 5 **– Logarithmic decrement and damping factor of a simple pendulum**

The period of a simple pendulum is 4 s and its amplitude is 5°. After 30 oscillations, its amplitude decreases to 3°. Calculate the logarithmic decrement and damping factor. Also calculate the number of oscillations in which its amplitude reduces by 25%.

18.3.2 Relaxation Time

Another parameter which is used to quantify damping is relaxation time. It is denoted by the symbol τ. *Relaxation time is defined as the time in which the amplitude of a damped oscillator decreases to* e^{-1} $(= 0.368)$ *of its initial value.* To understand this, recall that the amplitude of a weakly damped harmonic oscillator is given by Eq. (18.9):

$$a(t) = a_0 \exp(-bt)$$

So the amplitude after time $(t + \tau)$ can be written as

$$a(t + \tau) = a_0 \exp[-b(t + \tau)]$$

By taking the ratio of $a(t+\tau)$ and $a(t)$, we obtain

$$\frac{a(t+\tau)}{a(t)} = \exp(-b\tau)$$

If we assume that $b\tau = 1$, we have

$$\frac{a(t+\tau)}{a(t)} = e^{-1} = \frac{1}{e} \tag{18.16}$$

Thus, for $b = \tau^{-1}$, the amplitude drops to $1/e$ $(= 0.368)$ times of its initial value. **The relaxation time is, therefore, a measure of the rapidity with which oscillations of a weakly damped oscillator decrease.**

Yet another parameter which gives a quantitative measure of damping is quality factor (Q) of a system. You will learn about it now.

18.3.3 Quality Factor

The quality factor is a measure of the **rate of decay of energy**. The quality factor Q of a weakly damped oscillator is defined as

$$Q = \frac{\text{Energy stored in the system}}{\text{Energy dissipated per radian}} \tag{18.17}$$

From the definition, it is obvious that smaller the energy dissipated per radian, larger will be the value of Q for a damped system. It means that if a system has high value of Q, the damping in the system is weak and dissipation of energy is less.

You may now ask: Why have we defined quality factor in terms of **energy dissipated per radian?** To understand this, you may recall that in one complete oscillation, an oscillator traverses angular distance equal to 2π radians. So, energy dissipated in one cycle of oscillation is same as energy dissipated in 2π radians. Moreover, inclusion of *energy dissipated per radian* instead of energy *dissipated per cycle* facilitates mathematical convenience as will be evident shortly.

The power series expansion of the exponential function is

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots$$

For $x << 1$,

$$e^{-x} \cong 1 - x$$

For weak damping, we can assume that $2\,bT << 1$ and write

$\exp(-2\,bT) = 1 - 2bT$

To proceed further, we recall that average energy of a damped oscillator is given by Eq. (18.12):

$$<E> = E_0 \exp(-2bt)$$

where E_0 is the energy of the undamped oscillator.

Further, let us suppose that average energies of the damped oscillator at times t and $(t+T)$ are E_1 and E_2, respectively. Therefore, we can write

$$E_1 = E_0 \exp(-2bt) \quad \text{and} \quad E_2 = E_0 \exp\{-2b(t+T)\}$$

Hence, the ratio (see margin remark)

$$\frac{E_2}{E_1} = \exp(-2bT) \cong 1 - 2bT$$

or
$$\frac{E_1 - E_2}{E_1} = 2bT \tag{18.18}$$

The numerator of the LHS of Eq. (18.18) signifies the energy lost by a weakly damped oscillator in one cycle. Now to write an expression for Q as per Eq. (18.17) we need to know the energy dissipated per radian. So, we divide both sides of Eq. (18.18) by 2π and rewrite it as

$$\frac{E_1}{[(E_1 - E_2)/2\pi]} = \frac{1}{2bT} \times 2\pi = \frac{\omega_0}{2b} \tag{18.19}$$

On comparing Eqs. (18.17) and (18.19), we can write

$$Q = \frac{\omega_0}{2b} \tag{18.20}$$

Eq. (18.20) is the required expression for **quality factor** of a weakly damped oscillator. Note that Q is inversely proportional to b. It means that as damping in the system increases, the value of the quality factor decreases. Further, on substituting the values of $\omega_0 (= \sqrt{k/m})$ and $2b\ (= \gamma / m)$ in Eq. (18.20), we can express Q as

$$Q = \frac{\omega_0}{2b} = \frac{m}{\gamma} \times \sqrt{\frac{k}{m}} = \sqrt{\frac{km}{\gamma^2}} \tag{18.21}$$

Eq. (18.21) expresses the quality factor of a damped oscillator in terms of mass m, force constant k and damping constant γ of the system. It is also evident from Eq. (18.20) as well as Eq. (18.21) that Q is a dimensionless quantity. Due to this fact, it can be defined for any oscillator – mechanical or electrical.

We can also obtain the relation between the quality factor of an oscillator and its relaxation time, τ. In Sec. 18.3.2, you learnt that $b = 1/\tau$. On substituting this value of b in Eq. (18.20), we can write

$$Q = \frac{\omega_0 \tau}{2} \qquad (18.22)$$

So, we find that the quality factor and the relaxation time are directly proportional to each other. This is expected as both these parameters characterise the extent of damping in the system.

The quality factor of oscillating systems where damping is important, such as door closer or a shock absorber, is approximately 0.5. On the other hand, the value of Q of a tuning fork is approximately 1000. Can you guess the value of Q for an undamped system? From Eq. (18.20) you can easily conclude that Q is infinite for an undamped $(b = 0)$ oscillator.

We now summarise the results obtained in this section.

Recap

- The ratio of two successive amplitude separated by one period is called the decrement of motion. The logarithm of decrement is called logarithmic decrement of damped oscillation motion and it is given by

$$\lambda = \frac{1}{(n-1)} \ln\left(\frac{a_1}{a_n}\right)$$

  where n is the number of complete oscillations.

- Relaxation time is the time taken by the amplitude of a damped oscillator to decrease to e^{-1} $(= 0.368)$ times its initial value. It is given as $\tau = 1/b$

- Quality factor Q is defined as the rate of decay of energy of a damped oscillator. It is given by $Q = \omega_0 / 2b$.

You should go through the following example carefully to get a feel for the typical values of some of the parameters discussed above.

EXAMPLE 18.4 : RELAXATION TIME AND QUALITY FACTOR OF A STRINGED MUSICAL INSTRUMENT

The intensity of sound produced due to plucking of the string of a musical instrument decreases by half in 8 s. The natural frequency of the string is 512 Hz. Calculate the (i) relaxation time, (ii) quality factor and (iii) fractional loss of energy per cycle.

SOLUTION ■ We know that the intensity $I(t)$ of sound is proportional to the energy $E(t)$ of oscillation generating sound. So, we can write

$$I(t) \propto E(t)$$

i) Using the result $b = \tau^{-1}$, we can write Eq. (18.12) as

$$E(t) = E_0 \exp\left(-\frac{2t}{\tau}\right)$$

or
$$\frac{E(t)}{E_0} = \exp\left(-\frac{2t}{\tau}\right)$$

so that $\tau = \dfrac{2t}{\ln (E_0 / E(t))} = \dfrac{16\text{s}}{\ln(2)} = 23.1 \text{ s}$

ii) The quality factor Q is given by Eq. (18.20):

$$Q = \frac{\omega_0}{2b} = \frac{\omega_0 \tau}{2} = \pi \times 512 \times 23.1 = 37137.5$$

iii) Using Eq. (18.18) the fractional loss in energy can be expressed as

$$\frac{\Delta E}{E} = \frac{2\pi}{Q} = \frac{2\pi}{37137.5} = 1.69 \times 10^{-4}$$

You may now like to answer an SAQ.

***SAQ 6* – Parameters characterising damping**

The upper end of a massless spring is fixed to a rigid support. It carries a horizontal disc of mass 200 g at the lower end. It is observed that the system oscillates with frequency 10 Hz and the amplitude of the damped oscillations reduces to half its undamped value in one minute. Calculate (i) the damping constant, (ii) the relaxation time, (iii) quality factor of the system and (iv) the force constant of the spring.

Let us now summarise what you have learnt in this unit.

18.4 SUMMARY

Concept	Description
Equation of motion of a damped oscillator	■ The **equation of motion** of a **damped oscillator** is $$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + \omega_0^2 x = 0$$ where $2b = \gamma / m$ and $\omega_0^2 = k / m$
Solution for heavy damping	■ The **solution** of the equation of motion **for heavy damping** $(b > \omega_0)$ is $$x(t) = \frac{v_0}{2\beta} \exp(-bt)[\exp(\beta t) - \exp(-\beta t)]$$ where $\beta = \sqrt{b^2 - \omega_0^2}$

Critical damping

- For **critical damping** $(b = \omega_0)$, the instantaneous displacement of a damped oscillator is given by

$$x(t) = (p + qt)\exp(-bt)$$

Weak damping

- In case of **weak damping** $(b < \omega_0)$, the expression for instantaneous displacement of a damped oscillator is

$$x(t) = a_0 \exp(-bt)\cos\left(\omega_d t - \frac{\pi}{2}\right) = a(t)\sin\omega_d t$$

Amplitude and average energy

- The **amplitude** and **average energy** of a weakly damped oscillator decrease exponentially with time:

$$a(t) = a_0 \exp(-bt)$$

and $$\langle E \rangle = E_0 \exp(-2bt)$$

where a_0 is the initial amplitude and E_0 is the energy of the undamped oscillator.

Time period of weakly damped oscillator

- The **period** of a weakly damped oscillator is given by

$$T = \frac{2\pi}{\omega_d} = \frac{2\pi}{(\omega_0^2 - b^2)^{1/2}} = \frac{2\pi}{\left(\frac{k}{m} - \frac{\gamma^2}{4m^2}\right)^{1/2}}$$

Characteristics of weak damping; logarithmic decrement

- A weakly damped system is **characterised** by **logarithmic decrement, relaxation time** and **quality factor**. The logarithmic decrement is defined as the logarithm of the ratio of successive amplitudes separated by one period. It is given by

$$\lambda = \frac{1}{(n-1)}\ln\left(\frac{a_1}{a_n}\right)$$

Relaxation time

- The **relaxation time** (τ) is defined as the time taken by the amplitude of a weakly damped oscillator to decay to e^{-1} or 36.8% of its maximum amplitude. The relaxation time τ is related to damping factor b as $\tau = 1/b$.

Quality factor

- The **quality factor** Q of a weakly damped harmonic oscillator is defined as the ratio between the energy stored in the system and energy dissipated per radian. Its expression is given as

$$Q = \frac{\omega_0}{2b} = \frac{\omega_0 \tau}{2}$$

18.5 TERMINAL QUESTIONS

1. A damped harmonic oscillator is represented by the equation

$$m\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + kx = 0$$

with $m = 0.25$ kg, $\gamma = 0.070$ kg s^{-1} and $k = 85$ Nm^{-1}. Calculate (i) the period of oscillation, (ii) the number of oscillations in which amplitude of the oscillator will become half of its initial, (iii) the number of oscillations in which its mechanical energy will drop to half of its initial value and (iv) quality factor of the oscillator.

2. A block attached to a spring is made to oscillate with an initial amplitude of 12 cm. After 2.4 minutes, the amplitude decreases to 6 cm. Calculate (i) the time when the amplitude becomes 3 cm and (ii) the value of damping constant γ for this motion.

3. The period of a simple pendulum is 2 s and its amplitude is 5°. After 20 complete oscillations, its amplitude decreases to 4°. Calculate the damping constant and relaxation time.

4. The quality factor of a sonometer wire is 4,000. The wire vibrates at a frequency of 300 Hz. Calculate the time in which its amplitude will decrease to half of its initial value.

5. Starting from the definition $E(t) = K.E.(t) + P.E.(t)$, show that, for a damped oscillator

$$E(t) = E_0 \exp(-2bt)$$

where E_0 is the energy of an undamped oscillator.

6. The quality factor of a tuning fork of frequency 512 Hz is 6×10^4. Suppose that its initial energy is E_0. Calculate the time in which its energy drops to $E_0 e^{-1}$. How many oscillations will the tuning fork make in this time?

7. A box of mass 0.2 kg is attached to one end of a spring whose other end is fixed to a rigid support. When a mass of 0.8 kg is placed inside the box, the system executes 4 oscillations per second and the amplitude falls from 2 cm to 1 cm in 30 s. Calculate (i) the force constant, (ii) the relaxation time and (iii) the quality factor of the system.

8. A steady force of 60 N is required to vertically lift a mass of 1 kg through a viscous liquid at a constant speed of 5 ms^{-1}. Assuming that the effect of viscosity can be taken to be proportional to velocity, calculate the proportionality constant. The mass is suspended in the same liquid by a spring of force constant 50 Nm^{-1}. Calculate the equilibrium extension of the spring. The mass is pulled down and released from rest. It executes oscillatory motion of continuously decaying amplitude. Calculate damping constant and period of oscillation. Take $g = 10$ ms^{-2}.

18.6 SOLUTIONS AND ANSWERS

Self-Assessment Questions

1. On comparing the given equation with Eq. (18.3) – the equation of motion of a damped harmonic oscillator – we find that $b = 10$ and $\omega_0 = 5$. Since $b > \omega_0$, the given equation represents non-oscillatory motion characterising a heavily damped oscillator.

2. From Section 18.2.3, we know that

$$x(t) = a\ \exp(-bt)$$

does not represent complete solution of Eq. (18.3). Suppose that another solution of Eq. (18.3) is represented by

$$x(t) = qt\ \exp(-bt)$$

On calculating dx/dt and d^2x/dt^2, using the above expression for $x(t)$ and substituting them in Eq. (18.3), we note that it is satisfied. So, according to the superposition principle, we can write the complete solution of Eq. (18.3) as

$$x(t) = (p + qt)\exp(-bt)$$

Note that the factor a has been replaced by $(p + qt)$ where p and q are constants.

3. The time period of the given damped spring-mass system

$$T_d = \frac{200\ \text{s}}{100} = 2\,\text{s} \qquad \text{(i)}$$

Now, as per the problem, the system is oscillatory despite being damped. This is possible only if the system is weakly damped. Thus, we can write

$$T_d = \frac{2\pi}{\omega_d} = \frac{2\pi}{(\omega_0^2 - b^2)^{1/2}} \qquad \text{(ii)}$$

From Eqs. (i) and (ii), we get, $\omega_0^2 = \pi^2 + b^2$. Hence, we can write the time period of the oscillator without damping as

$$T = \frac{2\pi}{\omega_0} = \frac{2\pi}{(\pi^2 + b^2)^{1/2}} \qquad \text{(iii)}$$

To compute b, we use the relation for the instantaneous amplitude of the weakly damped system:

$$a(t) = a_0\ \exp(-\,bt)$$

Taking logarithm on both sides, this may be written as:

$$b = \frac{1}{t}\ln\left(\frac{a_0}{a(t)}\right)$$

Substituting the values of t, a_0 and a from the problem, we get:

$$b = \frac{1}{200\ \text{s}}\ln\left(\frac{10\ \text{cm}}{2.5\ \text{cm}}\right) = \frac{2.3}{200\ \text{s}}\log_{10} 4 = 6.9 \times 10^{-3}\ \text{s}^{-1} \qquad \text{(iv)}$$

Substituting Eq. (iv) in Eq. (iii), we get

$$T = \frac{2\pi}{[\pi^2 + (6.9\times10^{-3})^2]^{1/2}} \approx 2.0\ \text{s} \approx T_d$$

Thus, we find that the time periods with and without damping of a weakly damped system is nearly equal.

4. We are told that mass, $m = 1\,\text{kg}$, force constant, $k = 4\ \text{Nm}^{-1}$ and damping constant, $\gamma = 0.6\ \text{Nsm}^{-1}$. Therefore, the damping factor is given by

$$b = \frac{\gamma}{2m} = \frac{0.6\ \text{Ns m}^{-1}}{2\ \text{kg}} = 0.3\,\text{s}^{-1}$$

and the angular frequency, when damping is absent, is given by

$$\omega_0 = \sqrt{\frac{k}{m}} = 2\,\text{s}^{-1}$$

i) From these results, we note that $\omega_0 > b.$ This implies that the system is weakly damped and the motion will be oscillatory.

ii) The extent of damping may be changed by changing the damping factor, b. Further, we know that the condition for critical damping is, $\omega_0 = b.$ So, to attain critical damping, we must change b such that $b = 2\,\text{s}^{-1}\ (= \omega_0).$ To attain this value of b, the damping constant, γ should be

$$\gamma = 2mb = 2 \times (1\ \text{kg}) \times (2\ \text{s}^{-1}) = 4\ \text{Ns m}^{-1}$$

iii) The system may be made critically damped by changing m and keeping γ unchanged. If m' is the new value of mass for which the system is critically damped for the same value of the damping constant γ, then

$$\omega_0 = \sqrt{k/m'} \quad \text{and} \quad b = \frac{\gamma}{2m'}$$

And, as we know, for critically damped system, $\omega_0 = b.$ On substituting the values of ω_0 and b with new mass, m', we get

$$\sqrt{\frac{k}{m'}} = \frac{\gamma}{2m'}$$

$$m' = \frac{\gamma^2}{4k} = \frac{(0.6\ \text{Ns m}^{-1})^2}{4 \times (4\ \text{N m}^{-1})} = 0.0225\ \text{kg} = 22.5\ \text{g}$$

5. We know that the logarithmic decrement λ is given by Eq. (18.25):

$$\lambda = \frac{1}{(n-1)} \ln\left(\frac{a_1}{a_n}\right)$$

As per the problem, $a_1 = 5°,\ a_n = 3°$ and $n = 30.$ So, we can write

$$\lambda = \frac{1}{29} \ln\left(\frac{5°}{3°}\right) = \frac{1}{29} \ln\left(\frac{5}{3}\right)$$

$$= \frac{0.511}{29} = 0.018$$

Further, the damping factor, b and λ are related by Eq. (18.24):

$$\lambda = bT \quad \Rightarrow \quad b = \frac{\lambda}{T} = \frac{0.018}{4\ \text{s}} = 0.01\ \text{s}^{-1}$$

To compute the number of oscillations in which amplitude reduces by 25%, we note that the time variation of amplitude of a damped oscillator is given by Eq. (18.10):

$$a(t) = a_0 \exp(-bt)$$

Since the amplitude reduces by 25% in time, say t_1, we get

$$\frac{a(t_1)}{a_0} = \frac{3}{4}$$

So, $$\frac{3}{4} = \exp(-bt_1) \quad \Rightarrow \quad bt_1 = \ln\left(\frac{4}{3}\right)$$

$$\Rightarrow \qquad t_1 = \frac{1}{b}\ln\left(\frac{4}{3}\right) = \frac{0.285}{0.01\,\text{s}^{-1}} \cong 57\ \text{s}$$

That is, the amplitude will reduce by 25% in 57 s. Since the period of the pendulum is 4 s, the number of oscillations completed in this time is

$$\frac{57\,\text{s}}{4\,\text{s}} \cong 14$$

Thus, in 14 oscillations, amplitude of the pendulum will reduce by 25%.

6. The angular frequency of oscillation of a weakly damped oscillator is given by

$$\omega_d = \sqrt{\omega_0^2 - b^2} \qquad \Rightarrow \qquad \omega_0^2 = \omega_d^2 + b^2$$

Further, the angular frequency of the damped oscillator can also be expressed as

$$\omega_d = 2\pi f = 2\pi \times 10\ \text{s}^{-1} = 20\pi\ \text{s}^{-1}$$

Let the undamped amplitude be a_0. The damped amplitude $a(t)$ at time t is $a_0 \exp(-bt)$. In this case, $t = 60\,\text{s}$. So, we can write

$$a(60\,\text{s}) = \frac{a_0}{2} = a_0 \exp(-60b)$$

$$\therefore \qquad b = \frac{1}{60}\ln 2 = 0.01\,\text{s}^{-1}$$

i) The resistive force constant or the damping force constant is

$$\gamma = 2\,mb = 2 \times (0.2\,\text{kg}) \times (0.01\,\text{s}^{-1}) = 4.8 \times 10^{-3}\ \text{Nsm}^{-1}$$

ii) The relaxation time is $\tau = \frac{1}{b} = \frac{60}{\ln 2} = 86.6\,\text{s}$

iii) The quality factor is $Q = \frac{\omega_d \tau}{2}$

$$\therefore \quad Q = \frac{20\pi\ \text{s}^{-1} \times 86.6\ \text{s}}{2} = 2720$$

iv) The force constant of the spring is

$$k = \omega_0^2\, m = m\,(\omega_d^2 + b^2)$$

$$= (0.2\,\text{kg}) \times [(400\,\pi^2\,\text{s}^{-2}) + (.01\,\text{s}^{-1})^2] = 790\ \text{Nm}^{-1}$$

Terminal Questions

1. For the given damped harmonic oscillator, $m = 0.25$ kg, $\gamma = 0.070\ \text{kgs}^{-1}$ and $k = 85\ \text{Nm}^{-1}$. These values show that $b < \omega_0$ and hence the damped oscillator is weakly damped.

   i) For weakly damped oscillator, the period of oscillation is given by Eq. (18.11):

$$T_d = \frac{2\pi}{\sqrt{\frac{k}{m} - \left(\frac{\gamma}{2m}\right)^2}} = \frac{2\pi}{\sqrt{\frac{85\ \text{N m}^{-1}}{0.25\ \text{kg}} - \left(\frac{0.070\ \text{kg s}^{-1}}{2 \times 0.25\ \text{kg}}\right)^2}}$$

$$= \frac{2\pi}{\sqrt{(340 - 1.96 \times 10^{-2})\ \text{s}^{-2}}}$$

$$= \frac{2\pi}{18.44\ \text{s}^{-1}} = 0.34\ \text{s}$$

   ii) We know that amplitude of a damped oscillator is given by

$$a(t) = a_0 \exp(-bt) = a_0 \exp\left(-\frac{\gamma t}{2m}\right)$$

   As per the problem,

$$\frac{a(t)}{a_0} = \frac{1}{2} = \exp\left(-\frac{\gamma t}{2m}\right)$$

   On taking natural logarithm of both sides and rearranging terms, we get

$$t = \frac{2m \ln 2}{\gamma} = \frac{2 \times (0.25\ \text{kg}) \times 0.693}{0.070\ \text{kg s}^{-1}} = 4.95\ \text{s}$$

   Since the period of the oscillator is 0.34 s, the amplitude will reduce to half in $4.95\ \text{s} / 0.34\ \text{s} \approx 15$ oscillations.

   iii) From Eq. (18.12), we recall that average energy associated with a damped harmonic oscillator is given by

$$< E > = E_0 \exp(-bt) = E_0 \exp\left(-\frac{\gamma t}{2m}\right)$$

$$\therefore \qquad \frac{< E >}{E_0} = \exp\left(-\frac{\gamma t}{2m}\right)$$

   For $\frac{< E >}{E_0} = \frac{1}{2}$, we have

$$\frac{1}{2} = \exp\left(-\frac{\gamma t}{2m}\right)$$

Taking natural logarithm on both sides and rearranging the terms, we can rewrite it as

$$t = \frac{m \ln 2}{\gamma} = \frac{(0.25\,\text{kg}) \times 0.693}{0.070\,\text{kg s}^{-1}} = 2.48\,\text{s}$$

Since the period of the oscillator is $T = 0.34\,\text{s}$, we find that the energy of the oscillator will drop to half of its initial value in about $2.48\,\text{s}/0.34 \approx 7$ oscillations.

iv) From Eq. (18.22), we recall that quality factor Q of a damped harmonic oscillator is given by

$$Q = \frac{\omega_0 m}{\gamma}$$

since $\tau = \dfrac{1}{b} = \dfrac{2m}{\gamma}$. On substituting the values of various quantities, we get

$$Q = \frac{(18.44\,\text{s}^{-1}) \times (0.25\,\text{kg})}{0.070\,\text{kg s}^{-1}} \approx 66$$

The value of ω_0 has been taken from (i) above where we have used

$$T = \frac{2\pi}{\omega_d} \approx \frac{2\pi}{\omega_0}$$

2. Since the amplitude of the oscillator decreases with time, it is a damped oscillator. The amplitude of a damped oscillator is given by

$$a(t) = a_0 \exp(-bt) \qquad \text{(i)}$$

or

$$\exp(bt) = \frac{a_0}{a(t)} \quad \Rightarrow \quad b = \frac{1}{t} \ln\left(\frac{a_0}{a(t)}\right) \qquad \text{(ii)}$$

As per the problem,

$$a(t = 2.4\text{ min}) = 6\text{ cm} = 0.06\text{ m}$$

$$a_0 = 12\text{ cm} = 0.12\text{ m}$$

$$t = 2.4\text{ min} = 144\,\text{s}$$

On substituting these values in Eq. (ii), we get

$$b = \frac{1}{144\,\text{s}} \ln\left(\frac{0.12\text{ m}}{0.06\text{ m}}\right) = 4.81 \times 10^{-3}\,\text{s}^{-1}$$

i) To find the time in which amplitude becomes 3 cm, we have

$$a(t) = 3\text{ cm} = 0.03\text{ m}$$

So, from Eq. (i):

$$a(t) = a_0 \exp(-bt)$$

$$t = \frac{1}{b} \ln\left(\frac{a_0}{a(t)}\right) = \frac{1}{4.81 \times 10^{-3}\,\text{s}^{-1}} \ln\left(\frac{0.12\text{ m}}{0.03\text{ m}}\right)$$

$$= 288.2\text{ s} = 4.8\text{ min}$$

ii) The expression for damping constant is

$$\gamma = 2bm$$

$$= 2\times(4.81\times10^{-3}\,s^{-1})\times(1\,kg)$$

$$= 9.61\times10^{-3}\,Ns\,m^{-1}$$

3. The progressive decrease in the amplitude (angular displacement) of a simple pendulum due to damping (say, air resistance) can be expressed as:

$$\theta = \theta_0 \exp(-\,bt)$$

$$\Rightarrow \qquad b = \frac{1}{t}\ln\left(\frac{\theta_0}{\theta}\right)$$

As per the problem, $\theta_0 = 5°$, $\theta(t) = 4°$ and $T = 2\,s.$ Since the decrease in amplitude takes place in 20 oscillations, the time taken by the pendulum to complete these oscillations is $20\times2\,s=40\,s.$ So, $t=40\,s.$

On substituting the values of θ_0, $\theta(t)$ and t in the above expression, we get

$$b = \frac{1}{40\,s}\ln\left(\frac{5}{4}\right) = 5.58\times10^{-3}\,s^{-1}$$

and, the relaxation time, τ is given by

$$\tau = \frac{1}{b} = 179.3\,s$$

4. Since the quality factor, Q of a weakly damped system is given by $Q = \frac{\omega_0\tau}{2}$, we can write the relaxation time, τ as

$$\tau = \frac{2Q}{\omega_0} = \frac{2\times4000}{2\pi\times300\,s^{-1}} = 4.24\,s$$

Further, the amplitude, $a(t)$ of a damped oscillator at time t is given by

$$a(t) = a_0 \exp(-\,bt) = a_0\,\exp(-t/\tau)$$

So, $\quad t = \tau\,\ln\frac{a_0}{a} = (4.24\,s)\times\ln(2) = 2.94\,s$

5. We are given that

$$E(t) = K.E.(t) + P.E.(t)$$

$$= \left(\frac{1}{2}\right)m\left(\frac{dx}{dt}\right)^2 + \left(\frac{1}{2}\right)kx^2 \qquad \text{(i)}$$

where dx/dt denotes instantaneous velocity. For a weakly damped oscillator, the instantaneous displacement is given by

$$x(t) = a_0\,\exp(-bt)\cos(\omega_d + \phi)$$

By differentiating it with respect to time, we get instantaneous velocity:

$$\frac{dx}{dt} = a_0 \exp(-bt)\,[b\cos(\omega_d t + \phi) + \omega_d \sin(\omega_d t + \phi)] \qquad \text{(ii)}$$

Hence, kinetic energy of the given oscillator is

$$K.E. = \left(\frac{1}{2}\right) m \left(\frac{dx}{dt}\right)^2$$

$$= \left(\frac{1}{2}\right) m\, a_0^2 \exp(-2bt)\,[b\cos(\omega_d t + \phi) + \omega_d \sin(\omega_d t + \phi)]^2$$

$$= \left(\frac{1}{2}\right) m\, a_0^2 \exp(-2bt)\,[b^2 \cos^2(\omega_d t + \phi)$$

$$+ \omega_d^2 \sin^2(\omega_d t + \phi) + b\,\omega_d \sin 2(\omega_d t + \phi)] \qquad \text{(iii)}$$

Similarly, the potential energy of the oscillator is

$$P.E. = \left(\frac{1}{2}\right) kx^2 = \left(\frac{1}{2}\right) m\,\omega_0^2\, x^2$$

On substituting for x, we get

$$P.E. = \left(\frac{1}{2}\right) m a_0^2 \omega_0^2 \exp(-2bt) \cos^2(\omega_d t + \phi) \qquad \text{(iv)}$$

Hence, the total energy of the oscillator at any time t is given by

$$E(t) = \left(\frac{1}{2}\right) m a_0^2 \exp(-2bt)\,[(b^2 + \omega_0^2)\cos^2(\omega_d t + \phi)$$

$$+ \omega_d^2 \sin^2(\omega_d t + \phi) + b\omega_d \sin 2(\omega_d t + \phi)] \qquad \text{(v)}$$

When damping is weak, the amplitude of oscillation does not change much over one oscillation. So we may take the factor exp $(-2bt)$ as essentially constant. Further, since

$$< \sin^2(\omega_d t + \phi) > = < \cos^2(\omega_d t + \phi) > = \frac{1}{2}$$

and $< \sin 2(\omega_d t + \phi) > = 0,$ the energy of a weakly damped oscillator, when averaged over one cycle, is

$$< E > = \left(\frac{1}{2}\right) m a_0^2 \exp(-2bt) < [(b^2 + \omega_0^2)\cos^2(\omega_d t + \phi)$$

$$+ \omega_d^2 \sin^2(\omega_d t + \phi) + b\omega_d \sin 2(\omega_d t + \phi)] >$$

$$= \left(\frac{1}{2}\right) m a_0^2 \exp(-2bt) \left[\frac{b^2 + \omega_0^2}{2} + \frac{\omega_d^2}{2}\right]$$

Since $\omega_d^2 = \omega_0^2 - b^2$, this expression simplifies to

$$< E > = \left(\frac{1}{2}\right) m a_0^2 \ \omega_0^2 \ \exp(-2bt) \qquad \text{(vi)}$$

From Unit 16, we recall that $E_0 = (1/2) m a_0^2 \, \omega_0^2$ is the total energy of an undamped oscillator. Hence, we can write

$$< E > = E_0 \ \exp(-2bt)$$

6. The average energy of a weakly damped oscillator is given by

$$< E > = E_0 \ \exp(-2bt)$$

Since $b = \frac{1}{\tau}$, we can write

$$< E > = E_0 \ \exp\left(-\frac{2t}{\tau}\right)$$

When $t = \tau/2$, $< E > = \frac{E_0}{e}$

So, in time $(\tau/2)$, the energy of the oscillator will drop to $E_0 e^{-1}$. To calculate τ (the time for this to happen), we use the expression for quality factor, $Q = \frac{\omega_d \tau}{2}$. This gives

$$\tau = \frac{2Q}{\omega_d} = \frac{2 \times 6 \times 10^4}{2\pi \times 512 \text{ s}^{-1}}$$

$$= \frac{3 \times 10^4}{256\pi \text{ s}^{-1}} = 37.3 \text{ s}$$

Thus, energy will reduce to $1/e$ of its initial value in 18.7 s.

The number of oscillations, n made by the tuning fork in this time is given by

$$n = f_d \times t = 512 \text{ s}^{-1} \times 18.7 \text{ s} = 95.7 \times 10^2$$

7. i) Here $\omega_0 = 2\pi f = 2 \times (3.14) \times 4 \text{ s}^{-1} = 25.1 \text{ s}^{-1}$

We also know that

$$\omega_0 = \sqrt{\frac{k}{m}} \quad \Rightarrow \quad k = m\omega_0^2 = (1 \text{ kg}) \times (25.1 \text{ s})^2 = 630 \text{ Nm}^{-1}$$

since total mass, $m = 0.2 \text{ kg} + 0.8 \text{ kg} = 1 \text{ kg}$

ii) Since amplitude of oscillations decreases with time, the oscillator is damped. The amplitude of the damped oscillator is given by

$$a(t) = a_0 \ \exp(-\,bt) \qquad \text{(i)}$$

As per the problem, $a_0 = 2 \text{ cm} = 0.02 \text{ m}$, $a(t) = 1 \text{ cm} = 0.01 \text{ m}$ and $t = 30 \text{ s}$. On substituting these values in Eq. (i), we get

$$0.01 \text{ m} = (0.02 \text{ m}) \exp(-\,30b)$$

or $$b = \frac{\ln 2}{30} = 2.3 \times 10^{-2} \text{ s}^{-1}$$

Hence, relaxation time

$$\tau = \frac{1}{b} = \frac{1}{2.3\times10^{-2}\ \mathrm{s}^{-1}} = 43.3\,\mathrm{s}$$

iii) For a weakly damped system, the quality factor is given by

$$Q = \frac{\omega_0\tau}{2} = \frac{25\ \mathrm{s}^{-1}\times 43.3\,\mathrm{s}}{2} = 541$$

8. When the mass is lifted vertically, it moves at constant speed and no net force acts on it. The upward force of 60 N is, therefore, balanced by the downward forces, $mg + Cv$, where C is a **constant of proportionality** (called damping constant) and v is the velocity of mass with which it is being lifted upward. Thus, we can write

$$F = mg + Cv$$

or $$C = \frac{F - mg}{v}$$

$$= \frac{(60\,\mathrm{N}) - (1\,\mathrm{kg})\times(10\,\mathrm{m s}^{-2})}{(5\,\mathrm{m\,s}^{-1})} = 10\ \mathrm{N m}^{-1}\,\mathrm{s}$$

When the mass is suspended in the fluid by a spring of spring constant, k, no viscous force acts on it when it is in equilibrium. The weight, mg will be balanced by the restoring force kx of the spring. Thus,

$$x = \frac{mg}{k} = \frac{(1\,\mathrm{kg})\times(10\,\mathrm{m s}^{-2})}{50\,\mathrm{N\,m}^{-1}} = 0.2\ \mathrm{m}$$

Since the given mass executes oscillatory motion of decaying amplitude, it is clear that the system is damped. Therefore, we can write the damping factor

$$b = \frac{C}{2m} = \frac{10\ \mathrm{Nm}^{-1}\,\mathrm{s}}{2\times(1\ \mathrm{kg})} = 5\ \mathrm{s}^{-1}$$

and $$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{50\ \mathrm{Nm}^{-1}}{1\ \mathrm{kg}}} = 7.07\,\mathrm{s}^{-1}$$

So, we have $b = 5\,\mathrm{s}^{-1}$ and $\omega_0 = 7.07\,\mathrm{s}^{-1}$. Hence

$$\omega_d = \sqrt{\omega_0^2 - b^2} = \sqrt{50 - 25} = 5\,\mathrm{s}^{-1}$$

Therefore, the frequency of oscillation is given by

$$f = \frac{\omega_d}{2\pi} = \frac{5\ \mathrm{s}^{-1}}{2\times3.14} \cong 0.80\ \mathrm{Hz}$$

and the period of oscillations

$$T = \frac{1}{f} = \frac{1}{0.80\ \mathrm{s}^{-1}} = 1.25\,\mathrm{s}$$

One of the attractions of visiting sea beach is to enjoy the pleasure of watching the waves. These waves come from far off in the sea and break down at the shores. Can you imagine how much energy stored in nature is in the form of waves?
(Source of picture: wikimedia.org)

UNIT 19

WAVE MOTION

Structure

STUDY GUIDE

In previous three units of this block, you have learnt about free and damped oscillations. In this unit, you will learn about waves. As such, oscillations and waves are related concepts, but study of waves is more important because it provides you with the basis to understand a variety of natural phenomena.

We believe that you have studied about waves in your school physics course and are familiar with many concepts associated with waves. However, a better understanding of these concepts requires careful and attentive study. Therefore, we advise you to focus on the following important aspects associated with waves: i) how does wave motion arise? ii) how do we represent waves? and iii) how is energy carried by waves? The level of mathematical treatment in this unit is fairly simple. You are advised to solve all the SAQs, and TQs to get a feel for the qualitative difference between oscillatory motion and wave motion.

"An ocean traveller has even more vividly the impression that the ocean is made of waves than that it is made of water."

Arthur Stanley Eddington

19.1 INTRODUCTION

You have learnt about waves in your school physics course. The study of wave motion is interesting because waves are present all around us. For example, what we *hear* and *see* around us depends on waves. When we *speak*, our *vocal cords* inside our throat *vibrate*. Their vibrations cause the surrounding air molecules to vibrate and the effect is heard as ***sound***. When this sound reaches the ears of other persons near us, their ear drums begin to vibrate and the sound is heard by them. You know that sound is a form of energy and it is carried by ***sound waves**, which* enable us to hear what others speak. Sound waves are used in SONAR (**So**und **Na**vigation and **R**anging) and prospecting for mineral deposits and oil (commodities governing the economy of nations these days). Now-a-days we also use ***ultra-sound waves*** – waves of frequency greater than 20 kHz – to obtain images of soft tissues in human body.

Visible light enables us to see. It is an ***electromagnetic wave***. You also know about radio waves, *X*-rays and microwaves. These are all electromagnetic waves having different frequencies. Most modern communication technologies such as radio, television, telephone, fax, etc. are based on transmission and receipt of signals in the form of radio waves and microwaves. *X*-rays are used in medical diagnosis, e.g., for taking images of bones to diagnose fractures.

Seismic wave is other lesser known wave but it is equally important. It can cause immense destruction as seen due to earthquakes in Jan., 2001 in the state of Gujarat, in Oct., 2005 in J&K and in April 2015 in Nepal. The under-sea earthquake induced tsunami in Dec., 2004 caused huge destruction in Tamil Nadu in India and many other nations in the Indian Ocean. Earthquakes in Chile, China, Iran, Japan, Pakistan and several other countries have caused huge losses of life and resources. At the microscopic level, we learn about ***matter waves*** to understand the nature of atoms, molecules, electrons, protons and other elementary particles.

All these examples should convince you that understanding the physics of wave motion is of fundamental importance. The waves mentioned above can be broadly categorised into three main types: **mechanical waves**, **electromagnetic waves** and **matter waves**. Much of what we discuss in this unit applies to waves of all kinds. However, in this unit we shall discuss only mechanical waves on a string and sound waves. You will learn about electromagnetic waves in the second semester course on Electricity and Magnetism and the fourth semester course on Wave and Optics and about matter waves in the course on Modern Physics.

We begin our study of wave motion by describing, in Sec. 19.2, **how waves are formed and propagate**. We will consider the examples of waves on a string and sound waves. In Sec. 19.3, you will learn to depict wave motion graphically and describe it mathematically. In this section, we also define various **wave parameters** and discuss the concepts of the **phase of a wave**, **phase difference** and **phase velocity**.

Expected Learning Outcomes

After studying this unit, you should be able to:

- ❖ Explain how a wave is formed and how it propagates in a medium;
- ❖ Represent waves at a fixed position or at a fixed time graphically;
- ❖ Define the wave parameters (amplitude, time period, frequency, angular frequency, wavelength, wave number and phase);
- ❖ Write the mathematical expression for a wave;
- ❖ Calculate the values of wave parameters for a given wave; and
- ❖ Explain the concept of phase difference for waves.

19.2 WAVE FORMATION AND PROPAGATION

To discuss formation of waves and their propagation, we begin with water waves because we can easily observe them. If you drop small pebbles in still water, say, in a pond or in a tub (or a bucket), you will observe circular ripples spreading out on the water surface. These ripples spread out from the point at which the pebbles strike the water surface (Fig. 19.1). When you look casually at these ripples, you may get a feeling that water itself moves with them. However, if you observe the ripples carefully, you can see that this is not true: *water does not move along with the ripples*. You can verify this by placing a paper boat or a leaf on the water surface. You will observe that the paper boat (or the leaf) bounces up and down at the same place on the water surface, without moving with the ripples.

Fig. 19.1: **Water waves in still water in a container.**

You may now like to know: **If water does not move along the surface, what is it that moves in a water wave**? What moves is the **disturbance** caused in still water when we drop pebbles in it. This disturbance is transferred to the adjacent water molecules. To appreciate what we mean by disturbance and how it propagates, refer to Fig. 19.2 which shows N masses, each of mass m coupled to each other by identical springs. A push or pull at one end of the

The word **disturbance** has been used here as a general term which refers to the deformation in the shape of the water surface with respect to its undisturbed horizontal surface. It can also be used for a string.

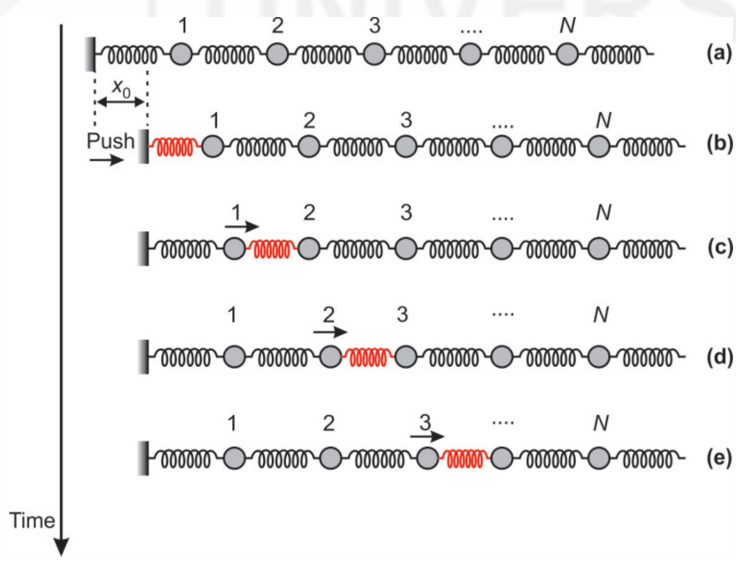


Fig. 19.2: a) A system of *N*-coupled masses. b), c), d) and e) shows the instantaneous configuration of the system as *disturbance* (compression of the spring), say, on the left of mass 1 is transferred by it to the left of mass 2, by mass 2 to the left of mass 3 and so on via adjacent masses.

chain of spring-mass systems makes a disturbance propagate along the chain.

When no force is applied to disturb the system, the masses are located at their respective equilibrium positions as shown in Fig. 19.2a. If we give a sudden push to the left end of the chain of coupled masses, as shown in Fig. 19.2b, so that it is displaced by a distance x_0 towards the right, the first spring is compressed. The compressed spring pushes mass 1 and displaces it from its equilibrium position towards the right. The mass 1 in turn, pushes the spring on its other side to its right. After mass 1 is displaced by a distance x_0, the spring on its left is relaxed while spring to its right is compressed and pushes mass 2. The second spring repeats what the first spring did to it. As a result, the disturbance, the compression of the spring, moves between masses 2 and 3. This sequence repeats itself as time passes, as shown in Fig. 19.2c, d and e. **From these observations, we can say that the net result of pushing one of the masses in the chain is that disturbance propagates in the form of compressions of springs along the chain of coupled spring-mass systems.** A similar sequence of events would be repeated if we pulled the spring at the left-end of the chain and created a disturbance in the form of extension of the spring. In both cases, the disturbance – compression or extension of the spring – propagates along the chain via spring-mass systems.

If we periodically disturb (displace) the first mass from its equilibrium position, individual masses will gradually begin to oscillate about their respective equilibrium positions. Note that **the masses and the springs or the system as a whole do not leave their positions; what moves instead is the disturbance – compression and extension of the spring.**

Perform the following simple activity with a string to appreciate this idea of 'propagation of disturbance' which will help you understand how waves are formed.

Activity

Take a long thin elastic string and fix one of its ends to a distant wall as shown in Fig. 19.3a.

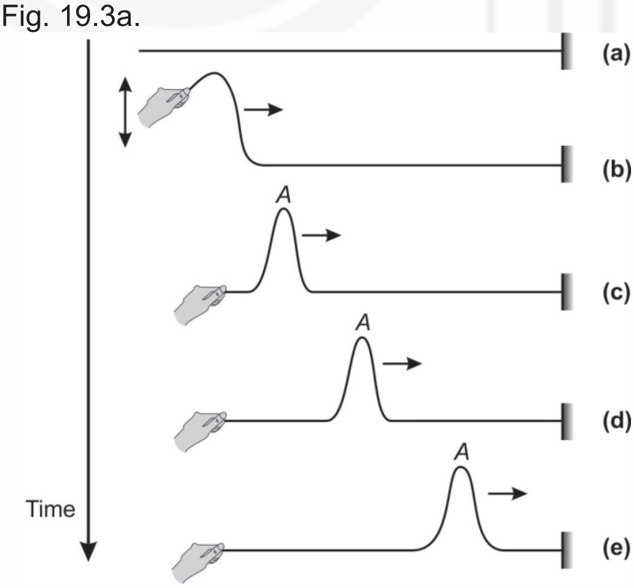


Fig. 19.3: **a) An elastic string fixed to a rigid wall; b) one quick up and down motion of the portion of the string held by hand; c) pulse *A* is generated; d) and e) show that the pulse *A* moves along the string as time passes.**

Hold the other end of the string with your hand so that the string is stretched and taut. Now quickly move your hand up and down, once. What do you observe? A disturbance marked *A* in Fig. 19.3c travels along the length of the string. An isolated disturbance like *A* is called a **pulse** and it is generated due to one quick up and down motion of the portion of the string held in your hand.

What will happen if you keep your hand moving up and down continuously? You will observe that a series of pulses move along the string giving rise to a **wave**. If the motion of the hand is sinusoidal, the wave will have sinusoidal shape at any given time. This is the simplest way of producing a mechanical wave.

You may ask: **How does the periodic motion of the string in your hand, as in the above activity, give rise to a wave moving along the string?** To answer this question, let us look a little more closely at the motion of the string. Refer to Fig. 19.4.

We can see the propagation of a disturbance in water or on a string. Can we see a sound wave propagating in air? We cannot. Then you may like to know as to how we detect sound waves. For this, we observe the motion at the source (like *sitar* string or *tabla* membrane) or at the receiver (microphone membrane).

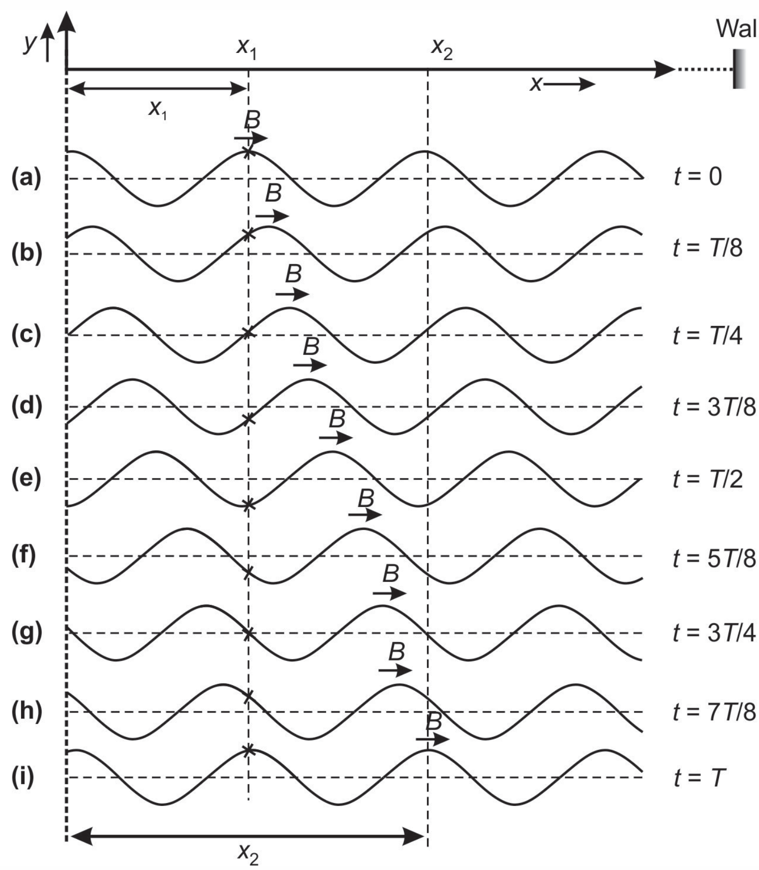


Fig. 19.4: **Periodic motion of the string element held in hand generates a disturbance with a sinusoidal profile. Parts (a)-(i) of the figure depict how the disturbance generated by the hand travels along the string to form a wave.**

Figs. 19.4a to 19.4i show nine **snapshots** of a wave on the string beginning at the instant $t = 0$. These have been taken at the intervals of $T / 8$ up to the instant $t = T$. Here T is the **time period**.

Fig. 19.4a shows the waveform at the instant $t = 0$. In this figure, look at the point marked B on the string. It is the maximum value of the disturbance (displacement of the element of the string from its equilibrium position) and is called the ***crest*** of the wave. Let us denote the position of the crest *B* on the *x*- axis at this instant by x_1. Next, we tie a ribbon at the point of the string at the position x_1. Note that *it coincides with the position of crest B at the instant* $t = 0$. The question we now ask is: **What happens to the crest *B* as we move the free end of the string up and down?**

Note from Fig. 19.4b that, in the time interval $t = 0$ to $t = T/8$, the crest *B* on the string has moved towards the right. The motion of *B* is indicated by the short arrow beside it. As *t* increases, crest *B* (and the arrow) moves further away from the oscillating hand (Figs. 19.4 c to h). At $t = T$, the crest *B* is at the position $x = x_2$ on the *x*-axis.

Now focus on the motion of the ribbon tied at the position $x = x_1$ on the string (Fig. 19.4a). Note that, as time passes and as the disturbance (the crest at *B*) moves along the string, the ribbon moves up and down just like the hand even as crest *B* moves from position x_1 to position x_2. From this we conclude that as wave (disturbance) propagates in a medium (string), two distinct motions are taking place: **the disturbance (represented by crest *B*) and the particles of the medium (represented by the ribbon) about their respective equilibrium positions**.

You may now like to ask: If the particles of the string do not move along the wave, then what is transported? To discover the answer to this question, you should perform the following activity using a spring-mass system.

Activity

Take a string and mark nine equidistant points (1 to 9) on it, as shown in Fig. 19.5a. Let each point represent a particle at that position. You can tie a ribbon at each point to observe their motion very distinctly. Tie one end of this string (at mark 1) to a vertical spring-mass system which can execute vertical oscillations and the other end to a rigid wall.

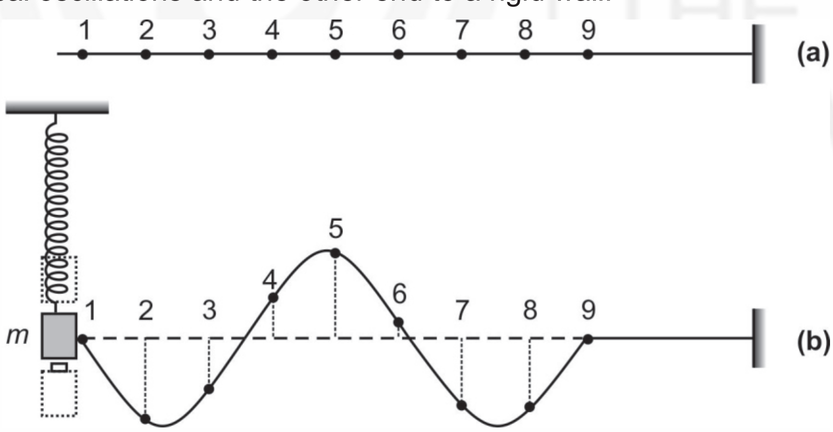


Fig. 19.5: **a) A string marked with equidistant points (1 to 9); b) snapshot of the string when its one end is fastened to a vertically oscillating spring-mass system and the other end is tied to a rigid wall.**

Pull (or push) the mass *m* of the spring-mass system vertically downwards and then release it. It will begin to oscillate. You will note that particles of the string at the 9 positions marked on it begin to oscillate one after the other. In a little while, a wave is set up on the string. If possible, take a snapshot (photograph), of the string. What do you observe in this photograph? We hope that you observe a waveform, as shown in Fig. 19.5b.

Let us now address the question. **Why do the particles in the string start oscillating leading to wave propagation**?

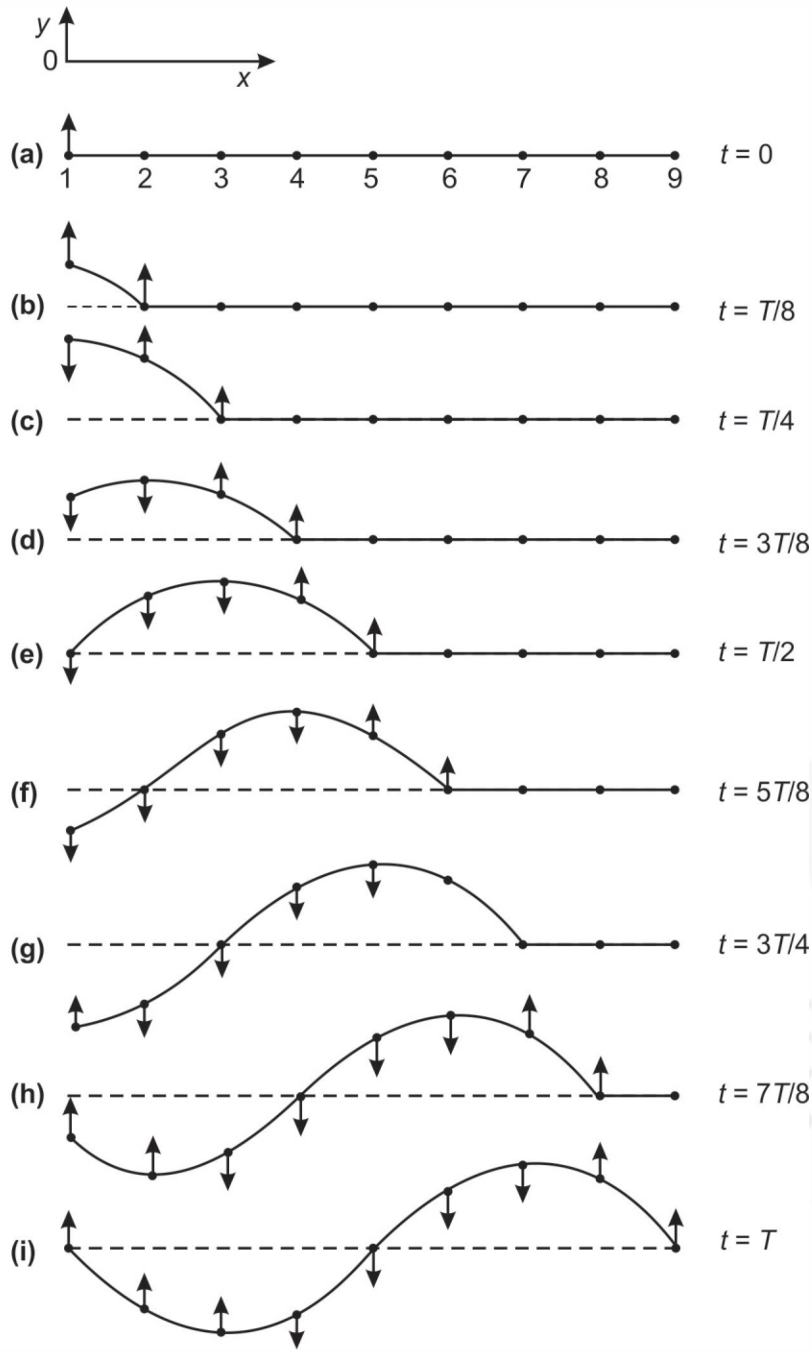


Fig. 19.6: **Snapshots of the motion of the particles at positions marked 1 to 9 of a string fastened to a vertically oscillating spring-mass system at intervals of $T/8$ in the time range $t = 0$ to $t = T$.**

Now let us refer to Fig. 19.6 which depicts snapshots of the positions of nine particles at intervals of $T/8$ from $t = 0$ to $t = T$. The arrow attached to each particle indicates the direction along which it is about to move at a particular instant. At $t = 0$, all particles are at their respective mean positions (Fig. 19.6a) but particle 1 is set to move upward. In the interval $t = 0$ to $t = T/8$, the disturbance initiated by the spring-mass system propagates from particle 1 to particle 2 (Fig. 19.6b). Similarly, in the next $T/8$ interval, the disturbance travels from particle 2 to particle 3 (Fig. 19.6c) and this process continues as the disturbance propagates from particle 1 to particle 9. Note that in this process energy from the spring-mass system is transferred to these particles. This is **how the disturbance moves in the medium**.

You may now like to know: When a wave propagates in a medium, **how do the particles of the medium move**? Study the motion of each particle at the marked positions in the time interval from $t = 0$ to $t = T$ shown in Figs. 19.6a to 19.6i. You will note that each particle executes oscillatory motion about its respective mean position. From Fig. 19.6a and 19.6i, we can conclude that

- at $t = 0$, all the particles are at their respective mean positions marked on the string (Fig. 19.6a); and
- at $t = T$ (Fig. 19.6i), particles 1, 5 and 9 are at their respective mean positions; particles 1 and 9 are about to move upward, whereas particle 5 is about to move downward. Particles 3 and 7 are at positions of maximum displacement from their respective mean positions but on opposite sides of the horizontal axis. Note that each particle oscillates about its mean position and does not move along the wave.

The curve (called the **envelope**) joining the positions of all the particles at the instant $t = T$ in Fig.19.6 (i) represents a **wave**.

Let us recapitulate important points discussed so far.

Recap

- A disturbance is generated in a medium when particles in that medium (driven by an oscillator) oscillate. The disturbance may take any shape from a finite width pulse to an infinitely long sinusoidal wave or any other shape (see, for example, Fig. 19.8) depending on the nature of the force driving the oscillations.
- The particles of the medium in which the disturbance travels, oscillate about their respective equilibrium positions (the mean position); **the particles themselves do not travel with the disturbance; they do not show any translational motion.**
- The disturbance/wave propagating in the medium **transfers energy** and **momentum, not matter** from one particle to another in the medium. (This conclusion is true for electromagnetic waves also.)

Waves carry huge energy; these can cause immense destruction and also can be used constructively for generating electricity (see the box below).

Energy Carried by Waves

A vivid demonstration of the energy carried by water waves is in the damage caused in coastal areas by **tidal waves** in stormy weather. You may know that the tidal waves generated in the super-cyclone in the Bay of Bengal in October, 1990 caused immense loss of life and property in coastal Orissa. More than ten thousand people lost their lives and millions were rendered homeless. Similar devastation by tidal waves was seen when typhoon Katrina struck US east coast in the year 2011. Tidal waves generated due to (an earthquake in Chile) carried huge amount of energy across 15,000 km of the Pacific Ocean and caused untold damage in Japan. *Do you know that a three metre high oceanic wave can lift* 30 *bags of wheat by about* 10 *ft*?

NOTE

Electromagnetic (e.m.) waves, which include radio and microwaves, infrared, visible and ultraviolet light, *X*-rays and gamma rays, travel with speed 3×10^8 ms^{-1}. Energy from the Sun reaches our planet in about 8 minutes in the form of electromagnetic radiations and sustains all forms of life.

Seismic waves can also cause untold damage. The earthquake in South Gujarat on 26 Jan., 2001 reduced the area to rubbles killing an estimated one hundred thousand people; high-rise buildings, houses and hospitals collapsed and roads developed huge cracks. Similar devastation was experienced by the people in Jammu and Kashmir in October, 2005 and people in Nepal in 2015.

An earthquake under sea bed near Indonesia in the Indian Ocean on Dec. 26, 2004 caused a **tsunami** of a height up to 30 feet and brought unimaginable misery in Indonesia, Thailand, Sri Lanka, Maldives, and India (Tamil Nadu).

You may also be aware how the energy of tidal waves is being harnessed the world over to meet the increasing electricity requirements.

Electromagnetic waves are transverse in nature.

We have taken the examples of mechanical waves on strings to introduce wave motion. **Mechanical waves can exist only in a material medium such as water, air, rocks, strings, etc.** Mechanical waves can be transverse or longitudinal. You must have learnt about these in your +2 classes. However, we now discuss these for completeness.

An isolated disturbance is called a **pulse**. When we drop a stone in still water, a pulse is generated and travels on the water surface. The sound produced by clapping of hands, a single spoken word of greeting or a command shouted by one person to another are also examples of a pulse of sound. Similarly, while standing on a railway platform, you may have seen that when a railway engine joins the compartments, the jerk produces a disturbance which is carried through the train as a pulse.

Transverse and Longitudinal Waves

Refer to Fig.19.6 again and note that the particles of the string **oscillate perpendicular to the direction in which the wave travels**. Such a wave is said to be **transverse**. Waves propagating on the strings of *ektara*, *sarangi*, *sitar*, *vina* and violin are transverse waves. As a child, you must have enjoyed playing a flute. **The musical sound produced by a flute is an example of longitudinal wave in which particles oscillate along the direction of propagation of the waves.** While a gaseous medium supports only longitudinal waves, liquids and solids support transverse waves also. Longitudinal waves arising due to the vibrations of a tuning fork are accompanied by alternate regions of compression and rarefaction. These are shown in Fig. 19.7a. You can also visualise the regions of compression and rarefaction by generating a longitudinal wave on a spring (Fig. 19.7b) by compressing or stretching it along its length.

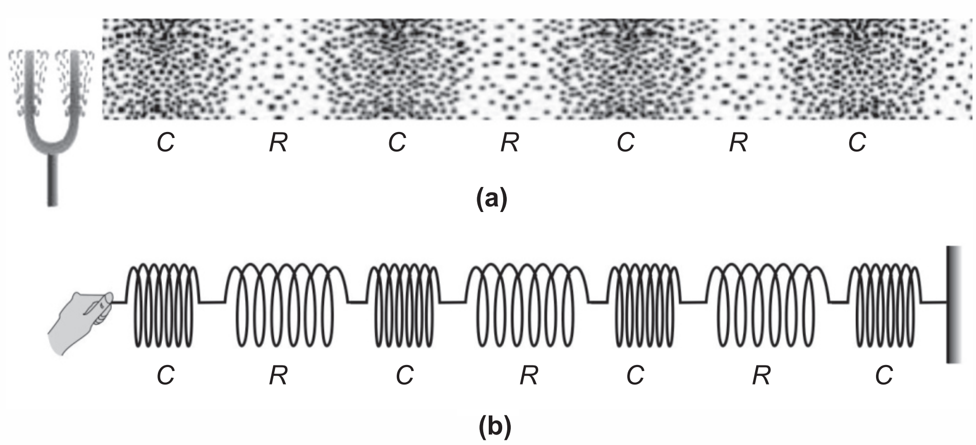


Fig. 19.7: **Longitudinal waves having alternate regions of compression (*C*) and rarefaction (*R*) in a) air; b) a spring.**

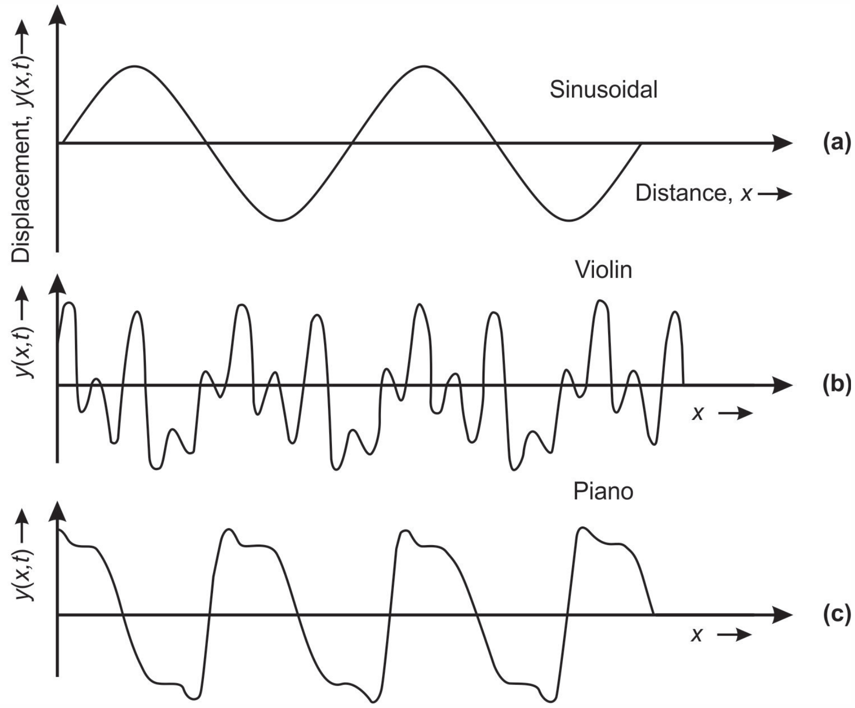


Fig. 19.8: **Waveforms for a) periodic (sinusoidal) wave; b) sound produced by a violin; c) sound produced by a piano.**

Waves produced in an continuous medium are known as **travelling waves**. In Fig. 19.8 we have shown three waveforms: a sinusoidal wave, a wave generated by a violin and a wave generated by a piano. If the oscillations of the driving force are simple harmonic, the travelling waves are said to be **harmonic waves**; these waves propagate with the frequency of the periodic driving force. In this unit we shall confine only to **harmonic waves in one-dimension**.

You may like to quickly check your understanding of waves by answering a simple SAQ.

SAQ 1 – Understanding waves

Give short answers to the following questions:

a) How does a pulse differ from a wave?

b) Give examples of waves (i) which require a medium for propagation; and (ii) which do not require a medium for their propagation.

c) State the difference between transverse and longitudinal waves. Give examples of both types of waves.

Before proceeding further, let us recall the important points discussed so far.

Recap

- When particles of a medium oscillate in a direction perpendicular to the direction in which a wave travels, the wave is called a **transverse wave**.
- When particles of a medium oscillate along the direction of propagation of the wave, the wave is called a **longitudinal wave**.

19.3 DESCRIBING WAVE MOTION

We hope that the activities and the discussion given in the preceding section have helped you to understand the role of the oscillatory motion in generation of mechanical waves. However, the description of wave motion presented so far is qualitative and as a student of physics, you may like to know: **How do we represent wave motion graphically as well as mathematically**? Now, let us discuss these aspects of wave motion. **For simplicity, we consider mechanical waves on a string,** though results will apply to other types of one-dimensional (1-D) waves as well.

19.3.1 Representation of Wave Motion

In Sec. 19.2, you have learnt that a wave is produced when a vibrating source (e.g., hand, spring-mass system or a tuning fork) creates a disturbance in a medium; the wave **travels** through the medium while the particles of the medium **oscillate** around their respective mean positions. To proceed further, we assume that

The frequency of the wave is equal to the number of vibrations completed by particles of the medium per second.

- The waves are sinusoidal.
- The medium is uniform and the wave propagates at a **constant speed**.

Under these assumptions, the wave will propagate with **the same time period** as the **period of vibration of each particle of the medium**. Since frequency is reciprocal of time period, we can write $f = 1/T$.

The **amplitude** of the wave is defined by the **maximum displacement** of the particles of the medium from their respective mean positions.

In Unit 16, you learnt the concept of phase for oscillatory motion: an oscillator is said to be in the same phase at any two instants when its states of motion are the same. For a wave, it implies that particles at two consecutive crests or troughs are in the same state of motion. **The distance between any two consecutive particles on a waveform in the same phase defines the wavelength of the wave.**

Using this information, we can arrive at graphical and mathematical representations of wave motion. Let us take the example of sinusoidal waves on a string (Figs. 19.5 and 19.6) and represent them graphically. When depicting waves, we show the displacement of the particle(s) of the medium on two types of graphs:

1. We keep the position of a particle of the medium fixed and plot its displacement as time passes.
2. We keep the time fixed and plot the displacement of the particles located at different positions.

The first type of graph is referred to as a **vibration graph.** It shows the wave behaviour at a **single location** along the path of the wave as time passes. You can obtain it by fixing a slit at a point in space along the path of the wave and observe wave motion as time passes. Recall from Fig. 19.6 that this is the same as the displacement-time graph for any oscillating particle on the string at a given position (say, $x = x_1$) on the string. It will be a sinusoidal graph, as shown in Fig. 19.9.

The graph shown in Fig. 19.9 **represents the motion of a wave with time at a given location in space**. You know that the equation representing such a motion is given by

$$y(t) = a \sin \omega t \quad (19.1a)$$

Here, a is **amplitude** of the wave (which is the same as the amplitude of oscillation of the particles) and ω is its **angular frequency**. It is related to the **frequency**, f by $\omega = 2\pi f$ and to the time period, T by $\omega = 2\pi / T$. Thus, Eq. (19.1a) can also be written as

$$y(t) = a \sin 2\pi(t/T) \quad (19.1b)$$

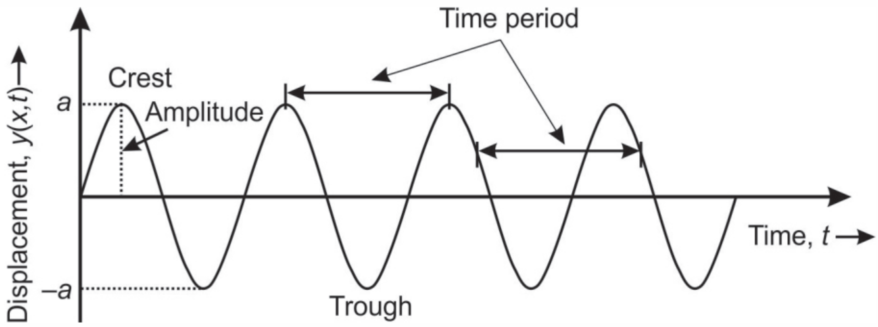


Fig. 19.9: **The vibration graph of a wave showing its motion at a *given position* as a function of time. The period of the wave is also shown.**

At this stage, you may like to recall the equation for oscillations of particles executing SHM (Unit 16). Even though we are using the sine function in this unit (Eq. 19.1a) to represent a sinusoidal function mathematically, we can as well use the cosine function and write

$$y(t) = a \cos \omega t$$

and sine would be replaced by cosine in other equations.

The **waveform graph** is obtained by keeping the time fixed and plotting displacement of particles with changing position. This graphical representation of wave motion is the same as the snapshot shown in Fig. 19.6i but taken at a much later instant of time. A **waveform graph** displays the wave behaviour at **different locations at a given time**. A typical waveform graph for waves is shown in Fig. 19.10.

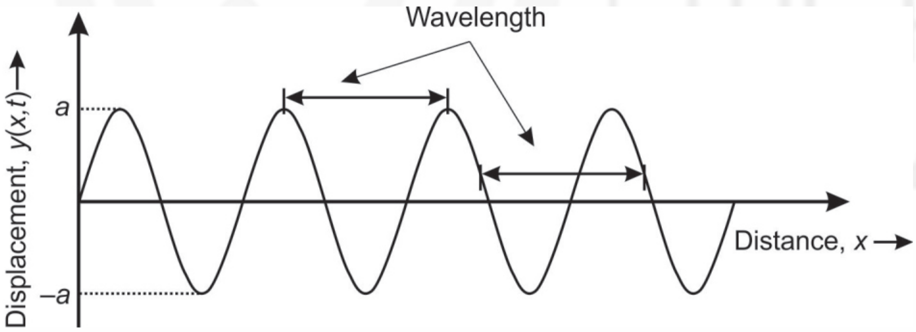


Fig. 19.10: **The waveform graph; this is like the snapshot at any instant of time, say at $t = t_1$. The distance equivalent to one wavelength is also marked.**

NOTE

Vibration graphs inform us of the wave's *shape*, *amplitude*, and *time period* and **waveform graphs** inform us of the wave's *shape*, *amplitude*, and *wavelength*.

From Figs. 19.9 and 19.10, you will note that the shapes of both types of graphs (vibration and waveform) are similar; the only difference is in the labels for the horizontal axis: For the *vibration graph*, we have **time** on the horizontal axis and for the *waveform* graph, we have **position** on the horizontal axis.

We can write the equation of the motion depicted in Fig. 19.10 as

$$y(x) = a \sin k x \quad (19.2)$$

The constant k used in Eq. (19.2) is called wave number. You should not confuse it with spring constant used in Unit 16.

where k is called **wave number**. It is related to wavelength which denotes the distance between any two consecutive points in the same state of motion along the distance-axis, as shown in Fig. 19.10. This helps us to express k in terms of λ. From the definition of the wavelength, it is clear that the displacement, $y(x)$ is the same at both ends of this wavelength, i.e., at $y(x = x_1) = y(x = x_1 + \lambda)$. Thus, using Eq. (19.2), we can write,

$$a \sin k\, x_1 = a \sin k\, (x_1 + \lambda)$$

$$= a \sin (kx_1 + k\lambda)$$

We know that for sinusoidal functions, we have $\sin(\theta + 2\pi) = \sin\theta$. Therefore, the above equality will be satisfied only when

$$k\lambda = 2\pi \quad \text{or} \quad k = \frac{2\pi}{\lambda} \qquad (19.3)$$

Eq. (19.3) gives the required relation between k and λ.

Using Eq. (19.3) in Eq. (19.2), we can write

$$y(x) = a \sin 2\pi(x/\lambda) \qquad (19.4)$$

The amplitude, time period, frequency and wavelength characterise a wave. Their definitions are given below.

WAVE PARAMETER	DEFINITION
Amplitude	The maximum positive (or negative) displacement of the particles of the medium from their respective equilibrium positions.
Time period	The time between two consecutive points in the same state of motion. These points can be two consecutive crests or two consecutive troughs.
Frequency	It is the number of oscillations in one second. It is reciprocal of the time period. We can also define frequency as the number of wavelengths that pass a given location each second along the wave's path.
Wavelength	The distance traversed by the wave in one time period. It is the distance between two consecutive particles of the medium in same state of motion.

In addition to the wave parameters defined so far, we also need to know the velocity of a wave. You may like to know: **How can we obtain the expression for the velocity with which the wave propagates**? Let us find out.

19.3.2 Relation between Wave Velocity, Frequency and Wavelength

We can establish the relation between the velocity of the wave, its frequency and its wavelength using their definitions. Recall that a wave moves a distance equal to one wavelength in one time period. Therefore, the wave velocity is given by

NOTE

Note that wave number k is defined as $2\pi / \lambda$ and it should refer to the number of waves in one meter. That is, it should be inverse of wavelength. And k (= $2\pi/\lambda$) should be called angular wave number. But we will follow the standard convention and call k the wave number. However, you should be mindful of the factor 2π used in defining angular frequency ω and wave number k.

$$v = \frac{\text{Wavelength}}{\text{Time Period}} = \frac{\lambda}{T} \quad (19.5)$$

Since frequency is reciprocal of time period, $(f = 1/T)$, we can write

$$v = f\lambda \quad (19.6)$$

You know that velocity is a vector quantity. In the discussions of wave motion here, we will consider waves travelling in 1-D only. For such waves, we can use the terms velocity and speed interchangeably.

That is, the velocity of wave is equal to the product of its frequency and its wavelength.

Note that we have obtained Eqs. (19.5) and (19.6) for one dimensional waves. But these equations hold for all kinds of waves, whether transverse or longitudinal, mechanical or electromagnetic. To give you an idea about the magnitude of *v*, we have given the values of the speed of some familiar waves) in some typical media in Table 19.1.

Table 19.1: Some Typical Wave Speeds

TYPE OF WAVE	SPEED (ms^{-1})	TYPE OF WAVE	SPEED (ms^{-1})
Sound waves in air (at STP)	332	Ripples on the surface of a pond	0.2
Sound waves in water (at STP)	1500	Seismic waves moving in the Earth's outer crust	6×10^3
Sound waves in steel (at STP)	5100	Light waves in vacuum	3×10^8

The speed of sound in air increases with temperature: $v \propto \sqrt{T}$. where temperature T is measured in Kelvin.

From Table 19.1, you can see that the speed of sound is maximum in solids, and minimum in air. But, the speed of light is much higher than that of sound. This explains why on a thundery day, we see the flash of light before the thunder.

You may now like to answer an SAQ to calculate the wave parameters.

SAQ 2 – Wave speed, frequency and wavelength

a) Light propagates with a speed of $3 \times 10^8 \text{ ms}^{-1}$. In the visible region of the electromagnetic spectrum, the wavelength, λ lies in the range 400 nm – 720 nm. Calculate the corresponding frequencies.

b) Sound propagates in air with a speed of 332 ms^{-1}. The audible range for human beings lies between 20 Hz to 20,000 Hz. Calculate the corresponding wavelengths.

We hope that now you know how to represent the time variation of a wave at a given position and how a wave changes at different points in space at a given instant of time. **However, as wave propagates in a medium, it changes both with time and position.** This complete information is contained neither in Eq. (19.1b) nor in Eq. (19.4). It means that neither of these equations describes a wave completely. To obtain complete mathematical description of a wave, we have to combine Eqs. (19.1b) and (19.4). Let us see how we do so.

19.3.3 Mathematical Description of Wave Motion

Refer to Fig. 19.11 which shows a snapshot of a wave travelling along the positive *x*-direction. Let us assume that the wave travels with velocity *v*.

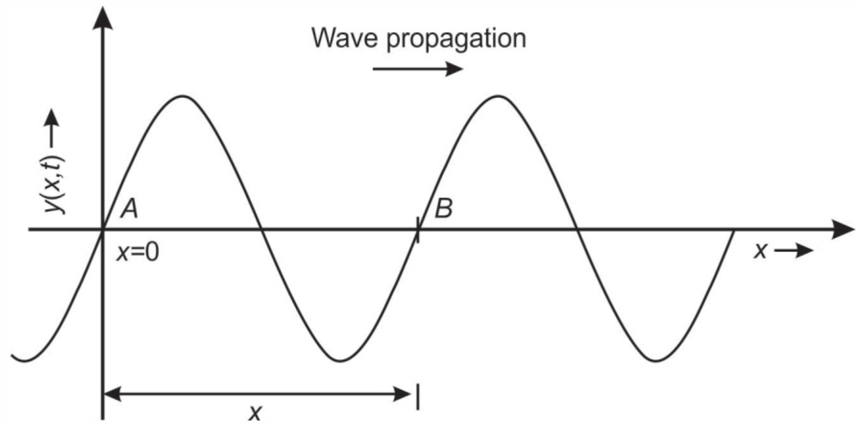


Fig. 19.11: **Snapshot of a wave travelling in the positive *x*-direction.**

Now consider two particles located at $A(x=0)$ and B on the snapshot separated by a distance *x*. You may note that the displacement, $y(x,t)$ of the particle at *B* at time *t* will be the same as the displacement, $y(x=0,t')$ of the particle at *A* at time t' if

$$t' = t - \frac{x}{v}$$

because the wave is travelling in positive *x*-direction and the time taken by the wave to travel a distance, *x* with velocity, *v* is (x/v). Therefore, we can write

$$y(x,t) = y(0,t') = y\left(x=0, t-\frac{x}{v}\right) \tag{19.7}$$

To determine the form of the function $y(0,t')$, we replace *t* by t' in Eq. (19.1b) and write the expression for displacement $y(0,t')$ as

$$y(0,t') = a\sin 2\pi(t'/T)$$

On substituting $t' = t - \dfrac{x}{v}$ in this expression, we get

$$y(0,t') = y(x=0, t-\frac{x}{v}) = a\sin\frac{2\pi}{T}\left(t-\frac{x}{v}\right)$$

$$= a\sin\left[\frac{2\pi}{\lambda}(vt-x)\right] \tag{19.8}$$

because, wave velocity, $v=(\lambda/T)$. On combining Eqs. (19.7) and (19.8), we have

$$\boxed{y(x,t) = a\sin\left[\frac{2\pi}{\lambda}(vt-x)\right]} \tag{19.9a}$$

Note that in the mathematical description of the wave given by Eq. (19.9a), the velocity and wavelength of the wave appear in the argument of sine function. It is possible to write Eq. (19.9a) in a few other equivalent forms. To do that, we note that $v=\lambda/T$. Then we can write Eq. (19.9a) in terms of *v* and *T* as

$$\boxed{y(x,t) = a\sin\left[\frac{2\pi}{T}\left(t-\frac{x}{v}\right)\right]} \tag{19.9b}$$

We can also rewrite this equation as:

$$y(x,t) = a\sin\left[2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right)\right] \qquad (19.9c)$$

Eqs. (19.9a to c) give complete mathematical description of a 1-D wave travelling along the positive x-direction. Note that these equations are equivalent representations of 1-D wave.

In terms of angular frequency $\omega\,(= 2\pi/T)$ and wave number $k\,(= 2\pi/\lambda)$, we can write Eq. (19.9c) as

$$y(x,\,t) = a\,\sin\,(\omega t - kx) \qquad (19.9d)$$

The simple way in which ω and k enter the mathematical expression for the wave explains why these quantities are so often used in the description of wave motion. Note that a wave described by Eq. (19.9d) has a single constant frequency and signifies a **monochromatic** wave.

Note that for a longitudinal wave, both x and y (x, t) are along the same direction. They, however, refer to two different quantities: x refers to the position of a particle of the medium and y (x, t) refers to the displacement of the particle located at x, with respect to its equilibrium position.

The use of any one of Eqs. (19.9a–d) to represent a wave mathematically depends on the specific situation. However, we will mostly use Eq. (19.9d). Note that the **waves represented by these equations are of an infinite extent**. That is, x can vary from $-\infty$ to ∞ for any fixed value of t, as there is no mathematical limit on the value of x.

It is important to mention here that these equations describe transverse as well as longitudinal waves travelling in the positive x-direction.

You may also like to know: **How do we represent a wave travelling in the negative x-direction**? In this case, the displacement $y(x = 0, t')$ at point A in Fig. 19.11 at time t' will be the same as displacement at point B at time t if

$$t' = t + \frac{x}{v}$$

and we can write

$$y(x,t) = y(x = 0, t')$$
$$= y\left(x = 0, t + \frac{x}{v}\right)$$

Now, you should repeat the steps used in arriving at Eq. (19.7 to 19.9d) and verify the following results for **a wave travelling in the negative x-direction**:

$$y(x,\,t) = a\,\sin\left[\frac{2\pi}{\lambda}(vt + x)\right] \qquad (19.10a)$$

$$y(x,\,t) = a\,\sin\left[\frac{2\pi}{T}\left(t + \frac{x}{v}\right)\right] \qquad (19.10b)$$

$$y(x,\,t) = a\,\sin\left[2\pi\left(\frac{t}{T} + \frac{x}{\lambda}\right)\right] \qquad (19.10c)$$

$$y(x,t) = a\,\sin(\omega t + kx) \qquad (19.10d)$$

Whenever you are given a function and asked to check whether or not it represents a wave, you should compare it with Eqs. (19.9a to d) or Eqs. (19.10a to d). If the given function is similar to anyone of these equations, you can say that the function represents a wave. By comparing corresponding terms, you can as well determine the values of wavelength, frequency, time period, etc. of the wave.

On comparing Eqs. (19.9a) and (19.9d) or Eqs. (19.10a) and (19.10d), we obtain the expression for wave velocity in terms of ω and k as:

$$v = \omega / k \qquad (19.11)$$

You should verify the result contained in Eq. (19.11). Before proceeding further, answer the following SAQ.

SAQ 3 – Direction of wave travel

For each of the following mathematical expressions representing a 1-D wave, identify the direction in which the wave is travelling:

a) $\phi\,(z,\,t) = a\,\sin(\omega t - kz)$

b) $z(x,\,t) = a\,\sin(kx - \omega t)$

c) $\psi\,(y,\,t) = a\,\sin(\omega t - ky)$

d) $\xi(z,t) = a\,\sin(kz + \omega t)$

The basic feature of Eqs. (19.9) and (19.10) representing waves is that the whole wave pattern **moves along the** x**-axis** as time changes. This leads to **one vital difference between the displacement of the particles of the medium (string) and the displacement** *y* (*x*, *t*) **of any point on the waveform**: **while the former changes periodically, the latter remains constant**. As the wave travels, **the entire waveform shifts**. Hence, the displacement of a point on the waveform remains the same and this holds for all points on the waveform. You can understand this point by studying Fig. 19.12.

When we say 'whole wave pattern' or 'the entire waveform', we mean the snapshot of the wave at some instant of time. For example, waveform *A* in Fig. 19.12 is the snapshot of the wave at time $t = 0$.

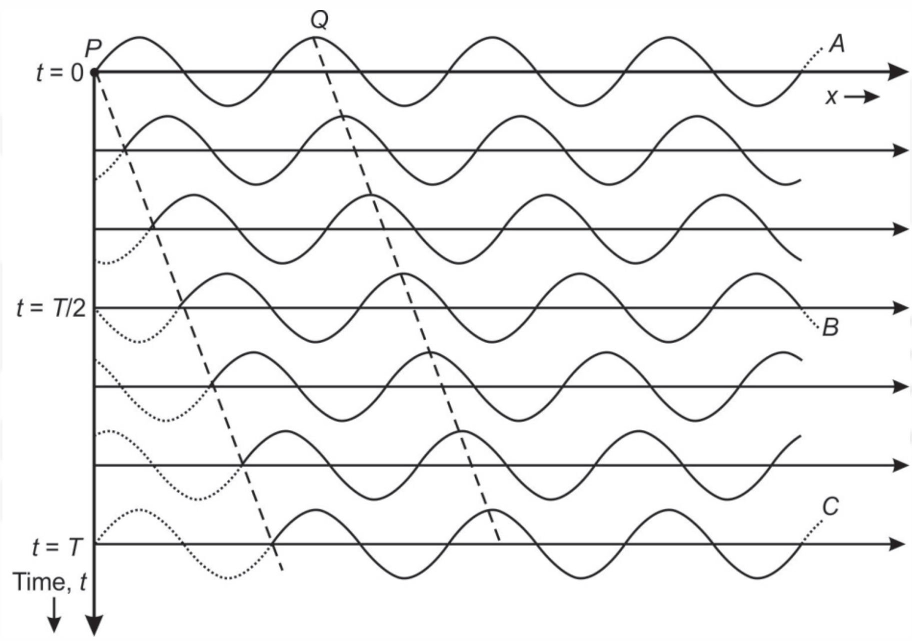


Fig. 19.12: **Plot of a sinusoidal wave travelling along the positive *x*-direction in the *x*-*t* plane.**

Fig. 19.12 shows a travelling wave represented by Eqs. (19.9 a to d) in the *x*-*t* plane (plane of the paper) propagating in the positive *x*-direction. **In the plot**, **the displacement**, *y* (*x*, *t*), **of the particles of the medium is normal to the *x*-*t* plane**. The plot shows the variation of *y* (*x*, *t*) with both *x* and *t*. You may now like to ask: What values the displacement *y* (*x*, *t*) takes for different values of *x* and *t* as the wave travels?

To seek answer to this question, focus on the curves *A*, *B* and *C* in Fig. 19.12, which show the wave at the instant $t = 0$ and after the time intervals $T/2$ and *T*, where *T* is the time period of the wave, respectively. Note that, with time, the entire waveform shifts to the right. Now focus on the vertical displacement of the points $P(x = 0,\, t = 0)$ and $Q(x = 5\lambda/4,\, t = 0)$ in all the three curves.

You can see that the displacements of the points P and Q remain unchanged at all times. This is true for every point you select on the wave for all values of t. Thus, we can conclude that as the wave travels, the displacements of all points on the waveform remain constant with time and are equal to their respective values at the reference position (in this case $t = 0$ and $x = 0$).

We now give examples showing you how to obtain wave parameters from the mathematical expression of a wave and how to write the mathematical expression for a wave on the basis of its parameters.

EXAMPLE 19.1 : PARAMETERS ASSOCIATED WITH A WAVE

A wave is represented by

$$y(x, t) = (4\,\text{cm})\ \sin\ [(20\,\text{s}^{-1})\ t + (10\,\text{cm}^{-1})\ x]$$

Determine the amplitude, wavelength, angular frequency, wave number and velocity of the wave.

SOLUTION ■ We compare the given expression for the wave with the standard forms and note that it is of the form of Eq. (19.10d):

$$y(x, t) = a\ \sin\ (\omega t + kx) \qquad \text{(i)}$$

Thus, we conclude that the wave is propagating in the **negative** x**-direction**. Comparing the corresponding terms in Eqs. (i) and the given expression, we obtain

Amplitude, $a = 4\,\text{cm}$; Wave number, $k = 10\ \text{cm}^{-1}$; Angular frequency, $\omega = 20\ \text{s}^{-1}$. On using Eq. (19.11), we get velocity v of the wave

$$v = \frac{\omega}{k} = \frac{20\,\text{s}^{-1}}{10\ \text{cm}^{-1}} = 2\ \text{cm}\,\text{s}^{-1}$$

And using the relation $k = 2\pi/\lambda$, we get the wavelength λ of the wave:

$$\lambda = \left(\frac{2\pi}{10\ \text{cm}^{-1}}\right) = 0.6\ \text{cm}$$

You may now like to answer an SAQ.

SAQ 4 – Wave parameters

A progressive wave is described by

$$y(x, t) = (1)\ \sin\left[1000\ \pi t - \frac{\pi x}{50}\right]\text{cm}$$

Determine its direction of propagation and calculate the amplitude, wave number, wavelength, angular frequency and frequency of the wave. Also calculate its velocity in cm s^{-1}.

EXAMPLE 19.2 : MATHEMATICAL EXPRESSION FOR WAVE FOR GIVEN WAVE PARAMETERS

A sound wave of frequency 275 Hz travels with speed 340 ms^{-1} along the positive x-axis. Each point of the medium moves to and fro through a total distance of 5.0 mm. Represent the wave mathematically.

SOLUTION ■ The amplitude of the wave is half the total distance that a point in the medium moves to and fro. Thus

$$a = 2.5\,\text{mm} = 2.5\times 10^{-3}\,\text{m}$$

Further, we are given the wave speed and its frequency. Since the wave is travelling in the positive x-direction, we have to write the wave equations in the forms given by Eqs. (19.9c) and (19.9d). For this, we need the time period, the wavelength, the angular frequency and the wave number of the wave. Using their definitions, we have

$$T = 1/f = (1/275)\,\text{s}$$

$$\lambda = v/f = 340\,\text{ms}^{-1}/275\,\text{s}^{-1}$$

$$= (340/275)\,\text{m} = 1.24\,\text{m}$$

$$\omega = 2\pi f = 2\pi\times(275\,\text{s}^{-1}) = 550\,\pi\,\text{s}^{-1}$$

$$= 1.73\times 10^{3}\,\text{s}^{-1}$$

$$k = 2\pi/\lambda = 2\pi/(1.24\text{ m})$$

$$= 5.07\text{ m}^{-1}$$

Hence, we can write the expression for wave in the form of Eq. (19.9c) as

$$y(x,\ t) = (2.5\times 10^{-3}\text{ m})\sin\left[2\pi\left(275\,t - \frac{x}{1.24}\right)\right]$$

We can also write this as

$$y(x,\ t) = (2.5\times 10^{-3}\text{ m})\sin\left[550\,\pi\left(t - \frac{x}{340}\right)\right]$$

The expression for the wave in the form of Eq. (19.9d) is

$$y(x,\ t) = (2.5\times 10^{-3}\text{ m})\ \sin\ (1.73\times 10^{3}\ t - 5.07\,x)$$

SAQ 5 – Expression for a wave for given wave parameters

A tidal wave having a maximum height of 7.4 m propagating in the negative x-direction with a speed of 93 ms^{-1} can be approximated by sine function. The distance between two successive crests is 5 cm. Write the expression for the wave.

Before proceeding further, we recapitulate what you have studied in this section till now.

Recap

- Wave motion can be depicted graphically on **vibration graph** (which shows a wave at a single location as time passes) and **waveform graph** (which shows a wave at different positions at a particular moment in time).
- **The frequency of a wave is a property of the source generating the wave** and it does not depend on the medium through which wave propagates.
- The velocity of the wave is given as: $v = f\lambda$. This shows that **for a given medium, velocity of a wave of given frequency is constant.**
- A wave travelling along the positive x-direction is represented by

  $$y(x,\, t) = a\ \sin(\omega t - kx)$$

  and the wave travelling in the negative x-direction is given by

  $$y(x,\, t) = a\ \sin(\omega t + kx)$$

So far you have learnt that waves propagate in a medium when the particles of the medium oscillate about their respective mean positions. You have also learnt about the **amplitude**, **time period**, **frequency** and **wavelength** of the wave which characterise a wave. For a complete mathematical description of a wave, we also need to know the **phase** of a wave. This forms the subject matter of discussion of the following section.

19.3.4 Phase of a Wave and Phase Difference

To explain the concept of phase of a wave, we consider a transverse wave propagating on a string and focus our attention on a particle at the equilibrium position at the start of the wave cycle. Recall that in one complete period, it will reach the crest, come back to the equilibrium position before moving downwards to the trough and finally attain its initial position. We use this information to arrive at the concept of the **phase** of a wave.

Refer to Fig. 19.13a, which shows a stretched string at time, t = 0. We have marked 17 particles on it. The positions of these 17 particles on a wave at time, t = 2T is shown in Fig. 19.13b. (In a way, it is extended form of Fig. 19.6.)

When the periodic force generating wave completes one oscillation, the wave covers a distance equal to one wavelength. If we represent the oscillatory motion on a reference circle, as shown in Fig. 19.14a, with uniform speed, then one complete

(Contd.)

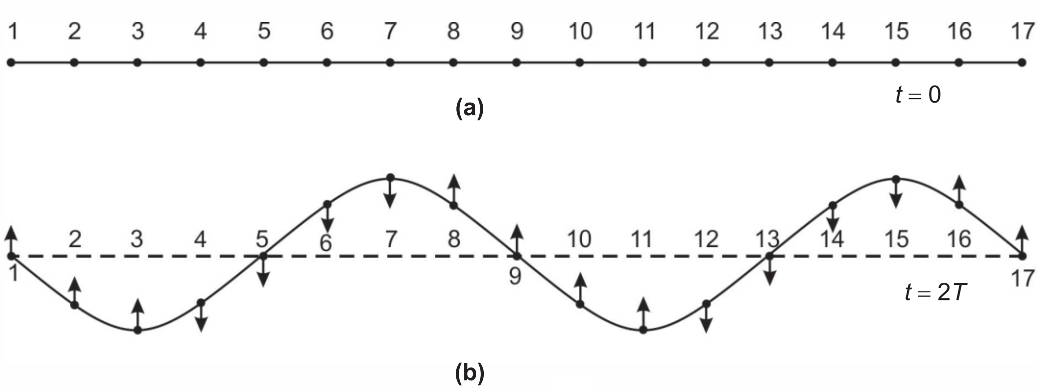


Fig. 19.13: **Snapshot of a) a stretched string at time,** t = 0**; b) waveform created due to transverse wave on the string at time,** t = 2T.

oscillation is equivalent to one complete rotation, that is, rotation by an angle 360° or 2π radians. Therefore, we can say that a distance equal to one wavelength, as shown in Fig. 19.14b, is equivalent to 360° or 2π radians in terms of angle on the references circle. This means that in terms of angle, two points on a waveform separated by a distance of (λ/2) are (360°/2) = 180° or (2π/2) = π radians apart. So, the phase difference between points *A* and *C* in Fig. 19.14b is 180° or π radian.

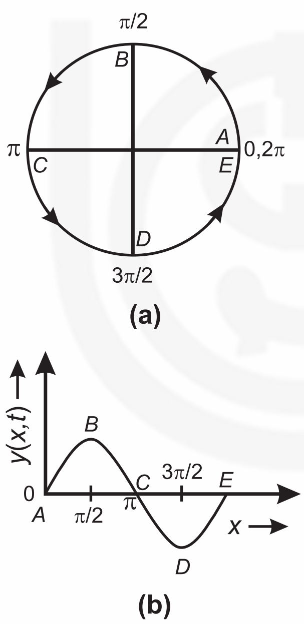


Fig. 19.14: **a) One complete oscillation generating a wave represented on a reference circle is equal to rotation by 360° or 2π radian; b) snapshot of a wave-form corresponding to one complete oscillation of the oscillator generating the wave.**

Refer to Fig. 19.13b and compare the displacements of particles at any two marked positions, say, at 1 and 2. We know that both these particles execute simple harmonic motion of the same amplitude and frequency. But particle 2 begins to oscillate at a time $t = T/8$ later than particle 1. We describe this situation as: particle 2 **lags behind** particle 1. Further, you may also note that particles 1, 9 and 17 in Fig. 19.13b are about to go up and **are in the same state of motion**. However, particle 1 is in a different state of motion than particles 2 to 8 and 10 to 16. Similarly, particles 2 and 10 are in the same state of motion and so on.

From the above discussion, we can say that even though all particles execute SHM and their displacements show the same sinusoidal variation in time, their **states of motion** may be different at any given instant of time. We denote this **difference in the states of motion of particles of the medium** in terms of the **phase angle** or simply ***phase***. We say that particles 1, 9 and 17 are in **the same phase** but the phase of particle 1 is not the same as that of particles 2, 3 ,….8 and 10, 11,….16 or it is **out of phase** with these particles.

Now you may logically ask: **How can we mathematically represent the phase of a wave on the basis of above qualitative description?** We know that wave motion arises due to periodic motion of particles of the medium around their respective mean positions. We make use of this fact to define the phase of a wave: **The argument of the sine (or cosine) function representing a periodic travelling wave** signifies the **phase** of the wave. We denote it by the symbol ϕ (*x*, *t*). Thus, the phase of a sinusoidal wave at position, *x* and time, *t* represented by Eq. (19.9d) is the argument of the sine function:

$$\phi(x,\ t) = \omega t - kx \tag{19.12a}$$

Note that the phase of a wave is an angle and is measured in degrees or in radians. You can easily convince yourself that a phase difference of 360º or 2π radians corresponds to one wavelength (see Fig. 19.14 and read the margin remark).

From Eq. (19.12a), we note that the phase of a wave changes with both space and time. Further, from the definition of the phase, it follows that **all points on the waveform separated by one wavelength or its integer multiples will be in the same phase**. To elaborate this point, let us recall that for a sinusoidal function at a given instant *t*

$$\sin\ (\omega t - kx \pm 2\pi) = \sin(\omega t - kx)$$

Now for a point $x' = x + \lambda,$ we have, (using the relation $k = 2\pi/\lambda$),

$$\sin(\omega t - kx') = \sin[\omega t - k(x + \lambda)] = \sin(\omega t - kx - k\lambda)$$

$$= \sin(\omega t - kx - 2\pi) = \sin(\omega t - kx)$$

Thus, at a given instant *t*, the phase at any point $x = \lambda$ on the wave is the same as the phase at point $x = 0.$ In the same way, you can convince yourself that **all other points on the waveform separated by integer multiples of the wavelength have the same phase**. (You have to substitute $x' = x \pm n\lambda$ and use the result $\sin(\theta \pm 2n\pi) = \sin\theta$ for $n = 0,\ 1,\ 2,\ 3, ...$).

We can now obtain the expression for phase difference between two arbitrary points on a wave. Let us consider two arbitrary points located at positions x_1 and x_2 respectively on a waveform. Mathematically, the phases ϕ_1 and ϕ_2 of particles at positions x_1 and x_2 at a fixed time t can be written as

$$\phi_1 = \omega t - kx_1$$

and

$$\phi_2 = \omega t - kx_2$$

Hence, the phase difference, $\Delta\phi$ between two positions x_1 and x_2 on the waveform at any given instant of time t is given by

$$\Delta\phi(x,t) = (\omega t - kx_2) - (\omega t - kx_1)$$

$$= k\,(x_1 - x_2)$$

$$= \frac{2\pi}{\lambda}(x_1 - x_2) \quad \text{(for fixed } t\text{)} \tag{19.12b}$$

Eq. (19.12b) gives the phase difference between two arbitrary points (located at x_1 and x_2, respectively) in terms of the wavelength. This relation shows that if the magnitude of separation $(x_1 - x_2)$ between the two points on the waveform is equal to one wavelength, the phase difference will be 2π. And as you know, two points having a phase difference of 2π are in the same phase. Therefore, we say that two points separated by one wavelength (or integral multiples of wavelength) are in the same phase or the phase difference between them is zero. This implies that if phase difference changes by 2π or its integral multiples, the waveform remains the same.

You must differentiate between the vertical displacement of the particles of the medium (string) and the vertical displacement of any point *P* (or all other points) on the waveform: while the former changes periodically the latter remains constant as the wave propagates.

Next we consider a wave at any fixed position. The expression for phase difference between two instants t_1 and t_2 of a wave readily follows from Eq. (19.12a):

$$\boxed{\Delta\phi(x,t) = \omega\Delta t} \quad \text{(for fixed } x\text{)} \tag{19.12c}$$

We hope that you now have a fairly good understanding as to what we mean by the phase difference between different points of a wave. We can extend this discussion of phase of different points on a single waveform with respect to some reference point to a situation where more than one wave is travelling in space. **Two waves are said to be in phase when the corresponding points of each wave reach their respective maximum or minimum displacements at the same time.** Thus, if the crests and troughs of the two waves coincide, they are said to be **in phase** (Fig. 19.15a). If the crest of one wave coincides with the trough of the other wave, as shown in Fig. 19.15b, their phases are said to differ by 180° and the waves are said to have **opposite phase**. The **phase difference** between two waves can vary from 0 to 360°. Fig. 19.15c shows two out of phase waves *A* and *B* with phase difference of an arbitrary angle θ. From your +2 physics course, you may recall that phase difference between two or more waves plays a critical role in deciding the outcome of their superposition.

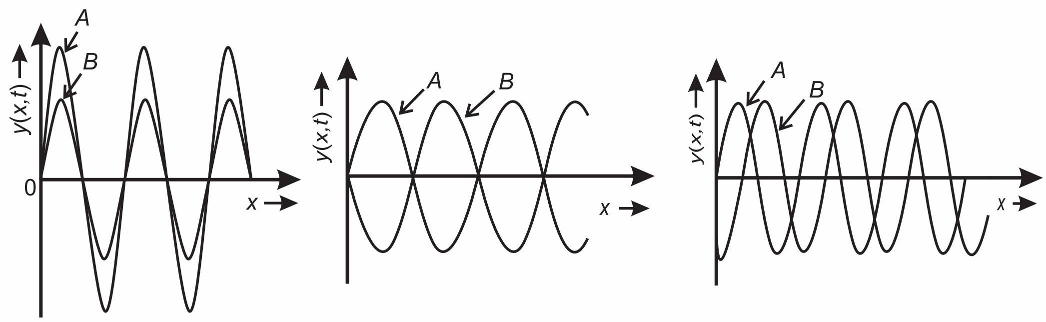


Fig. 19.15: **Two waves** *A* **and** *B* **are a) in-phase; b) completely out of phase; c) out of phase by an arbitrary angle.**

Before we end this unit, it is important to point out that displacement of a wave given by Eqs. (19.9a-d) and travelling with velocity v is solution of the wave equation:

$$\frac{\partial^2 y}{\partial x^2} = \left(\frac{1}{v^2}\right)\frac{\partial^2 y}{\partial t^2}$$

where y represents the displacement of a particle of the medium in which a wave propagates. To visualise wave propagation in a medium (strings, fluids or a solid) consider a large number of masses connected through identical springs (Fig. 19.2). When one of the masses is disturbed either longitudinally or in a transverse direction, the bonding between them is disturbed and while getting back to the original state, energy exchange takes place between successive masses. Propagation of energy in the infinitely long spring-mass system leads to wave propagation in one dimension. This analogy can be extended to two and three dimensions.

Let us now summarise what you have learnt in this unit.

19.4 SUMMARY

Concept	Description
Wave is a disturbance	■ Wave is a **disturbance** which propagates in a medium progressively. **Waves do not transport matter; these only transport energy and momentum.** Mechanical waves can exist only in material medium.
Transverse and longitudinal wave	■ In a **transverse wave**, such as a wave on a stretched string, the particles of the medium oscillate in a direction perpendicular to the direction of propagation of the wave. On the other hand, in a **longitudinal wave**, such as sound wave in air, the particles of the medium oscillate along the direction of propagation of the wave.
Graphical representation	■ The **graphical representation** of wave motion can be done by two types of graphs: **vibration graph** and **waveform graph**. A vibration graph shows the wave behaviour at a single location along the wave path as time passes. It is snap shot in space. A waveform graph depicts the wave behaviour at different locations along the wave path at a particular time. It is a snapshot at a given time.

Wave frequency	■ The **frequency** of a wave is a property of the source responsible for generation of wave.
Wave velocity	■ The velocity, v of a wave is given as $v = f\lambda$, where f **is the frequency** of the wave and λ **is its wavelength**. Thus, for a given medium, the velocity of a wave of given frequency is constant. In terms of angular frequency, ω and wave number, k, the wave velocity is given as $v = \omega / k$.
Progressive wave	■ A one-dimensional **progressive** wave propagating along the positive x-direction is described mathematically as $$y(x, t) = a \sin(\omega t - kx)$$ On using the relations $\omega = 2\pi/T$ and $k = 2\pi/\lambda$, we can rewrite the above expression in other equivalent forms as $$y(x, t) = a \sin\left[\frac{2\pi}{\lambda}(vt - x)\right] = a \sin\left[\frac{2\pi}{T}\left(t - \frac{x}{v}\right)\right] = a \sin\left[2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right)\right]$$
Phase of wave	■ Two points on the vibration graph of a wave separated by T or its integer multiples are in the same **phase**. Similarly, two points on the waveform graph of a wave separated by λ or its integer multiples are in the same phase.
In-phase wave	■ Two waves are said to be **in-phase** when the corresponding points of each wave reach their respective maximum or minimum displacements at the same time.
One-dimensional wave	■ One dimensional wave can be represented by the wave equation $$\frac{\partial^2 y}{\partial x^2} = \left(\frac{1}{v^2}\right)\frac{\partial^2 y}{\partial t^2}$$

19.5 TERMINAL QUESTIONS

1. A progressive transverse wave is described by

$$y(x,t) = 0.02 \sin(1886t - 42x) \text{ m}$$

where x is in meters and t is in seconds. Determine the direction of propagation of the wave and calculate its amplitude, wavelength, frequency and velocity.

2. A transverse wave on a string is represented by

$$y(x,t) = 0.03 \sin(2t - 3x) \text{ m}$$

where x is in metres and t is in seconds. a) Calculate the displacement at

$x = 0$, 0.1 m and 0.3 m when $t = 0$. b) Calculate the values of displacement at $x = 0.1$ m at $t = 0$, 0.1s and 0.2 s. c) Obtain the expression for the velocity of oscillation of the particles of the string. d) What is the maximum velocity of oscillation of a given particle of the string?

3. A transverse wave travelling in the positive *x*-direction is represented as

$$y(x,t) = 5 \sin(4.0t - 0.02x) \text{cm}$$

where *x* is in cm and *t* is in seconds. Calculate the velocity of the wave, maximum particle velocity and acceleration.

4. A travelling wave is given by

$$y(x,t) = 0.26 \sin(12.1t - 2.3x + 1.6) \text{ m}$$

where *x* is measured in meters and *t* is measured in seconds. Determine the distance by which the origin on the *x*-axis should be shifted so that the expression for the wave becomes

$$y(x',t) = 0.26 \sin(12.1t - 2.3x) \text{ m}$$

5. A transverse wave of amplitude 1 cm is generated at one end $(x = 0)$ of a long string by an oscillator of frequency of 500 Hz connected to the string. At some instant of time, the displacements of the particles situated at $x = 10$ cm and at $x = 20$ cm are + 0.5 cm and – 0.5 cm, respectively. Calculate the velocity and wavelength of the wave. If the wave is travelling along the positive *x*-direction and the end $x = 0$ is at the equilibrium position at $t = 0$, write the displacement in terms of wave speed.

19.6 SOLUTIONS AND ANSWERS

Self-Assessment Questions

1. a) When a single disturbance propagates in a medium, it is called a pulse. A wave is generated by continuous oscillations of medium particles.

   b) i) Mechanical waves such as sound and waves on a string.
   ii) Electromagnetic waves.

   c) In a transverse wave, the particles of the medium oscillate in a direction perpendicular to the direction of wave propagation. A wave on water surface is an example of transverse wave in 2-D. In a longitudinal wave, the direction of oscillation of the particles is along the direction of the wave motion. Sound waves are the most familiar example of longitudinal waves.

2. a) From Eq. (19.6), we recall that frequency of light wave can be written as

$$f = \frac{c}{\lambda}$$

Now, the value of the wavelength, λ for the violet-end of the range is $4000\,\text{Å} = 4\times10^{-7}\,\text{m}$. So, the corresponding frequency of light is

$$f_v = \frac{3\times10^{8}\,\text{ms}^{-1}}{4\times10^{-7}\,\text{m}} = 7.5\times10^{14}\,\text{s}^{-1}$$

Similarly, the wavelength for the red-end of the visible range is 7.2×10^{-7} m. So, the corresponding frequency is

$$f_r = \frac{3\times10^{8}\,\text{ms}^{-1}}{7.2\times10^{-7}\,\text{m}} = 4.2\times10^{14}\,\text{s}^{-1}$$

b) As per the problem, we have, for sound, $v = 332\,\text{ms}^{-1}$ and frequency of the sound wave at the lower end of audible range is 20 Hz. Again, using Eq. (19.6), we can write

$$\lambda = \frac{v}{f} = \frac{332\,\text{ms}^{-1}}{20\,\text{s}^{-1}} = 16.6\,\text{m}$$

Similarly, for the upper end of the audible range, we get

$$\lambda = \frac{332\,\text{ms}^{-1}}{2\times10^{4}\,\text{s}^{-1}} = 16.6\times10^{-4}\,\text{m} = 1.66\,\text{mm}$$

3. a) positive *z*-direction; b) positive *x*-direction; c) positive *y*-direction; d) negative *z*-direction

4. The equation of the given wave travelling in positive *x*-direction is

$$y(x, t) = 1\sin\left[1000\,\pi t - \frac{\pi x}{50}\right]\text{cm} \qquad \text{(i)}$$

We recall that the standard form of a wave travelling along + *x*-direction is given by Eq. (19.9d):

$$y(x, t) = a\sin(\omega t - kx) \qquad \text{(ii)}$$

On comparing Eqs. (i) and (ii), we get

Amplitude $= 1$ cm; Wave number $k = \pi/50\,\text{cm}^{-1}$;

Wavelength $\lambda = 2\pi/k = 2\pi\times(50/\pi) = 100$ cm;

Angular frequency $\omega = 1000\,\pi\,\text{s}^{-1}$; Frequency $f = \omega/2\pi = 1000\pi/2\pi = 500$ Hz;

Wave velocity $= f\lambda = (500\,\text{s}^{-1})\times(100\,\text{cm}) = 5\times10^{4}\,\text{cm}\,\text{s}^{-1}$

5. A wave propagating along negative *x*-direction is described by

$$y(x,t) = a\sin(\omega t + kx)$$

Since the maximum height of the wave is 7.4 m, its amplitude $a = 7.4\,\text{m}$. Further, the angular frequency, ω of the wave is related to velocity, v and wavelength, λ by the relation

$$\omega = \frac{2\pi v}{\lambda}$$

As per the problem $v = 93\ \text{m s}^{-1}$ and $\lambda = 5\ \text{m}$ because the distance between two successive crests is 5 m. Hence

$$\omega = \frac{2\pi \times 93\,\text{m s}^{-1}}{5\,\text{m}} = 116.8\,\text{s}^{-1}$$

Similarly, the wave number, k is related to wavelength, λ as

$$k = \frac{2\pi}{\lambda} = \frac{2 \times 22}{7 \times 5\,\text{m}} = 1.26\ \text{m}^{-1}$$

Hence, the required equation of the wave is

$$y(x,\ t) = (7.4\,\text{m})\ \sin\ (1.26\,x + 116.8\,t)$$

Terminal Questions

1. The standard expression for a progressive transverse wave is of the form

$$y(x,\ t) = a\ \sin(\omega t - kx)$$

On comparing the given expression with this expression, we note that the direction of propagation of the wave is positive x-direction. Further, we find that

Amplitude of the wave $= 0.02\text{m} = 2 \times 10^{-2}\,\text{m}$

Wavelength of the wave, $\lambda = \dfrac{2\pi}{k} = \dfrac{2\pi}{42} = 0.15\ \text{m}$

Frequency of the wave, $f = \dfrac{\omega}{2\pi} = \dfrac{1886}{2\pi} = 300\ \text{Hz}$

Velocity of the wave, $v = f\lambda = (300\,\text{Hz}) \times (0.15\text{m}) = 45\ \text{ms}^{-1}$

2. a) To calculate the values of displacement, $y(x, t)$ for different values of x, and t, we substitute the given values of x and t in the expression for the wave:

$$y(x,t) = 0.03\ \sin\ (2t - 3x)\ \text{m}$$

So, displacement $y(x,t)$ at different values of x at $t = 0$ are:

at $x = 0$ m, $\quad y(0,\ 0) = 0.03\ \sin(0)\ \text{m} = 0$

at $x = 0.3\,\text{m}$, $\quad y(0.3,\ 0) = 0.03\ \sin(-0.9)\ \text{m} = -2.35 \times 10^{-2}\ \text{m}$

b) At position $x = 0.1$ m, displacement at different times are:

at $t = 0$ s, $\quad y(0.1,\ 0) = 0.03\ \sin(-0.3)\ \text{m} = -8.8 \times 10^{-3}\ \text{m}$

at $t = 0.1$ s, $\quad y(0.1,\ 0.1) = 0.03\ \sin(0.2 - 0.3)\ \text{m} = -3.0 \times 10^{-3}\ \text{m}$

at $t = 0.2$ s, $\quad y(0.1,\ 0.2) = 0.03\ \sin(0.4 - 0.3)\ \text{m} = 3.0 \times 10^{-3}\ \text{m}$

c) Velocity of oscillating particles of the string is given by

$$v = \frac{dy}{dt} = 0.06\ \cos(2t - 3x) = 6 \times 10^{-2}\ \cos(2t - 3x)\ \text{ms}^{-1}$$

d) Maximum velocity of oscillation will be when

$$\cos(2t - 3x) = 1$$

So, $|v_{max}| = 6.0 \times 10^{-2}\, ms^{-1}$

3. As per the problem, the expression for transverse wave travelling along the positive x-direction is

$$y(x,t) = 5\sin(4.0t - 0.02x)$$

On comparing it with the standard expression for a wave travelling in +ve x-direction

$$y(x,t) = a\sin(\omega t - kx)$$

we note that $\omega = 4.0\ s^{-1}$, $k = 0.02\ cm^{-1}$ and $a = 5\ cm.$ Hence, velocity of the wave is

$$v = \frac{\omega}{k} = \frac{4.0\, s^{-1}}{0.02\, cm^{-1}} = 200\ cm\, s^{-1}$$

The velocity of the particles of the medium is

$$\frac{dy(x,t)}{dt} = 5 \times 4\cos(4.0t - 0.02x)\ cm\, s^{-1}$$

$$= 20\cos(4.0t - 0.02x)\ cm\, s^{-1}$$

So, the maximum velocity of the particle is $20\ cm\, s^{-1}$.

Further, the acceleration of the particles of the medium is

$$\frac{d^2y(x,t)}{dt^2} = -20 \times 4\sin(4.0t - 0.02x)\ cm\, s^{-2}$$

$$= -80\sin(4.0t - 0.02x)\ cm\, s^{-2}$$

Thus, the maximum particle acceleration is $80\ cms^{-2}$.

4. We know that the relation between phase difference and the spatial distance of two points x_1 and x_2 on the waveform is given by

$$\Delta\phi = \frac{2\pi}{\lambda}(x_1 - x_2)$$

As per the given problem,

$$1.6 = \frac{2\pi}{\lambda}(x' - x) \quad \Rightarrow \quad x' - x = \frac{1.6}{k} = \frac{1.6}{2.3\ m^{-1}} = 0.7\ m$$

or $\quad x' = x + 0.7\, m$

Thus, the origin of x' is 0.7 m right to the origin of x.

5. We know that a transverse wave can be described by

$$y(x,t) = a\sin\left[\frac{2\pi}{\lambda}(vt - x)\right] \qquad \text{(i)}$$

As per the problem, we have amplitude, $a = 1\, cm.$ Further, at $x = 10\ cm,\ y(10,t) = 0.5\ cm.$ So, we can write from Eq. (i):

$$+0.5 \text{ cm} = (1.0 \text{ cm}) \sin\left[\frac{2\pi}{\lambda}(vt - 10)\right]$$

or $$\sin\left[\frac{2\pi}{\lambda}(vt-10)\right] = +\frac{1}{2} = \sin\left(\frac{\pi}{6}\right)$$

This equality implies that

$$\frac{2\pi}{\lambda}(vt-10) = \frac{\pi}{6} \quad \Rightarrow \quad vt - 10 = \frac{\lambda}{12} \qquad \text{(ii)}$$

We are also given that at $x = 20$ cm; $y(20, t) = -0.5$ m. So, we can again write from Eq. (i):

$$-0.5 \text{ cm} = (1.0 \text{ cm}) \sin\left[\frac{2\pi}{\lambda}(vt-20)\right]$$

$$\Rightarrow \qquad -0.5 = \sin\frac{7\pi}{6} = \sin\left[\frac{2\pi}{\lambda}(vt-20)\right]$$

or $$vt - 20 = \frac{7\lambda}{12} \qquad \text{(iii)}$$

From Eqs. (ii) and (iii), we have

$$\lambda = 20 \text{ cm} = 0.2 \text{ m}$$

Now, we know that the velocity

$$v = f\lambda = (500 \text{ s}^{-1}) \times (0.2 \text{ m}) = 100 \text{ ms}^{-1}$$

So we can write the expression for the wave, in terms of wave velocity as

$$y(x,t) = (0.01 \text{ m}) \sin\left[\frac{2\pi}{0.2}(100t - x)\right]$$

$$= (0.01 \text{ m}) \sin\left[10\pi(100t - x)\right] \text{ m}$$

FURTHER READING

1. **Principles of Physics**; D. Halliday, R. Resnick and J. Walker, Tenth Edition, Wiley India Ltd. (2015).
2. **Mechanics** (**Berkeley Physics Course**, **Volume I**); C. Kittel, W. D. Knight, M. A. Ruderman, A. C. Helmholz, B. J. Moyer; McGraw Hill International Book Company (2017).

TABLE OF PHYSICAL CONSTANTS

Symbol	Quantity	Value
c	Speed of light in vacuum	$3.00 \times 10^{8}\ \mathrm{ms^{-1}}$
μ_0	Permeability of free space	$1.26 \times 10^{-6}\ \mathrm{NA^{-2}}$
ε_0	Permittivity of free space	$8.85 \times 10^{-12}\ \mathrm{C^2 N^{-1} m^{-2}}$
$1/4\pi\varepsilon_0$		$8.99 \times 10^{9}\ \mathrm{Nm^2 C^{-2}}$
e	Charge of the proton	$1.60 \times 10^{-19}\ \mathrm{C}$
$-e$	Charge of the electron	$-1.60 \times 10^{-19}\ \mathrm{C}$
h	Planck's constant	$6.63 \times 10^{-34}\ \mathrm{Js}$
$\hbar$	$h / 2\pi$	$1.05 \times 10^{-34}\ \mathrm{Js}$
m_e	Electron rest mass	$9.11 \times 10^{-31}\ \mathrm{kg}$
$-e/m_e$	Electron charge to mass ratio	$-1.76 \times 10^{11}\ \mathrm{C\,kg^{-1}}$
m_p	Proton rest mass	$1.67 \times 10^{-27}\ \mathrm{kg}$ (1 amu)
m_n	Neutron rest mass	$1.68 \times 10^{-27}\ \mathrm{kg}$
a_0	Bohr radius	$5.29 \times 10^{-11}\ \mathrm{m}$
N_A	Avogadro constant	$6.02 \times 10^{23}\ \mathrm{mol^{-1}}$
R	Universal gas constant	$8.31\ \mathrm{Jmol^{-1} K^{-1}}$
k_B	Boltzmann constant	$1.38 \times 10^{-23}\ \mathrm{J\,K^{-1}}$
G	Universal gravitational constant	$6.67 \times 10^{-11}\ \mathrm{Nm^2 kg^{-2}}$

Astrophysical Data

Celestial Body	Mass (kg)	Mean radius (m)	Mean distance from the centre of Earth (m)
Sun	1.99×10^{30}	6.96×10^{8}	1.50×10^{11}
Moon	7.35×10^{22}	1.74×10^{6}	3.84×10^{8}
Earth	5.97×10^{24}	6.37×10^{6}	0

LIST OF BLOCKS AND UNITS: BPHCT-131

BLOCK 1:	**MATHEMATICAL PRELIMINARIES**
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Unit 2	Vector Algebra-II
Unit 3	First Order Ordinary Differential Equations
Unit 4	Second Order Ordinary Differential Equations with Constant Coefficients
BLOCK 2:	**BASIC CONCEPTS OF MECHANICS**
Unit 5	Newton's Laws of Motion and Force
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BLOCK 3:	**ROTATIONAL MOTION AND MANY-PARTICLE SYSTEMS**
Unit 11	Kinematics of Angular Motion
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Unit 14	Dynamics of Many-particle Systems
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BLOCK 4:	**HARMONIC OSCILLATIONS**
Unit 16	Simple Harmonic Motion
Unit 17	Superposition of Harmonic Oscillations
Unit 18	Damped Oscillations
Unit 19	Wave Motion

SYLLABUS: MECHANICS (BPHCT-131) 4 Credits

Vector Algebra: Geometrical and algebraic representation of vectors, Vector algebra; Scalar and vector products; Derivatives of a vector with respect to a scalar.

First Order Ordinary Differential Equations: First order homogeneous differential equations (separable and linear first order differential equations).

Second Order Ordinary Differential Equations: 2^{nd} order homogeneous differential equations with constant coefficients.

Laws of Motion: Frames of reference; Newton's Laws of motion; Straight line motion; Motion in a plane; Uniform circular motion; 3-d motion.

Applications of Newton's Laws of Motion: Friction; Tension; Gravitation; Spring-mass system – Hooke's law; Satellite in circular orbit and applications; Geosynchronous orbits; Basic idea of global positioning system (GPS); Weight and Weightlessness.

Linear Momentum and Impulse: Conservation of momentum; Impulse; impulse-momentum Theorem; Motion of rockets.

Work and Energy: Work and energy; Conservation of energy; Head-on and 2-d collisions.

Kinematics of Angular Motion: Kinematics of angular motion: Angular displacement, angular velocity and angular acceleration; General angular motion.

Dynamics of Rotational Motion: Torque; Rotational inertia; Kinetic energy of rotation; Angular momentum; Conservation of angular momentum and its applications.

Motion under Central Force Field: Motion of a particle in a central force field (motion in a plane, conservation of angular momentum, constancy of areal velocity); Kepler's Laws (statement only).

Dynamics of Many Particle Systems: Dynamics of a system of particles; Centre of Mass, determination of the centre of mass of discrete mass distributions, centre of mass of a rigid body (qualitative).

Conservation Laws: Linear momentum, angular momentum and energy conservation for many-particle systems.

Simple Harmonic Motion: Simple Harmonic Motion; Differential equation of SHM and its solutions; Kinetic Energy, Potential Energy, and Total Energy of SHM and their time averages.

Superposition of Harmonic Oscillations: Linearity and Superposition Principle; Superposition of Collinear Oscillations having equal frequency and having different frequencies (beats); Superposition of Orthogonal Oscillations with equal and unequal frequency; Lissajous Figures and their uses.

Damped Oscillations: Equation of Motion of Damped Oscillations and its solution (without derivation); Qualitative description of the solution for Heavy, Critical and Weak Damping; Characterising Damped Oscillations – Logarithmic Decrement, Relaxation Time and Quality Factor.

Wave Motion: Qualitative Description (Wave formation and Propagation; Describing Wave Motion, Frequency, Wavelength and Velocity of Wave; Mathematical Description of Wave Motion).

NOTE