CHAPTER

14

OSCILLATIONS

14.1 PERIODIC MOTION

1. *What is periodic motion ? Give some of its examples.*

Periodic motion. *Any motion that repeats itself over and over again at regular intervals of time is called periodic or harmonic motion.* The smallest interval of time after which the motion is repeated is called its time period. The time period is denoted by T and its SI unit is second.

Examples of periodic motion :

(*i*) The motion of any planet around the sun in an elliptical orbit is periodic. The period of revolution of Mercury is 87.97 days.

(*ii*) The motion of the moon around the earth is periodic. Its time period is 27.3 days.

(*iii*) The motion of Halley's comet around the sun is periodic. It appears on the earth after every 76 years.

(*iv*) The motion of the hands of a clock is periodic.

(*v*) The heart beats of a human being are periodic. The periodic time is about 0.8 second for a normal person.

14.2 OSCILLATORY OR HARMONIC MOTION

2. *What is oscillatory motion ? Give some of its examples.*

Oscillatory motion. *If a body moves back and forth repeatedly about its mean position, its motion is said to be oscillatory or vibratory or harmonic motion.* Such a motion repeats itself over and over again about a mean position such that it remains confined within well defined limits (known as extreme positions) on either side of the mean position.

Examples of oscillatory motion :

(*i*) The swinging motion of the pendulum of a wall clock.

(*ii*) The oscillations of a mass suspended from a spring.

(*iii*) The motion of the piston of an automobile engine.

(*iv*) The vibrations of the string of a guitar.

(*v*) When a freely suspended bar magnet is displaced from its equilibrium position along north-south line and released, it executes oscillatory motion.

14.3 PERIODIC MOTION VS. OSCILLATORY MOTION

3. *Every oscillatory motion is necessarily periodic but every periodic motion need not be oscillatory. Justify.*

Distinction between periodic and oscillatory motions. Every oscillatory motion is necessarily periodic because it is repeated at regular intervals of time. In addition, it is bounded about one mean position. But every periodic motion need not be oscillatory. For example, the earth completes one revolution around the sun in 1 year but it is not a to and fro motion about any mean position. Hence its motion is periodic but not oscillatory.

14.4 PERIODIC FUNCTIONS AND FOURIER ANALYSIS

4. *With suitable examples, explain the meaning of a periodic function. Construct two infinite sets of periodic functions with period T. Hence state Fourier theorem.*

Periodic function. *Any function that repeats itself at regular intervals of its argument is called a periodic function.* Consider the function $f(\theta)$ satisfying the property,

$$f(\theta+T)=f(\theta)$$

This indicates that the value of the function f remains same when the argument is increased or decreased by an integral multiple of T for all values of θ. A function f satisfying this property is said to be periodic having a period T. For example, trigonometric functions like $\sin\theta$ and $\cos\theta$ are periodic with a period of 2π radians, because

$$\sin(\theta+2\pi)=\sin\theta$$

$$\cos(\theta+2\pi)=\cos\theta$$

If the independent variable θ stands for some dimensional quantity such as time t, then we can construct periodic functions with period T as follows :

$$f_1(t)=\sin\frac{2\pi t}{T} \quad \text{and} \quad g_1(t)=\cos\frac{2\pi t}{T}$$

We can check the periodicity by replacing t by $t+T$. Thus

$$f_1(t+T)=\sin\frac{2\pi}{T}(t+T)=\sin\left(\frac{2\pi t}{T}+2\pi\right)$$

$$=\sin\frac{2\pi t}{T}=f_1(t)$$

Similarly, $g_1(t+T)=g_1(t)$

It can be easily seen that functions with period T/n, where $n=1,2,3,\ldots..$ also repeat their values after a time T. Hence it is possible to construct two infinite sets of periodic functions such as

$$f_n(t)=\sin\frac{2\pi nt}{T} \qquad n=1,2,3,4,\ldots..$$

$$g_n(t)=\cos\frac{2\pi nt}{T} \qquad n=0,1,2,3,4,\ldots..$$

In the set of cosine functions we have included the constant function $g_0(t)=1$.

The constant function 1 is periodic for any value of T and hence does not alter the periodicity of $g_n(t)$.

Fourier theorem. *This theorem states that any arbitrary function F (t) with period T can be expressed as the unique combination of sine and cosine functions $f_n(t)$ and $g_n(t)$ with suitable coefficients.* Mathematically, it can be expressed as

$$F(t)=b_0+b_1\cos\frac{2\pi t}{T}+b_2\cos\frac{4\pi t}{T}+b_3\cos\frac{6\pi t}{T}+\ldots..$$

$$+a_1\sin\frac{2\pi t}{T}+a_2\sin\frac{4\pi t}{T}+a_3\sin\frac{6\pi t}{T}+\ldots..$$

$$=b_0+b_1\cos\omega t+b_2\cos2\omega t+b_3\cos3\omega t+\ldots..$$

$$+a_1\sin\omega t+a_2\sin2\omega t+a_3\sin3\omega t+\ldots..$$

or $\quad F(t)=b_0+\sum\limits_n b_n\cos n\omega t+\sum\limits_n a_n\sin n\omega t$

where $\omega=2\pi/T$.

The coefficients $b_0, b_1, b_2, \ldots.., a_1, a_2, a_3, \ldots..$ are called ***Fourier coefficients***. These coefficients can be determined uniquely by a mathematical method called ***Fourier analysis***. Suppose all the Fourier coefficients except a_1 and b_1 are zero, then

$$F(t)=a_1\sin\frac{2\pi t}{T}+b_1\cos\frac{2\pi t}{T}$$

This equation is a special periodic motion called ***simple harmonic motion*** (S.H.M.).

14.5 PERIODIC, HARMONIC AND NON-HARMONIC FUNCTIONS

5. *Distinguish between periodic, harmonic and non-harmonic functions. Give examples of each.*

Periodic, harmonic and non-harmonic functions. *Any function that repeats itself at regular intervals of its argument is called a periodic function.* The following sine and cosine functions are periodic with period T :

$$f(t)=\sin\omega t=\sin\frac{2\pi t}{T}$$

and $$g(t)=\cos\omega t=\cos\frac{2\pi t}{T}$$

Fig. 14.1. shows how these functions vary with time t.

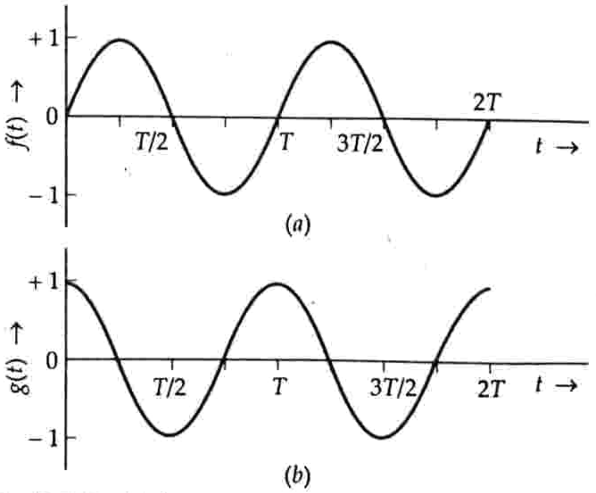


Fig. 14.1 Periodic functions which are harmonic.

Obviously, these functions vary between a maximum value + 1 and minimum value − 1 passing through zero in between. *The periodic functions which can be represented by a sine or cosine curve are called* **harmonic functions**.

All harmonic functions are necessarily periodic but all periodic functions are not harmonic. The periodic functions which cannot be represented by single sine or cosine function are called non-harmonic functions. Fig. 14.2 shows some periodic functions which repeat themselves in a period T but are not harmonic.

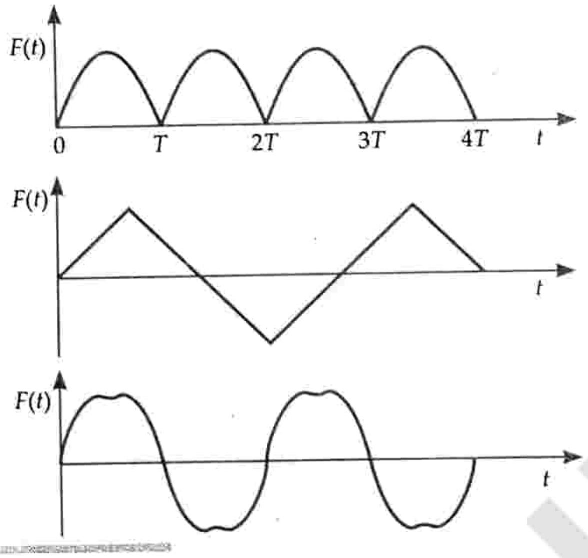


Fig. 14.2 Some non-harmonic periodic functions.

Any non-harmonic periodic function can be constructed from two or more harmonic functions.

One such function is : $F(t) = a_1 \sin \omega t + a_2 \sin 2\omega t$

It can be easily checked that the functions $\tan \omega t$ and $\cot \omega t$ are periodic with period $T = \pi/\omega$ while $\sec \omega t$ and $\operatorname{cosec} \omega t$ are periodic with period $T = 2\pi/\omega$. Thus

$$\tan\left\{\omega\left(t+\frac{\pi}{\omega}\right)\right\} = \tan(\omega t + \pi) = \tan \omega t$$

$$\sec\left\{\omega\left(t+\frac{2\pi}{\omega}\right)\right\} = \sec(\omega t + 2\pi) = \sec \omega t$$

But such functions take values between zero and infinity. So these functions cannot be used to represent displacement functions in periodic motions because displacement always takes a finite value in any physical situation.

Examples Based on Periodic and Harmonic Functions

CONCEPTS USED

1. A function which can be represented by a single sine or cosine function is a harmonic function otherwise non-harmonic.
2. A periodic function can be expressed as the sum of sine and cosine functions of different time periods with suitable coefficients.

EXAMPLE 1. *On an average a human heart is found to beat 75 times in a minute. Calculate its beat frequency and period.* **[NCERT]**

Solution. Beat frequency of the heart,

$$\nu = \frac{\text{No. of beats}}{\text{Time taken}} = \frac{75}{1 \text{ min}}$$

$$= \frac{75}{60 \text{ s}} = 1.25 \text{ s}^{-1} = \mathbf{1.25\ Hz.}$$

Beat period, $T = \frac{1}{\nu} = \frac{1}{1.25 \text{ s}^{-1}} = \mathbf{0.8\ s.}$

EXAMPLE 2. *Which of the following functions of time represent (a) periodic and (n) non-periodic motion ? Give the period for each case of periodic motion. [ω is any positive constant].* **[NCERT]**

(*i*) $\sin \omega t + \cos \omega t$ (*ii*) $\sin \omega t + \cos 2\omega t + \sin 4\omega t$

(*iii*) $e^{-\omega t}$ (*iv*) $\log(\omega t)$.

Solution. (*i*) Here $x(t) = \sin \omega t + \cos \omega t$

$$= \sqrt{2}\left[\sin \omega t \cos \frac{\pi}{4} + \cos \omega t \sin \frac{\pi}{4}\right]$$

$$= \sqrt{2} \sin(\omega t + \pi/4)$$

Moreover,

$$x\left(t + \frac{2\pi}{\omega}\right) = \sqrt{2} \sin[\omega(t + 2\pi/\omega) + \pi/4]$$

$$= \sqrt{2} \sin\left(\omega t + 2\pi + \frac{\pi}{4}\right)$$

$$= \sqrt{2} \sin\left(\omega t + \frac{\pi}{4}\right) = x(t).$$

Hence $\sin \omega t + \cos \omega t$ is a ***periodic function*** with time period equal to $2\pi/\omega$

(*ii*) Here $x(t) = \sin \omega t + \cos 2\omega t + \sin 4\omega t$

$\sin \omega t$ is a periodic function with period

$$= 2\pi/\omega = T$$

$\cos 2\omega t$ is a periodic function with period

$$= 2\pi/2\omega = \pi/\omega = T/2$$

$\sin 4\omega t$ is a periodic function with period

$$= 2\pi/4\omega = \pi/2\omega = T/4$$

Clearly, the entire function $x(t)$ repeats after a minimum time $T = 2\pi/\omega$. Hence the given function is ***periodic***.

(*iii*) The function $e^{-\omega t}$ decreases monotonically to zero as $t \to \infty$. It is an exponential function with a negative exponent of e, where $e \simeq 2.71828$. It never repeats its value. So it is ***non-periodic***.

(*iv*) The function $\log(\omega t)$ increases monotonically with time. As $t \to \infty$, $\log(\omega t) \to \infty$. It never repeats its value. So it is ***non-periodic***.

PROBLEMS FOR PRACTICE

Which of the following functions of time represent (a) simple harmonic motion, (b) periodic but not simple harmonic and (c) non-periodic motion ? Find the period of each periodic motion. Here ω is a positive real constant.

1. $\sin \omega t + \cos \omega t$. (**Ans.** Simple harmonic)
2. $\sin \pi t + 2\cos 2\pi t + 3\sin 3\pi t$. (**Ans.** Periodic but not simple harmonic)
3. $\cos (2\omega t + \pi / 3)$. (**Ans.** Simple harmonic)
4. $\sin^2 \omega t$. (**Ans.** Periodic but not simple harmonic)
5. $\cos \omega t + 2\sin^2 \omega t$. (**Ans.** Periodic but not simple harmonic)

HINTS

1. $\sin \omega t + \cos \omega t = \sqrt{2} \sin (\omega t + \pi / 4)$, $T = 2\pi / \omega$.
2. Each term represents S.H.M.

   Period of $\sin \pi t$, $T = \frac{2\pi}{\pi} = 2\text{s}$

   Period of $2\cos 2\pi t = \frac{2\pi}{2\pi} = 1\text{s} = T/2$

   Period of $3\sin 3\pi t = \frac{2\pi}{3\pi} = \frac{2}{3}\text{s} = T/3$

   The sum is not simple harmonic but periodic with $T = 2\text{s}$.
3. $\cos (2\omega t + \pi / 3)$ represents S.H.M. with

   $$T = 2\pi / 2\omega = \pi / \omega.$$
4. $\sin^2 \omega t = 1/2 - (1/2)\cos 2\omega t$.

   The function does not represent S.H.M. but is periodic with $T = 2\pi / 2\omega = \pi / \omega$.
5. $\cos \omega t + 2\sin^2 \omega t = \cos \omega t + 1 - \cos 2\omega t$

   $$= 1 + \cos \omega t - \cos 2\omega t$$

   $\cos \omega t$ represents S.H.M. with $T = 2\pi / \omega$.

   $\cos 2\omega t$ represents S.H.M. with period

   $$= 2\pi / 2\omega = \pi / \omega = T/2$$

   The combined function does not represent S.H.M. but is periodic with $T = 2\pi / \omega$.

14.6 SIMPLE HARMONIC MOTION

6. *What is meant by simple harmonic motion ? Give some examples.*

Simple harmonic motion. *A particle is said to execute simple harmonic motion if it moves to and fro about a mean position under the action of a restoring force which is directly proportional to its displacement from the mean position and is always directed towards the mean position.*

If the displacement of the oscillating body from the mean position is small, then

Restoring force ∝ Displacement

$$F \propto x \quad \text{or} \quad F = -kx$$

This equation defines S.H.M. Here k is a positive constant called ***force constant*** or ***spring factor*** and is *defined as the restoring force produced per unit displacement*. The SI unit of k is Nm^{-1}. The negative sign in the above equation shows that the restoring force F always acts in the opposite direction of the displacement x.

Now, according to Newton's second law of motion,

$$F = ma$$

$$\therefore \quad ma = -kx$$

$$\text{or} \quad a = -\frac{k}{m}x \quad \text{i.e.,} \quad a \propto x$$

Hence simple harmonic motion may also be defined as follows :

A particle is said to possess simple harmonic motion if it moves to and fro about a mean position under an acceleration which is directly proportional to its displacement from the mean position and is always directed towards that position.

Examples of simple harmonic motion :

(*i*) Oscillations of a loaded spring.
(*ii*) Vibrations of a tuning fork.
(*iii*) Vibrations of the balance wheel of a watch.
(*iv*) Oscillations of a freely suspended magnet in a uniform magnetic field.

7. *State some important features of simple harmonic motion.*

Some important features of S.H.M.

(*i*) The motion of the particle is periodic.
(*ii*) It is the oscillatory motion of simplest kind in which the particle oscillates back and forth about its mean position with constant amplitude and fixed frequency.
(*iii*) Restoring force acting on the particle is proportional to its displacement from the mean position.
(*iv*) The force acting on the particle always opposes the increase in its displacement.
(*v*) A simple harmonic motion can always be expressed in terms of a single harmonic function of sine or cosine.

14.7 DIFFERENTIAL EQUATION FOR S.H.M.

8. *Write down the differential equation for S.H.M. Give its solution. Hence obtain expression for time period of S.H.M.*

Differential equation of S.H.M. In S.H.M., the restoring force acting on the particle is proportional to its displacement. Thus

$$F = -kx.$$

The negative sign shows that F and x are oppositely directed. Here k is spring factor or force constant.

By Newton's second law,

$$F = m\frac{d^2x}{dt^2}$$

where m is the mass of the particle and $\frac{d^2x}{dt^2}$ is its acceleration.

$$\therefore \quad m\frac{d^2x}{dt^2} = -kx \quad \text{or} \quad \frac{d^2x}{dt^2} = -\frac{k}{m}x$$

$$\text{Put} \quad \frac{k}{m} = \omega^2, \quad \text{then} \quad \frac{d^2x}{dt^2} = -\omega^2 x$$

$$\text{or} \quad \frac{d^2x}{dt^2} + \omega^2 x = 0 \qquad ...(1)$$

This is the ***differential equation of*** **S.H.M.** Here ω is the angular frequency. Clearly, x should be such a function whose second derivative is equal to the function itself multiplied with a negative constant. So a possible solution of equation (1) may be of the form

$$x = A\cos(\omega t + \phi_0)$$

$$\text{Then} \quad \frac{dx}{dt} = -\omega A\sin(\omega t + \phi_0)$$

$$\text{and} \quad \frac{d^2x}{dt^2} = -\omega^2 A\cos(\omega t + \phi_0) = -\omega^2 x$$

$$\text{or} \quad \frac{d^2x}{dt^2} + \omega^2 x = 0$$

which is same as equation (1). Hence the solution of the equation (1) is

$$x = A\cos(\omega t + \phi_0) \qquad ...(2)$$

It gives displacement of the harmonic oscillator at any instant t.

Here A is the *amplitude* of the oscillating particle.

$\phi = \omega t + \phi_0$, is the phase of the oscillating particle.

ϕ_0 is the initial *phase* (at $t = 0$) or epoch.

Time period of S.H.M. If we replace t by $t + \frac{2\pi}{\omega}$ in equation (2), we get

$$x = A\cos\left[\omega\left(t + \frac{2\pi}{\omega}\right) + \phi_0\right]$$

$$= A\cos(\omega t + 2\pi + \phi_0) = A\cos(\omega t + \phi_0)$$

i.e., the motion repeats after time interval $\frac{2\pi}{\omega}$. Hence $\frac{2\pi}{\omega}$ is the time period of S.H.M.

$$\therefore \quad T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{k/m}} \qquad \left[\because \ \omega^2 = \frac{k}{m}\right]$$

$$\text{or} \quad T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{\text{Inertia factor}}{\text{Spring factor}}}$$

In general, m is called inertia factor and k the spring factor.

14.8 SOME IMPORTANT TERMS CONNECTED WITH S.H.M.

9. *Define the terms harmonic oscillator, displacement, amplitude, cycle, time period, frequency, angular frequency, phase and epoch with reference to oscillatory motion.*

Some important terms connected with S.H.M.

(*i*) **Harmonic oscillator.** *A particle executing simple harmonic motion is called harmonic oscillator.*

(*ii*) **Displacement.** *The distance of the oscillating particle from its mean position at any instant is called its displacement.* It is denoted by x.

There can be other kind of displacement variables. These can be voltage variations in time across a capacitor in an a.c. circuit, pressure variations in time in the propagation of a sound wave, the changing electric and magnetic fields in the propagation of a light wave, etc.

(*iii*) **Amplitude.** *The maximum displacement of the oscillating particle on either side of its mean position is called its amplitude.* It is denoted by A. Thus $x_{\max} = \pm A$.

(*iv*) **Oscillation or cycle.** *One complete back and forth motion of a particle starting and ending at the same point is called a cycle or oscillation or vibration.*

(*v*) **Time period.** *The time taken by a particle to complete one oscillation is called its time period. Or, it is the smallest time interval after which the oscillatory motion repeats.* It is denoted by T.

(*vi*) **Frequency.** *It is defined as the number of oscillations completed per unit time by a particle.* It is denoted by ν (nu). Frequency is equal to the reciprocal of time period. That is,

$$\nu = \frac{1}{T}$$

Clearly, the unit of frequency is $(\text{second})^{-1}$ or s^{-1}. It is also expressed as *cycles per second* (cps) or *hertz* (Hz).

SI unit of frequency = s^{-1} = cps = Hz.

(*vii*) **Angular frequency.** *It is the quantity obtained by multiplying frequency* ν *by a factor of* 2π. It is denoted by ω.

$$\text{Thus,} \qquad \omega = 2\pi\nu = \frac{2\pi}{T}$$

SI unit of angular frequency = rad s^{-1}.

(*viii*) **Phase.** *The phase of a vibrating particle at any instant gives the state of the particle as regards its position and the direction of motion at that instant.* It is equal to the argument of sine or cosine function occurring in the displacement equation of the S.H.M. Suppose a simple harmonic equation is represented by

$$x = A\cos(\omega t + \phi_0)$$

Then phase of the particle is : $\phi = \omega t + \phi_0$

Clearly, the phase ϕ is a function of time t. It is usually expressed either as the fraction of the time period T or fraction of angle 2π that has elapsed since the vibrating particle last passed its mean position in the positive direction.

$\phi = \omega t + \phi_0$	0	$\pi/2$	π	$3\pi/2$	2π
$x = A\cos(\omega t + \phi_0)$	$+A$	0	$-A$	0	$+A$

Thus the phase ϕ gives an idea about the position and the direction of motion of the oscillating particle.

(*ix*) **Initial phase or epoch.** *The phase of a vibrating particle corresponding to time $t = 0$ is called initial phase or epoch.*

At $t = 0$, $\quad \phi = \phi_0$

The constant ϕ_0 is called initial phase or epoch. It tells about the initial state of motion of the vibrating particle.

14.9 UNIFORM CIRCULAR MOTION AND S.H.M.

10. *Show that simple harmonic motion may be regarded as the projection of uniform circular motion along a diameter of the circle. Hence derive an expression for the displacement of a particle in S.H.M.*

Relation between S.H.M. and uniform circular motion. As shown in Fig. 14.3, consider a particle P moving along a circle of radius A with uniform angular velocity ω. Let N be the foot of the perpendicular drawn from the point P to the diameter XX'. Then N is called the *projection* of P on the diameter XX'. As P moves along the circle from X to Y, Y to X', X' to Y' and Y' to X ; N moves from X to O, O to X', X' to O and O to X. Thus, as P revolves along the circumference of the circle, N moves to and fro about the point O along the diameter XX'. The motion of N about O is said to be simple harmonic. Hence ***simple harmonic motion** may be defined as the projection of uniform circular motion upon a diameter of a circle.* The particle P is called *reference particle* or *generating particle* and the circle along which the particle P revolves is called *circle of reference.*

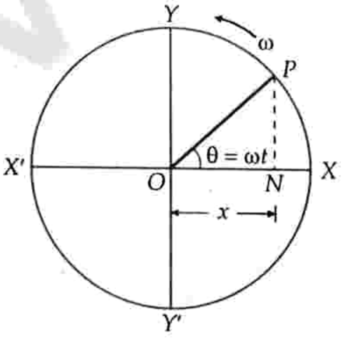


Fig. 14.3 Reference circle.

Displacement in simple harmonic motion. As shown in Fig. 14.4, consider a particle moving in anticlockwise direction with uniform angular velocity ω along a circle of radius A and centre O. Suppose at time $t = 0$, the reference particle is at point A such that $\angle XOA = \phi_0$. At any time t, suppose the particle reaches the point P such that $\angle AOP = \omega t$. Draw $PN \perp XX'$.

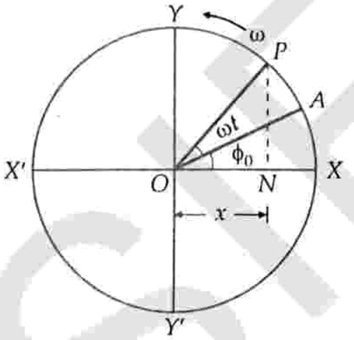


Fig. 14.4 Displacement in S.H.M., epoch ($+\phi_0$)

Clearly, displacement of projection N from centre O at any instant t is $\quad x = ON.$

In right-angled ΔONP,

$$\angle PON = \omega t + \phi_0$$

$$\therefore \quad \frac{ON}{OP} = \cos(\omega t + \phi_0)$$

or

$$\frac{x}{A} = \cos(\omega t + \phi_0)$$

or

$$x = A\cos(\omega t + \phi_0).$$

This equation gives displacement of a particle in S.H.M. at any instant t. The quantity $\omega t + \phi_0$ is called phase of the particle and ϕ_0 is called *initial phase* or *phase constant or epoch* of the particle. The quantity A is called *amplitude* of the motion. It is a positive constant whose value depends on how the motion is initially started. Thus

$$\underset{\text{Displacement}}{\underset{\Uparrow}{x}} = \underset{\text{Amplitude}}{\underset{\Uparrow}{A}} \overbrace{\cos(\underset{\text{Angular frequency}}{\underset{\Uparrow}{\omega t}} + \underset{\text{Initial phase}}{\underset{\Uparrow}{\phi_0}})}^{\text{Phase}}$$

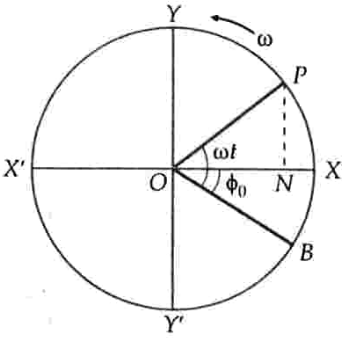


Fig. 14.5 Epoch ($-\phi_0$)

As shown in Fig. 14.5, if the reference particle starts motion from the point P such that $\angle BOX = \phi_0$ and $\angle BOP = \omega t$, then

$$\angle PON = \omega t - \phi_0$$

$$\therefore \qquad x = A\cos(\omega t - \phi_0)$$

Here $-\phi_0$ is the initial phase of the S.H.M.

11. *Show that a linear combination of sine and cosine functions like*

$$x(t) = a \sin \omega t + b \cos \omega t$$

represents a simple harmonic motion. Determine its amplitude and phase constant.

General expression for S.H.M. We are given

$$x = a \sin \omega t + b \cos \omega t \qquad \text{...(1)}$$

Differentiating w.r.t. time t, we get

$$\frac{dx}{dt} = \omega a \cos \omega t - \omega b \sin \omega t$$

Again, differentiating w.r.t. time t, we get

$$\frac{d^2x}{dt^2} = -\omega^2 a \sin \omega t - \omega^2 b \cos \omega t$$

$$= -\omega^2 (a \sin \omega t + b \cos \omega t)$$

or $$\frac{d^2x}{dt^2} = -\omega^2 x$$

i.e., acceleration ∝ displacement

Hence the equation (1) represents S.H.M.

To determine its amplitude and phase constant, we put

$$a = A\cos\phi \qquad \text{...(2)}$$

and $$b = A\sin\phi \qquad \text{...(3)}$$

Then $$x = A\cos\phi \sin\omega t + A\sin\phi\cos\omega t$$

$$= A(\sin\omega t\cos\phi + \cos\omega t\sin\phi)$$

or $$x = A\sin(\omega t + \phi)$$

This again shows that equation (1) represents S.H.M. of amplitude A and phase constant ϕ.

Squaring and adding (2) and (3), we get

$$a^2 + b^2 = A^2(\cos^2\phi + \sin^2\phi) = A^2 \times 1$$

$$\therefore \text{ Amplitude, } A = \sqrt{a^2 + b^2}$$

Dividing (3) by (2), we get : $\dfrac{b}{a} = \dfrac{A\sin\phi}{A\cos\phi} = \tan\phi$

$$\therefore \text{ Phase constant, } \phi = \tan^{-1}\frac{b}{a}.$$

14.10 VELOCITY IN S.H.M.

12. *Deduce an expression for the velocity of a particle executing S.H.M. When is the particle velocity (i) maximum and (ii) minimum ?*

Expression for the velocity of a particle executing S.H.M. As shown in Fig. 14.6, consider a particle P moving with uniform angular speed ω in a circle of radius A. Its velocity vector $\vec{v}$ is directed along the tangent and the magnitude of this velocity vector is

$$v = \text{Angular velocity} \times \text{radius} = \omega A$$

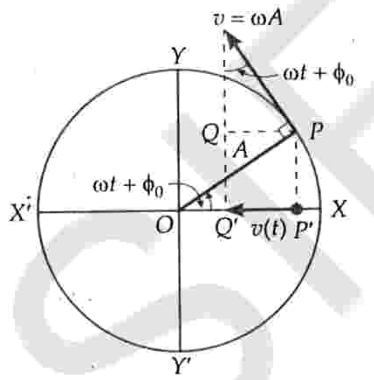


Fig. 14.6 Velocity of a particle in S.H.M.

Draw PP' and QQ' perpendiculars to the diameter XX'. The motion of P' is simple harmonic. Clearly, the instantaneous velocity of a particle executing S.H.M. will be

$v(t)$ = Velocity of the particle P' at any instant t

= Projection of the velocity v of the reference particle P

$$= P'Q' = PQ = -v\sin(\omega t + \phi_0)$$

or $$v(t) = -\omega A \sin(\omega t + \phi_0).$$

The negative sign shows that the velocity of P' is directed towards left *i.e.*, in the negative X-direction.

Moreover,

$$v(t) = -\omega A\sqrt{1 - \cos^2(\omega t + \phi_0)} = -\omega A\sqrt{1 - \frac{x^2}{A^2}}$$

or $$v(t) = -\omega\sqrt{A^2 - x^2} \qquad [\because x = A\cos(\omega t + \phi_0)]$$

Special cases. (*i*) *When the particle is at the mean position*, then $x = 0$, so

$$v(t) = -\omega\sqrt{A^2 - 0^2} = -\omega A$$

This is the maximum velocity which a particle in S.H.M. can execute and is called velocity amplitude, denoted by $v_{\max}$

$$\therefore \qquad v_{\max} = \omega A = \frac{2\pi}{T}A$$

(*ii*) *When the particle is at the extreme position*, then $x = \pm A$, so

$$v = -\omega\sqrt{A^2 - A^2} = 0.$$

Thus the velocity of a particle in S.H.M. is zero at either of its extreme positions.

14.11 ACCELERATION OF A PARTICLE IN S.H.M.

13. *Show that the acceleration of a particle in S.H.M. is proportional to its displacement from the mean position. Hence write the expression for the time period of S.H.M.*

Expression for the acceleration of a particle executing S.H.M. As shown in Fig. 14.7, consider a particle P moving with uniform angular speed ω in a circle of radius A. The particle has the centripetal acceleration $\vec{a}_c$ acting radially towards the centre O. The magnitude of this acceleration is $a_c = \omega^2 A$.

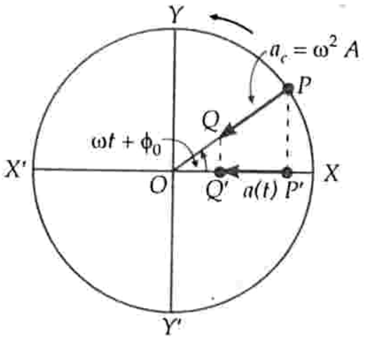


Fig. 14.7 Acceleration of a particle in S.H.M.

Draw PP' and QQ' perpendiculars to the diameter XX'. The motion of P' is simple harmonic. Clearly, the instantaneous acceleration of a particle executing S.H.M. will be

$a(t)$ = Acceleration of particle P' at any instant t

= Projection of the acceleration a_c of the reference particle P

= Projection of PQ on diameter XX'

$= P'Q' = -a_c \cos(\omega t + \phi_0)$

or $\quad a(t) = -\omega^2 A\cos(\omega t + \phi_0) = -\omega^2 x$

This equation expresses the acceleration of a particle executing S.H.M. It shows that *the acceleration of a particle in S.H.M. is proportional to its displacement from the mean position and acts in the opposite direction of the displacement.*

Special cases. (*i*) *When the particle is at the mean position,* then $x = 0$, so, acceleration $= -\omega^2(0) = 0$.

Hence *the acceleration of a particle in S.H.M. is zero at the mean position.*

(*ii*) *When the particle is at the extreme position,* then $x = A$, so, acceleration $= -\omega^2 A$

This is the maximum value of acceleration which a particle in S.H.M. can possess and is called **acceleration** amplitude, denoted by $a_{\max}$.

$$\therefore \quad a_{\max} = \omega^2 A = \left(\frac{2\pi}{T}\right)^2 A$$

14.12 PHASE RELATIONSHIP BETWEEN DISPLACEMENT, VELOCITY AND ACCELERATION

14. *Draw displacement-time, velocity-time and acceleration-time graphs for a particle executing simple harmonic motion. Discuss their phase relationship.*

Inter-relationship between particle displacement, velocity and acceleration in S.H.M. If a particle executing S.H.M. passes through its positive extreme position ($x = +A$) at time $t = 0$, then its displacement equation can be written as

$$x(t) = A\cos\omega t$$

Velocity, $\quad v(t) = \frac{dx}{dt} = -\omega A\sin\omega t$

$$= \omega A\cos\left(\omega t + \frac{\pi}{2}\right)$$

Acceleration, $\quad a(t) = \frac{dv}{dt} = -\omega^2 A\cos\omega t$

$$= \omega^2 A\cos(\omega t + \pi).$$

Using the above relations, we determine the values of displacement, velocity and acceleration at various instant t for one complete cycle as illustrated below.

Time, t	***0***	$\frac{T}{4}$	$\frac{T}{2}$	$\frac{3T}{4}$	***T***
Phase angle, $\omega t = \frac{2\pi}{T}t$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
Displacement, $x(t)$	$+A$ max.	0 min.	$-A$ max.	0 min.	$+A$ max.
Velocity, $v(t)$	0 min.	$-\omega A$ max.	0 min.	$+\omega A$ max.	0 min.
Acceleration, $a(t)$	$-\omega^2 A$ max.	0 min.	$+\omega^2 A$ max.	0 min.	$-\omega^2 A$ max.

In Fig. 14.8, we have plotted separately the x versus t, v versus t and a versus t curves for a simple harmonic motion.

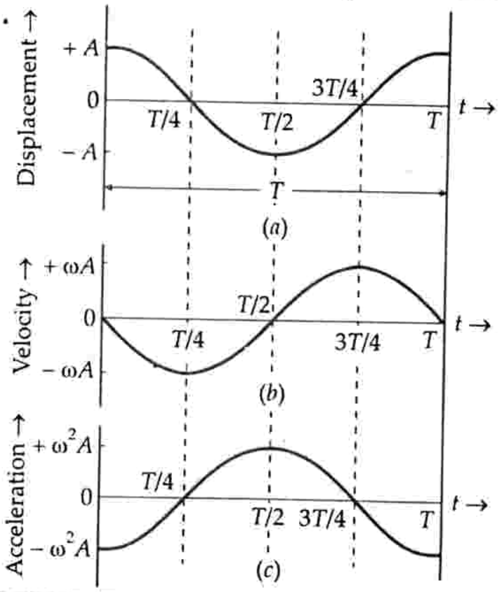


Fig. 14.8 Relation between velocity, displacement and acceleration in S.H.M.

Conclusions. From the above graphs, we can draw the following conclusions about simple harmonic motion :

(*i*) Displacement, velocity and acceleration, all vary harmonically with time.

(*ii*) The velocity amplitude is ω times ; and acceleration amplitude is ω^2 times the displacement amplitude A.

(*iii*) Clearly, the velocity curve lies shifted to the left of the displacement curve by an interval of $T/4$. Thus the *particle velocity is ahead of its displacement by a phase angle of $\pi/2$ rad*. This means that whichever value displacement attains at any instant, velocity attains a similar value a $T/4$ time (a quarter of cycle) earlier. When the particle velocity is maximum, the displacement is minimum and vice versa.

(*iv*) Clearly, the acceleration curve lies shifted to the left of the displacement curve by an interval of $T/2$. *Thus the particle acceleration is ahead of its displacement by a phase angle of π rad.* Or, acceleration is ahead of velocity in phase by $\pi/2$ rad. When acceleration has maximum positive value, displacement has maximum negative value and vice versa. When the displacement is zero, the acceleration is also zero.

Examples based on Displacement, Velocity, Acceleration and Time Period of SHM

FORMULAE USED

1. Displacement, $x = A\cos(\omega t + \phi_0)$
   where A = amplitude, ω = angular frequency and ϕ_0 = initial phase of particle in SHM.
2. Velocity, $v = \frac{dx}{dt} = -\omega A \sin(\omega t + \phi_0)$
   $= -\omega\sqrt{A^2 - x^2}$
   Maximum velocity, $v_{max} = \omega A$.
3. Acceleration, $a = \frac{dv}{dt} = -\omega^2 A\cos(\omega t + \phi_0) = -\omega^2 x$
   Maximum acceleration, $a_{max} = \omega^2 A$
4. Restoring force, $F = -kx = -m\omega^2 x$
   where k = force constant and $\omega^2 = k/m$.
5. Angular frequency, $\omega = 2\pi\nu = 2\pi/T$.
6. Time period, $T = 2\pi\sqrt{\frac{\text{Displacement}}{\text{Acceleration}}} = 2\pi\sqrt{\frac{x}{a}}$.
7. Time period, $T = 2\pi\sqrt{\frac{\text{Inertia factor}}{\text{Spring factor}}} = 2\pi\sqrt{\frac{m}{k}}$.

UNITS USED

Displacement x and amplitude A are in m or cm, force constant k in Nm^{-1}, frequency ν in Hz, angular frequency ω in rad s^{-1}.

EXAMPLE 3. *The following figures depict two circular motions. The radius of the circle, the period of revolution, the initial position and the sense of revolution are indicated on the figures. Obtain the simple harmonic motions of the x-projection of the radius vector of the rotating particle P in each case.* **[NCERT]**

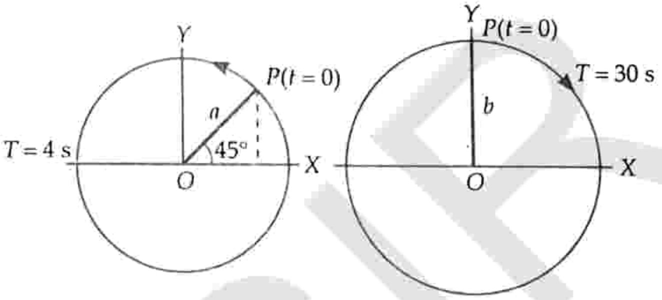


Fig. 14.9

Solution. (*a*) As shown in Fig. 14.10(*a*), suppose the particle moves in the anticlockwise sense from P to P' in time t.

Angle swept by the radius vector,

$$\theta = \omega t = \frac{2\pi}{T}t = \frac{2\pi}{4}t \qquad [\because T = 4\text{ s}]$$

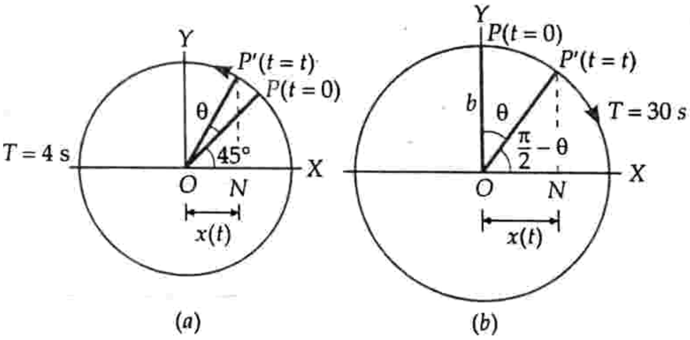
N is the foot of perpendicular drawn from P' on the x-axis.

Displacement,

$$ON = OP'\cos(\theta + \pi/4)$$

or $$x(t) = a\cos\left(\frac{2\pi}{4}t + \frac{\pi}{4}\right)$$

This represents S.H.M. of amplitude a, period 4 s and an initial phase = $\pi/4$ rad.



(*a*) (*b*)

Fig. 14.10

(*b*) As shown in Fig. 14.10(*b*), suppose the particle moves in the clockwise sense from P to P' in time t.

Angle swept by the radius vector,

$$\theta = \omega t = \frac{2\pi}{T}t = \frac{2\pi}{30}t \qquad [\because T = 30\text{ s}]$$

Displacement,

$$ON = OP' \cos\left(\frac{\pi}{2} - \theta\right)$$

or $$x(t) = b \cos\left(\frac{\pi}{2} - \frac{2\pi}{30} t\right)$$

or $$x(t) = b \cos\left(\frac{2\pi}{30} t - \frac{\pi}{2}\right) \quad [\because \cos(-\theta) = \cos\theta]$$

This represents S.H.M. of amplitude b, period 30 s and an initial phase $= -\pi/2$ rad.

Example 4. *A simple harmonic motion is represented by*

$$x = 10 \sin(20\,t + 0.5)$$

Write down its amplitude, angular frequency, frequency, time period and initial phase, if displacement is measured in metres and time in seconds. **[Himachal 09C]**

Solution. Given $x = 10 \sin(20\,t + 0.5)$

Standard equation for displacement in SHM is

$$x = A \sin(\omega t + \phi_0)$$

Comparing the above two equations, we get

(*i*) Amplitude, $A = \mathbf{10\ m}$.

[$\because$ A and x have same units]

(*ii*) Angular frequency, $\omega = \mathbf{20\ rad\ s^{-1}}$.

(*iii*) Frequency, $\nu = \frac{\omega}{2\pi} = \frac{20}{2\pi} = \frac{10}{\pi} = \mathbf{3.18\ Hz}$.

(*iv*) Time period, $T = \frac{2\pi}{\omega} = \frac{2\pi}{20} = \frac{\pi}{10} = \mathbf{0.314\ s}$.

(*v*) Initial phase, $\phi_0 = \mathbf{0.5\ rad}$.

Example 5. *A particle executes SHM with a time period of 2 s and amplitude 5 cm. Find (i) displacement (ii) velocity and (iii) acceleration, after 1/3 second ; starting from the mean position.*

Solution. Here $T = 2$ s, $A = 5$ cm, $t = 1/3$ s

(*i*) For the particle starting from mean position,

Displacement, $x = A \sin\omega t = A \sin\frac{2\pi}{T} t$

$$= 5 \sin\frac{2\pi}{2} \times \frac{1}{3} = 5 \sin\frac{\pi}{3} = 5 \times \frac{\sqrt{3}}{2} = \mathbf{4.33\ cm.}$$

(*ii*) Velocity, $v = \frac{dx}{dt} = \frac{2\pi A}{T} \cos\frac{2\pi}{T} t$

$$= \frac{2\pi \times 5}{2} \cos\frac{\pi}{3} = 5 \times 3.14 \times 0.5 = \mathbf{7.85\ cm\ s^{-1}}.$$

(*iii*) Acceleration, $a = \frac{dv}{dt} = \frac{4\pi^2 A}{T^2} \sin\frac{2\pi}{T} t$

$$= \frac{4 \times 9.87 \times 5}{4} \sin\frac{\pi}{3}$$

$$= 9.87 \times 5 \times \frac{\sqrt{3}}{2} = \mathbf{42.77\ cm\ s^{-2}}.$$

Example 6. *A body oscillates with SHM according to the equation :*

$$x(t) = 5 \cos(2\pi t + \pi/4),$$

where t is in sec. and x in metres. Calculate

(*a*) *Displacement at* $t = 0$ (*b*) *Time period*

(*c*) *Initial velocity* **[Central Schools 08]**

Solution. Given $x(t) = 5 \cos(2\pi t + \pi/4)$

We compare with standard equation,

$$x(t) = A \cos(\omega t + \phi_0)$$

(*a*) Displacement at $t = 0$,

$$x(0) = 5 \cos\frac{\pi}{4} = 5 \times \frac{1}{\sqrt{2}} = \mathbf{\frac{5}{\sqrt{2}}\ m}.$$

(*b*) Clearly, $\omega = 2\pi$ or $\frac{2\pi}{T} = 2\pi$

$\therefore$ Time period, $T = \mathbf{1\ s}$.

(*c*) Velocity, $v = \frac{dx}{dt} = -5 \sin\left(2\pi t + \frac{\pi}{4}\right) \times 2\pi$

Initial velocity at $t = 0$,

$$v = -10\pi \sin\frac{\pi}{4} = -\mathbf{\frac{10\pi}{\sqrt{2}}\ m/s}.$$

Example 7. *A body oscillates with SHM according to the equation,*

$$x = (5.0\ m) \cos[(2\pi\ rad\ s^{-1})\,t + \pi/4].$$

At $t = 1.5$ s, calculate (a) displacement, (b) speed and (c) acceleration of the body. **[NCERT]**

Solution. Here $\omega = 2\pi$ rad s^{-1}, $T = 2\pi/\omega = 1$ s,

$$t = 1.5\ \text{s}$$

(*a*) Displacement,

$$x = 5.0 \cos(2\pi \times 1.5 + \pi/4) = 5.0 \cos(3\pi + \pi/4)$$

$$= -5.0 \cos\pi/4 = -5.0 \times 0.707 = \mathbf{-3.535\ m}.$$

(*b*) Velocity,

$$v = \frac{dx}{dt} = \frac{d}{dt}[5.0 \cos(2\pi t + \pi/4)]$$

$$= -5.0 \times 2\pi \sin(2\pi t + \pi/4)$$

$$= -5.0 \times 2\pi \sin(2\pi \times 1.5 + \pi/4)$$

$$= +5.0 \times 2\pi \sin\pi/4 = 5.0 \times 2 \times \frac{22}{7} \times 0.707$$

$$= \mathbf{22.22\ m\ s^{-1}}.$$

(*c*) Acceleration,

$$a = \frac{dv}{dt} = \frac{d}{dt}[-10\pi \sin(2\pi t + \pi/4)]$$

$$= -20\pi^2 \cos(2\pi t + \pi/4)$$

$$= -4\pi^2 [5.0 \cos(2\pi \times 1.5 + \pi/4)]$$

$$= -4 \times 9.87 \times (-3.535) \quad \text{[Using (a)]}$$

$$= \mathbf{139.56\ m\ s^{-1}}.$$

EXAMPLE 8. *The equation of a simple harmonic motion is given by $y = 6 \sin 10\pi t + 8 \cos 10\pi t$, where y is in cm and t in sec. Determine the amplitude, period and initial phase.*

Solution. Given $y = 6 \sin 10\pi t + 8 \cos 10\pi t$ ...(1)

The general equation of SHM is

$$y = a \sin(\omega t + \phi) = a \sin \omega t \cos \phi + a \cos \omega t \sin \phi$$

$$= (a \cos \phi) \sin \omega t + (a \sin \phi) \cos \omega t \qquad ...(2)$$

Comparing equations (1) and (2), we get

$$a \cos \phi = 6 \qquad ...(3)$$

$$a \sin \phi = 8 \qquad ...(4)$$

and $\omega t = 10\pi t$ or $\omega = 10\pi$

$\therefore$ Time period, $T = \frac{2\pi}{\omega} = \frac{2\pi}{10\pi} = \mathbf{0.2\ s.}$

Squaring and adding (3) and (4), we get

$$a^2(\cos^2 \phi + \sin^2 \phi) = 6^2 + 8^2$$

$$= 36 + 64 = 100 \quad \text{or} \quad a^2 = 100$$

$\therefore$ Amplitude, $a = \mathbf{10\ cm}$

Dividing (4) by (3), we get

$$\tan \phi = \frac{8}{6} = 1.3333$$

$\therefore$ Initial phase, $\phi = \tan^{-1}(1.3333) = \mathbf{53°8'}$.

EXAMPLE 9. *A particle executes S.H.M. of amplitude 25 cm and time period 3 s. What is the minimum time required for the particle to move between two points 12.5 cm on either side of the mean position ?*

Solution. When the particle starts from mean position, its displacement at instant t is given by

$$y = A \sin \omega t$$

Given $A = 25$ cm, $T = 3$ s, $y = 12.5$ cm

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{3} \text{ rad s}^{-1}$$

$$\therefore \quad 12.5 = 25 \sin \frac{2\pi}{3} t$$

or $\sin \frac{2\pi}{3} t = \frac{12.5}{25} = \frac{1}{2}$

$$\therefore \quad \frac{2\pi t}{3} = \frac{\pi}{6} \quad \text{or} \quad t = \frac{1}{4} \text{ s}$$

$\therefore$ Time taken by the particle to move between two points 12.5 cm on either side of mean position is given by

$$2t = 2 \times \frac{1}{4} = \frac{1}{2} \text{ s} = \mathbf{0.5\ s.}$$

EXAMPLE 10. *The shortest distance travelled by a particle executing SHM from mean position in 2 s is equal to $(\sqrt{3}/2)$ times its amplitude. Determine its time period.*

Solution. Here $t = 2$ s, $y = (\sqrt{3}/2)$ A, $T = ?$

As $y = A \sin \omega t = A \sin \frac{2\pi}{T} t$

$$\therefore \quad \frac{\sqrt{3}}{2} A = A \sin \frac{2\pi \times 2}{T}$$

or $\sin \frac{4\pi}{T} = \frac{\sqrt{3}}{2} = \sin \frac{\pi}{3} \quad \therefore \quad \frac{4\pi}{T} = \frac{\pi}{3}$

or $T = \mathbf{12\ s.}$

EXAMPLE 11. *The time-period of a simple pendulum is 2 s and it can go to and fro from equilibrium position at a maximum distance of 5 cm. If at the start of the motion the pendulum is in the position of maximum displacement towards the right of the equilibrium position, then write the displacement equation of the pendulum.*

Solution. The displacement in SHM is given by

$$y = A \sin(\omega t + \phi_0)$$

Given $T = 2$ s, $A = 5$ cm

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi \text{ rad s}^{-1}$$

$$\therefore \quad y = 5 \sin(\pi t + \phi_0)$$

At time $t = 0$, displacement $y = 5$ cm. Therefore,

$$5 = 5 \sin(\pi \times 0 + \phi_0)$$

or $\sin \phi_0 = 1 \quad \therefore \quad \phi_0 = \pi/2$

Hence displacement equation for the pendulum is

$$y = 5 \sin\left(\pi t + \frac{\pi}{2}\right) = \mathbf{5 \cos \pi t.}$$

EXAMPLE 12. *A particle executes S.H.M. of time period 10 seconds. The displacement of particle at any instant is given by : $x = 10 \sin \omega t$ (in cm). Find (i) the velocity of body 2 s after it passes through mean position (ii) the acceleration 2 s after it passes the mean position.* **[Central Schools 04]**

Solution. Here $T = 10$ s, $x = 10 \sin \omega t$ cm

(*i*) Velocity, $v = \frac{dx}{dt} = 10\, \omega \cos \omega t$ cm s^{-1}

$$= 10\left(\frac{2\pi}{T}\right) \cos \frac{2\pi}{T} t \text{ cm s}^{-1}$$

Velocity of the body 2s after it passes through the mean position,

$$v = 10\left(\frac{2\pi}{10}\right) \cos\left(\frac{2\pi}{10} \times 2\right) \text{cm s}^{-1}$$

$$= 2\pi \cos 72° = 2 \times 3.14 \times 0.309 = \mathbf{1.94\ cm\ s^{-1}}.$$

(*ii*) Acceleration of the body 2s after it passes through the mean position,

$$a = -\omega^2 x = -\left(\frac{2\pi}{T}\right)^2 \times 10 \sin\left(\frac{2\pi}{T} t\right)$$

$$= -\frac{4\pi^2}{(10)^2} \times 10 \sin 72°$$

$$= -\frac{4 \times 9.87 \times 0.951}{10} = \mathbf{-3.75\ cm\ s^{-2}}.$$

EXAMPLE 13. *For a particle in SHM, the displacement x of the particle as a function of time t is given as*

$$x = A \sin(2\pi t)$$

Here x is in cm and t is in seconds.

Let the time taken by the particle to travel from x = 0 to x = A/2 be t_1 and the time taken to travel from x = A/2 to x = A be t_2. Find t_1 / t_2. **[Delhi 04]**

Solution. Here $x = 0$ at $t = 0$.

Also $\omega = \frac{2\pi}{T} = 2\pi \quad \therefore \quad T = 1 \text{ s}$

At $t = t_1$, $x = A/2$. Therefore

$$\frac{A}{2} = A \sin(2\pi t_1) \quad \text{or} \quad \frac{1}{2} = \sin(2\pi t_1)$$

$$\therefore \quad 2\pi t_1 = \frac{\pi}{6} \quad \text{or} \quad t_1 = \frac{1}{12} \text{ s}$$

Time taken from $x = 0$ to $x = A$ is $\frac{T}{4} = \frac{1}{4}$ s

or $$t_1 + t_2 = \frac{T}{4} = \frac{1}{4} \text{ s}$$

or $$t_2 = \frac{1}{4} - \frac{1}{12} = \frac{1}{6} \text{ s}$$

Hence $$\frac{t_1}{t_2} = \frac{1/12}{1/6} = \mathbf{\frac{1}{2}}.$$

EXAMPLE 14. *In a HCl molecule, we may treat Cl to be of infinite mass and H alone oscillating. If the oscillation of HCl molecule shows frequency 9×10^{13} s^{-1}, deduce the force constant. The Avogadro number $= 6 \times 10^{26}$ per kg-mole.*

Solution. Frequency, $\nu = 9 \times 10^{13}$ s^{-1}

Mass of a H-atom, $m = \frac{M}{N} = \frac{1}{6 \times 10^{26}}$ kg

As $\nu = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$ or $\nu^2 = \frac{1}{4\pi^2} \cdot \frac{k}{m}$

$$\therefore \quad k = 4\pi^2 m \nu^2$$

$$= 4\left(\frac{22}{7}\right)^2 \times \frac{1}{6 \times 10^{26}} \times (9 \times 10^{13})^2$$

$$= \mathbf{533.4 \ Nm^{-1}}.$$

EXAMPLE 15. *A particle is moving with SHM in a straight line. When the distance of the particle from the equilibrium position has values x_1 and x_2, the corresponding values of velocities are u_1 and u_2. Show that the time period of oscillation is given by*

$$T = 2\pi\left[\frac{x_2^2 - x_1^2}{u_1^2 - u_2^2}\right]^{1/2}.$$

Solution. When $x = x_1, \quad v = u_1$

When $x = x_2, \quad v = u_2$

As $v = \omega\sqrt{A^2 - x^2}$

$$\therefore \quad u_1 = \omega\sqrt{A^2 - x_1^2} \quad \text{or} \quad u_1^2 = \omega^2(A^2 - x_1^2) \quad ...(1)$$

and $$u_2 = \omega\sqrt{A^2 - x_2^2} \quad \text{or} \quad u_2^2 = \omega^2(A^2 - x_2^2) \quad ...(2)$$

Subtracting (2) from (1), we get

$$u_1^2 - u_2^2 = \omega^2(A^2 - x_1^2) - \omega^2(A^2 - x_2^2) = \omega^2(x_2^2 - x_1^1)$$

or $$\omega = \left[\frac{u_1^2 - u_2^2}{x_2^2 - x_1^2}\right]^{1/2}$$

$$T = \frac{2\pi}{\omega} = 2\pi\left[\frac{x_2^2 - x_1^2}{u_1^2 - u_2^2}\right]^{1/2}.$$

EXAMPLE 16. *If the distance y of a point moving on a straight line measured from a fixed origin on it and velocity v are connected by the relation $4v^2 = 25 - y^2$, then show that the motion is simple harmonic and find its time period.*

Solution. Given $4v^2 = 25 - y^2$

or $$v = \frac{1}{2}\sqrt{25 - y^2}$$

Also velocity in SHM, $v = \omega\sqrt{A^2 - y^2}$

Comparing the above two equations, we find that the given equation represents SHM of amplitude $A = 5$ and $\omega = 1/2$ rad s^{-1}.

Time-period, $T = \frac{2\pi}{\omega} = \frac{2\pi \times 2}{1} = \mathbf{4\pi \ s}.$

EXAMPLE 17. *A particle executing SHM along a straight line has a velocity of 4 ms^{-1} when at a distance 3 m from the mean position and 3 ms^{-1} when at a distance of 4 m from it. Find the time it takes to travel 2.5 m from the positive extremity of its oscillation.*

Solution. When $y_1 = 3$ m, $\quad v_1 = 4$ ms^{-1}

When $y_2 = 4$ m, $\quad v_2 = 3$ ms^{-1}

As $$v = \omega\sqrt{A^2 - y^2} \quad \therefore \quad 4 = \omega\sqrt{A^2 - 3^2}$$

or $$16 = \omega^2(A^2 - 9) \quad ...(1)$$

and $$3 = \omega\sqrt{A^2 - 4^2} \text{ or } 9 = \omega^2(A^2 - 16) \quad ...(2)$$

Dividing (1) by (2), we get

$$\frac{16}{9} = \frac{A^2 - 9}{A^2 - 16} \quad \text{or} \quad 16A^2 - 256 = 9A^2 - 81$$

or $$7A^2 = 256 - 81 = 175 \quad \text{or} \quad A^2 = 25$$

$$\therefore \quad A = \sqrt{25} = 5 \text{ m}$$

From (1), $4 = \omega\sqrt{5^2 - 3^2} = \omega \times 4$

or $$\omega = 1 \text{ rad s}^{-1}$$

When the particle is 2.5 m from the positive extreme position, its displacement from the mean position is

$$y = 5 - 2.5 = 2.5 \text{ m}$$

When the time is noted from the extreme position, we can write

$$y = A \cos \omega t$$

$$\therefore \quad 2.5 = 5 \cos (1 \times t)$$

or $$\cos t = \frac{2.5}{5} = \frac{1}{2} = \cos \frac{\pi}{3}$$

Hence $$t = \frac{\pi}{3} = \frac{3.142}{3} = \mathbf{1.047\ s.}$$

Example 18. *A particle executing linear SHM has a maximum velocity of 40 $cm\,s^{-1}$ and a maximum acceleration of 50 $cm\,s^{-2}$. Find its amplitude and the period of oscillation.*

Solution. Maximum velocity,

$$v_{\max} = \omega A = 40 \text{ cm s}^{-1}$$

Maximum acceleration,

$$a_{\max} = \omega^2 A = 50 \text{ cm s}^{-2}$$

$$\therefore \quad \frac{a_{\max}}{v_{\max}} = \frac{\omega^2 A}{\omega A} = \frac{50}{40}$$

or $$\omega = \frac{5}{4} \text{ rad s}^{-1}.$$

Amplitude, $$A = \frac{v_{\max}}{\omega} = \frac{40 \times 4}{5} = \mathbf{32\ cm.}$$

Period of oscillation,

$$T = \frac{2\pi}{\omega} = \frac{2 \times 3.142 \times 4}{5} = \mathbf{5.03\ s.}$$

Example 19. *The vertical motion of a huge piston in a machine is approximately simple harmonic with a frequency of 0.50 s^{-1}. A block of 10 kg is placed on the piston. What is the maximum amplitude of the piston's SHM for the block and the piston to remain together ?*

Solution. Here $\nu = 0.5 \text{ s}^{-1}$, $g = 9.8 \text{ m s}^{-2}$

The maximum acceleration in SHM is given by

$$a_{\max} = \omega^2 A = (2\pi\nu)^2 A = 4\pi^2 \nu^2 A$$

The block will remain in contact with the piston if

$$a_{\max} \le g \quad \text{or} \quad 4\pi^2 \nu^2 A \le g$$

Hence the maximum amplitude of the piston will be

$$A_{\max} = \frac{g}{4\pi^2 \nu^2} = \frac{9.8}{4\pi^2 (0.5)^2} = \mathbf{0.99\ m.}$$

Example 20. *A block of mass one kg is fastened to a spring with a spring constant 50 Nm^{-1}. The block is pulled to a distance $x = 10$ cm from its equilibrium position at $x = 0$ on a frictionless surface from rest at $t = 0$. Write the expression for its $x(t)$ and $v(t)$.* **[Central Schools 03]**

Solution. Here $m = 1$ kg, $k = 50 \text{ Nm}^{-1}$,

$$A = 10 \text{ cm} = 0.10 \text{ m}.$$

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{50}{1}} = 7.07 \text{ rad s}^{-1}.$$

As the motion starts from the mean position, so the displacement equation can be written as

$$x(t) = A \sin \omega t \quad \text{or} \quad x(t) = 0.10 \sin 7.07\, t$$

and $$v(t) = \frac{dx}{dt} = 0.10 \times 7.07 \cos 7.07\, t$$

or $$v(t) = \mathbf{0.707 \cos 7.07\, t\ ms^{-1}}.$$

Example 21. *A person normally weighing 60 kg stands on a platform which oscillates up and down harmonically at a frequency of 2.0 s^{-1} and an amplitude 5.0 cm. If a machine on the platform gives the person's weight against time, deduce the maximum and minimum readings it will show. Take $g = 10\ ms^{-2}$.*

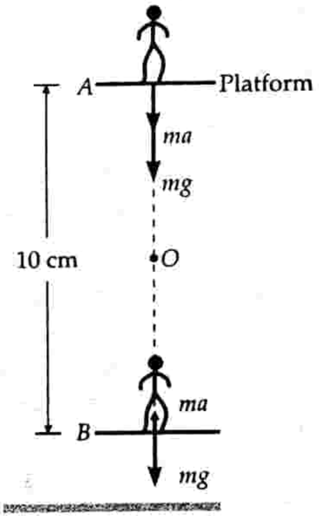


Fig. 14.11

Solution. The platform vibrates between the positions A and B about the mean position O, as shown in Fig. 14.11.

Given $A = 5.0$ cm,

$m = 60$ kg, $\nu = 2$ Hz

At A and B, the acceleration is maximum and is directed towards the mean position.

It is given by

$$a_{\max} = \omega^2 A$$
$$= 4\pi^2 \nu^2 A$$
$$= 4 \times 9.87 \times (2)^2 \times 0.05 = 7.9 \text{ ms}^{-2}$$

At A, both the weight mg and the restoring force F are directed towards O. Therefore, the weight at A is maximum and is given by

$$W_1 = (mg + F) = (mg + ma_{\max}) = m(g + a_{\max})$$
$$= 60\,(10 + 7.9) = 60 \times 17.9 = 1074 \text{ N}$$
$$= \frac{1074}{g} = \frac{1074}{10} = \mathbf{107.4\ kg\ f.}$$

At B, mg and F are opposed to each other so that the weight is minimum. It is given by

$$W_2 = (mg - F) = (mg - ma_{\max}) = m(g - a_{\max})$$
$$= 60\,(10 - 7.9) = (60 \times 2.1) \text{ N} = 126 \text{ N}$$
$$= \frac{126}{10} = \mathbf{12.6\ kg\ f.}$$

EXAMPLE 22. *A body of mass 0.1 kg is executing SHM according to the equation*

$$y = 0.5 \cos\left(100\,t + \frac{3\pi}{4}\right) \text{ metre}$$

Find (i) the frequency of oscillation (ii) initial phase (iii) maximum velocity (iv) maximum acceleration and (v) total energy.

Solution. Given $x = 0.5 \cos\left(100\,t + \frac{3\pi}{4}\right)$ metre

For any SHM, $x = A \cos(\omega t + \phi_0)$

Comparing the above two equations, we get

$$A = 0.5 \text{ m}, \quad \omega = 100 \text{ rad s}^{-1}, \quad \phi_0 = \frac{3\pi}{4} \text{ rad}$$

(*i*) Frequency, $\nu = \frac{\omega}{2\pi} = \frac{100}{2\pi} = \mathbf{\frac{50}{\pi}\ Hz.}$

(*ii*) Initial phase, $\phi_0 = \mathbf{\frac{3\pi}{4}\ rad.}$

(*iii*) $v_{max} = \omega A = 100 \times 0.5 = \mathbf{50\ ms^{-1}}.$

(*iv*) $a_{max} = \omega^2 A = (100)^2 \times 0.5 = \mathbf{5000\ ms^{-2}}.$

(*v*) Total energy

$$= \frac{1}{2} m v_{max}^2 = \frac{1}{2} \times 0.1 \times (50)^2 = \mathbf{125\ J.}$$

PROBLEMS FOR PRACTICE

1. A simple harmonic oscillation is represented by the equation, $y = 0.40 \sin(440\,t + 0.61)$
Here y and t are in m and s respectively. What are the values of (*i*) amplitude (*ii*) angular frequency (*iii*) frequency of oscillations (*iv*) time period of oscillations and (*v*) initial phase ?
[**Ans.** (*i*) 0.40 m (*ii*) 440 rad s^{-1} (*iii*) 70 Hz (*iv*) 0.0143 s (*v*) 0.61 rad]

2. The periodic time of a body executing SHM is 2 s. After how much time interval from $t = 0$, will its displacement be half of its amplitude ? (**Ans.** 1/6 s)

3. A particle executes SHM represented by the equation : $10y = 0.1 \sin 50\,\pi\,t$, where the displacement y is in metre and time t in second. Find the amplitude and frequency of the particle. (**Ans.** $A = 0.01$ m, $\nu = 25$ Hz)

4. The displacement of a particle executing periodic motion is given by $y = 4\cos^2(t/2)\sin(1000\,t)$. Find the independent constituent SHM's. **[IIT 93]**
[**Ans.** $\sin(1001\,t)$, $\sin(1000\,t)$, $\sin(999t)$]

5. A particle executing SHM completes 1200 oscillations per minute and passes through the mean position with a velocity of 31.4 ms^{-1}. Determine the maximum displacement of the particle from the mean position. Also obtain the displacement equation of the particle if its displacement be zero at the instant $t = 0$.
[**Ans.** $A = 0.025$ m, $y = 0.025 \sin(40\pi t)$ metre]

6. The acceleration of a particle performing SHM is 12 cm s^{-2} at a distance of 3 cm from the mean position. Calculate its time-period. (**Ans.** 3.142 s)

7. In a pendulum, the amplitude is 0.05 m and a period of 2 s. Compute the maximum velocity. (**Ans.** 0.1571 ms^{-1})

8. In what time after its motion begins, will a particle oscillating according to the equation, $y = 7 \sin 0.5\,\pi\,t$, move from the mean position to maximum displacement ? **[Himachal 08C]** (**Ans.** 1s)

9. A particle executes SHM on a straight line path. The amplitude of oscillation is 2 cm. When the displacement of the particle from the mean position is 1 cm, the magnitude of its acceleration is equal to its velocity. Find the time period, maximum velocity and maximum acceleration of SHM.
(**Ans.** 3.63 s, 3.464 cms^{-1}, 6 cms^{-2})

10. The velocity of a particle describing SHM is 16 cm s^{-1} at a distance of 8 cm from mean position and 8 cm s^{-1} at a distance of 12 cm from mean position. Calculate the amplitude of the motion. (**Ans.** 13.06 cm)

11. A particle is executing SHM. If u_1 and u_2 are the speeds of the particle at distances x_1 and x_2 from the equilibrium position, show that the frequency of oscillation,
$$f = \frac{1}{2\pi}\left(\frac{u_1^2 - u_2^2}{x_2^2 - x_1^2}\right)^{1/2}.$$

12. If a particle executes SHM of time period 4 s and amplitude 2 cm, find its maximum velocity and that at half its full displacement. Also find the acceleration at the turning points and when the displacement is 0.75 cm. (**Ans.** 3.14 cm s^{-1}, 2.72 cm s^{-1}, 4.93 cm s^{-2}, 1.85 cm s^{-2})

13. Show that if a particle is moving in SHM, its velocity at a distance $\sqrt{3}/2$ of its amplitude from the central position is half its velocity in central position. **[Chandigarh 03 ; Central Schools 09]**

14. A particle executes SHM of period 12 s. Two seconds after it passes through the centre of oscillation, the velocity is found to be 3.142 cm s^{-1}. Find the amplitude and the length of the path. (**Ans.** 12 cm, 24 cm)

15. A block lying on a horizontal table executes SHM of period 1 second, horizontally. What is the maximum

amplitude for which the block does not slide ? Coefficient of friction between block and surface is 0.4, $\pi^2 = 10$. **(Ans.** 9.8 cm)

16. A horizontal platform moves up and down simple harmonically, the total vertical movement being 10 cm. What is the shortest period permissible, if objects resting on the platform are to remain in contact with it throughout the motion ? Take $g = 980\text{ cm s}^{-2}$. **(Ans.** 0.449 s)

17. In a gasoline engine, the motion of the piston is simple harmonic. The piston has a mass of 2 kg and stroke (twice the amplitude) of 10 cm. Find maximum acceleration and the maximum unbalanced force on the piston, if it is making 50 complete vibrations each minute. **(Ans.** 1.371 ms^{-2}, 2.742 N)

18. A man stands on a weighing machine placed on a horizontal platform. The machine reads 50 kg. By means of a suitable mechanism the platform is made to execute harmonic vibrations up and down with a frequency of 2 vibrations per second. What will be the effect on the reading of the weighing machine ? The amplitude of vibration of the platform is 5 cm. Take $g = 10\text{ ms}^{-2}$.

(Ans. Max. reading = 89.5 kg f,
Min. reading = 10.5 kg f)

✖ HINTS

1. Comparing $y = 0.40 \sin(440t + 0.61)$ with $y = A\sin(\omega t + \phi_0)$, we get

(*i*) Amplitude, $A = \mathbf{0.40\ m.}$

(*ii*) Angular frequency, $\omega = \mathbf{440\ rad\ s^{-1}}$.

(*iii*) Frequency, $\nu = \frac{\omega}{2\pi} = \frac{440 \times 7}{2 \times 22} = \mathbf{70\ Hz.}$

(*iv*) Time period, $T = \frac{1}{\nu} = \frac{1}{70} = \mathbf{0.0143\ s.}$

(*v*) Initial phase, $\phi_0 = \mathbf{0.61\ rad.}$

2. Here $T = 2\text{ s}$, $y = A/2$, $t = ?$

As $y = A\sin\omega t = A\sin\frac{2\pi}{T}t$

$$\therefore \quad \frac{A}{2} = A\sin\frac{2\pi}{2}t = A\sin\pi t$$

$$\text{or} \quad \sin\pi t = \frac{1}{2} = \sin\frac{\pi}{6} \quad \text{or} \quad \pi t = \frac{\pi}{6}$$

$$\therefore \quad \mathbf{t = 1/6\ s.}$$

4. $y = 4\cos^2(t/2)\sin(1000t)$

$= 2(1 + \cos t)\sin(1000t)$ $\quad [\because 1 + \cos 2\theta = 2\cos^2\theta]$

$= 2\sin(1000t) + 2\sin(1000t)\cos t$

$= 2\sin(1000t) + [\sin(1000t + t) + \sin(1000t - t)]$

$[\because 2\sin A\cos B = \sin(A + B) + \sin(A - B)]$

$= 2\sin(1000t) + \sin(1001t) + \sin(999t)$

Thus motion y is composed of three independent SHMs which are $\sin(1000t)$, $\sin(1001t)$ and $\sin(999t)$.

5. Here $\nu = \frac{1200}{60} = 20\text{ Hz}$, $v_{max} = 3.14\text{ ms}^{-1}$

But $v_{max} = \omega A = 2\pi\nu A$

$$\therefore \quad A = \frac{v_{max}}{2\pi\nu} = \frac{3.14}{2 \times 3.14 \times 20} = \mathbf{0.025\ m.}$$

As displacement is zero at $t = 0$, so we can write

$$y = A\sin\omega t = A\sin(2\pi\nu t) = \mathbf{0.025\sin(40\pi t).}$$

6. Here $a = 12\text{ cm s}^{-2}$, $y = 3\text{ cm}$

$$\omega = \sqrt{\frac{a}{y}} = \sqrt{\frac{12}{3}} = 2\text{ rad s}^{-1}$$

Time period, $T = \frac{2\pi}{\omega} = \frac{2 \times 3.142}{2} = \mathbf{3.142\ s.}$

7. $v_{max} = \omega A = \frac{2\pi}{T}A = \frac{2\pi}{2} \times 0.05$

$= 3.142 \times 0.05 = \mathbf{0.1571\ ms^{-1}}$.

8. Given $y = 7\sin 0.5\pi t$

On comparing with the standard equation, $y = a\sin\omega t$, we get : $a = 7$, $\omega = 0.5\pi$

Let t be the time taken by the particle in moving from mean position to maximum displacement.

Then $7 = 7\sin 0.5\pi t$ or $\sin 0.5\pi t = 1 = \sin\frac{\pi}{2}$

$$\therefore \quad 0.5\pi t = \frac{\pi}{2} \quad \text{or} \quad t = \frac{1}{2 \times 0.5} = \mathbf{1\ s.}$$

9. Here $A = 2\text{ cm}$. When displacement $y = 1\text{ cm}$,

magnitude of velocity = magnitude of acceleration

$$\text{or} \quad \omega\sqrt{A^2 - y^2} = \omega^2 y$$

$$\text{or} \quad A^2 - y^2 = \omega^2 y^2$$

$$\text{or} \quad 2^2 - 1^2 = \omega^2 \times 1^2 \quad \text{or} \quad \omega = \sqrt{3}\text{ rad s}^{-1}$$

$$\therefore \text{ Time period,} \quad T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{3}} = \mathbf{3.63\ s.}$$

$$v_{max} = \omega A = \sqrt{3} \times 2 = 1.732 \times 2 = \mathbf{3.464\ cm\ s^{-1}}.$$

$$a_{max} = \omega^2 A = 3 \times 2 = \mathbf{6\ cm\ s^{-2}}.$$

10. As $v = \omega\sqrt{A^2 - y^2}$

$\therefore$ In first case : $16 = \omega\sqrt{A^2 - 8^2}$ ...(1)

In second case : $8 = \omega\sqrt{A^2 - 12^2}$ ...(2)

Dividing (1) by (2),

$$\frac{16}{8} = \frac{\omega\sqrt{A^2 - 8^2}}{\omega\sqrt{A^2 - 12^2}} \quad \text{or} \quad 2 = \sqrt{\frac{A^2 - 64}{A^2 - 144}}$$

On solving $3A^2 = 512$ or $A^2 = 170.6$

or $\mathbf{A = 13.06\ cm.}$

12. Here $T = 4\,\text{s}$, $\quad A = 2\,\text{cm}$,

$$\omega = \frac{2\pi}{T} = \frac{2 \times 3.14}{4} = 1.57 \text{ rad s}^{-1}$$

$$v_{\max} = \omega A = 1.57 \times 2 = \mathbf{3.14\ cm\ s^{-1}}.$$

At $\quad y = A/2 = 1\,\text{cm}$,

$$v = \omega\sqrt{A^2 - y^2} = 1.57\sqrt{2^2 - 1^2} = \mathbf{2.72\ cm\ s^{-1}}.$$

At the turning points, acceleration is maximum

$$\therefore \quad a_{\max} = \omega^2 A = (1.57)^2 \times 2 = \mathbf{4.93\ cm\ s^{-2}}.$$

At $\quad y = 0.75\,\text{cm}$,

$$a = \omega^2 y = (1.57)^2 \times 0.75 = \mathbf{1.85\ cm\ s^{-2}}.$$

13. Here $y = \sqrt{3}/2\ A$

$$\therefore \quad v = \omega\sqrt{A^2 - y^2} = \omega\sqrt{A^2 - 3/4\ A^2}$$

$$= \frac{1}{2}\,\omega A = \frac{1}{2}\,v_{\max}.$$

14. Let $\quad y = A \sin \omega t$

$$\text{Then } v = \frac{dy}{dt} = \omega A \cos \omega t = \frac{2\pi}{T} A \cos \frac{2\pi}{T} t$$

$$\therefore \quad 3.142 = \frac{2 \times 3.142}{12} A \cos \frac{2\pi}{12} \times 2$$

or $\quad A = \mathbf{12\ cm}$ and length of path $= 2A = \mathbf{24\ cm.}$

15.

$$a_{\max} = \omega^2 A = \mu g$$

$$\therefore \quad A = \frac{\mu g}{\omega^2} = \frac{\mu g T^2}{4\pi^2}$$

$$= \frac{0.4 \times 9.8 \times (1)^2}{4 \times 10} = 0.098 \text{ m} = \mathbf{9.8\ cm.}$$

16. Take $a_{\max} = \omega^2 A = \left(\frac{2\pi}{T}\right)^2 A = g.$

17. Length of stroke $= 2\,A = 10\,\text{cm}$.

18. Here $\quad m = 50\,\text{kg}$, $\nu = 2\,\text{Hz}$,

$$A = 5\,\text{cm} = 0.05\text{ m} = 4 \times 9.87 \times 2^2 \times 0.05 = 7.9 \text{ ms}^{-2}.$$

$$a_{\max} = \omega^2 A = 4\pi^2 \nu^2 A$$

Max. force on the man

$$= m(g + a_{max}) = 50\,(10 + 7.9) = 895.0 \text{ N} = \mathbf{89.5\ kg\ f.}$$

Min. force on the man

$$= m(g - a_{\max}) = 50\,(10 - 7.9) = 105.0 \text{ N} = \mathbf{10.5\ kg\ f.}$$

14.13 ENERGY IN S.H.M. : KINETIC AND POTENTIAL ENERGIES

15. *Derive expressions for the kinetic and potential energies of a simple harmonic oscillator. Hence show that the total energy is conserved in S.H.M. In which positions of the oscillator, is the energy wholly kinetic or wholly potential ?*

Total energy in S.H.M. The energy of a harmonic oscillator is partly kinetic and partly potential. When a body is displaced from its equilibrium position by doing work upon it, it acquires potential energy. When the body is released, it begins to move back with a velocity, thus acquiring kinetic energy.

(*i*) **Kinetic energy.** At any instant, the displacement of a particle executing S.H.M. is given by

$$x = A\cos(\omega t + \phi_0)$$

$$\therefore \quad \text{Velocity,} \quad v = \frac{dx}{dt} = -\,\omega A \sin(\omega t + \phi_0)$$

Hence kinetic energy of the particle at any displacement x is given by

$$K = \frac{1}{2} m v^2 = \frac{1}{2} m\omega^2 A^2 \sin^2(\omega t + \phi_0)$$

$$\text{But} \quad A^2 \sin^2(\omega t + \phi_0) = A^2\,[1 - \cos^2(\omega t + \phi_0)]$$

$$= A^2 - A^2\cos^2(\omega t + \phi_0) = A^2 - x^2$$

$$\therefore \quad K = \frac{1}{2} m\omega^2 A^2 \sin^2(\omega t + \phi_0)$$

$$\text{or} \quad K = \frac{1}{2} m\omega^2 (A^2 - x^2) = \frac{1}{2} k\,(A^2 - x^2)$$

(*ii*) **Potential energy.** When the displacement of a particle from its equilibrium position is x, the restoring force acting on it is

$$F = -\,kx$$

If we displace the particle further through a small distance dx, then work done against the restoring force is given by

$$dW = -\,F\,dx = +\,k\,x\,dx$$

The total work done in moving the particle from mean position ($x = 0$) to displacement x is given by

$$W = \int dW = \int_0^x k\,x\,dx = k\left[\frac{x^2}{2}\right]_0^x = \frac{1}{2} kx^2$$

This work done against the restoring force is stored as the potential energy of the particle. Hence potential energy of a particle at displacement x is given by

$$U = \frac{1}{2} kx^2 = \frac{1}{2} m\omega^2 x^2 = \frac{1}{2} m\omega^2 A^2 \cos^2(\omega t + \phi_0).$$

(*iii*) **Total energy.** At any displacement x, the total energy of a harmonic oscillatory is given by

$$E = K + U = \frac{1}{2} k\,(A^2 - x^2) + \frac{1}{2} kx^2$$

$$\text{or} \quad E = \frac{1}{2} k\,A^2 = \frac{1}{2} m\omega^2 A^2 = 2\pi^2 m\nu^2 A^2$$

$$[\because \omega = 2\pi\nu]$$

Thus the total mechanical energy of a harmonic oscillator is independent of time or displacement. Hence *in the absence of any frictional force, the total energy of a harmonic oscillator is conserved.*

Obviously, the total energy of particle in S.H.M. is

(*i*) directly proportional to the mass m of the particle,

(*ii*) directly proportional to the square of its frequency ν, and

(*iii*) directly proportional to the square of its vibrational amplitude A.

Graphical representation. *At the mean position*, $x = 0$

Kinetic energy, $K = \frac{1}{2} k (A^2 - 0^2) = \frac{1}{2} kA^2$

Potential energy, $U = \frac{1}{2} k (0^2) = 0$

Hence at the mean position, the energy is all kinetic.

At the extreme positions, $x = \pm A$

Kinetic energy, $K = \frac{1}{2} k (A^2 - A^2) = 0$

Potential energy, $U = \frac{1}{2} kA^2$

Hence at the two extreme positions, the energy is all potential.

Figure 14.12 shows the variations of kinetic energy K, potential energy U and total energy E with displacement x. The graphs for K and U are parabolic while that for E is a straight line parallel to the displacement axis. At $x = 0$, the energy is all kinetic and for $x = \pm A$, the energy is all potential.

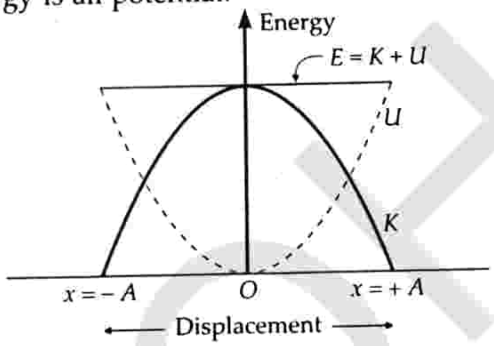


Fig. 14.12 *K*, *U* and *E* as functions of displacement *x* for a harmonic oscillator.

Figure 14.13 shows the variations of energies K, U and E of a harmonic oscillator with time t. Clearly, twice in each cycle, both kinetic and potential energies assume their peak values. Both of these energies are periodic functions of time, the time period of each being $T/2$.

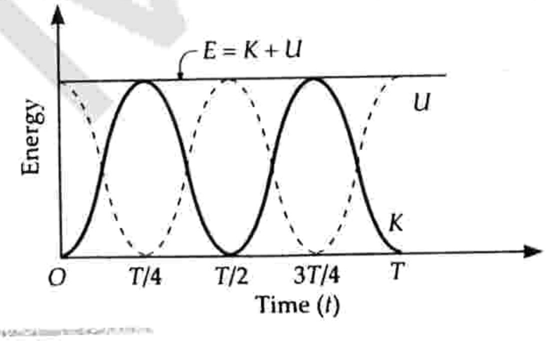


Fig. 14.13 *K*, *U* and *E* as functions of time *t* for a harmonic oscillator.

Examples based on Energy of S.H.M.

FORMULAE USED

1. P.E. at displacement y from the mean position,

$$E_p = \frac{1}{2} ky^2 = \frac{1}{2} m\,\omega^2\, y^2 = \frac{1}{2} m\,\omega^2\, A^2 \sin^2 \omega t$$

2. K.E. at displacement y from the mean position,

$$E_k = \frac{1}{2} k (A^2 - y^2) = \frac{1}{2} m\,\omega^2 (A^2 - y^2)$$

$$= \frac{1}{2} m\,\omega^2\, A^2 \cos^2 \omega t$$

3. Total energy at any point,

$$E = \frac{1}{2} kA^2 = \frac{1}{2} m\,\omega^2\, A^2 = 2\pi^2\, m\, A^2\, \nu^2$$

UNITS USED

Energies E_p, E_k and E are in joule, displacement in metre, force constant k in Nm^{-1} and angular frequency ω in rad s^{-1}.

EXAMPLE 23. *A block whose mass is 1 kg is fastened to a spring. The spring has a spring constant of 50 N m^{-1}. The block is pulled to a distance $x = 10$ cm from its equilibrium position at $x = 0$ on a frictionless surface from rest at $t = 0$. Calculate the kinetic, potential and total energies of the block when it is 5 cm away from the mean position.* **[NCERT]**

Solution. Here $m = 1$ kg, $k = 50 \text{ N m}^{-1}$,

$A = 10 \text{ cm} = 0.10 \text{ m}$, $y = 5 \text{ cm} = 0.05 \text{ m}$

Kinetic energy,

$$E_k = \frac{1}{2} k (A^2 - y^2) = \frac{1}{2} \times 50\,[(0.10)^2 - (0.05)^2]$$

$$= \mathbf{0.1875\ J.}$$

Potential energy,

$$E_p = \frac{1}{2} ky^2 = \frac{1}{2} \times 50 \times (0.05)^2 = \mathbf{0.0625\ J.}$$

Total energy,

$$E = E_k + E_p = 0.1875 + 0.0625 = \mathbf{0.25\ J.}$$

EXAMPLE 24. *A body executes SHM of time period 8 s. If its mass be 0.1 kg, its velocity 1 second after it passes through its mean position be 4 ms^{-1}, find its (i) kinetic energy (ii) potential energy and (iii) total energy.*

Solution. Here $m = 0.1$ kg, $T = 8$ s

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{8} = \frac{\pi}{4} \text{ rad s}^{-1}$$

When $t = 1 \text{ s}, v = 4 \text{ ms}^{-1}$

But $v = \omega A \cos \omega t$

$\therefore$ $4 = \frac{\pi}{4} \times A \cos\left(\frac{\pi}{4} \times 1\right) = \frac{\pi}{4} \times A \times \frac{1}{\sqrt{2}}$

or $A = \frac{16\sqrt{2}}{\pi}$ m

Total energy,

$$E = \frac{1}{2} m\omega^2 A^2 = \frac{1}{2} \times 0.1 \times \left(\frac{\pi}{4}\right)^2 \times \left(\frac{16\sqrt{2}}{\pi}\right)^2 = \mathbf{1.6\ J.}$$

Kinetic energy,

$$E_k = \frac{1}{2} mv^2 = \frac{1}{2} \times 0.1 \times (4)^2 = \mathbf{0.8\ J.}$$

Potential energy,

$$E_p = E - E_k = 1.6 - 0.8 = \mathbf{0.8\ J.}$$

EXAMPLE 25. *A spring of force constant 800 Nm^{-1} has an extension of 5 cm. What is the work done in increasing the extension from 5 cm to 15 cm?* **[AIEEE 02]**

Solution. Here $k = 800\ \text{Nm}^{-1}, x_1 = 5\ \text{cm} = 0.05\ \text{m}$,

$$x_2 = 15\ \text{cm} = 0.15\ \text{m}$$

W = Increase in P.E. of the spring

$$= \tfrac{1}{2} k (x_2^2 - x_1^2)$$

$$= \tfrac{1}{2} \times 800\,[(0.15)^2 - (0.05)^2]\ \text{J}$$

$$= \mathbf{8\ J.}$$

EXAMPLE 26. *A particle of mass 10 g is describing SHM along a straight line with a period of 2 s and amplitude of 10 cm What is the kinetic energy when it is (i) 2 cm (ii) 5 cm, from its equilibrium position? How do you account for the difference between its two values?*

Solution. Velocity at displacement y is

$$v = \omega\sqrt{A^2 - y^2}$$

Given $A = 10\ \text{cm},\ T = 2\ \text{s}$

Angular frequency,

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi\ \text{rad s}^{-1}$$

(*i*) When $y = 2$ cm,

$$v = \pi\sqrt{100 - 4} = \pi\sqrt{96}\ \text{cm s}^{-1}$$

$$\therefore \quad \text{K.E.} = \frac{1}{2} mv^2 = \frac{1}{2} \times 10 \times \pi^2 \times 96$$

$$= \mathbf{480\ \pi^2\ erg.}$$

(*ii*) When $y = 5$ cm,

$$v = \pi\sqrt{100 - 25} = \pi\sqrt{75}\ \text{cm s}^{-1}$$

$$\therefore \quad \text{K.E.} = \frac{1}{2} mv^2$$

$$= \frac{1}{2} \times 10 \times \pi^2 \times 75 = \mathbf{375\ \pi^2\ erg.}$$

The K.E. decreases when the particle moves from $y = 2$ cm to $y = 5$ cm. This is due to the increase in the potential energy of the particle.

EXAMPLE 27. *At a time when the displacement is half the amplitude, what fraction of the total energy is kinetic and what fraction is potential in S.H.M.?*

Solution.

Displacement $= \frac{1}{2}$ amplitude or $y = \frac{1}{2} A$

Total energy of SHM, $E = \frac{1}{2} m\omega^2 A^2$

Kinetic energy of SHM, $E_k = \frac{1}{2} m\omega^2 (A^2 - y^2)$

$$= \frac{1}{2} m\omega^2 \left[A^2 - \left(\frac{A}{2}\right)^2\right]$$

$$= \frac{1}{2} m\omega^2 \left(\frac{3A^2}{4}\right) = \frac{3}{4} \cdot \frac{1}{2} m\omega^2 A^2 = \mathbf{\frac{3}{4}\ E}$$

Potential energy of SHM,

$$E_p = \frac{1}{2} m\omega^2 y^2 = \frac{1}{2} m\omega^2 \left(\frac{A}{2}\right)^2$$

$$= \frac{1}{4} \cdot \frac{1}{2} m\omega^2 A^2 = \mathbf{\frac{1}{4}\ E.}$$

EXAMPLE 28. *A particle is executing SHM of amplitude A. At what displacement from the mean position, is the energy half kinetic and half potential?*

Solution. As $\quad E_K = E_P$

$$\therefore \quad \frac{1}{2} m\omega^2 (A^2 - y^2) = \frac{1}{2} m\omega^2 y^2$$

or $\quad A^2 - y^2 = y^2 \quad$ or $\quad 2y^2 = A^2$

or $\quad y^2 = \frac{A^2}{2} \quad$ or $\quad y = \pm \frac{A}{\sqrt{2}}$

Thus the energy will be half kinetic and half potential at displacement $\frac{A}{\sqrt{2}}$ on either side of the mean position.

EXAMPLE 29. *A particle executes simple harmonic motion of amplitude A. (i) At what distance from the mean position is its kinetic energy equal to its potential energy? (ii) At what points is its speed half the maximum speed?*

Solution. The potential energy and kinetic energy of a particle at a displacement y are given by

$$E_p = \frac{1}{2} ky^2$$

and $\quad E_k = \frac{1}{2} k (A^2 - y^2) \qquad ...(1)$

where A is the amplitude

and $\quad k$ is the force constant.

(*i*) As $E_k = E_p$

$$\therefore \quad \frac{1}{2}k(A^2 - y^2) = \frac{1}{2}ky^2 \quad \text{or} \quad 2y^2 = A^2$$

or
$$y = \pm \frac{A}{\sqrt{2}} = \pm 0.71\,A$$

= **0.71 times the amplitude on either side of mean position.**

(*ii*) Here, $v = \frac{1}{2}v_{max}$

In general, kinetic energy

$$= \frac{1}{2}mv^2 = \frac{1}{2}m\left(\frac{1}{2}v_{max}\right)^2 = \frac{1}{4}\cdot\frac{1}{2}mv_{max}^2$$

$$= \frac{1}{4} \times \text{Maximum kinetic energy}$$

or
$$E_k = \frac{1}{4} \times (E_k)_{max} \qquad \text{...(2)}$$

From equation (1),

$$E_k = \frac{1}{2}k(A^2 - y^2)$$

$$\therefore \quad (E_k)_{max} = \frac{1}{2}kA^2 \qquad [\text{Put } y = 0]$$

Putting these values in equation (2), we get

$$\frac{1}{2}k(A^2 - y^2) = \frac{1}{4} \times \frac{1}{2}kA^2$$

or
$$4y^2 = 3A^2$$

or
$$y = \pm\frac{\sqrt{3}}{2}A = \pm\,\mathbf{0.86\,A}$$

= **0.86 times the amplitude on either side of mean position.**

PROBLEMS FOR PRACTICE

1. A bob of simple pendulum of mass 1 g is oscillating with a frequency 5 vibrations per second and its amplitude is 3 cm. Find the kinetic energy of the bob in the lowest position. (**Ans.** 4441.5 erg)
2. A body weighing 10 g has a velocity of 6 cm s^{-1} after one second of its starting from mean position. If the time period is 6 seconds, find the kinetic energy, potential energy and the total energy. (**Ans.** 180 erg, 540 erg, 720 erg)
3. A particle executes SHM of period 8 seconds. After what time of its passing through the mean position will the energy be half kinetic and half potential ? **[Chandigarh 08]** (**Ans.** 1 s)
4. The total energy of a particle executing SHM of period 2π seconds is 1.024×10^{-3} J. The displacement of the particle at $\pi/4$ s is $0.08\sqrt{2}$ m. Calculate the amplitude of motion and mass of the particle. (**Ans.** 0.16 m ; 0.08 kg)
5. A particle which is attached to a spring oscillates horizontally with simple harmonic motion with a frequency of $1/\pi$ Hz and total energy of 10 J. If the maximum speed of the particle is 0.4 ms^{-1}, what is the force constant of the spring ? What will be the maximum potential energy of the spring during the motion ? (**Ans.** $k = 500$ Nm^{-1}, $U_{max} = 10$ J)
6. The length of a weightless spring increases by 2 cm when a weight of 1.0 kg is suspended from it. The weight is pulled down by 10 cm and released. Determine the period of oscillation of the spring and its kinetic energy of oscillation. Take $g = 10$ ms^{-2}. (**Ans.** 0.28 s, 2.5 J)

HINTS

1. At the lowest or the mean position, energy of the bob is entirely kinetic and maximum.

$$(E_k)_{max} = \frac{1}{2}m\omega^2 A^2$$

$$= \frac{1}{2}m(2\pi\nu)^2 A^2 = 2\pi^2 m\nu^2 A^2$$

$$= 2 \times 9.87 \times 1 \times 5^2 \times 3^2 = \mathbf{4441.5\ erg.}$$

2. Here $m = 10\,g$, $T = 6\,s$,

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{6} = \frac{\pi}{3}\ \text{rad s}^{-1}$$

When $t = 1\,s$, $v = 6\,\text{cm s}^{-1}$

As $v = A\,\omega\cos\omega t$

$$\therefore \quad 6 = A \times \frac{\pi}{3}\cos\frac{\pi}{3} \times 1 = A \times \frac{\pi}{3}\cos 60^\circ$$

$$= A \times \frac{\pi}{3} \times \frac{1}{2} = \frac{\pi A}{6} \quad \text{or} \quad A = \frac{36}{\pi}\ \text{cm}$$

Total energy, $E = \frac{1}{2}mA^2\omega^2$

$$= \frac{1}{2} \times 10 \times \left(\frac{36}{\pi}\right)^2 \times \left(\frac{\pi}{3}\right)^2 = \mathbf{720\ erg.}$$

Kinetic energy $= \frac{1}{2}mv^2 = \frac{1}{2} \times 10 \times 6^2 = \mathbf{180\ erg.}$

∴ Potential energy

= Total energy − Kinetic energy

= 720 − 180 = **540 erg.**

3. As P.E. = K.E.

$$\therefore \quad \frac{1}{2}ky^2 = \frac{1}{2}k(A^2 - y^2)$$

$$\text{or} \quad y^2 = A^2 - y^2 \quad \text{or} \quad y = \frac{A}{\sqrt{2}}$$

Now $y = A \sin \omega t = A \sin \frac{2\pi}{T} t$

∴ $\frac{A}{\sqrt{2}} = A \sin \frac{2\pi}{8} t$

or $\sin \frac{\pi t}{4} = \frac{1}{\sqrt{2}} = \sin \frac{\pi}{4}$

or $\frac{\pi t}{4} = \frac{\pi}{4}$ or $t = \mathbf{1}$ **s.**

4. When $t = \frac{\pi}{4}$ s, $y = 0.08\sqrt{2}$ m

As $y = A \sin \omega t = A \sin \frac{2\pi}{T} t$

∴ $0.08\sqrt{2} = A \sin \frac{2\pi}{2\pi} \times \frac{\pi}{4} = A \sin \frac{\pi}{4}$

or $0.08\sqrt{2} = A \times \frac{1}{\sqrt{2}}$

∴ $A = 0.08\sqrt{2} \times \sqrt{2} = \mathbf{0.16}$ **m.**

$$\text{Total energy} = \frac{1}{2} m \omega^2 A^2 = \frac{1}{2} m \left(\frac{2\pi}{T}\right)^2 A^2$$

∴ $1.024 \times 10^{-3} = \frac{1}{2} m \left(\frac{2\pi}{2\pi}\right)^2 \times (0.16)^2$

or $m = \frac{2 \times 1.024 \times 10^{-3}}{(0.16)^2} = \mathbf{0.08}$ **kg.**

5. Here $\nu = 1/\pi$ Hz, $E = 10$ J, $v_{\max} = 0.4 \text{ ms}^{-1}$

Now $v_{\max} = \omega A = 2\pi \nu A$

∴ $A = \frac{v_{\max}}{2\pi\nu} = \frac{0.4 \times \pi}{2\pi \times 1} = 0.2$ m

As $E = \frac{1}{2} k A^2$

∴ $k = \frac{2E}{A^2} = \frac{2 \times 10}{(0.2)^2} = \mathbf{500 \ Nm^{-1}}$.

$(E_p)_{\max} = E = \mathbf{10}$ **J.**

6. Here $F = mg = 1.0 \times 10$ N, $y = 2$ cm $= 0.02$ m

∴ $k = \frac{F}{y} = \frac{1.0 \times 10}{0.02} = 500 \text{ Nm}^{-1}$

$T = 2\pi \sqrt{\frac{m}{k}} = 2 \times 3.14 \times \sqrt{\frac{1.0}{500}} = \mathbf{0.28}$ **s.**

E_k = Work done in pulling the spring through 10 cm or 0.1 m

$= \frac{1}{2} k x^2 = \frac{1}{2} \times 500 \times (0.1)^2 = \mathbf{2.5}$ **J.**

14.14 OSCILLATIONS DUE TO A SPRING

16. *Derive an expression for the time-period of the horizontal oscillations of a massless loaded spring.*

Horizontal oscillations of a body on a spring. Consider a massless spring lying on a frictionless horizontal table. Its one end is attached to a rigid support and the other end to a body of mass m. If the body is pulled towards right through a small distance x and released, it starts oscillating back and forth about its equilibrium position under the action of the restoring force of elasticity,

$$F = -kx$$

where k is the force constant (restoring force per unit compression or extension) of the spring. The negative sign indicates that the force is directed oppositely to x.

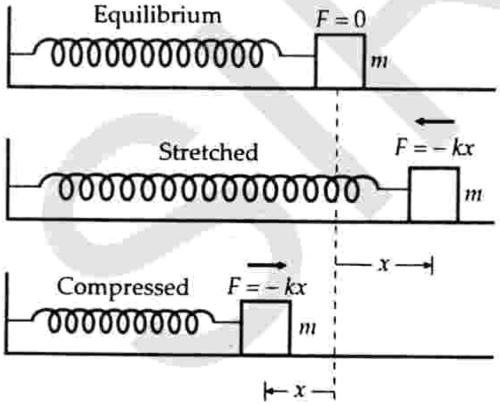


Fig. 14.14 Horizontal oscillations of a loaded spring.

If d^2x/dt^2 is the acceleration of the body, then

$$m \frac{d^2x}{dt^2} = -kx$$

or
$$\frac{d^2x}{dt^2} = -\frac{k}{m} x = -\omega^2 x$$

This shows that the acceleration is proportional to displacement x and acts opposite to it. Hence the body executes simple harmonic motion. Its time period is given by

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{k/m}}$$

or
$$T = 2\pi \sqrt{\frac{m}{k}}$$

Frequency of oscillation will be

$$\nu = \frac{1}{T} = \frac{1}{2\pi} \sqrt{\frac{k}{m}}$$

Clearly, the time period T will be small or frequency ν large if the spring is stiff (high k) and attached body is light (small m).

17. *Deduce an expression for the time-period of the vertical oscillations of a massless loaded spring. Does it depend on acceleration due to gravity ?*

Vertical oscillations of a body on a spring. If a spring is suspended vertically from a rigid support and a body of mass m is attached to its lower end, the spring gets stretched to a distance d due to the weight mg of the body. Because of the elasticity of the spring, a restoring force equal to kd begins to act in the upward direction. Here k is the spring factor of the spring. In the position of equilibrium,

$$mg = kd$$

If the body be pulled vertically downwards through a small distance x from its equilibrium position and then released, it begins to oscillate up and down about this position. The weight mg of the body acts vertically downwards while the restoring force $k(d + x)$ due to elasticity acts vertically upwards. Therefore, the resultant force on the body is

$$F = mg - k(d + x)$$

$$= kd - kd - kx \qquad [\because mg = kd]$$

or $$F = -kx$$

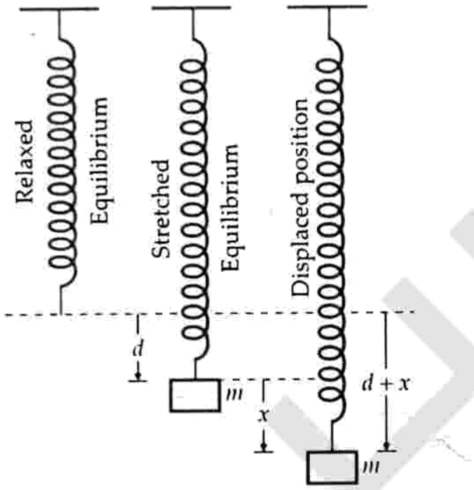


Fig. 14.15

If d^2x/dt^2 is the acceleration of the body, then

$$m\frac{d^2x}{dt^2} = -kx \quad \text{or} \quad \frac{d^2x}{dt^2} = -\frac{k}{m}x = -\omega^2 x$$

Thus acceleration is proportional to displacement x and is directed opposite to it. Hence the body executes S.H.M. and its time period is

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{k/m}} \quad \text{or} \quad T = 2\pi\sqrt{\frac{m}{k}}$$

Obviously, *the force of gravity has no effect on the force constant k and hence the time period of the oscillating mass.*

14.15 OSCILLATIONS OF LOADED SPRING COMBINATION

18. *Fig. 14.16 shows four different spring arrangements. If the mass of each arrangement is displaced from its equilibrium position and released, what is the resulting frequency of vibration in each case ? Neglect the mass of each spring.* **[NCERT]**

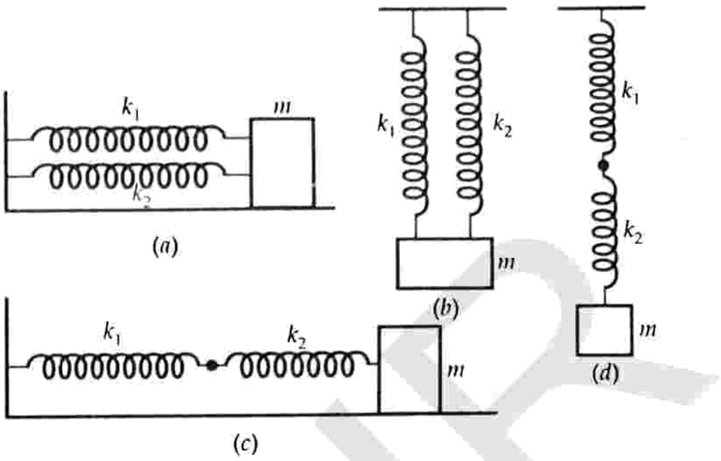


Fig. 14.16

Springs connected in parallel. Figs. 14.16(*a*) and (*b*) show two springs connected in parallel. Let k_1 and k_2 be their spring constants. Let y be the extension produced in each spring. Restoring forces produced in the two springs will be

$$F_1 = -k_1 y \text{ and } F_2 = -k_2 y$$

The total restoring force is

$$F = F_1 + F_2 = -(k_1 + k_2)y \qquad \text{...(1)}$$

Let k_p be the force constant of the parallel combination. Then

$$F = -k_p y \qquad \text{...(2)}$$

From (1) and (2), $\boldsymbol{k_p = k_1 + k_2}$

Frequency of vibration of the parallel combination is

$$\nu_p = \frac{1}{2\pi}\sqrt{\frac{k_p}{m}} = \frac{1}{2\pi}\sqrt{\frac{k_1 + k_2}{m}}$$

Springs connected in series. Figs. 14.16(*c*) and (*d*) represent two springs connected in series. Let x_1 and x_2 be the extensions produced in the two springs. The restoring force F acting in the two springs is same.

$$\therefore \quad F = -k_1x_1 = -k_2x_2$$

or $$x_1 = -\frac{F}{k_1} \text{ and } x_2 = -\frac{F}{k_2}$$

Total extension, $$x = x_1 + x_2 = -\frac{F}{k_1} - \frac{F}{k_2}$$

$$= -F\left(\frac{1}{k_1} + \frac{1}{k_2}\right) = -F\left(\frac{k_1 + k_2}{k_1k_2}\right)$$

or $$F = -\left(\frac{k_1k_2}{k_1 + k_2}\right)x \qquad \text{...(3)}$$

Let k_s be the force constant of the series combination. Then

$$F = -k_s x \qquad \text{...(4)}$$

From (3) and (4), $k_s = \dfrac{k_1 k_2}{k_1 + k_2}$.

Frequency of oscillation of the series combination is

$$\nu_s = \frac{1}{2\pi}\sqrt{\frac{k_s}{m}} = \frac{1}{2\pi}\sqrt{\frac{k_1 k_2}{m(k_1 + k_2)}}.$$

Examples based on Oscillations of a Loaded Spring

FORMULAE USED

1. Spring factor or force constant, $k = \dfrac{F}{y}$
2. Period of oscillation of a mass m suspended from massless spring of force constant k,
$$T = 2\pi\sqrt{\frac{m}{k}}$$
3. For two springs of springs factors k_1 and k_2 connected in parallel, effective spring factor,
$$k = k_1 + k_2 \quad \therefore \quad T = 2\pi\sqrt{\frac{m}{k_1 + k_2}}$$
4. For two springs connected in series, effective spring factor k is given by
$$\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2} \quad \text{or} \quad k = \frac{k_1 k_2}{k_1 + k_2}$$
$$\therefore \quad T = 2\pi\sqrt{\frac{m(k_1 + k_2)}{k_1 k_2}}$$
5. When length of a spring is made n times, its spring factor becomes $1/n$ times and hence time period increases $\sqrt{n}$ times.
6. When a spring is cut into n equal pieces, spring factor of each part becomes nk.
$$\therefore \quad T = 2\pi\sqrt{\frac{m}{nk}}$$

UNITS USED

Spring factors k, k_1, k_2 are in Nm^{-1}, mass m is in kg and time period T in second.

EXAMPLE 30. *The pan attached to a spring balance has a mass of 1 kg. A weight of 2 kg when placed on the pan stretches the spring by 10 cm. What is the frequency with which the empty pan will oscillate ?*

Solution. Applied force,

$$F = 2 \text{ kg wt} = 2 \times 9.8 \text{ N} = 19.6 \text{ N}$$

Displacement, $y = 10 \text{ cm} = 0.1 \text{ m}$

$\therefore$ Force constant, $k = \dfrac{F}{y} = \dfrac{19.6}{0.1} = 196 \text{ Nm}^{-1}$

For the empty pan, $m = 1$ kg

Hence the frequency of oscillation of the empty pan will be

$$\nu = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{196}{1}}$$
$$= \frac{1}{2\pi} \times 14 = \frac{7}{\pi} \textbf{ Hz.}$$

EXAMPLE 31. *A spring compressed by 0.2 m develops a restoring force of 25 N. A body of mass 5 kg is placed over it. Deduce :*

(i) force constant of the spring

(ii) the depression of the spring under the weight of the body and

(iii) the period of oscillation, if the body is disturbed.

Take $g = 10 \text{ N kg}^{-1}$.

Solution. (*i*) Here $y = 0.2$ m, $F = 25$ N

$\therefore$ Force constant,

$$k = \frac{F}{y} = \frac{25}{0.2} = \mathbf{125\ Nm^{-1}}.$$

(*ii*) Here $F = 5 \text{ kg wt} = 5 \times 10 \text{ N} = 50 \text{ N}$

$\therefore$ Displacement,

$$y = \frac{F}{k} = \frac{50}{125} = \mathbf{0.4\ m.}$$

(*iii*) Here $m = 5$ kg, $k = 125 \text{ Nm}^{-1}$

Time period,

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{5}{125}} = \frac{2\pi}{5} \textbf{ s.}$$

EXAMPLE 32. *A 0.2 kg of mass hangs at the end of a spring. When 0.02 kg more mass is added to the end of the spring, it stretches 7 cm more. If the 0.02 kg mass is removed, what will be the period of vibration of the system ?*

[Central Schools 04]

Solution. When 0.02 kg mass is added, the spring stretches by 7 cm.

As $mg = kx$

$$\therefore \quad k = \frac{mg}{x} = \frac{0.02 \times 10}{7 \times 10^{-2}} = \frac{20}{7} \text{ Nm}^{-1}$$

When 0.02 kg mass is removed, the period of vibration will be

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{0.2}{20/7}}$$
$$= 2\pi\sqrt{\frac{7}{100}} = \frac{2\pi \times 2.645}{10} = \mathbf{1.66\ s.}$$

EXAMPLE 33. *A body of mass 12 kg is suspended by a coil spring of natural length 50 cm and force constant $2.0 \times 10^3 \text{ Nm}^{-1}$. What is the stretched length of the spring ?*

If the body is pulled down further stretching the spring to a length of 59 cm and then released, what is the frequency of oscillation of the suspended mass ? (Neglect the mass of the spring).

Solution. Here $m = 12$ kg, $k = 2.0 \times 10^3$ Nm^{-1}

Natural length, $l = 50$ cm

Extension produced in the spring due to 12 kg mass,

$$y = \frac{F}{k} = \frac{mg}{k} = \frac{12 \times 9.8}{2.0 \times 10^3} = 0.0588 \text{ m} = 5.88 \text{ cm}$$

Stretched length of the spring

$$= l + y = 50 + 5.88 = \mathbf{55.88\ cm.}$$

When the loaded spring is further stretched, its frequency of oscillation does not change and is given by

$$\nu = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2 \times 3.14}\sqrt{\frac{2 \times 10^3}{12}} = \mathbf{2.06\ Hz.}$$

EXAMPLE 34. *An impulsive force gives an initial velocity of* $-1.0\ m s^{-1}$ *to the mass in the unstretched spring position [see Fig. 14.17(a)]. What is the amplitude of motion ? Give x as a function of time t for the oscillating mass. Given* $m = 3$ *kg and* $k = 1200$ *Nm*$^{-1}$.

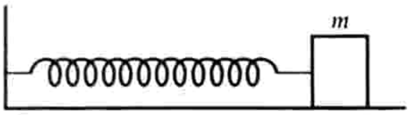


Fig. 14.17 (*a*)

Solution. Here initial velocity in unstretched position,

$$v = -1.0 \text{ ms}^{-1}$$

Clearly, $v_{\max} = 1.0 \text{ ms}^{-1}$

Also, $$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{1200}{3}} = 20 \text{ rad s}^{-1}$$

Amplitude, $$A = \frac{v_{\max}}{\omega} = \frac{1.0}{20} = \frac{1}{20} \text{ m} = \mathbf{5\ cm.}$$

As the motion starts from the unstretched position, the expression for the displacement can be written as

$$x = A \sin \omega t = 5 \sin 20\, t$$

As initial impulse is negative, the displacement is towards negative X-axis. So

$$\mathbf{x = -5 \sin 20\, t.}$$

EXAMPLE 35. *A 5 kg collar is attached to a spring of force constant 500 Nm*$^{-1}$*. It slides without friction on a horizontal rod as shown in Fig. 14.17(b). The collar is displaced from its equilibrium position by 10.0 cm and released.*

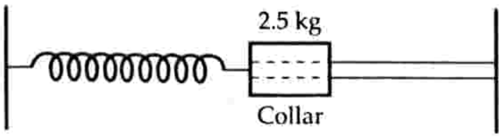


Fig. 14.17 (*b*)

Calculate

(i) the period of oscillation, (ii) the maximum speed, and (iii) the maximum acceleration of the collar.

[NCERT ; Delhi 03C]

Solution. Here $m = 5$ kg, $k = 500$ Nm^{-1},

$$A = 10.0 \text{ cm} = 0.10 \text{ m}$$

(*i*) Period of oscillation,

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{5}{500}}$$

$$= 2 \times 3.14 \times \frac{1}{10} \text{ s} = \mathbf{0.628\ s.}$$

(*ii*) The maximum speed of the collar,

$$v_{\max} = \omega A = \sqrt{\frac{k}{m}}\, A = \sqrt{\frac{500}{5}} \times 0.10$$

$$= \mathbf{1.0\ m\ s^{-1}}.$$

(*iii*) The maximum acceleration of the collar,

$$a_{\max} = \omega^2 A = \frac{k}{m} A = \frac{500}{5} \times 0.10 = \mathbf{10\ m\ s^{-2}}.$$

EXAMPLE 36. *A small trolley of mass 2.0 kg resting on a horizontal turn table is connected by a light spring to the centre of the table. When the turn table is set into rotation at a speed of 300 rpm, the length of the stretched spring is 40 cm. If the original length of the spring is 35 cm, determine the force constant of the spring.*

Solution. Mass of trolley, $m = 2.0$ kg

Frequency of rotation of turn table,

$$\nu = \frac{300}{60} = 5 \text{ rps}$$

Extension produced in the string,

$$y = 40 - 35 = 5 \text{ cm} = 5 \times 10^{-2} \text{ m}$$

When the turn-table is set into rotation, the tension (restoring force) in spring is equal to the centripetal force. Thus

Restoring force = Centripetal force

$$F = ky = mr\,\omega^2 = mr\,(2\pi\nu)^2$$

or $$k = \frac{4\pi^2 \nu^2\, mr}{y}$$

[r = length of stretched spring = 40 cm]

$$= \frac{4 \times 9.87 \times 5^2 \times 2.0 \times 40 \times 10^{-2}}{5 \times 10^{-2}}$$

$$= \mathbf{15795\ Nm^{-1}}.$$

EXAMPLE 37. *Two masses* $m_1 = 1.0$ *kg and* $m_2 = 0.5$ *kg are suspended together by a massless spring of force constant,* $k = 12.5$ *Nm*$^{-1}$*. When they are in equilibrium position,* m_1 *is gently removed. Calculate the angular frequency and the amplitude of oscillation of* m_2*. Given* $g = 10$ *ms*$^{-2}$.

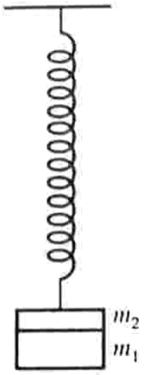


Fig. 14.18

Solution. Let y be the extension in the length of the spring when both m_1 and m_2 are suspended. Then

$$F = (m_1 + m_2)g = ky$$

or $$y = \frac{(m_1 + m_2)g}{k}$$

Let the extension be reduced to y' when m_1 is removed, then

$$m_2 g = ky'$$

or $$y' = \frac{m_2 g}{k}$$

$$\therefore \quad y - y' = \frac{(m_1 + m_2)g}{k} - \frac{m_2 g}{k}$$

$$= \frac{m_1 g}{k}$$

This will be the amplitude of oscillation of m_2.

$\therefore$ Amplitude, $A = \frac{m_1 g}{k} = \frac{1.0 \times 10}{12.5} = \mathbf{0.8\ m.}$

Angular frequency,

$$\omega = \sqrt{\frac{k}{m_2}} = \sqrt{\frac{12.5}{0.5}} = \mathbf{5\ rad\ s^{-1}}.$$

EXAMPLE 38. *Two identical springs, each of spring factor k, may be connected in the following ways. Deduce the spring factor of the oscillation of the body in each case.*

Solution. For each spring,

$$F = -ky \qquad ...(1)$$

where F = restoring force, k = spring factor, and
y = displacement of the spring.

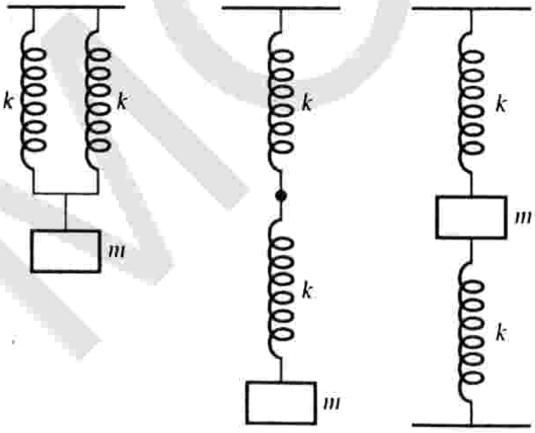


Fig. 14.19

(*i*) In Fig. 14.19(*a*), let the mass m produce a displacement y in each spring and F be the restoring force in each spring. If k_1 be the spring factor of the combined system, then

$$2F = -k_1 y$$

or $$F = -\frac{k_1}{2} y \qquad ...(2)$$

Comparing (1) and (2), we get

$$\frac{k_1}{2} = k \qquad \text{or} \qquad k_1 = 2k.$$

(*ii*) In Fig. 14.19(*b*), as the length of the spring is doubled, the mass m will produce double the displacement ($2y$). If k_2 be the spring factor of the combined system, then

$$F = -k_2(2y) = -2k_2 y \qquad ...(3)$$

Comparing (1) and (3),

$$2k_2 = k \quad \text{or} \quad k_2 = \frac{k}{2}.$$

(*iii*) In Fig. 14.19(*c*), the mass m stretches the upper spring and compresses the lower spring, each giving rise to a restoring force F in the same direction. If k_3 be the spring factor of the combined system, then

$$2F = -k_3 y$$

or $$F = -\frac{k_3}{2} y \qquad ...(4)$$

Comparing (1) and (4),

$$\frac{k_3}{2} = k \quad \text{or} \quad k_3 = 2k.$$

EXAMPLE 39. *Two identical springs, each of force constant k are connected in (a) series (b) parallel, and they support a mass m. Calculate the ratio of the time periods of the mass in the two systems.* **[Central Schools 07]**

Solution. (*a*) For series combination, the effective force constant is

$$k_s = \frac{k \times k}{k + k} = \frac{k}{2}$$

$$\therefore \quad T_s = 2\pi\sqrt{\frac{m}{k_s}} = 2\pi\sqrt{\frac{m}{k/2}}$$

(*b*) For parallel combination, the effective force constant is

$$k_p = k + k = 2k$$

$$\therefore \quad T_p = 2\pi\sqrt{\frac{m}{k_p}} = 2\pi\sqrt{\frac{m}{2k}}$$

Required ratio of the time periods,

$$\frac{T_s}{T_p} = \sqrt{\frac{2k}{k/2}} = \mathbf{2}$$

EXAMPLE 40. *A tray of mass 12 kg is supported by two identical springs as shown in Fig. 14.20. When the tray is pressed down slightly and released, it executes SHM with a time period of 1.5 s. What is the force constant of each spring? When a block of mass M is placed on the tray, the period of SHM changes to 3.0 s. What is the mass of the block ?*

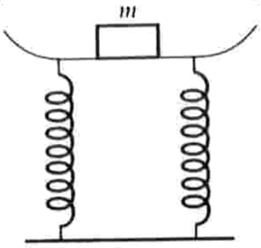


Fig. 14.20

Solution. Let k be the force constant of each spring. As the two springs are connected in parallel, so the force constant of the combination is

$$k' = k + k = 2k$$

Now $$T = 2\pi\sqrt{\frac{m}{k'}}$$

or $$k' = \frac{4\pi^2 m}{T^2} = \frac{4 \times (3.14)^2 \times 12}{(1.5)^2}$$

$$= 210.34 \text{ Nm}^{-1}$$

$\therefore$ $$k = k'/2 = \mathbf{105.17\ Nm^{-1}}.$$

When a block of mass M is placed in the tray, the period of oscillation becomes

$$T' = 2\pi\sqrt{\frac{M+m}{k'}}$$

Hence $$\frac{T'}{T} = \sqrt{\frac{M+m}{m}} \quad \text{or} \quad \frac{3.0}{1.5} = \sqrt{\frac{M+12}{12}}$$

or $$\sqrt{\frac{M+12}{12}} = 2 \quad \text{or} \quad \frac{M+12}{12} = 4$$

$\therefore$ $$M = 48 - 12 = \mathbf{36\ kg}.$$

EXAMPLE 41. *The identical springs of spring constant k are attached to a block of mass m and to fixed supports as shown below.*

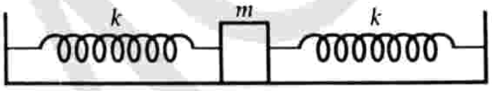


Fig. 14.21

Show that when the mass is displaced from its equilibrium position on either side, it executes a simple harmonic motion. Find the period of oscillations. **[NCERT]**

Solution. As shown in Fig. 14.22, suppose the mass m is displaced by a small distance x to the right side of the equilibrium position O. Then the left spring gets elongated by length x and the right spring gets compressed by the same length x.

Force exerted by the left spring,

$$F_1 = -kx, \text{ towards left}$$

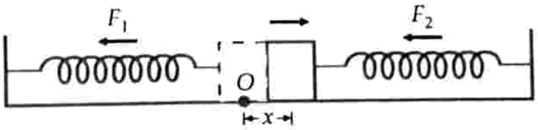


Fig. 14.22

Force exerted by the right spring,

$$F_2 = -kx, \text{ towards left}$$

The net force acting on mass m is

$$F = F_1 + F_2 = -2kx$$

Thus the force acting on the mass m is proportional to its displacement x and is directed towards its mean position. Hence the motion of the mass m is simple harmonic. Force constant is

$$k' = 2k$$

The period of oscillation is

$$T = 2\pi\sqrt{\frac{m}{k'}} = 2\pi\sqrt{\frac{m}{2k}}.$$

EXAMPLE 42. *A trolley of mass 3.0 kg is connected to two identical springs each of force constant 600 Nm^{-1}, as shown in Fig. 14.23. If the trolley is displaced from its equilibrium position by 5.0 cm and released, what is (i) the period of ensuing oscillations, (ii) the maximum speed of the trolley ? (iii) How much is the total energy dissipated as heat by the time the trolley comes to rest due to damping forces ?*

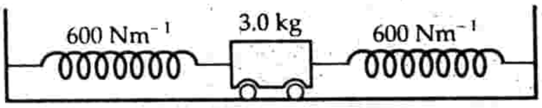


Fig. 14.23

Solution. When the trolley is displaced from the mean position, it stretches one spring and compresses the other by the same amount. The restoring forces developed in the two springs are in the same direction. If the trolley is displaced through distance y, then total restoring force is

$$F = F_1 + F_2 = -ky - ky = -2ky$$

If k' is the force constant of the combination, then

$$F = -k'y$$

Clearly, $k' = 2k = 2 \times 600 = 1200 \text{ Nm}^{-1}$

Also, $m = 3.0$ kg

amplitude, $A = 5.0 \times 10^{-2}$ m

(*i*) Period of oscillation,

$$T = 2\pi\sqrt{\frac{m}{k'}}$$

$$= 2 \times 3.14\sqrt{\frac{3.0}{1200}} = \mathbf{0.314\ s.}$$

(*ii*) Maximum speed,

$$v_{\max} = \omega A = \sqrt{\frac{k'}{m}} \times A$$

$$= \sqrt{\frac{1200}{3.0}} \times 5.0 \times 10^{-2} = \mathbf{1.0\ ms^{-1}}.$$

(*iii*) Total energy dissipated as heat

= Initial maximum K.E. of the trolley

$$= \frac{1}{2} mv_{\max}^2 = \frac{1}{2} \times 3.0 \times (1.0)^2 = \mathbf{1.5\ J.}$$

✖ PROBLEMS FOR PRACTICE

1. A spring compressed by 0.1 m develops a restoring force of 10 N. A body of mass 4 kg is placed on it. Deduce (*i*) the force constant of the spring (*ii*) the depression of the spring under the weight of the body and (*iii*) the period of oscillation, if the body is disturbed.
[**Ans.** (*i*) 100 Nm^{-1} (*ii*) 0.4 m (*iii*) 1.26 s]

2. The period of oscillation of a mass *m* suspended by an ideal spring is 2 s. If an additional mass of 2 kg be suspended, the time period is increased by 1 s. Find the value of *m*. (**Ans.** 1.6 kg)

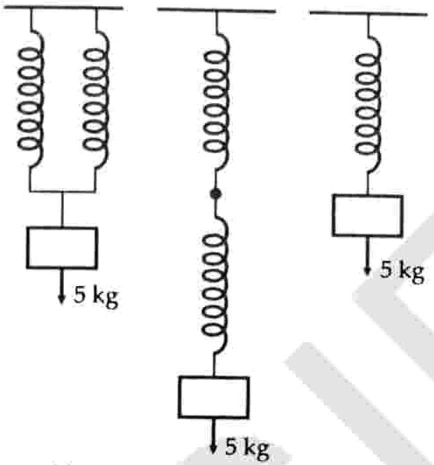
3. An uncalibrated spring balance is found to have a period of oscillation of 0.314 s, when a 1 kg weight is suspended from it ? How does the spring elongate, when a 1 kg weight is suspended from it ? [Take $\pi = 3.14$] (**Ans.** 2.45 cm)

4. The frequency of oscillations of a mass *m* suspended by a spring is ν_1. If the length of the spring is cut to one-half, the same mass oscillates with frequency ν_2. Determine the value of ν_2 / ν_1.
[Chandigarh 03]
(**Ans.** $\sqrt{2}$)

5. The periodic time of a mass suspended by a spring (force constant *k*) is *T*. If the spring is cut in three equal pieces, what will be the force constant of each part ? If the same mass be suspended from one piece, what will be the periodic time ?
(**Ans.** 3*k*, $T/\sqrt{3}$)

6. The time period of a body suspended by a spring be *T*. What will be the new period, if the spring is cut into two equal parts and when (*i*) the body is suspended from one part (*ii*) the body is suspended from both the parts connected in parallel.
[**Ans.** (*i*) $T/\sqrt{2}$ (*ii*) $T/2$]

7. Two identical springs have the same force constant of 147 Nm^{-1}. What elongation will be produced in each spring in each case shown in Fig. 14.24 ? Take $g = 9.8\ ms^{-2}$.
[**Ans.** (*a*) 1/6 m (*b*) 1/3 m, 1/3 m (*c*) 1/3 m]



(*a*) (*b*) (*c*)

Fig. 14.24

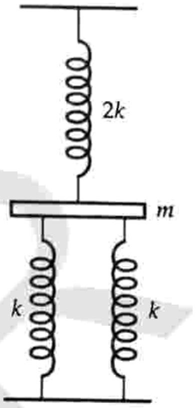


Fig. 14.25

8. Three springs are connected to a mass *m* as shown in Fig. 14.25. When mass *m* oscillates, what is the effective spring constant and time period of vibration ? Given $k = 2\ Nm^{-1}$ and $m = 80\ g$.
(**Ans.** 8 Nm^{-1}, 0.628 s)

9. Two springs are joined and connected to a mass *m* as shown in Fig. 14.26. If the force constants of the two springs are k_1 and k_2, show that frequency of oscillation of mass *m* is

$$\nu = \frac{1}{2\pi} \sqrt{\frac{k_1 k_2}{(k_1 + k_2)\, m}}.$$

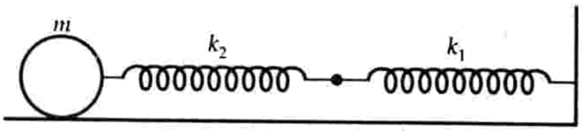


Fig. 14.26

✖ HINTS

1. (*i*) $k = \frac{F}{y} = \frac{10}{0.1} = \mathbf{100\ Nm^{-1}}.$

(*ii*) $y = \frac{mg}{k} = \frac{4 \times 10}{100} = \mathbf{0.4\ m.}$

(*iii*) $T = 2\pi \sqrt{\frac{m}{k}} = 2 \times \frac{22}{7} \times \sqrt{\frac{4}{100}} = \mathbf{1.26\ s.}$

2. As $T = 2\pi \sqrt{\frac{m}{k}}$

∴ In first case,

$$2 = 2\pi \sqrt{\frac{m}{k}} \quad \text{or} \quad 4 = 4\pi^2 \times \frac{m}{k} \qquad ...(1)$$

In second case,

$$3 = 2\pi \sqrt{\frac{m+2}{k}} \text{ or } 9 = 4\pi^2 \times \frac{m+2}{k} \qquad ...(2)$$

Dividing (2) by (1), we get

$$\frac{9}{4}=\frac{m+2}{m} \quad \text{or} \quad m=\frac{8}{5}=\mathbf{1.6\ kg.}$$

4. Let k be the force constant of the full spring. Then frequency of oscillation of mass m will be

$$\nu_1=\frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

When the spring is cut to one-half of its length, its force constant is doubled ($2k$). Frequency of oscillation of mass m will be

$$\nu_2=\frac{1}{2\pi}\sqrt{\frac{2k}{m}} \qquad \therefore \quad \nu_2/\nu_1=\sqrt{2}.$$

5. Time period of mass m when suspended from the full spring is

$$T=2\pi\sqrt{\frac{m}{k}}$$

When the spring is cut into three equal parts, the force constant of each part becomes $3k$. Time period of mass m when suspended from one such piece will be

$$T'=2\pi\sqrt{\frac{m}{3k}}=\frac{T}{\sqrt{3}}.$$

6. For full spring, $T=2\pi\sqrt{\frac{m}{k}}$.

If the spring is cut into two equal parts, then the force constant of each part becomes $2k$.

(*i*) When the body is suspended from one part, its period of oscillation is

$$T'=2\pi\sqrt{\frac{m}{2k}}=\frac{T}{\sqrt{2}}.$$

(*ii*) For the two parts connected in parallel, force constant

$$=2k+2k=4k.$$

The period of oscillation becomes

$$T''=2\pi\sqrt{\frac{m}{4k}}=\frac{T}{2}.$$

7. Here $k = 147\ \text{Nm}^{-1}$. In Fig. 14.24(*a*), the effective spring constant,

$$K=k+k=2k=2\times 147=294\ \text{Nm}^{-1}$$

$\therefore$ Elongation in the spring,

$$y_1=\frac{mg}{K}=\frac{5\times 9.8}{294}=\frac{1}{6}\ \mathbf{m.}$$

In Fig. 14.24(*b*), the effective spring constant,

$$K=\frac{k\times k}{k+k}=\frac{k}{2}=\frac{147}{2}\ \text{Nm}^{-1}$$

$\therefore$ Total elongation in the spring,

$$y_2=\frac{5\times 9.2\times 2}{147}=\frac{2}{3}\ \text{m}$$

$\therefore$ Elongation in each spring $=\mathbf{\frac{1}{3}\ m.}$

In Fig. 14.24(*c*), the effective spring constant,

$$K=147\ \text{Nm}^{-1}$$

$\therefore$ Elongation in the spring, $y_3=\frac{5\times 9.8}{147}=\frac{1}{3}$ **m.**

8. The given arrangement is equivalent to the three springs connected in parallel. The effective spring constant is

$$K=k+2k+k=4k=4\times 2=\mathbf{8\ Nm^{-1}}.$$

Time period,

$$T=2\pi\sqrt{\frac{m}{4k}}=2\times\frac{22}{7}\times\sqrt{\frac{0.08}{8}}=\mathbf{0.628\ s.}$$

9. Let a force F applied on the body produce displacements x_1 and x_2 in the two springs. Then

$$F=-k_1x_1=-k_2x_2$$

$$\therefore \qquad x_1=-\frac{F}{k_1} \text{ and } x_2=-\frac{F}{k_2}$$

Total extension,

$$x=x_1+x_2=-F\left[\frac{1}{k_1}+\frac{1}{k_2}\right]=-F\left(\frac{k_1+k_2}{k_1k_2}\right)$$

$$\text{or} \qquad F=-\frac{k_1k_2}{k_1+k_2}\,x$$

Clearly, force constant of the system, $k=\frac{k_1k_2}{k_1+k_2}$

Frequency, $\nu=\frac{1}{2\pi}\sqrt{\frac{k}{m}}=\frac{1}{2\pi}\sqrt{\frac{k_1k_2}{(k_1+k_2)\,m}}$.

14.16 SIMPLE PENDULUM

19. *Show that for small oscillations the motion of a simple pendulum is simple harmonic. Derive an expression for its time period. Does it depend on the mass of the bob ?*

Simple pendulum. *An ideal simple pendulum consists of a point-mass suspended by a flexible, inelastic and weightless string from a rigid support of infinite mass.* In practice, we can neither have a point-mass nor a weightless string.

In practice, a simple pendulum is obtained by suspending a small metal bob by a long and fine cotton thread from a rigid support.

Expression for time period. In the equilibrium position, the bob of a simple pendulum lies vertically below the point of suspension. If the bob is slightly displaced on either side and released, it begins to oscillate about the mean position.

Suppose at any instant during oscillation, the bob lies at position A when its displacement is $OA = x$ and the thread makes angle θ with the vertical. The forces acting on the bob are

(*i*) Weight mg of the bob acting vertically downwards.

(*ii*) Tension T along the string.

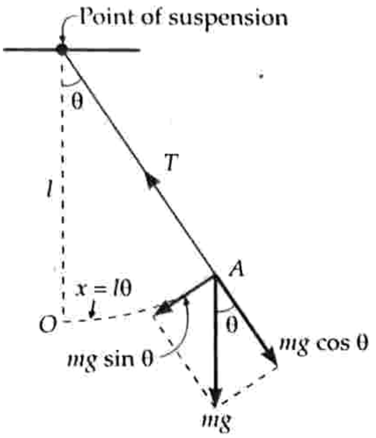


Fig. 14.27 Force acting on the bob of a pendulum.

The force mg has two rectangular components (*i*) the component $mg \cos \theta$ acting along the thread is balanced by the tension T in the thread and (*ii*) the tangential component $mg \sin \theta$ is the net force acting on the bob and tends to bring it back to the mean position. Thus, the restoring force is

$$F = -mg \sin \theta = -mg\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - .....\right)$$

$$= -mg\,\theta\left(1 - \frac{\theta^2}{6} + \frac{\theta^4}{120} - .....\right)$$

where θ is in radians. Clearly, oscillations are not simple harmonic because the restoring force F is not proportional to the angular displacement θ.

However, if θ is so small that its higher powers can be neglected, then

$$F = -mg\,\theta$$

If l is the length of the simple pendulum, then

$$\theta\,(\text{rad}) = \frac{\text{arc}}{\text{radius}} = \frac{x}{l}$$

$$\therefore \quad F = -mg\,\frac{x}{l}$$

$$\text{or} \quad ma = -\frac{mg}{l}\,x$$

$$\text{or} \quad a = -\frac{g}{l}\,x = -\omega^2 x$$

Thus, the acceleration of the bob is proportional to its displacement x and is directed opposite to it. Hence for small oscillations, the motion of the bob is simple harmonic. Its time period is

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{g/l}} \quad \text{or} \quad T = 2\pi\sqrt{\frac{l}{g}}$$

Obviously, the time period of a simple pendulum depends on its length l and acceleration due to gravity g but is independent of the mass m of the bob.

Examples based on Oscillations of a Simple Pendulum

FORMULAE USED

1. Time period, $T = 2\pi\sqrt{\frac{l}{g}}$
2. Frequency, $\nu = \frac{1}{2\pi}\sqrt{\frac{g}{l}}$

UNITS USED

Length l of the pendulum is in metre and acceleration due to gravity g in ms^{-2}.

EXAMPLE 43. *What is the length of a simple pendulum, which ticks seconds ?* **[NCERT ; Delhi 09]**

Solution. The simple pendulum which ticks seconds is a second pendulum whose time period is 2 s. Thus

$$T = 2 \text{ s}, \quad g = 9.8 \text{ ms}^{-2}$$

$$\text{As} \quad T = 2\pi\sqrt{\frac{l}{g}} \quad \text{or} \quad T^2 = 4\pi^2\,\frac{l}{g}$$

$$\therefore \quad l = \frac{T^2 g}{4\pi^2} = \frac{(2)^2 \times 9.8}{4 \times 9.87} = \mathbf{0.992\ m}.$$

EXAMPLE 44. *A pendulum clock shows accurate time. If the length increases by 0.1%, deduce the error in time per day.* **[Delhi 95]**

Solution. Correct number of seconds per day,

$$\nu = 24 \times 60 \times 60 = 86400.$$

Let error introduced per day = x seconds

Then incorrect number of seconds per day,

$$\nu' = 86400 + x$$

If l is the original length of the pendulum, then its new length will be

$$l' = l + 0.1\% \text{ of } l = l + \frac{0.1 \times l}{100} = (1 + 0.001)\,l$$

$$\text{Now frequency,} \quad \nu = \frac{1}{2\pi}\sqrt{\frac{g}{l}} \quad i.e., \quad \nu \propto \frac{1}{\sqrt{l}}$$

$$\therefore \quad \frac{\nu'}{\nu} = \sqrt{\frac{l}{l'}} \quad \text{or} \quad \frac{86400 + x}{86400} = \sqrt{\frac{l}{(1 + 0.001)\,l}}$$

$$\text{or} \quad 1 + \frac{x}{86400} = (1 + 0.001)^{-1/2}$$

$$= 1 - \frac{1}{2} \times 0.001 = 1 - 0.0005$$

$$\text{or} \quad \frac{x}{86400} = -0.0005$$

$$\text{or} \quad x = -0.0005 \times 86400 = \mathbf{-43.2\ s.}$$

The negative sign shows that the clock will run slow and it will lose 43.2 seconds per day.

Example 45. Two pendulums of lengths 100 cm and 110.25 cm start oscillating in phase. After how many oscillations will they again be in same phase ?

Solution. The two pendulums will be in same phase again when large pendulum completes ν oscillations and small pendulum completes (ν + 1) oscillations.

For larger pendulum,

$$\nu = \frac{1}{2\pi}\sqrt{\frac{g}{l}} = \frac{1}{2\pi}\sqrt{\frac{g}{110.25}}$$

For smaller pendulum, $\nu + 1 = \frac{1}{2\pi}\sqrt{\frac{g}{100}}$

$$\therefore \quad \frac{\nu+1}{\nu} = \sqrt{\frac{110.25}{100}}$$

$$= \sqrt{\frac{100+10.25}{100}} = \left(1 + \frac{10.25}{100}\right)^{1/2}$$

or $$1 + \frac{1}{\nu} = 1 + \frac{1}{2} \times \frac{10.25}{100} = 1 + 0.05$$

or $$\nu = \frac{1}{0.05} = 20$$

Thus the two pendulums will be in same phase when the larger pendulum completes 20 oscillations or smaller pendulum completes 21 oscillations.

Example 46. A second's pendulum is taken in a carriage. Find the period of oscillation when the carriage moves with an acceleration of 4 ms^{-2} (i) vertically upwards (ii) vertically downwards, and (iii) in a horizontal direction.

Solution. Time period of a pendulum,

$$T = 2\pi\sqrt{\frac{l}{g}}$$

For second's pendulum, $T = 2$ s

$$\therefore \quad 2 = 2\pi\sqrt{\frac{l}{g}} \quad \text{or} \quad 1 = \pi\sqrt{\frac{l}{g}}$$

or $$1 = \pi^2\frac{l}{g} \quad \therefore \quad l = \frac{g}{\pi^2} = \frac{9.8}{\pi^2}.$$

(*i*) When the carriage moves up with an acceleration $a = 4$ ms^{-2}, the time period is

$$T_1 = 2\pi\sqrt{\frac{l}{g+a}} = 2\pi\sqrt{\frac{9.8}{\pi^2\,(9.8+4)}}$$

$$= \frac{2\pi}{\pi}\sqrt{\frac{9.8}{13.8}} = 2 \times 0.843 = \mathbf{1.69\ s.}$$

(*ii*) When the carriage moves down with an acceleration $a = 4$ ms^{-2}, the time period is

$$T_2 = 2\pi\sqrt{\frac{l}{g-a}} = 2\pi\sqrt{\frac{9.8}{\pi^2\,(9.8-4)}}$$

$$= 2\sqrt{\frac{9.8}{5.8}} = 2 \times 1.299 = \mathbf{2.59\ s.}$$

(*iii*) When the carriage moves horizontally, both g and a are at right angle to each other, hence the net acceleration is

$$a' = \sqrt{g^2 + a^2} = \sqrt{(9.8)^2 + (4)^4}$$

$$= \sqrt{96.04 + 16} = \sqrt{112.04} = 10.58 \text{ ms}^{-2}$$

Time period will be

$$T_3 = 2\pi\sqrt{\frac{l}{a'}} = 2\pi\sqrt{\frac{9.8}{\pi^2 \times 10.58}}$$

$$= 2 \times 0.96 = \mathbf{1.92\ s.}$$

Example 47. The bottom of a dip on a road has a radius of curvature R. A rickshaw of mass M left a little away from the bottom oscillates about the dip. Deduce an expression for the period of oscillation. **[Chandigarh 02]**

Solution. As shown in Fig. 14.28, let the rickshaw of mass M be at position A at any instant and $\angle AOB = \theta$.

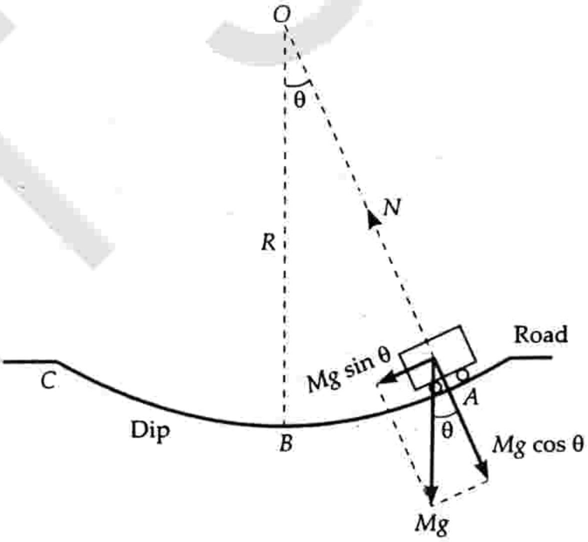


Fig. 14.28

Forces acting on the rickshaw at position A are

(*i*) Weight Mg acting vertically downwards.

(*ii*) The normal reaction N of the road.

The weight Mg can be resolved into two rectangular components :

(*i*) $Mg \cos\theta$ perpendicular to the road. It balances the normal reaction N.

(*ii*) $Mg \sin\theta$ tangential to the road. It is the only unbalanced force acting on the rickshaw which acts towards the mean position B. Hence the restoring force is

$$F = -Mg\sin\theta$$

For small θ, $$\sin\theta \simeq \theta = \frac{\text{Arc}}{\text{Radius}} = \frac{AB}{R} = \frac{y}{R}$$

$$\therefore \quad F = -\frac{Mg}{R}y \quad i.e., \quad F \propto y$$

Hence the motion of the rickshaw is simple harmonic with force constant,

$$k = \frac{Mg}{R}$$

Time period,

$$T = 2\pi\sqrt{\frac{M}{k}} = 2\pi\sqrt{\frac{M}{Mg/R}} = 2\pi\sqrt{\frac{R}{g}}.$$

PROBLEMS FOR PRACTICE

1. The time taken by a simple pendulum to perform 100 vibrations is 8 minutes 9 seconds in Bombay and 8 minutes 20 seconds in Pune. Calculate the ratio of acceleration due to gravity in Bombay and Pune. (**Ans.** 1.0455)
2. If the length of a pendulum is decreased by 2%, find the gain or loss in time per day. (**Ans.** Gain of 864 s)
3. If the length of a second's pendulum is increased by 1%, how many seconds will it lose or gain in a day ? (**Ans.** Loss of 432 s)
4. If the length of a simple pendulum is increased by 45%, what is the percentage increase in its time period ? (**Ans.** 22.5%)
5. What will be the time period of second's pendulum if its length is doubled ? (**Ans.** 2.828 s)
6. If the acceleration due to gravity on moon is one-sixth of that on the earth, what will be the length of a second pendulum there ? Take $g = 9.8\ \text{ms}^{-2}$. (**Ans.** 16.5 cm)

HINTS

1. Let g_1 and g_2 be the values of acceleration due to gravity in Bombay and Pune and T_1 and T_2 be the values of the time-periods at the respective places. Then

$$T_1 = \frac{8\text{ min } 9\text{ s}}{100} = \frac{489}{100}\text{ s} = 4.89\text{ s}$$

$$T_2 = \frac{8\text{ min } 20\text{ s}}{100} = \frac{500\text{ s}}{100} = 5\text{ s}$$

As $\quad \dfrac{T_2}{T_1} = \sqrt{\dfrac{g_1}{g_2}}$

$$\therefore \quad \frac{g_1}{g_2} = \frac{T_2^2}{T_1^2} = \frac{(5)^2}{(4.89)^2} = \mathbf{1.0455.}$$

2. As $v \propto 1/\sqrt{l}$, so the number of seconds gained per day on decreasing the length by 2%

$$= \frac{1}{2}\frac{\Delta l}{l} \times 86400 = \frac{1}{2} \times \frac{2}{100} \times 86400 = \mathbf{864\ s.}$$

4. As $T \propto \sqrt{l}$, so the percentage increase in time period on increasing the length by 45%

$$= \frac{1}{2}\frac{\Delta l}{l} \times 100 = \frac{1}{2} \times \frac{45}{100} \times 100 = \mathbf{22.5\%.}$$

6. On the moon, $g_m = \dfrac{g}{6} = \dfrac{9.8}{6}\ \text{ms}^{-2}, \quad T = 2\text{ s}.$

As $\quad T = 2\pi\sqrt{\dfrac{l}{g_m}}$

$$\therefore \quad l = \frac{T^2 g_m}{4\pi^2} = \frac{2^2 \times 9.8}{4 \times 9.87 \times 6} = 0.165\text{ m} = \mathbf{16.5\ cm.}$$

14.17 OTHER EXAMPLES OF S.H.M.

20. *One end of a U-tube containing mercury is connected to a suction pump and the other end is connected to the atmosphere. A small pressure difference is maintained between the two columns. Show that when the suction pump is removed, the liquid in the U-tube executes SHM.* **[NCERT]**

Oscillations of a liquid column in a U-tube. Initially, suppose the U-tube of cross-section A contains liquid of density ρ upto height h. Then mass of the liquid in the U-tube is

$$m = \text{Volume} \times \text{density} = A \times 2h \times \rho$$

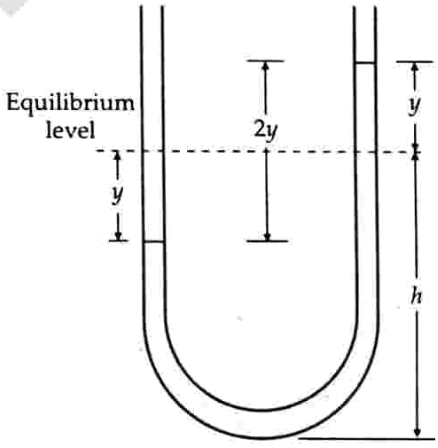


Fig. 14.29 Oscillations of a liquid column in a *U*-tube.

If the liquid in one arm is depressed by distance y, it rises by the same amount in the other arm. If left to itself, the liquid begins to oscillate under the restoring force,

$$F = \text{Weight of liquid column of height } 2y$$

$$F = -A \times 2y \times \rho \times g = -2\,A\rho g y$$

i.e., $\quad F \propto y$

Thus the force on the liquid is proportional to displacement and acts in its opposite direction. Hence the liquid in the U-tube executes SHM with force constant,

$$k = 2\,A\rho g$$

The time-period of oscillation is

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{A \times 2h \times \rho}{2\,A\rho\,g}} = 2\pi\sqrt{\frac{h}{g}}.$$

If l is the length of the liquid column, then

$$l = 2h \quad \text{and} \quad T = 2\pi\sqrt{\frac{l}{2g}}.$$

21. *If the earth were a homogeneous sphere of radius R and a straight hole bored in it through its centre, show that a body dropped into the hole will execute SHM and find its time period.*

Oscillations of a body dropped in a tunnel along the diameter of the earth. As shown in Fig. 14.30, consider earth to be a sphere of radius R and centre O. A straight tunnel is dug along the diameter of the earth. Let g be the value of acceleration due to gravity at the surface of the earth.

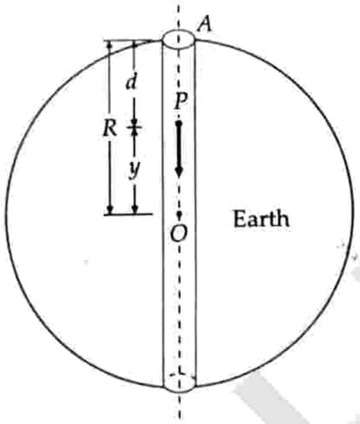


Fig. 14.30 A body dropped in a tunnel along the diameter of the earth.

Suppose a body of mass m is dropped into the tunnel and it is at point P *i.e.*, at a depth d below the surface of the earth at any instant. If g' is acceleration due to gravity at P, then

$$g' = g\left(1 - \frac{d}{R}\right) = g\left(\frac{R-d}{R}\right)$$

If y is distance of the body from the centre of the earth (displacement from mean position), then

$$R - d = y \quad \therefore \quad g' = g\,\frac{y}{R}$$

Force acting on the body at point P is

$$F = -mg' = -\frac{mg}{R}\,y \quad \textit{i.e.,} \quad F \propto y$$

Negative sign shows that the force F acts in the opposite direction of displacement *i.e.*, it acts towards the mean position O. Thus the body will execute SHM with force constant,

$$k = \frac{mg}{R}$$

The period of oscillation of the body will be

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{m}{mg/R}} = 2\pi\sqrt{\frac{R}{g}}.$$

22. *A cylindrical piece of cork of base area A and height h floats in a liquid of density* ρ_1. *The cork is depressed slightly and then released. Show that the cork oscillates up and down simple harmonically with a period* $T = 2\pi\sqrt{\frac{h\rho}{\rho_1 g}}$, *where* ρ *is the density of cork. (Ignore damping due to viscosity of the liquid).* **[NCERT]**

Oscillations of a floating cylinder. In equilibrium, weight of the cork is balanced by the upthrust of the liquid.

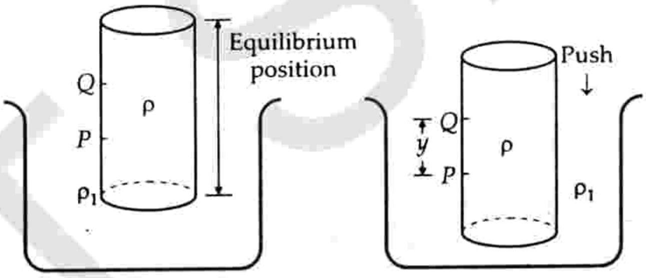


Fig. 14.31 Oscillations of a floating cylinder.

Let the cork be slightly depressed through distance y from the equilibrium position and left to itself. It begins to oscillate under the restoring force,

$$F = \text{Net upward force}$$
$$= \text{Weight of liquid column of height } y$$

or $$F = -A\,y\,\rho_1\,g = -A\rho_1\,g\,y \quad \textit{i.e.,} \quad F \propto y.$$

Negative sign shows that F and y are in opposite directions. Hence the cork executes SHM with force constant, $k = A\rho_1\,g$

Also, mass of cork $= A\rho\,h$

$\therefore$ Period of oscillation of the cork is

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{A\rho\,h}{A\rho_1\,g}} = 2\pi\sqrt{\frac{\rho\,h}{\rho_1\,g}}.$$

23. *An air chamber of volume V has a neck of area of cross-section A into which a ball of mass m can move without friction. Show that when the ball is pressed down through some distance and released, the ball executes SHM. Obtain the formula for the time period of this SHM, assuming pressure-volume variations of the air to be (i) isothermal and (ii) adiabatic.* **[NCERT]**

Oscillations of a ball in the neck of an air chamber. Fig. 14.32 shows an air chamber of volume V, having a neck of area of cross-section A and a ball of mass m fitting smoothly in the neck. If the ball be pressed down a little and released, it starts oscillating up and down about the equilibrium position.

If the ball be depressed by distance y, then the decrease in volume of air in the chamber is $\Delta V = Ay$.

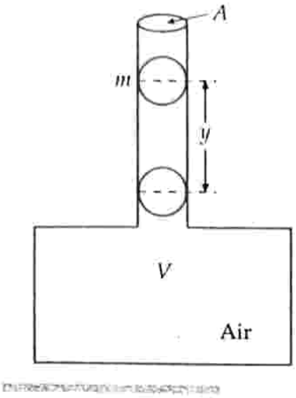


Fig. 14.32

$$\therefore \quad \text{Volume strain} = \frac{\Delta V}{V} = \frac{Ay}{V}$$

If pressure P is applied to the ball, then hydrostatic stress $= P$

$\therefore$ Bulk modulus of elasticity of air,

$$E = -\frac{P}{\Delta V / V} = -\frac{P}{Ay / V} \quad \text{or} \quad P = -\frac{EA}{V}\, y$$

$$\text{Restoring force,} \quad F = PA = -\frac{EAy}{V}\, A = -\frac{EA^2}{V}\, y$$

Thus F is proportional to y and acts in its opposite direction. Hence the ball executes SHM with force constant,

$$k = \frac{EA^2}{V}$$

Period of oscillation of the ball is

$$T = 2\pi \sqrt{\frac{m}{k}} = 2\pi \sqrt{\frac{m}{EA^2 / V}} = 2\pi \sqrt{\frac{mV}{EA^2}}$$

(*i*) If the P-V variations are *isothermal*, then $E = P$,

$$\therefore \qquad T = 2\pi \sqrt{\frac{mV}{PA^2}}.$$

(*ii*) If the P-V variations are adiabatic, then $E = \gamma P$

$$\therefore \qquad T = 2\pi \sqrt{\frac{mV}{\gamma PA^2}}.$$

24. *Show that the angular oscillations of a balance wheel of a watch are simple harmonic. Hence derive an expression for its period of oscillation.*

Oscillations of the balance-wheel of a watch. In a watch, a balance-wheel controls the movement of its hands. An axle passing through its centre is held between two diamond points. A hair-spring controls its oscillations.

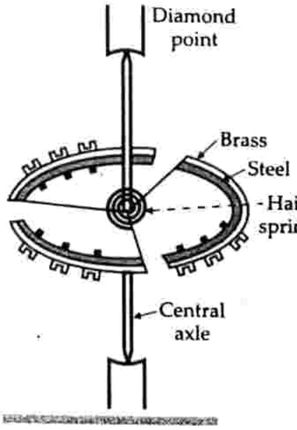


Fig. 14.33

For an angular displacement θ, the hair-spring develops a restoring torque $C\theta$, which tends to bring back the wheel into its equilibrium position. Here C is the restoring torque produced per unit angular displacement. Now

Torque = Moment of inertia × angular acceleration

$$\therefore \qquad C\,\theta = -I \times \frac{d^2\,\theta}{dt^2}$$

$$\text{or} \qquad \frac{d^2\theta}{dt^2} = -\frac{C}{I}\,\theta = -\,\omega^2\,\theta$$

where I is the moment of inertia of the wheel about its axis of rotation. Clearly, angular acceleration $\frac{d^2\theta}{dt^2}$ is proportional to angular displacement θ and acts in its opposite direction. Hence oscillations of the balance-wheel are simple harmonic.

$$\text{Angular frequency,} \quad \omega = \sqrt{\frac{C}{I}}$$

$$\text{Period of oscillation, } T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{C / I}} = 2\pi \sqrt{\frac{I}{C}}.$$

Examples based on
Other Examples of SHM

FORMULAE USED

1. For a liquid of density ρ contained in a U-tube upto height h, $T = 2\pi \sqrt{\frac{h}{g}}$
2. For a body dropped in a tunnel along the diameter of the earth,
   $$T = 2\pi \sqrt{\frac{R}{g}}, \quad \text{where } R = \text{radius of the earth}$$
3. For a cylinder of density ρ floating with length h submerged in a liquid of density ρ, $T = 2\pi \sqrt{\frac{\rho\, h}{\sigma\, g}}$
4. For a ball of mass m oscillating in the neck of air chamber of volume V, $T = 2\pi \sqrt{\frac{mV}{EA^2}}$
   where A = area of cross-section of the neck,
   E = bulk modulus of elasticity of air
5. For a balance-wheel of a watch of moment of inertia I and torsional constant C, $T = 2\pi \sqrt{\frac{I}{C}}$

UNITS USED

Here h and R are in metre, densities ρ and σ in kg m^{-3}, bulk modulus E in Nm^{-2}, moment of inertia I in kg m^2, torsional constant C in Nm rad^{-1}.

Example 48. A vertical U-tube of uniform cross-section contains water upto a height of 2.45 cm. If the water on one side is depressed and then released, its up and down motion in tube is simple harmonic. Calculate its time period. Given $g = 980\ cm\,s^{-2}$.

Solution. Here $h = 2.45$ cm, $g = 980$ cm s^{-2}

$$\therefore \quad T = 2\pi\sqrt{\frac{h}{g}} = 2\pi\sqrt{\frac{2.45}{980}} = 2\pi \times 0.05 = \mathbf{0.314\ s.}$$

Example 49. A test tube weighing 10 g and external diameter 2 cm is floated vertically in water by placing 10 g of mercury at its bottom. The tube is depressed in water a little and then released. Find the time of oscillation. Take $g = 10\ ms^{-2}$.

Solution. Total mass of test tube and mercury,

$$m = 10 + 10 = 20 \text{ g} = 0.02 \text{ kg}$$

Area of cross-section of the test-tube,

$$A = \pi r^2 = \frac{22}{7} \times \left(\frac{1}{100}\right)^2 = \frac{22}{7} \times 10^{-4} \text{ m}^2$$

Density of water, $\rho = 10^3$ kg m^{-3}

Let the tube be depressed in water by a little distance y and then released.

Spring factor,

$$k = \frac{F}{y} = \frac{Ay \cdot \rho \cdot g}{y} = A\rho g$$

$$= \frac{22}{7} \times 10^{-4} \times 10^3 \times 10 = \frac{22}{7} \text{ Nm}^{-1}.$$

Inertia factor, $m = 0.02$ kg

$$\therefore \quad T = 2\pi\sqrt{\frac{m}{k}} = 2 \times \frac{22}{7} \times \sqrt{\frac{0.02 \times 7}{22}} = \mathbf{0.5\ s.}$$

Example 50. A cylindrical wooden block of cross-section 15.0 cm^2 *and mass 230 g is floated over water with an extra weight of 50 g attached to its bottom. The cylinder floats vertically. From the state of equilibrium, it is slightly depressed and released. If the specific gravity of wood is 0.30 and* $g = 9.8\ ms^{-2}$*, deduce the frequency of the block,* $A = 15.0\ cm^2 = 15 \times 10^{-4}\ m^2$.

Solution. Area of cross-section of the block,

$$A = 15.0 \text{ cm}^2 = 15 \times 10^{-4} \text{ m}^2$$

Total mass of the block,

$$m = 230 + 50 = 280 \text{ g} = 0.28 \text{ kg}$$

Density of water,

$$\sigma = 10^3 \text{ kg m}^{-3}$$

Density of wood,

$$\rho = 0.30 \times 10^3 \text{ kg m}^{-3} = 300 \text{ kg m}^{-3}$$

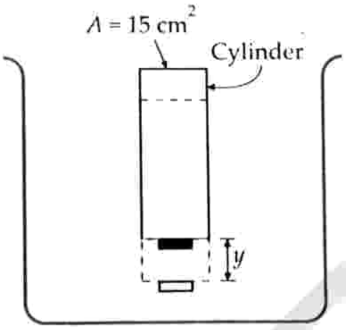


Fig. 14.34

Let the cylinder be depressed through a small distance y. Then

Restoring force = Weight of water displaced

or $$F = Ay\sigma g$$

Force constant,

$$k = \frac{F}{y} = A\sigma g = 15 \times 10^{-4} \times 10^3 \times 9.8 = \mathbf{14.7\ Nm^{-1}}$$

$$\text{Frequency, } \nu = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{14.7}{0.28}}$$

$$= \frac{7}{44} \times \sqrt{52.5} = \mathbf{1.15\ Hz.}$$

Example 51. The balance wheel of a watch has a moment of inertia of $2 \times 10^{-8}\ kg\ m^2$ *and the torsional constant of its hair spring is* $9.8 \times 10^{-6}\ Nm\ rad^{-1}$*. Calculate its frequency.*

Solution. Here $I = 2 \times 10^{-8}$ kg m^2,

$$C = 9.8 \times 10^{-6} \text{ Nm rad}^{-1}$$

Frequency,

$$\nu = \frac{1}{2\pi}\sqrt{\frac{C}{I}} = \frac{7 \times \sqrt{10}}{2 \times 3.14} = \frac{7 \times 3.17}{2 \times 3.14} = \mathbf{3.53\ Hz.}$$

Example 52. A sphere is hung with a wire. 30° rotation of the sphere about the wire generates a restoring torque of 4.6 Nm. If the moment of inertia of the sphere is 0.082 $kg\ m^2$*, deduce the frequency of angular oscillations.*

Solution. Here $\theta = 30° = \frac{\pi}{6}$ rad, $\tau = 4.6$ Nm,

$$I = 0.082 \text{ kg m}^2$$

Restoring torque per unit angular displacement,

$$C = \frac{\tau}{\theta} = \frac{4.6}{\pi/6}$$

$$= \frac{4.6 \times 6 \times 7}{22} = 8.78 \text{ Nm rad}^{-1}$$

$\therefore$ Frequency,

$$\nu = \frac{1}{2\pi}\sqrt{\frac{C}{I}}$$

$$= \frac{7}{2 \times 22}\sqrt{\frac{8.78}{0.082}} = \mathbf{1.65\ Hz.}$$

PROBLEMS FOR PRACTICE

1. If the earth were a homogeneous sphere and a straight hole was bored in it through its centre, show that a body dropped in the hole will execute SHM and calculate the time period of its vibration. Radius of earth is 6.4×10^5 m and $g = 9.8 \text{ ms}^{-2}$. **(Ans.** 5077.6 s)
2. A weighted glass tube is floating in a liquid with 20 cm of its length immersed. It is pushed down through a certain distance and then released. Show that up and down motion executed by the glass tube is SHM and find the time period of vibration. Given, $g = 980 \text{ cm s}^{-2}$. **(Ans.** 0.898 s)
3. A sphere is hung with a wire. 60° rotation of the sphere about the wire produces a restoring torque of 4.1 Nm. If the moment of inertia of the sphere is 0.082 kg m^2, find the frequency of angular oscillations. **(Ans.** 1.1 Hz)
4. A lactometer whose mass is 0.2 kg is floating vertically in a liquid of relative density 0.9. Area of cross-section of the marked portion of lactometer is $0.5 \times 10^{-4} \text{ m}^2$. If it is dipped down in the liquid slightly and released, what type of motion will it execute ? What will be its time-period ?

   (Ans. Motion is simple harmonic, 4.2 s)

HINTS

1. Here $R = 6.4 \times 10^5$ m, $g = 9.8 \text{ ms}^{-2}$

$$T = 2\pi\sqrt{\frac{R}{g}} = 2\pi\sqrt{\frac{6.4 \times 10^6}{9.8}} = 2\pi \times 808.1 = \mathbf{5077.6 \text{ s.}}$$

2. Here $h = 20$ cm, $g = 980 \text{ cm s}^{-2}$

$$T = 2\pi\sqrt{\frac{h}{g}} = 2\pi\sqrt{\frac{20}{980}} = 2\pi \times 0.143 = \mathbf{0.898 \text{ s.}}$$

3. Restoring torque, $\tau = 4.1$ Nm

Angular displacement, $\theta = 60° = \frac{\pi}{3}$ rad

Torsion constant, $C = \frac{\tau}{\theta} = \frac{4.1}{\pi/3} = \frac{4.1 \times 3}{\pi} \text{ Nm rad}^{-1}$

Moment of inertia, $I = 0.082 \text{ kg m}^2$

Frequency, $\nu = \frac{1}{2\pi}\sqrt{\frac{C}{I}} = \frac{1}{2\pi}\sqrt{\frac{3 \times 4.1}{\pi \times 0.082}} = \mathbf{1.1 \text{ Hz.}}$

4. When the lactometer is depressed through distance y,

F = upthrust of the liquid $= -A\,y\,\rho \times g = -A\,\rho\,g\,y$

As $F \propto y$, so motion of lactometer is SHM with

$k = A\,\rho\,y$

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{m}{A\rho g}}$$

$$= 2 \times 3.14\sqrt{\frac{0.2}{0.5 \times 10^{-4} \times 0.9 \times 10^3 \times 9.8}} = \mathbf{4.2 \text{ s.}}$$

14.18 FREE, DAMPED AND MAINTAINED OSCILLATIONS

25. *What are free, damped and maintained oscillations ? Give examples.*

(*a*) **Free oscillations.** *If a body, capable of oscillation, is slightly displaced from its position of equilibrium and left to itself, it starts oscillating with a frequency of its own. Such oscillations are called free oscillations.* The frequency with which a body oscillates freely is called natural frequency and is given by

$$\nu_0 = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

Some important features of free oscillations are

(*i*) In the absence of dissipative forces, such a body vibrates with a constant amplitude and fixed frequency, as shown in Fig. 14.35. Such oscillations are also called **undamped oscillations**.

(*ii*) The amplitude of oscillation depends on the energy supplied initially to the oscillator.

(*iii*) The natural frequency of an oscillator depends on its mass, dimensions and restoring force *i.e.*, on its inertial and elastic properties (m and k).

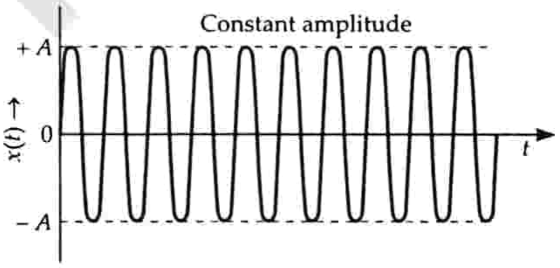


Fig. 14.35 Free or undamped oscillations.

Examples. (*i*) The vibrations of the prongs of tunning fork struck against a rubber pad.

(*ii*) The vibrations of the string of a sitar when pulled aside and released.

(*iii*) The oscillations of the bob of a pendulum when displaced from its mean position and released.

(*b*) **Damped oscillations.** *The oscillations in which the amplitude decreases gradually with the passage of time are called damped oscillations.*

In actual practice, most of the oscillations occur in viscous media, such as air, water, etc. A part of the energy of the oscillating system is lost in the form of heat, in overcoming these resistive forces. As a result, the amplitude of such oscillations decreases exponentially with time, as shown in Fig. 14.36. Eventually, these oscillations die out.

In an oscillatory motion, friction produces three effects :

(*i*) It changes the simple harmonic motion into periodic motion.

(*ii*) It decreases the amplitude of oscillation.

(*iii*) It slightly reduces the frequency of oscillation.

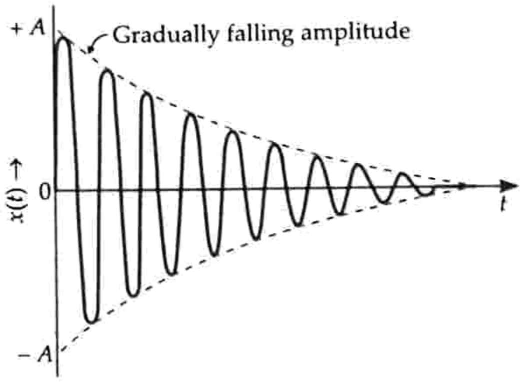


Fig. 14.36 Damped oscillations.

Examples. (*i*) As shown in Fig. 14.37, consider a block of mass m that oscillates vertically on a spring with spring constant k. The block is connected to a vane through a rod. The vane is submerged in a liquid. As the block oscillates up and down, the vane also oscillates in a similar manner inside the liquid. The liquid exerts an opposing force of viscosity on the vane. The energy of the oscillating system is lost in the liquid as heat. The amplitude of oscillation decreases continuously with time.

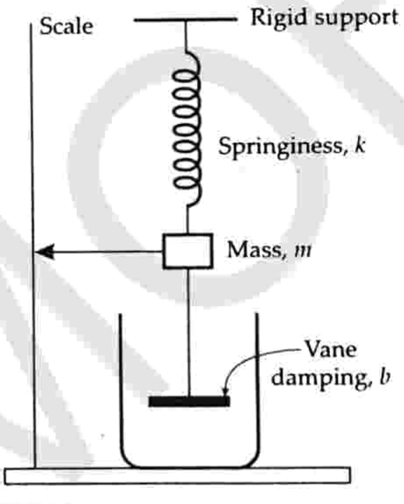


Fig. 14.37 A damped simple harmonic oscillator.

(*ii*) The oscillations of a swing in air.

(*iii*) The oscillations of the bob of a pendulum in a fluid.

(*c*) **Maintained oscillations.** *If to an oscillating system, energy is continuously supplied from outside at the same rate at which the energy is lost by it, then its amplitude can be maintained constant. Such oscillations are called maintained oscillations.* Here, the system oscillates with its own natural frequency.

Examples. (*i*) The oscillations of the balance wheel of a watch in which the main spring provides the required energy.

(*ii*) An electrically maintained tuning fork.

(*iii*) A child's swing in which energy is continuously fed to maintain constant amplitude.

For Your Knowledge

▲ **Differential equation for damped oscillators and its Solution.** In a real oscillator, the damping force is proportional to the velocity v of the oscillator.

$$F_d = -bv$$

where b is damping constant which depends on the characteristics of the fluid and the body that oscillates in it. The negative sign indicates that the damping force opposes the motion.

∴ Total restoring force $= -kx - bv$

or $$m\frac{d^2x}{dt^2} = -kx - b\frac{dx}{dt} \qquad \left[v = \frac{dx}{dt}\right]$$

or $$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$$

This is the differential equation for damped S.H.M. The solution of the equation is

$$x(t) = a\,e^{-bt/2m}\cos(\omega' t + \phi)$$

The amplitude of the damped S.H.M. is

$$a' = ae^{-bt/2m}$$

where a is amplitude of undamped S.H.M. Clearly, a' decreases exponentially with time.

The angular frequency of the damped oscillator is

$$\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$

Time period, $$T' = \frac{2\pi}{\omega'} = \frac{2\pi}{\sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}}$$

Clearly, damping increases the time period (due to the presence of the term $b^2/4\,m^2$ in the denominator).

The mechanical energy of the damped oscillator at any instant t will be

$$E(t) = \frac{1}{2}ka'^2 = \frac{1}{2}ka^2 e^{-bt/m}$$

Obviously, the total energy decreases exponentially with time.

As damping constant, $b = F/v$

∴ SI unit of $$b = \frac{\text{N}}{\text{ms}^{-1}} = \frac{\text{kg ms}^{-2}}{\text{ms}^{-1}} = \text{kg s}^{-1}$$

CGS unit of $b = \text{g s}^{-1}$.

EXAMPLE 53. *For the damped oscillator shown in earlier Fig. 14.37, the mass m of the block is 200 g, k = 90 Nm^{-1} and the damping constant b is 40 g s^{-1}. Calculate (a) the period of oscillation, (b) time taken for its amplitude of vibrations to drop to half of its initial value and (c) the time taken for its mechanical energy to drop to half its initial value.* **[NCERT]**

Solution. (*a*) Here $\sqrt{km} = \sqrt{90 \times 0.200} = 4.24 \text{ kg s}^{-1}$

Damping constant, $b = 40 \text{ g s}^{-1}$

As the damping constant, $b << \sqrt{km}$, is small, so the time period T is given by

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{0.2 \text{ kg}}{90 \text{ Nm}^{-1}}} = \mathbf{0.3 \text{ s}}.$$

(*b*) The time, $T_{1/2}$ for the amplitude to drop to half of its initial value is given by

$$\frac{A}{2} = A\, e^{-(bT_{1/2})/2m}$$

$$\therefore \quad T_{1/2} = -\frac{\ln(1/2)}{b/2\,m} = \frac{0.693}{40} \times 2 \times 200 \text{ s} = \mathbf{6.93 \text{ s}}.$$

(*c*) The time, $t_{1/2}$ for the mechanical energy to drop to half its initial value is given by

$$E(t_{1/2}) = E(0)\, e^{-(bt_{1/2})/m}$$

or $$E(t_{1/2}) / E(0) = \exp(-bt_{1/2}/m)$$

or $$1/2 = \exp(-bt_{1/2}/m)$$

$$\ln(1/2) = -(bt_{1/2}/m)$$

or $$t_{1/2} = \frac{0.693}{40 \text{ g s}^{-1}} \times 200 \text{ g} = \mathbf{3.4 \text{ s}}.$$

14.19 FORCED AND RESONANT OSCILLATIONS

26. *Distinguish between forced and resonant oscillations. Give an experimental illustration in support of your answer. Give examples.*

Forced oscillations. *When a body oscillates under the influence of an external periodic force, not with its own natural frequency but with the frequency of the external periodic force, its oscillations are said to be forced oscillations.* The external agent which exerts the periodic force is called the *driver* and the oscillating system under consideration is called the *driven body.*

Examples. (*i*) When the stem of a vibrating tuning fork is pressed against a table, a loud sound is heard. This is because the particles of table are forced to vibrate with the frequency of the tuning fork.

(*ii*) When the free end of the string of a simple pendulum is held in hand and the pendulum is made to oscillate by giving jerks by the hand, the pendulum executes forced oscillations. Its frequency is same as that of the periodic force exerted by the hand.

(*iii*) The sound boards of all stringed musical instruments like sitar, violin, etc. execute forced oscillations and the frequency of oscillation is equal to the natural frequency of the vibrating string.

Suppose an external periodic force of frequency ν is applied to an oscillator of natural frequency ν_0. Initially, the body tries to vibrate with its own natural frequency, while the applied force tries to drive the body with its own frequency. But soon the free vibrations of the body die out and finally the body vibrate a with a constant amplitude and with the frequency of the driving force. In this steady state, the rate of loss of energy through friction equals the rate at which energy is fed to the oscillator by the driver.

Fig. 14.38 shows the variation of the amplitude of forced oscillations as the frequency of the driver varies from zero to a large value. Clearly, the amplitude of forced oscillations is very small for $\nu << \nu_0$ and $\nu >> \nu_0$. But when $\nu \simeq \nu_0$, the amplitude of the forced oscillations becomes very large. In this condition, the oscillator responds most favourably to the driving force and draws maximum energy from it. The case $\nu = \nu_0$ is called *resonance* and the oscillations are called *resonant oscillations.*

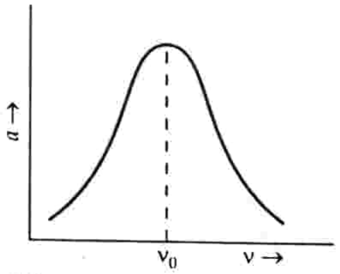


Fig. 14.38 Amplitude *a* of a forced oscillator as a function of the frequency ν of the driver.

Resonant oscillations and resonance. *It is a particular case of forced oscillations in which the frequency of the driving force is equal to the natural frequency of the oscillator itself and the amplitude of oscillations is very large. Such oscillations are called resonant oscillations and phenomenon is called resonance.*

Examples. (*i*) An aircraft passing near a building shatters its window panes, if the natural frequency of the window matches the frequency of the sound waves sent by the aircraft's engine.

(*ii*) The air-column in a reasonance tube produces a loud sound when its frequency matches the frequency of the tuning fork.

(*iii*) A glass tumbler or a piece of china-ware on shelf is set into resonant vibrations when some note is sung or played.

Experimental illustration. As shown in Fig. 14.39, suspend four pendulums *A*, *B*, *C* and *D* from an elastic string *PQ*. Set the pendulum *A* into oscillation. It executes free oscillations. The energy from this pendulum is transferred to other pendulums through the elastic string. Initially, the motions of *B*, *C* and *D* are irregular. But soon all these pendulums start oscillating with the frequency of *A*. The oscillations of *B*, *C* and *D* are forced oscillations. But pendulums *B* and *D* have small amplitudes. This is because the frequency of *B* is much larger than that of *A* (due to shorter length) and the frequency of *D* is much smaller than that of *A* (due to larger length). The pendulum *C* which has same length as the pendulum *A* (and hence the same frequency) oscillates with largest amplitude. Hence the oscillations of *C* are resonant oscillations.

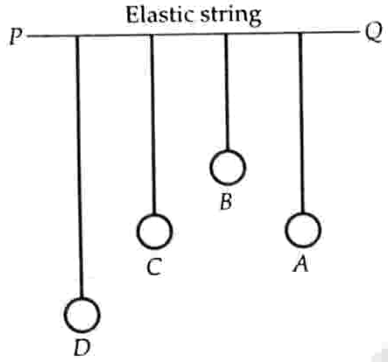


Fig. 14.39 Illustrating free, forced and resonant oscillations.

27. *Briefly explain the principle underlying the tuning of a radio receiver.*

Principle of tuning of a radio receiver. Tuning of the radio receiver is based on the principle of resonance. Waves from all stations are present around the antenna. When we tune our radio to a particular station, we produce a frequency of the radio circuit which matches with the frequency of that station. When this condition of resonance is achieved, the radio receives and responds selectively to the incoming waves from that station and thus gets tuned to that station.

14.20 COUPLED OSCILLATIONS

28. *What are coupled oscillations ? Give examples.*

Coupled oscillations. *A system of two or more oscillators linked together in such a way that there is mutual exchange of energy between them is called a coupled oscillator. The oscillations of such a system are called coupled oscillations.*

Examples. (*i*) Two masses attached to each other by three springs between two rigid supports. The middle spring provides the coupling between the driver and the driven system [Fig. 14.40(*a*)].

(*ii*) Two simple pendulums coupled by a spring [Fig. 14.40(*b*)].

(*iii*) Two *LC*-circuits placed close to each other. The circuits are linked by each other through the magnetic lines of force [Fig. 14.40(*c*)].

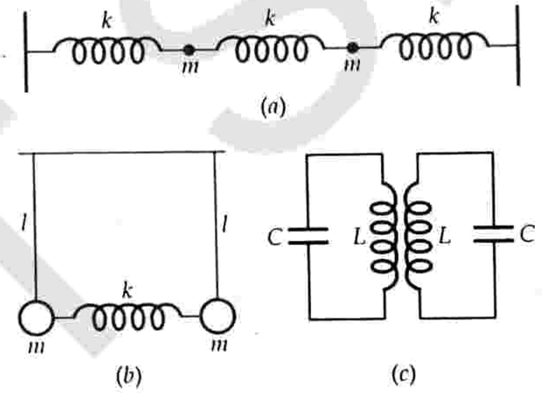


Fig. 14.40 Coupled oscillators.

When two identical oscillators are coupled together, the general motion of such a system is complex. It is periodic but not simple harmonic. It can be viewed as the superposition of two independent simple harmonic motions, called normal modes having angular frequencies ω_1 and ω_2. The constituent oscillators execute fast oscillations of average angular frequency, $\omega_{av} = (\omega_1 + \omega_2)/2$. The amplitude of either oscillator varies with an angular frequency $(\omega_1 - \omega_2)$. This phenomenon of variation of amplitudes is known as *beats* and the frequency $(\omega_1 - \omega_2)$ is called *beat frequency*.

Very Short Answer Conceptual Problems

Problem 1. Can a motion be periodic and not oscillatory ?

Solution. Yes. For example, uniform circular motion is periodic but not oscillatory.

Problem 2. Can a motion be oscillatory but not simple harmonic ? If your answer is yes, give an example and if not, explain why.

Solution. Yes; when a ball is dropped from a height on a perfectly elastic surface, the motion is oscillatory but not simple harmonic as restoring force $F = mg$ = constant and not $F \propto - x$, which is an essential condition for S.H.M.

Problem 3. Every simple harmonic motion is periodic motion, but every periodic motion need not be simple harmonic motion. Do you agree ? Give one example to justify your statement.

Solution. Yes, every periodic motion need not be simple harmonic motion. For example, the motion of the earth round the sun is a periodic motion, but not simple harmonic motion as the back and forth motion is not taking place.

Problem 4. The rotation of the earth about its axis is periodic but not simple harmonic. Justify.

Solution. The earth takes 24 hours to complete its rotation about its axis, but the concept of to and fro motion is absent, and hence the rotation of the earth is periodic and not simple harmonic.

Problem 5. What is the basic condition for the motion of a particle to be S.H.M.? *[Delhi 02]*

Solution. The motion of a particle will necessarily be simple harmonic if the restoring force acting on it is proportional to its displacement from the mean position *i.e.*, $F = -kx$.

Problem 6. Which of the following conditions is not sufficient for simple harmonic motion and why ?

(*i*) acceleration $\propto$ displacement,

(*ii*) restoring force $\propto$ displacement.

Solution. Condition (*i*) is not sufficient because it does not mention the direction of acceleration. In S.H.M. the acceleration is always in a direction opposite to that of the displacement.

Problem 7. Are the functions $\tan \omega t$ and $\cot \omega t$ periodic ? Are they harmonic ?

Solution. Both $\tan \omega t$ and $\cot \omega t$ are periodic functions each with period $T = \pi / \omega$, because

$$\tan\left[\omega\left(t + \frac{\pi}{\omega}\right)\right] = \tan(\omega t + \pi) = \tan \omega t$$

and $$\cot\left[\omega\left(t + \frac{\pi}{\omega}\right)\right] = \cot(\omega t + \pi) = \cot \omega t$$

But these functions are not harmonic because they can take any value between 0 and ∞.

Problem 8. What provides the restoring force for simple harmonic oscillations in the following cases :

(*i*) Simple pendulum (*ii*) Spring

(*iii*) Column of Hg in U-tube ?

Solution. (*i*) Gravity (*ii*) Elasticity (*iii*) Weight of difference column.

Problem 9. When are the displacement and velocity in the same direction in S.H.M. ?

Solution. When a particle moves from mean position to extreme position, its displacement and velocity are in the same direction.

Problem 10. When are the velocity and acceleration in the same direction in S.H.M. ?

Solution. When a particle moves from extreme position to mean position, its velocity and acceleration are in the same direction.

Problem 11. Can displacement and acceleration be in the same direction in S.H.M. ?

Solution. No. In S.H.M., acceleration is always in the opposite direction of displacement.

Problem 12. The relation between the acceleration a and displacement x of a particle executing S.H.M. is $a = -(p/q)\,y$, where p and q are constants. What will be the time period T of the particle ?

Solution. Here $a = -\frac{p}{q} y = -\omega^2 y$, where $\omega = \sqrt{\frac{p}{q}}$

$\therefore$ Time period, $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{q}{p}}$.

Problem 13. The maximum acceleration of a simple harmonic oscillator is a_0 and the maximum velocity is v_0. What is the displacement amplitude ? *[Delhi 99]*

Solution. Let A be the displacement amplitude and ω be the angular frequency of S.H.M. Then

Maximum velocity, $v_0 = \omega A \quad \therefore \quad \omega = v_0 / A$

Maximum acceleration, $a_0 = \omega^2 A = \left(\frac{v_0}{A}\right)^2 A = \frac{v_0^2}{A}$

$\therefore$ Displacement amplitude, $A = \frac{v_0^2}{a_0}$.

Problem 14. The time period of an oscillating body is given by $T = 2p\sqrt{m/adg}$. What would be the force equation for the body ?

Solution. On comparing the given equation $T = 2\pi\sqrt{m/adg}$ with the standard equation $T = 2\pi\sqrt{m/k}$, we get $k = adg$, which gives the force equation $F = -adg\,(y)$.

Problem 15. Two simple pendulums of unequal length meet each other at mean position while oscillating. What is their phase difference ?

Solution. If both pendulums are moving in the same direction, then $\phi = 0°$ and if they are moving in opposite directions, then $\phi = 180°$ or π radian.

Problem 16. Velocity and displacement of a body executing S.H.M. are out of phase by $\pi/2$. How ?

Solution. Displacement, $x = a \cos \omega t$

Velocity, $v = \frac{dx}{dt} = -\omega a \sin \omega t = \omega a \cos(\omega t + \pi/2)$

Clearly, velocity leads the displacement by $\pi/2$ rad.

Problem 17. A particle executes S.H.M. of amplitude A. At what positions of its displacement (x), will its (*i*) velocity be zero and maximum and (*ii*) acceleration be zero and maximum ?

Solution. (*i*) Zero velocity at $x = \pm A$, maximum velocity at $x = 0$.

(*ii*) Zero acceleration at $x = 0$, maximum acceleration at $x = A$.

Problem 18. At what points along the path of a simple pendulum is the tension in the string (*i*) maximum and (*ii*) minimum ?

Solution. (*i*) The tension is maximum at the mean position and is equal to mg, where m is the mass of the bob.

(*ii*) The tension is minimum at either extreme position and is equal to $mg \cos \theta$, where θ is the angle through which the string gets displaced to reach the extreme position.

Problem 19. Is the statement "the bob of a simple pendulum moves faster at the lowest position for larger amplitude" true ? Justify your answer.

Solution. We know that velocity of a simple pendulum is maximum at the lowest position (mean position) and is given by

$$v_{\max} = \omega A.$$

i.e. for larger amplitude (A), the bob of simple pendulum would move faster.

Problem 20. Can we use a pendulum watch in an artificial satellite ?

Solution. No. In an artificial satellite, a body is in a state of weightlessness, *i.e.* $g = 0$.

$$\therefore \qquad T = 2\pi \sqrt{\frac{l}{g}} = \infty$$

Inside the satellite, the pendulum does not oscillate. Hence a pendulum watch cannot be used in an artificial satellite.

Problem 21. A girl is swinging in the sitting position. How will the period of the swing change if she stands up ? *[AIEEE 02 ; Central Schools 09]*

Solution. The girl and the swing together constitute a pendulum of time period,

$$T = 2\pi \sqrt{\frac{l}{g}}$$

As the girl stands up, her C.G. is raised. The distance between the point of suspension and the C.G. decreases *i.e.*, length l decreases. Hence the time period T decreases.

Problem 22. Will a pendulum clock lose or gain time when taken to the top of a mountain ? *[Himachal 04]*

Solution. On the top of the mountain, the value of g is less than that on the surface of the earth. The decrease in the value of g increases the time period of the pendulum on the top of the mountain. So the pendulum clock loses time.

Problem 23. What will be the period of oscillation, if the length of a second's pendulum is halved ?

Solution. $\frac{l_1}{l_2} = \frac{T_1^2}{T_2^2}$ or $\frac{l}{l/2} = \frac{2 \times 2}{T_2^2}$

or $\qquad T_2^2 = 2 \quad$ or $\quad T_2 = \sqrt{2}$ s.

Problem 24. The length of a second's pendulum on the surface of earth is 1 m. What will be the length of a second's pendulum on the surface of moon ?

Solution. $T = 2\pi \sqrt{\frac{l}{g}}$

In both the cases, T is same so that

$$l \propto g$$

On the moon, the value of acceleration due to gravity is one-sixth of that on the surface of earth. So the length of second's pendulum is $\frac{1}{6}$ m.

Problem 25. The bob of a simple pendulum is made of wood. What will be the effect on the time period if the wooden bob is replaced by an identical bob of iron ?

Solution. There will be no effect because the time period does not depend upon the nature of material of the bob.

Problem 26. If a hollow pipe passes across the centre of gravity of the earth, then what changes would take place in the velocity and acceleration of a ball dropped in the pipe ?

Solution. The ball will execute S.H.M. to and fro about the centre of the earth. At the centre, the velocity of the ball will be maximum (acceleration zero) and at the earth's surface the velocity will be zero (acceleration maximum).

Problem 27. The bob of a simple pendulum of length l is negatively charged. A positively-charged metal plate is placed just below the bob and the pendulum is made to oscillate. What will be the effect on the time-period of the pendulum ?

Solution. The positively charged metal plate attracts the negatively charged bob. This increases the effective value of g. Hence the time period will decrease.

Problem 28. A simple pendulum of length l and with a bob of mass m is moving along a circular arc of angle θ in a vertical plane. A sphere of mass m is placed at the end of the circle. What momentum will be given to the sphere by the moving bob ?

Solution. Zero. This is because the velocity of the bob at the end of the arc will be zero.

Problem 29. A body moves along a straight line OAB simple harmonically. It has at zero velocity at the points A and B which are at distances a and b respectively from O and has velocity v when half way between them. Find the period of S.H.M.

Solution. Clearly, C is the mean position of S.H.M., as shown in Fig. 14.41

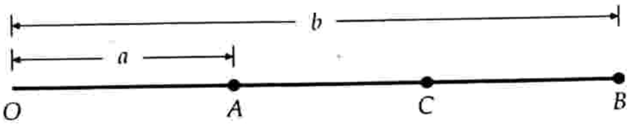


Fig. 14.41

The amplitude of S.H.M. is

$$A = AC = CB = \frac{AB}{2} = \frac{b-a}{2}$$

The velocity at the mean position C will be

$$v = \omega A = \frac{2\pi}{T} \cdot \frac{b-a}{2}$$

$$\therefore \qquad T = \frac{\pi(b-a)}{v}.$$

Problem 30. When a 2.0 kg body is suspended by a spring, the spring is stretched. If the body is pulled down slightly and released, it oscillates up and down. What force is applied on the body by the spring when it passes through the mean position? (g = 9.8 newton/kg).

Solution. There is no acceleration in the body at the mean position, hence the resultant force applied by the spring will be exactly equal to the weight of the body *i.e.,* 2×9.8 or 19.6 newton.

Problem 31. A spring having a force constant k is divided into three equal parts. What would be the force constant for each individual part ?

Solution. Force constant of the spring $k = \frac{F}{x}$, where F is the restoring force. When the spring is divided into three parts, the displacement for the same force reduces to $x / 3$, therefore, the force constant for each individual part is

$$k' = \frac{F}{x/3} = 3\left(\frac{F}{x}\right) = 3k$$

Problem 32. How would the time period of a spring mass system change, when it is made to oscillate horizontally, and then vertically ? *[Himachal 04]*

Solution. Time period will remain the same for both the cases.

Problem 33. Alcohol in a U-tube executes S.H.M. of time period T. Now, alcohol is replaced by water up to the same height in the U-tube. What will be the effect on the time period ?

Solution. The time period T remains same. This is because the period of oscillation of a liquid in a U-tube does not depend on the density of the liquid.

Problem 34. There are two springs, one delicate and another stiffer one. Which spring will have a greater frequency of oscillation for a given load ?

Solution. Frequency, $\nu = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$

Force constant k is larger for the stiffer spring, so its frequency of oscillation will be greater than that of delicate spring.

Problem 35. What is the ratio between the potential energy and the total energy of a particle executing S.H.M., when its displacement is half of its amplitude ?

Solution. $\frac{\text{Potential energy}}{\text{Total energy}} = \frac{\frac{1}{2} m\omega^2 y^2}{\frac{1}{2} m\omega^2 a^2}$

$$= \frac{y^2}{a^2} = \frac{(a/2)^2}{a^2} = \frac{1}{4} = \mathbf{1:4}$$

Problem 36. What fraction of the total energy is kinetic when the displacement of a simple harmonic oscillator is half of its amplitude ?

Solution. $\frac{\text{Kinetic energy}}{\text{Total energy}} = \frac{\frac{1}{2} m\omega^2 \left(A^2 - \frac{A^2}{4}\right)}{\frac{1}{2} m\omega^2 A^2} = \frac{3}{4}.$

Problem 37. Why is restoring force necessary for a body to execute S.H.M. ?

Solution. A body in S.H.M. oscillates about its mean position. At the mean position, it possesses kinetic energy because of which it moves from mean position to extreme position. Then the body can return to the mean position only if it is acted upon by a restoring force.

Problem 38. What would happen to the motion of the oscillating system if the sign of the force term in the equation $F = -kx$ is changed ?

Solution. The force will not be the restoring nature. The back and forth nature of the motion is lost. The body will continue to move in a particular sense.

Problem 39. What determines the natural frequency of a body ?

Solution. Natural frequency of a body depends upon (*i*) elastic properties of the material of the body and (*ii*) dimensions of the body.

Problem 40. Why does the amplitude of an oscillating pendulum go on decreasing ?

Solution. Due to frictional resistance between air and bob, the amplitude of oscillations of the pendulum gradually decreases and finally the bob stops.

Problem 41. Why are army troops not allowed to march in steps while crossing a bridge ? *[Himachal 05]*

Solution. Army troops are not allowed to march in steps while crossing a bridge because it is quite likely that the frequency of the foot steps may match with the natural frequency of the bridge, and due to resonance the bridge may pick up large amplitude and break.

Problem 42. A passing aeroplane sometimes causes the rattling of the windows of a house. Why ?

Solution. When the frequency of the sound waves from the engine of an aeroplane matches with the natural frequency of a window, resonance takes place which causes the rattling of window.

Problem 43. How can earthquakes cause disaster sometimes ? *[Himachal 05C]*

Solution. The resonance may cause disaster during the earthquake, if the frequency of oscillations present within the earth per chance coincides with the natural frequency of some building, which may start vibrating with large amplitude due to resonance and may get damaged.

Problem 44. Sometimes a wine glass is broken by the powerful voice of a celebrated singer. Why ?

Solution. When the natural frequency of the wine glass becomes equal to that of the singer's voice, the resulting resonance due to the powerful voice of the singer may break the glass.

Problem 45. Glass windows may be broken by a far away explosion. Explain why.

[Himachal 05 ; Central Schools 08]

Solution. A distant explosion sends out sound waves of large amplitude in all directions. As these sound waves strike the glass windows, they set them into forced oscillations. Since glass is brittle, so the glass windows break as soon as they start oscillating due to forced oscillations.

Problem 46. The body of a bus begins to rattle sometimes, when the bus picks up a certain speed. Why ? *[Himachal 05]*

Solution. At a particular speed, the frequency of the engine of the bus becomes equal to the natural frequency of the body of the bus. The frame of the bus begins to vibrate strongly due to resonance.

Problem 47. What will be the change in time period of a loaded spring, when taken to moon ? *[Himachal 03]*

Solution. Time period of a loaded spring,

$$T = 2\pi\sqrt{\frac{m}{k}}$$

As T is independent of g, it will not be affected when the loaded spring is taken to the moon.

Problem 48. A spring of force constant k is cut into two pieces, such that one piece is double the length of the other. What is the force constant of the longer piece of the spring ? *[IIT 99]*

Solution. Force constant,

$$k = \frac{F}{x}$$

The length of longer part is $2x/3$. So its force constant is

$$k' = \frac{F}{2x/3} = \frac{3}{2}\frac{F}{x} = \frac{3}{2}\boldsymbol{k}.$$

Problem 49. In forced oscillation of a particle, the amplitude is maximum for a frequency ω_1 of the force, while the energy is maximum for a frequency ω_2 of the force. What is relation between ω_1 and ω_2 ? *[AIEEE 04]*

Solution. Only in case of resonance, both amplitude and energy of oscillation are maximum. In the condition of resonance,

$$\omega_1 = \omega_2.$$

Problem 50. The maximum velocity of a particle, executing simple harmonic motion with an amplitude of 7 mm, is 4.4 m s^{-1}. What is the period of oscillation ?

[AIEEE 06]

Solution. $v_{\max} = \omega A = \dfrac{2\pi}{T} A$

$$\therefore \quad T = \frac{2\pi A}{v_{\max}}$$

$$= \frac{2 \times 22 \times 7 \times 10^{-3}}{7 \times 4.4} = \mathbf{0.01\ s.}$$

Short Answer Conceptual Problems

Problem 1. Justify the following statements :

(*i*) The motion of an artificial satellite around the earth cannot be taken as S.H.M.

(*ii*) The time period of a simple pendulum will get doubled if its length is increased four times.

[Himachal 06]

Solution. (*i*) The motion of an artificial satellite around the earth is periodic as it repeats after a regular interval of time. But it cannot be taken as S.H.M. because it is not a to-and-fro motion about any mean position.

(*ii*) Time period of simple pendulum,

$$T = 2\pi\sqrt{\frac{l}{g}} \quad \text{i.e.,} \quad T \propto \sqrt{l}.$$

Clearly, if the length is increased four times, the time period gets doubled.

Problem 2. (*i*) What is meant by simple harmonic motion (S.H.M.) ?

(*ii*) At what points is the energy entirely kinetic and potential in S.H.M. ?

(*iii*) What is the total distance travelled by a body executing S.H.M. in a time equal to its time period, if its amplitude is A ? *[Delhi 09]*

Solution. (*i*) Refer to point 5 of Glimpses.

(*ii*) The energy is entirely kinetic at mean position *i.e.*, at $y = 0$. The energy is entirely potential at extreme positions, *i.e.*,

$$y = \pm A.$$

(*iii*) Total distance travelled in time period T

$$= 2A + 2A = \mathbf{4A}.$$

Problem 3. A simple pendulum consisting of an inextensible length l and mass m is oscillating in a stationary lift. The lift then accelerates upwards with a constant acceleration of 4.5 m/s^2 . Write expression for the time period of simple pendulum in two cases. Does the time period increase, decrease or remain the same, when lift is accelerated upwards ? *[Central Schools 08]*

Solution. When the lift is stationary,

$$T = 2\pi\sqrt{\frac{l}{g}}$$

(*ii*) When the lift accelerates upwards with an acceleration of 4.5 m / s^2,

$$T' = 2\pi\sqrt{\frac{l}{g+a}} = 2\pi\sqrt{\frac{l}{g+4.5}}$$

Clearly, the time period decreases when the lift accelerates upwards.

Problem 4. What is meant by restoring force ? Give one example.

Solution. The force which tends to bring a vibrating body from its displaced position to the equilibrium position is called restoring force. When the bob of a simple pendulum is displaced through an θ from the vertical, a restoring force equal to $mg \sin\theta$ due to gravity acts on it.

Problem 5. Two particles execute simple harmonic motions of the same amplitude and frequency along the same straight line. They cross one another when going in opposite directions. What is the phase difference between them when their displacements are half of their amplitudes ?

Solution. The general equation for S.H.M. is

$$y = A\sin(\omega t + \phi_0)$$

As the displacement is half of the amplitude ($y = A/2$), so

$$A/2 = A\sin(\omega t + \phi_0)$$

or $$\sin(\omega t + \phi_0) = \frac{1}{2}$$

$$\therefore \quad \omega t + \phi_0 = 30° \text{ or } 150°.$$

As the two particles are going in opposite directions, the phase of one is 30° and that of the other 150°.

Hence the phase difference between the two particles = 150 – 30 = **120°**.

Problem 6. A simple pendulum is hung in a stationary lift and its periodic time is T. What will be the effect on its periodic time T if

(*i*) the lift goes up with uniform velocity v,

(*ii*) the lift goes up with uniform acceleration a, and

(*iii*) the lift comes down with uniform acceleration a ?

Solution. (*i*) When the lift goes up [Fig. 14.42(*a*)] with uniform velocity v, tension in the string, $T = mg$.

The value of g remains unaffected.

The period T remains same as that in stationary lift, *i.e.*,

$$T = 2\pi\sqrt{\frac{l}{g}}$$

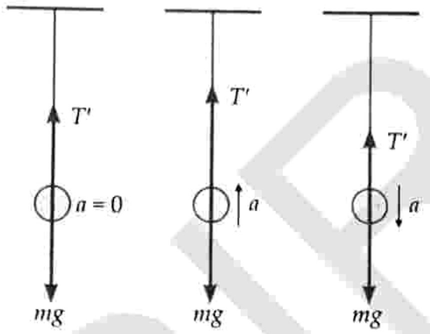


Fig. 14.42

(*ii*) When the lift goes up with acceleration a [Fig. 14.42(*b*)], the net upward force on the bob is

$$T' - mg = ma$$

$$\therefore \quad T' = m(g + a)$$

The effective value of g is $(g + a)$ and time period is

$$T_1 = 2\pi\sqrt{\frac{l}{g+a}}$$

Clearly, $T_1 < T$ *i.e.*, time period decreases.

(*iii*) When lift comes down with acceleration a [Fig. 14.42(*c*)], the net downward force on the bob is

$$mg - T' = ma \quad \therefore \quad T' = m(g - a)$$

The effective value of g becomes $(g - a)$ and time period is

$$T_2 = 2\pi\sqrt{\frac{l}{g-a}}$$

Clearly, $T_2 > T$ *i.e.*, time period increases.

Problem 7. The bob of a vibrating pendulum is made of ice. How will the time period change when the ice starts melting ?

Solution. If the ice bob is of very small size, the position of its C.G. will remain same as the ice melts. Hence its time period will remain same.

If the size of the ice bob is large, then

$$T = 2\pi\sqrt{\frac{\frac{2r^2}{5l} + l}{g}}$$

As ice melts, the radius r and hence the time period T will decrease. The pendulum will oscillate faster.

Problem 8. The amplitude of a simple harmonic oscillator is doubled. How does this affect (*i*) periodic time, (*ii*) maximum velocity, (*iii*) maximum acceleration and (*iv*) maximum energy ? *[Chandigarh 03]*

Solution.

(*i*) $T = 2\pi\sqrt{\dfrac{1}{\text{Acceleration per unit displacement}}}$

As the acceleration per unit displacement is a constant quantity, T is not affected on changing the amplitude.

(*ii*) $v_{\max} = \omega A$

When amplitude is doubled, maximum velocity is also doubled.

(*iii*) $a_{\max} = \omega^2 A$

When amplitude is doubled, the maximum acceleration is also doubled.

(*iv*) $E = 2\pi^2\, m v^2\, A^2$ *i.e.,* $E \propto A^2$

When amplitude is doubled, the energy of the oscillator becomes four times.

Problem 9. You have a light spring, a metre scale and a known mass. How will you find the time period of oscillation of mass without the use of a clock ?

Solution. Suspend the known mass m form the spring and note the extension l of the spring with the metre scale. If k is the force constant of the spring, then in equilibrium

$$kl = mg \quad \text{or} \quad \frac{m}{k} = \frac{l}{g}$$

Time period of the loaded spring, $T = 2\pi\sqrt{\dfrac{m}{k}} = 2\pi\sqrt{\dfrac{l}{g}}$

So by knowing the value of extension l, time period T can be determined.

Problem 10. A man is standing on a platform which oscillates up and down simple harmonically. How will the weight of the man change as recorded by a weighing machine on the platform ?

Solution. As the platform moves from the mean position to the upper extreme position or from upper extreme position to mean position, the acceleration of the oscillating system acts vertically downwards and hence weight of the man will decrease.

On the other hand, as the platform moves from mean position to lower extreme position and then back to mean position, the acceleration acts vertically upwards. Hence weight of the man increases.

Problem 11. The frequency of oscillations of a mass *m* suspended by a spring is ν_1. If the length of the spring is cut to one-half, the same mass oscillates with frequency ν_2. Determine the value of ν_2 / ν_1.

[Chandigarh 03]

Solution. Let k be the force constant of the full spring. Then frequency of oscillation of mass m will be

$$\nu_1 = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

When the spring is cut to one-half of its length, its force constant is doubled ($2k$). Frequency of oscillation of mass m will be

$$\nu_2 = \frac{1}{2\pi}\sqrt{\frac{2k}{m}}$$

$$\therefore \quad \nu_2 / \nu_1 = \sqrt{2}.$$

Problem 12. All trigonometric functions are periodic, but only sine or cosine functions are used to define S.H.M. Why ? *[Central Schools 03]*

Solution. All trigonometric functions are periodic. The sine and cosine functions can take value between − 1 and + 1 only. So they can be used to represent a bounded motion like S.H.M. But the functions such as tangent, cotangent, secant and cosecant can take value between 0 and ∞ (both positive and negative). So those functions cannot be used to represent bounded motion like S.H.M.

Problem 13. A simple harmonic motion is represented by $\dfrac{d^2x}{dt^2} + \alpha x = 0$. What is its time period ? *[AIEEE 05]*

Solution. Clearly, $\dfrac{d^2x}{dt^2} = -\alpha x$ or $a = -\alpha x$

Time period, $T = 2\pi\sqrt{\dfrac{x}{a}} = 2\pi\sqrt{\dfrac{x}{\alpha x}} = \dfrac{2\pi}{\sqrt{\alpha}}.$

Problem 14. Does the function $y = \sin^2 \omega t$ represent a periodic or a simple harmonic motion ? What is the period of the motion ? *[AIEEE 05]*

Solution. Displacement, $y = \sin^2 \omega t$

Velocity, $v = \dfrac{dy}{dt} = 2\sin\omega t \times \cos\omega t \times \omega$

$= \omega \sin 2\omega t$

Acceleration, $a = \dfrac{dv}{dt} = \omega \times \cos 2\omega t \times 2\omega$

$= 2\omega^2 \cos 2\omega t$

As the acceleration a is not proportional to displacement y, the given function does not represent SHM. It represents a periodic motion of angular frequency 2ω.

$\therefore$ Time period,

$$T = \frac{2\pi}{\text{Angular frequency}} = \frac{2\pi}{2\omega} = \frac{\pi}{\omega}.$$

Problem 15. The length of a simple pendulum executing SHM is increased by 21%. What is the percentage increase in the time period of the pendulum of increased length. *[AIEEE 03]*

Solution. Time period,

$$T = 2\pi\sqrt{\frac{l}{g}} \quad \textit{i.e.,} \quad T \propto l^{1/2}$$

The percentage increase in time period is given by

$$\frac{\Delta T}{T} \times 100 = \frac{1}{2}\frac{\Delta l}{l} \times 100$$

$$= \frac{1}{2} \times 21\% = \mathbf{10.5\%}$$

HOTS

Problems on Higher Order Thinking Skills

Problem 1. *Two simple harmonic motions are represented by the equations :*

$$x_1 = 5\sin(2\pi t + \pi/4),\quad x_2 = 5\sqrt{2}\,(\sin 2\pi t + \cos 2\pi t)$$

What is the ratio of their amplitudes ? **[Roorkee 96]**

Solution. $x_1 = 5\sin(2\pi t + \pi/4) \quad \therefore \quad A_1 = 5$

$$x_2 = 5\sqrt{2}\,(\sin 2\pi t + \cos 2\pi t)$$

$$= 10\sin(\sin 2\pi t\cos\pi/4 + \cos 2\pi t\sin\pi/4)$$

or $\quad x_2 = 10\sin(2\pi t + \pi/4)$

$\therefore \quad A_2 = 10$

Hence $\dfrac{A_1}{A_2} = \dfrac{5}{10} = \mathbf{1:2.}$

Problem 2. *The bob of a simple pendulum is a hollow sphere filled with water. How will the period of oscillation change if the water begins to drain out of the hollow sphere from a fine hole at its bottom ?*

Or

The bob of a simple pendulum is a spherical hollow ball filled with water. A plugged hole near the bottom of the oscilalting body gets suddenly unplugged. How would the time period of oscillation of the pendulum change, till water is coming out ? **[AIEEE 05]**

Solution. Time period, $T = 2\pi\sqrt{\dfrac{l}{g}}$

As water flows out of the sphere, the time period first increases and then decreases. Initially when the sphere is completely filled with water, its C.G. lies at its centre. As water flows out, the C.G. begins to shift below the centre of the sphere. The effective length of the pendulum increases and hence its time period increases.

When the sphere becomes more than half empty, its C.G. begins to rise up. The effective length of the pendulum increases and time period T decreases.

When the entire water is drained out of the sphere, the C.G. is once again shifted to centre of the sphere and the time period T attains its initial value.

Problem 3. *The period of vibration of a mass m suspended by a spring is T. The spring is cut into n equal parts and the body is again suspended by one of the pieces. Find the time period of oscillation of the mass.* **[AIEEE 02]**

Solution. The force constant is inversely proportional to the length. If k is the force constant of the original spring, then the force constant of each part will be nk.

$$\therefore \quad T = 2\pi\sqrt{\frac{m}{k}} \quad \text{and} \quad T' = 2\pi\sqrt{\frac{m}{nk}}$$

Hence $T' = \dfrac{T}{\sqrt{n}}$.

Problem 4. *Two simple harmonic motions are represented by the equations :*

$$y_1 = 0.1\sin(100\pi t + \pi/3) \text{ and } y_2 = 0.1\cos\pi t$$

What is the phase difference of the velocity of the particle 1 with respect to the velocity of particle 2 ? **[AIEEE 05]**

Solution. Velocity of particle 1,

$$v_1 = \frac{dy_1}{dt} = 0.1\cos(100\pi t + \pi/3)\times 100\pi$$

$$= 10\pi\cos(100\pi t + \pi/3)$$

Velocity of particle 2,

$$v_2 = \frac{dy_2}{dt} = 0.1(-\sin\pi t)\times\pi = -0.1\pi\sin\pi t$$

$$= 0.1\cos(\pi t + \pi/2)$$

Phase difference of the velocity of particle 1 with respect to the velocity of particle 2 is

$$\Delta\phi = \phi_1 - \phi_2 = \frac{\pi}{3} - \frac{\pi}{2} = -\frac{\pi}{6}.$$

Problem 5. *A particle of mass m is attached to a spring (of spring constant k) and has a natural angular frequency ω_0. An external force,*

$$f(t) \propto \cos\omega t \qquad (\omega \neq \omega_0)$$

is applied to the oscillator. How does the time displacement of oscillator vary ? **[AIEEE 04]**

Solution. With natural angular frequency ω_0, the acceleration of the particle at displacement y is

$$a_0 = -\omega_0^2\,y$$

The external force $F(t) \propto \cos\omega t$ has an angular frequency ω. The acceleration produced by this force at displacement y is

$$a' = \omega^2 y$$

The net acceleration of the particle at displacement y is

$$a = a_0 + a' = -\omega_0^2\,y + \omega^2\,y = -(\omega_0^2 - \omega^2)\,y$$

The resultant force on the particle at displacement y is

$$F = ma = -m(\omega_0^2 - \omega^2)\,y \quad \text{or} \quad y = -\frac{F}{m(\omega_0^2 - \omega^2)}$$

Clearly, $\quad y \propto \dfrac{1}{m(\omega_0^2 - \omega^2)}$

Problem 6. *A simple pendulum has time period T_1. The point of suspension is now moved upward according to the relation $y = Kt^2$ $(K = 1\ ms^{-2})$, where y is the vertical displacement. The time period now becomes T_2. What is the ratio T_1^2/T_2^2 ? Given $g = 10\ ms^{-2}$* **[IIT 05]**

Solution. In first case,

$$T_1 = 2\pi\sqrt{\frac{l}{g}} \qquad ...(1)$$

In second case, displacement $y = Kt^2$

Upward velocity, $\quad v = \frac{dy}{dt} = 2\,Kt$

Upward acceleration, $a = 2\ K = 2 \times 1\text{ms}^{-2} = 2\ \text{ms}^{-2}$

$$\therefore \quad T_2 = 2\pi\sqrt{\frac{l}{g+a}} = 2\pi\sqrt{\frac{l}{g+2}} \qquad ...(2)$$

Hence $\quad \frac{T_1^2}{T_2^2} = \frac{4\pi^2 l}{g} \times \frac{g+2}{4\pi^2 l}$

$$= \frac{g+2}{g} = \frac{10+2}{10} = \frac{\mathbf{6}}{\mathbf{5}}.$$

Problem 7. *The bob of simple pendulum executes SHM in water with a period t, while the period of oscillation of the bob is t_0 in air. Neglecting frictional force of water and given that the density of the bob is $\frac{4000}{3}$ kg m^{-3}, find the relationship between t and t_0 ?* **[AIEEE 04]**

Solution. In air, $t_0 = 2\pi\sqrt{\frac{l}{g}}$

Let V be the volume of the bob. Then

Apparent weight of bob in water

= Weight of bob in air – Upthrust

or $\quad V\rho g' = V\rho g - V\sigma g$

or $\quad g' = \left(1 - \frac{\sigma}{\rho}\right) g$

Density of bob, $\quad \rho = \frac{4000}{3}$ kg m^{-3}

Density of water, $\sigma = 1000$ kg m^{-3}

$$\therefore \quad g' = \left(1 - \frac{1000 \times 3}{4000}\right) g = \frac{g}{4}$$

Time period of the pendulum in water,

$$t = 2\pi\sqrt{\frac{l}{g'}} = 2\pi\sqrt{\frac{l}{g/4}} = 2 \times 2\pi\sqrt{\frac{l}{g}} = \mathbf{2t_0}.$$

Problem 8. *A mass M is suspended from a spring of negligible mass. The spring is pulled a little and then released so that the mass executes SHM of time period T. If the mass is increased by m, the time period becomes 5T/3. What is the ratio m/M ?* **[AIEEE 03]**

Solution. With mass M, the time period of the spring is

$$T = 2\pi\sqrt{\frac{M}{k}}$$

With mass $M + m$, the time period becomes

$$\frac{5T}{3} = 2\pi\sqrt{\frac{M+m}{k}}$$

or $\quad \frac{5}{3} \times 2\pi\sqrt{\frac{M}{k}} = 2\pi\sqrt{\frac{M+m}{k}}$

or $\quad \frac{25}{9} M = M + m \quad$ or $\quad \frac{16}{9} M = m$

or $\quad \frac{m}{M} = \frac{\mathbf{16}}{\mathbf{9}}.$

Problem 9. *Two bodies M and N of equal masses are suspended from two separate massless springs of spring constants k_1 and k_2 respectively. If the two bodies oscillate vertically, such that their maximum velocities are equal, then find the ratio of the amplitude of M to that of N.* **[AIEEE 03]**

Solution. The maximum velocity of body in SHM is given by

$$v_{\max} = A\omega = A\sqrt{\frac{k}{m}} \qquad \left[\because \omega^2 = \frac{k}{m}\right]$$

Given $\quad v_{\max}(M) = v_{\max}(N)$

or $\quad A_1\sqrt{\frac{k_1}{m}} = A_2\sqrt{\frac{k_2}{m}} \qquad [m_M = m_N = m]$

or $\quad \frac{A_1}{A_2} = \sqrt{\frac{k_2}{k_1}}.$

Problem 10. *A particle at the end of a spring executes simple harmonic motion with a period t_1, while the corresponding period for another spring is t_2. What is the period of oscillation when the two springs are connected in series ?* **[AIEEE 04]**

Solution. If a force F applied to the series combination produces displacements t_1 and t_2 in the two springs, then

$$F = -k_1 x_1 = -k_2 x_2$$

$$\therefore \quad x_1 = -\frac{F}{k_1} \quad \text{and} \quad x_2 = -\frac{F}{k_2}$$

Total extension,

$$x = x_1 + x_2 = -F\left[\frac{1}{k_1} + \frac{1}{k_2}\right] = -F\left[\frac{k_1 + k_2}{k_1 k_2}\right]$$

or $\quad F = -\frac{k_1 k_2}{k_1 + k_2}\,x$

$\therefore$ Force constant of the series combination,

$$k_s = \frac{k_1 k_2}{k_1 + k_2}$$

Period of oscillation for the series combination,

$$T = 2\pi\sqrt{\frac{m}{k_s}} = 2\pi\sqrt{\frac{m(k_1+k_2)}{k_1k_2}} = 2\pi\sqrt{\frac{m}{k_1}+\frac{m}{k_2}}$$

or $$T^2 = 4\pi^2\left(\frac{m}{k_1}+\frac{m}{k_2}\right) = \left(2\pi\sqrt{\frac{m}{k_1}}\right)^2 + \left(2\pi\sqrt{\frac{m}{k_2}}\right)^2$$

or $$T^2 = t_1^2 + t_2^2.$$

Problem 11. *A particle executes simple harmonic motion between $x = -A$ and $x = +A$. The time taken for it to go from 0 to $A/2$ is T_1 and to go from $A/2$ to A is T_2. Then how are T_1 and T_2 related ?* **[IIT Screening 01]**

Solution. The displacement equation for S.H.M. is $x = A \sin \omega t$

At $t = T_1$, $x = A/2$

$$\therefore \quad \frac{A}{2} = A \sin \omega T_1 \quad \text{or} \quad \frac{1}{2} = \sin \omega T_1$$

$$\text{or} \quad \omega T_1 = \frac{\pi}{6} \quad \text{or} \quad T_1 = \frac{\pi}{6\omega}$$

At $t = T_1 + T_2$, $x = A$

$$\therefore \quad A = A \sin \omega (T_1 + T_2) \quad \text{or} \quad 1 = \sin \omega (T_1 + T_2)$$

$$\text{or} \quad \omega (T_1 + T_2) = \frac{\pi}{2} \quad \text{or} \quad T_1 + T_2 = \frac{\pi}{2\omega}$$

$$\therefore \quad T_2 = \frac{\pi}{2\omega} - T_1 = \frac{\pi}{2\omega} - \frac{\pi}{6\omega} = \frac{\pi}{3\omega} = \mathbf{2T_1}.$$

Problem 12. *Two simple harmonic motions are represented by the equations :*

$$y_1 = 10 \sin \frac{\pi}{4}(12\,t + 1), \quad y_2 = 5\,(\sin 3\pi\,t + \sqrt{3} \cos 3\pi\,t)$$

Find the ratio of their amplitudes. What are time periods of the two motions ? **[IIT 86 ; MNREC 90]**

Solution. $y_1 = 10 \sin \frac{\pi}{4}(12t + 1)$

$$= 10 \sin\left(3\pi\,t + \frac{\pi}{4}\right) \quad \text{...(1)}$$

$$y_2 = 5\,(\sin 3\pi\,t + \sqrt{3} \cos 3\pi\,t)$$

$$= 10\left(\sin 3\pi\,t \times \frac{1}{2} + \cos 3\pi\,t \times \frac{\sqrt{3}}{2}\right)$$

$$= 10\left(\sin 3\pi\,t \cos \frac{\pi}{3} + \cos 3\pi\,t \sin \frac{\pi}{3}\right)$$

$$\text{or} \quad y_2 = 10 \sin\left(3\pi\,t + \frac{\pi}{3}\right) \quad \text{...(2)}$$

The general equation for SHM is

$$y = A \sin(\omega t + \phi_0) = A \sin\left(\frac{2\pi}{T}t + \phi_0\right) \quad \text{...(3)}$$

Comparing equations (1) and (2) with (3), we get

$$A_1 = 10, \quad A_2 = 10, \quad \frac{2\pi}{T_1} = \frac{2\pi}{T_2} = 3\pi$$

$$\therefore \quad \frac{A_1}{A_2} = \mathbf{1:1} \;;\; T_1 = T_2 = \mathbf{\frac{2}{3}\ s}$$

Problem 13. *A point particle of mass 0.1 kg is executing SHM of amplitude 0.1 m. When the particle passes through the mean position, its kinetic energy is 8×10^{-3} J. Obtain the equation of motion of the particle if the initial phase of oscillation is 45°.* **[Roorkee 91]**

Solution. Here $m = 0.1$ kg, $A = 0.1$ m, $E = 8 \times 10^{-3}$ J, $\phi_0 = 45° = \frac{\pi}{4}$ rad

K.E. at the mean position $= (E_k)_{\max} = \frac{1}{2} m \omega^2 A^2$

$$\therefore \quad 8 \times 10^{-3} = \frac{1}{2} \times 0.1 \times \omega^2 \times (0.1)^2$$

$$\text{or} \quad \omega^2 = 16 \quad \text{or} \quad \omega = 4 \text{ rad s}^{-1}$$

The equation of motion for the particle is

$$y = A \sin \omega t = \mathbf{0.1 \sin (4\,t + \pi/4)}.$$

Problem 14. *A simple harmonic motion has an amplitude A and time period T. What is the time taken to travel from $x = A$ to $x = A/2$?* **[REC 92]**

Solution. Displacement from mean position

$$= A - \frac{A}{2} = \frac{A}{2}$$

When the motion starts from the positive extreme position,

$$y = A \cos \omega\,t \quad \therefore \quad \frac{A}{2} = A \cos \frac{2\pi}{T}t$$

$$\text{or} \quad \cos \frac{2\pi}{T}t = \frac{1}{2} = \cos \frac{\pi}{3} \quad \text{or} \quad \frac{2\pi}{T}t = \frac{\pi}{3}$$

$$\therefore \quad t = \mathbf{\frac{T}{6}}.$$

Problem 15. *A block is resting on a piston which is moving vertically with simple harmonic motion of period 1.0 second. At what amplitude of motion will the block and piston separate ? What is the maximum velocity of the piston at this amplitude ?* **[Roorkee 85]**

Solution. The block and piston will just separate when

$$a_{\max} = g \quad \text{or} \quad \omega^2 A = \left(\frac{2\pi}{T}\right)^2 A = g$$

$$\therefore \quad A = \frac{gT^2}{4\pi^2} = \frac{9.8 \times (1.0)^2}{4 \times 9.87} = 0.248 \text{ m}$$

Maximum velocity of the block,

$$v_{\max} = \omega A = \frac{2\pi}{T} A = \frac{2 \times 3.142}{1.0} \times 0.248 = \mathbf{1.56\ ms^{-1}}.$$

Problem 16. *A block is kept on a horizontal table. The table is undergoing simple harmonic motion of frequency 3 Hz in a horizontal plane. The coefficient of static friction between the block and the table surface is 0.72. Find the maximum amplitude of the table at which the block does not slip on the surface. Take $g = 10\ ms^{-1}$.* **[Roorkee 96]**

Solution. Maximum acceleration of the block,

$$a_{max} = \omega^2 A$$

$\therefore$ Maximum force on the block, $F = ma_{max} = m\omega^2 A$

Frictional force on the block $= \mu\ mg$

The block will not slip on the surface of the table if

$$m\omega^2 A = \mu\ mg$$

$\therefore$ Amplitude,

$$A = \frac{\mu\ g}{\omega^2} = \frac{0.72 \times 10}{(2\pi\nu)^2} = \frac{0.72 \times 10}{(2 \times 3.14 \times 3)^2} = \mathbf{0.02\ m.}$$

Problem 17. *Springs of spring constants k, 2k, 4k, 8k, ... are connected in series. A mass m kg is attached to the lower end of the last spring and the system is allowed to vibrate. What is the time period of oscillations ?*

Given $m = 40$ g and $k = 2.0\ N\,cm^{-1}$

Solution. Here $m = 40\ \text{g} = 0.04\ \text{kg}$,

$$k = 2.0\ \text{N cm}^{-1} = 2.0 \times 100\ \text{Nm}^{-1}$$

The effective spring constant k of the series combination is given by

$$\frac{1}{k'} = \frac{1}{k} + \frac{1}{2k} + \frac{1}{4k} + \frac{1}{8k} + \ldots = \frac{1}{k}\left[1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots\right]$$

$$= \frac{1}{k}\left[\frac{1}{1 - 1/2}\right] = \frac{2}{k} \qquad \left[\text{Sum of finite G.P.} = \frac{a}{1-r}\right]$$

or $k' = k/2$

$$\therefore \qquad T = 2\pi\sqrt{\frac{m}{k'}} = 2\pi\sqrt{\frac{2m}{k}}$$

$$= 2 \times \frac{22}{7} \times \sqrt{\frac{2 \times 0.04}{2.0 \times 100}}$$

$$= \mathbf{0.126\ s.}$$

Problem 18. *A uniform spring whose unstretched length is l has a force constant k. The spring is cut into two pieces of unstretched lengths l_1 and l_2, where $l_1 = nl_2$ and n is an integer. What are the corresponding force constants k_1 and k_2 in terms of n and k ? What is the ratio k_1/k_2 ?*

Solution. Here $l = l_1 + l_2$ and $l_1 = nl_2$ or $\frac{l_1}{l_2} = n$

As $\qquad k = \frac{mg}{l}$

$\therefore \qquad k_1 = \frac{mg}{l_1}$ and $k_2 = \frac{mg}{l_2}$

Hence $\frac{k_1}{k} = \frac{mg}{l_1} \times \frac{l}{mg} = \frac{l}{l_1} = \frac{l_1 + l_2}{l_1} = 1 + \frac{l_2}{l_1} = 1 + \frac{1}{n}$

or $\qquad k_1 = \left(\frac{\boldsymbol{n+1}}{\boldsymbol{n}}\right)\mathbf{k.}$

Also $\frac{k_2}{k} = \frac{mg}{l_2} \times \frac{l}{mg} = \frac{l}{l_2} = \frac{l_1 + l_2}{l_2} = \frac{l_1}{l_2} + 1 = n + 1$

or $\qquad \mathbf{k_2 = (n + 1)\ k.}$

Clearly, $\frac{k_1}{k_2} = \frac{\mathbf{1}}{\boldsymbol{n}}.$

Problem 19. *A horizontal spring block system of mass M executes simple harmonic motion. When the block is passing through its equilibrium position, an object of mass m is put on it and the two move together. Find the new amplitude and frequency of vibration.* **[Roorkee 88]**

Solution. Original frequency,

$$\nu = \frac{1}{2\pi}\sqrt{\frac{k}{M}}$$

Let A = Initial amplitude of oscillation

v = Velocity of mass M when passing through mean position

Maximum K.E. = Total energy

or $\qquad \frac{1}{2}Mv^2 = \frac{1}{2}kA^2$

$\therefore \qquad v = \sqrt{\frac{k}{M}}\ A$

When mass m is put on the system, total mass $= (M + m)$. If v' is the velocity of the combination in equilibrium position, then by the conservation of linear momentum,

$$Mv = (M + m)\,v' \qquad \text{or} \qquad v' = \frac{Mv}{M + m}$$

If A' is the new amplitude, then

$$\frac{1}{2}(M + m)\,v'^2 = \frac{1}{2}kA'^2$$

or $\qquad A' = \sqrt{\frac{M+m}{k}}\,.\,v' = \sqrt{\frac{M+m}{k}} \times \frac{Mv}{M+m}$

$$= \sqrt{\frac{M+m}{k}} \times \frac{M}{M+m} \times \sqrt{\frac{k}{M}}\ A = \sqrt{\frac{\mathbf{M}}{\mathbf{M+m}}}\ .\ \mathbf{A.}$$

New frequency, $\qquad \nu' = \frac{1}{2\pi}\sqrt{\frac{k}{M+m}}.$

Problem 20. *The bob of pendulum of length l is pulled aside from its equilibrium position through an angle θ and then released. Find the speed v with which the bob passes through the equilibrium position.* **[Kurukshetra CEE 96]**

Solution. The situation is shown in Fig. 14.43.

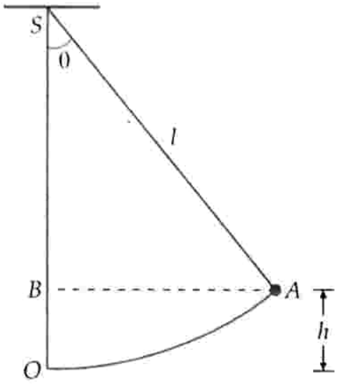


Fig. 14.43

Clearly, $h = OB = OS - BS = l - l \cos \theta$

$$= l\,(1 - \cos \theta)$$

Let v and v' be the velocities of the bob at position O and A respectively. Then by the conservation of energy,

$$\frac{1}{2}mv^2 = \frac{1}{2}mv'^2 + mgh$$

or $$v' = \sqrt{v^2 - 2gh}$$

$$= \sqrt{v^2 - 2gl\,(1 - \cos \theta)}$$

Guidelines to NCERT Exercises

14.1. *Which of the following examples represent periodic motion ?*

(*i*) *A swimmer completing one (return) trip from one bank of a river to the other and back.*

(*ii*) *A freely suspended bar magnet displaced from its N – S direction and released.*

(*iii*) *A hydrogen molecule rotating about its center of mass.*

(*iv*) *An arrow released from a bow.* (*v*) *Halley's comet.*

Ans. (*i*) ***Not periodic.*** Because the motion of the swimmer is not repeated over and over again after any fixed time interval.

(*ii*) ***Periodic.*** As the magnet is released from its displaced position, it oscillates about the $N - S$ direction with a definite time period.

(*iii*) ***Periodic.*** The motion of the hydrogen molecule rotating about its centre of mass repeats after a fixed time interval.

(*iv*) ***Not periodic.*** The motion of the arrow does not repeat itself after a fixed time interval.

(*v*) ***Periodic.*** Halley's comet appears after every 76 years.

14.2. *Which of the following examples represent (nearly) simple harmonic motion and which represent periodic but not simple harmonic motion ?*

(*i*) *The rotation of earth about its axis.*

(*ii*) *Motion of an oscillating mercury column in a U-tube.*

(*iii*) *Motion of a ball bearing inside a smooth curved bowl, when released from a point slightly above the lower most point.*

(*iv*) *General vibrations of a polyatomic molecule about its equilibrium position.*

Ans. (*i*) ***Periodic but not simple harmonic.*** The motion of the earth about its axis repeats after every 24 hours but it is not a to and fro motion.

(*ii*) ***Simple harmonic.*** The restoring force is proportional to the displacement of the mercury column from the equilibrium level.

(*iii*) ***Simple harmonic.*** The motion of the ball bearing is to and fro about the lower most point and the restoring force is proportional to its displacement from that point.

(*iv*) ***Periodic but not simple harmonic.*** A polyatomic molecule has a number of natural frequencies. In general, its vibration is a superposition of SHM's of a number of different frequencies. This superposition is periodic but not simple harmonic.

14.3. *Fig. 14.44 depicts four x-t plots for linear motion of a particle. Which of the plots represent periodic motion ? What is the period of motion (in case of periodic motion) ?*

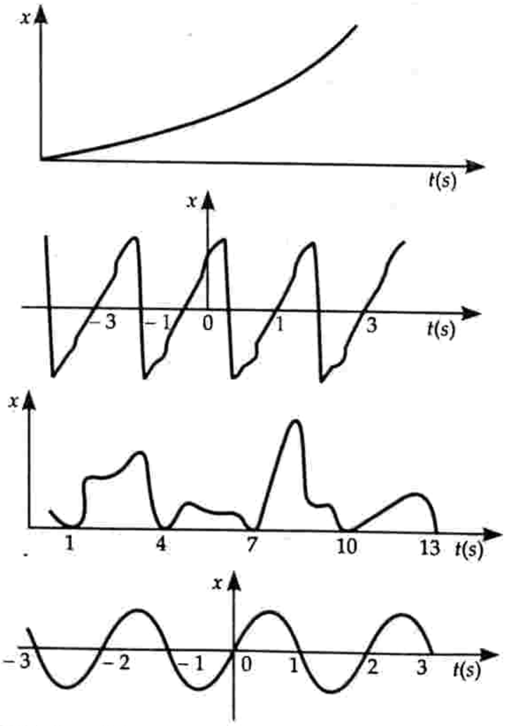


Fig. 14.44

Ans. (*i*) Plot (*a*) does not represent periodic motion because the motion is not repeated after a fixed interval.

(*ii*) Plot (*b*) represents periodic motion with $T = 2s$.

(*iii*) Plot (*c*) does not represent periodic motion. The repetition of merely one position is not enough for the motion to be periodic. The entire motion during one period must be repeated successively.

(*iv*) Plot (*d*) represents periodic motion with $T = 2s$.

14.4. *Which of the following functions of time represent (a) simple harmonic, (b) periodic but not simple harmonic, and (c) non-periodic motion ? Give period for each case of periodic motion [ω is any positive constant)*

(*i*) $\sin \omega t - \cos \omega t$ (*ii*) $\sin^3 \omega t$

(*iii*) $3 \cos (\pi/4 - 2\omega t)$ (*iv*) $\cos \omega t + \cos 3\omega t + \cos 5\omega t$

(*v*) $\exp(-\omega^2 t^2)$ (*vi*) $1 + \omega t + \omega^2 t^2$.

Ans.

(*i*) Here $x(t) = \sin \omega t - \cos \omega t$

$$= \sqrt{2} (\sin \omega t \cos \pi/4 - \cos \omega t \sin \pi/4)$$
$$= \sqrt{2} \sin (\omega t - \pi/4)$$

Moreover,

$$x(t + 2\pi/\omega) = \sqrt{2} \sin [\omega(t + 2\pi/\omega) - \pi/4]$$
$$= \sqrt{2} \sin (\omega t + 2\pi - \pi/4)$$
$$= \sqrt{2} \sin (\omega t - \pi/4) = x(t)$$

Hence the given function represents a ***simple harmonic motion*** with $T = 2\pi/\omega$ and phase angle $= -\pi/4$ or $7\pi/4$.

(*ii*) $x(t) = \sin^3 \omega t = \frac{1}{4}(3 \sin \omega t - \sin 3\omega t)$

$$[\because \sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta]$$

It represents two separate simple harmonic motions but their combination does not represent SHM.

$$\text{Period of } \frac{3}{4} \sin \omega t = \frac{2\pi}{\omega} = T$$

$$\text{Period of } \frac{1}{4} \sin 3\omega t = \frac{2\pi}{3\omega} = \frac{T}{3}$$

Thus the minimum time after which the combined function repeats is $T = 2\pi/\omega$. Hence the given function is ***periodic but not simple harmonic.***

(*iii*) Here $x(t) = 3 \cos (\pi/4 - 2\omega t)$

$$= 3 \cos [-(2\omega t - \pi/4)]$$
$$= 3 \cos (2\omega t - \pi/4)$$
$$[\because \cos(-\theta) = \cos \theta]$$

It represents S.H.M. with period $T = \frac{2\pi}{2\omega} = \frac{\pi}{\omega}$.

(*iv*) $x(t) = \cos \omega t + \cos 3\omega t + \cos 5\omega t$

$\cos \omega t$ represents S.H.M. with period $= \frac{2\pi}{\omega} = T$

$\cos 3\omega t$ represents S.H.M. with period $= \frac{2\pi}{3\omega} = \frac{T}{3}$

$\cos 5\omega t$ represents S.H.M. with period $= \frac{2\pi}{5\omega} = \frac{T}{5}$

The minimum time after which the combined function repeats its value is T. The given function is ***periodic but not simple harmonic.***

(*v*) $x(t) = \exp(-\omega^2 t^2) = e^{-\omega^2 t^2}$

It is an exponential function. It decreases monotonically to zero as $t \to \infty$. It never repeats its value. It is a ***non-periodic function.***

(*vi*) $x(t) = 1 + \omega t + \omega^2 t^2$

As t increases, $x(t)$ increases monotonically. Again, as $t \to \infty$, $x(t) \to \infty$. The function never repeats its value. So $x(t)$ is ***non-periodic.***

14.5. *A particle is in linear simple harmonic motion between two points, A and B, 10 cm apart. Take the direction from A to B as the positive direction and give the signs of velocity, acceleration and force on the particle when it is*

(*a*) *at the end A,* (*b*) *at the end B,*

(*c*) *at the mid-point of AB going towards A,*

(*d*) *at 2 cm away from B going towards A.*

(*e*) *at 3 cm away from A going towards B, and*

(*f*) *at 4 cm away from A going towards A.*

Ans.

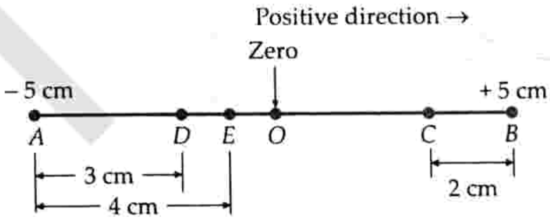


Fig. 14.45

Position	***Velocity***	***Acceleration***	***Force***
(*a*) At *A*	0 (at extreme position)	+ ve (acts from *A* to *O*)	+ ve (acts from *A* to *O*)
(*b*) At *B*	0 (at extreme position)	– ve (acts from *B* to *O*)	– ve (acts from *B* to *O*)
(*c*) At midpoint *O*, going towards *A*	– ve and maximum (acts from *O* to *A*)	0 (at mid-point)	0 (at mid-point)
(*d*) At *C*, going towards *A*	– ve (acts from *C* to *O*)	– ve (acts from *C* to *O*)	– ve (acts from *C* to *O*)
(*e*) At *D*, going towards *B*	+ ve (acts from *D* to *O*)	+ ve (acts from *D* to *O*)	+ ve (acts from *D* to *O*)
(*f*) At *E*, going towards *A*	– ve (acts from *E* to *A*)	+ ve (acts from *E* to *O*)	+ ve (acts from *E* to *O*)

14.6. *Which of the following relationships between the acceleration a and the displacement x of a particle involve simple harmonic motion ?*

(*a*) $a = 0.7x$ (*b*) $a = -200x^2$ (*c*) $a = -10x$ (*d*) $a = 100x^3$.

Ans. Only (*c*) represents S.H.M. because here $a \propto x$ and a acts in the opposition direction of x.

14.7. *(a) A particle in SHM is described by the displacement function,*

$$x(t) = A\cos(\omega t + \phi),\ \omega = \frac{2\pi}{T}.$$

If the initial ($t = 0$) position of the particle is 1 cm and its initial velocity is π cm s^{-1}, what are its amplitude and initial phase angle ? The angular frequency of the particle is π s^{-1}.

(b) A particle in SHM is described by the displacement function,

$$x(t) = B\sin(\omega t + \alpha),\ \omega = \frac{2\pi}{T}.$$

If the initial ($t = 0$) position of the particle is 1 cm and its initial velocity is π cm s^{-1}, what are its amplitude and initial phase angle ? The angular frequency of the particle is π s^{-1}.

Ans. (*a*) At $t = 0$, $x = 1\,\text{cm}$ and $v = \pi\,\text{cm s}^{-1}$. Also, $\omega = \pi\,\text{s}^{-1}$

In SHM, displacement at any time t is given by

$$x = A\cos(\omega t + \phi)$$

Since, at $t = 0$, $x = 1$, therefore

$$1 = A\cos(\omega \times 0 + \phi)$$

or $$A\cos\phi = 1 \qquad ...(i)$$

Now velocity,

$$v = \frac{dx}{dt} = \frac{d}{dt}[A\cos(\omega t + \phi)]$$

$$= -A\omega\sin(\omega t + \phi)$$

Again at $t = 0$, $v = \pi$ cm s^{-1}, so we have

$$\pi = -A(\pi)\sin(\omega \times 0 + \phi)$$

or $$A\sin\phi = -1 \qquad ...(ii)$$

Squaring and adding equations (*i*) and (*ii*), we get

$$A^2\cos^2\phi + A^2\sin^2\phi = 1^2 + (-1)^2$$

or $$A^2(\cos^2\phi + \sin^2\phi) = 2 \quad \text{or} \quad A^2(1) = 2$$

$\therefore$ $$A = \sqrt{2}\ \textbf{cm.}$$

Dividing equation (*ii*) by (*i*), we get

$$\frac{A\sin\phi}{A\cos\phi} = \frac{-1}{1} \quad \text{or} \quad \tan\phi = -1$$

or $$\phi = \frac{3\pi}{4}\ \textbf{or}\ \frac{7\pi}{4}.$$

(*b*) At $t = 0$, $x = 1\,\text{cm}$ and $v = \pi\,\text{cm s}^{-1}$.

Also, $\omega = \pi\,\text{s}^{-1}$

Given $x = B\sin(\omega t + \alpha)$

Since, at $t = 0$, $x = 1$, therefore

$$1 = B\sin(\omega \times 0 + \alpha)$$

or $$B\sin\alpha = 1 \qquad ...(i)$$

Now velocity,

$$v = \frac{dx}{dt} = \frac{d}{dt}\{B\sin(\omega t + \alpha)\} = -B\omega\cos(\omega t + \alpha)$$

Again, at $t = 0$, $v = \pi$ cm s^{-1}, so we have

$$\pi = B(\pi)\cos(\omega \times 0 + \alpha)$$

or $$B\cos\alpha = 1 \qquad ...(ii)$$

Squaring and adding equations (*i*) and (*ii*), we get

$$B^2\sin^2\alpha + B^2\cos^2\alpha = 1^2 + 1^2 \text{ or } B^2 = 2$$

or $$B = \sqrt{2}\ \textbf{cm.}$$

Dividing equation (*i*) by (*ii*), we get

$$\frac{B\sin\alpha}{B\cos\alpha} = \frac{1}{1} \quad \text{or} \quad \tan\alpha = 1$$

or $$\alpha = \frac{\pi}{4}\ \text{or}\ \frac{5\pi}{4}.$$

14.8. *A spring balance has a scale that reads from 0 to 50 kg. The length of the scale is 20 cm. A body suspended from this spring, when displaced and released, oscillates with a period of 0.60 s. What is the weight of the body ?*

Ans. The 20 cm length of the scale reads upto 50 kg, so

$$F = mg = 50 \times 9.8\ \text{N},\quad y = 20\ \text{cm} = 0.20\ \text{m}$$

Force constant, $$k = \frac{F}{y} = \frac{50 \times 9.8}{0.20} = 2450\ \text{Nm}^{-1}$$

Suppose the spring oscillates with time period of 0.60 s when loaded with a mass of M kg. Then

$$T = 2\pi\sqrt{\frac{M}{k}}$$

or $$T^2 = 4\pi^2\frac{M}{k}$$

$\therefore$ $$M = \frac{T^2k}{4\pi^2} = \frac{(0.60)^2 \times 2450}{4 \times (3.14)^2} = 22.36\ \text{kg}$$

$$\text{Weight} = Mg = 22.36 \times 9.8 = \textbf{219.13 N.}$$

14.9. *A spring of force constant 1200 Nm^{-1} is mounted horizontally on a horizontal table. A mass of 3.0 kg is attached to the free end of the spring, pulled sideways to a distance of 2.0 cm and released. (i) What is the frequency of oscillation of the mass ? (ii) What is the maximum acceleration of the mass ? (iii) What is the maximum speed of the mass ?*

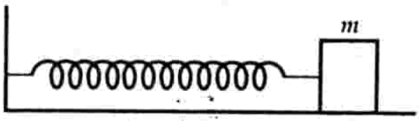


Fig. 14.46

Ans. Here $k = 1200\ \text{Nm}^{-1}$, $m = 3.0$ kg,

$$A = 2.0\ \text{cm} = 2.0 \times 10^{-2}\ \text{m}$$

(*i*) Frequency of oscillation of the mass,

$$\nu = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2 \times 3.14}\sqrt{\frac{1200}{3.0}}$$

$$= \frac{1}{2 \times 3.14} \times 20 = \textbf{3.2 s}^{-1}\textbf{.}$$

(*ii*) Angular frequency,

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{1200}{3.0}} = 20\ \text{s}^{-1}.$$

$\therefore$ Maximum acceleration of the mass

$$= \omega^2 A = (20)^2 \times 2.0 \times 10^{-2}$$

$$= \mathbf{8.0\ ms^{-2}}.$$

(*iii*) Maximum speed of the mass

$$= \omega A = 20 \times 2.0 \times 10^{-2}$$

$$= \mathbf{0.40\ ms^{-1}}.$$

14.10. *In Exercise 14.9, let us take the position of the mass, when the spring is unstretched, as $x = 0$, and the direction from left to right as the positive direction of X-axis. Give x as a function of time t for the oscillating mass, if at the moment we start the stop watch ($t = 0$), the mass is (i) at the mean position (ii) at the maximum stretched position (iii) at the maximum compressed position.*

In what do these different functions of SHM differ ? Frequency, amplitude or initial phase ?

Ans. When the mass starts motion from mean position, the displacement of SHM is given by

$$x = A \sin \omega t$$

And, when the mass starts motion from extreme position, the displacement of SHM is given by

$$x = \pm A \cos \omega t$$

From above exercise, we have

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{1200}{3}} = 20 \text{ rad s}^{-1}$$

(*i*) At $t = 0$, when the mass is at *mean position*

$\therefore$ Displacement is given by

$$x = A \sin \omega t = \mathbf{2 \sin 20t}.$$

(*ii*) At $t = 0$, when the mass is at the *maximum stretched position*. The motion starts from positive extreme position, thus

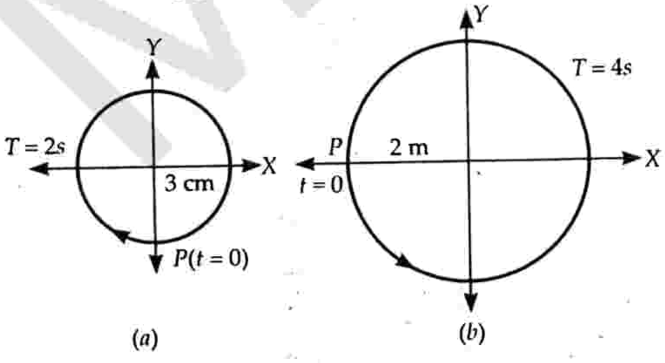
$$x = + A \cos \omega t = \mathbf{2 \cos 20t}.$$

(*iii*) At $t = 0$, when the mass is at the *maximum compressed position*. The mass starts its motion from negative extreme position, thus

$$x = - A \cos \omega t = \mathbf{-2 \cos 20t}.$$

14.11. *Fig. 14.47 corresponds to two circular motions. The radius of the circle, the period of revolution, the initial position, and the sense of revolution (i.e., clockwise or anti-clockwise) are indicated on each figure.*

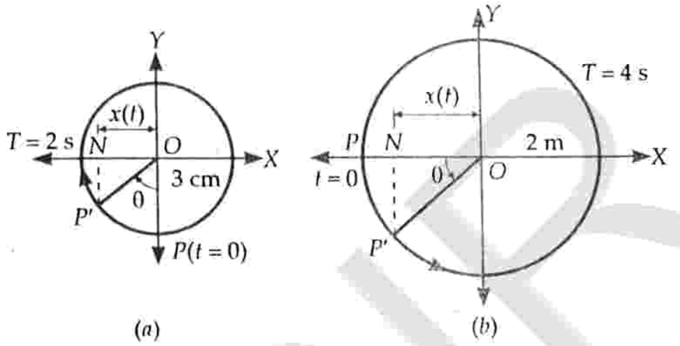
Obtain the corresponding simple harmonic motions of the x-projection of the radius vector of the revolving particle P, in each case.



(*a*) (*b*)

Fig. 14.47

Ans.



(*a*) (*b*)

Fig. 14.48

(*a*) As shown in Fig. 14.48(*a*), suppose the particle moves from P to P' in time t.

Angle swept by the radius vector,

$$\theta = \omega t = \frac{2\pi}{T} t = \frac{2\pi}{2} t = \pi t \text{ rad}$$

Displacement,

$$ON = OP' \cos\left(\frac{\pi}{2} - \theta\right) = OP' \sin \theta$$

or $\quad -x(t) = 3 \sin \theta$

[Displacement being to the left O]

or $\quad x(t) = -3 \sin \pi t$

(*b*) As shown in Fig. 14.48(*b*), suppose the particle moves from P to P' in time t.

Angle swept by the radius vector,

$$\theta = \omega t = \frac{2\pi}{T} t = \frac{2\pi}{4} t = \frac{\pi t}{2} \text{ rad}$$

Displacement,

$$ON = OP' \cos \theta$$

or $\quad -x(t) = 2 \cos \frac{\pi t}{2} \qquad \left[\because OP' = 2\text{m}, \theta = \frac{\pi t}{2}\right]$

or $\quad x(t) = -2 \cos \frac{\pi t}{2}.$

14.12. *Plot the corresponding reference circle for each of the following simple harmonic motions. Indicate the initial ($t = 0$) position of the particle, the radius of the circle, and the angular speed of the rotating particle. For simplicity, the sense of rotation may be fixed to be anticlockwise in every case (x is in cm and t is in s).*

(*i*) $x = -2 \sin(3t + \pi/3)$ (*ii*) $x = \cos(\pi/6 - t)$

(*iii*) $x = 3 \sin(2\pi t + \pi/4)$ (*iv*) $x = 2 \cos \pi t$.

Ans. (*i*) $x = -2 \sin(3t + \pi/3)$

$$= 2 \cos(3t + \pi/3 + \pi/2)$$

or $\quad x = 2 \cos(3t + 5\pi/6) \quad [-\sin\theta = \cos(\pi/2 + \theta)]$

Comparing with $x = A\cos(\omega t + \phi_0)$, it follows that

$$A = 2\text{ cm},\ \omega = 3\text{ rad s}^{-1},\ \phi_0 = 5\pi/6\text{ rad}$$

The reference circle is shown in Fig. 14.49(*a*).

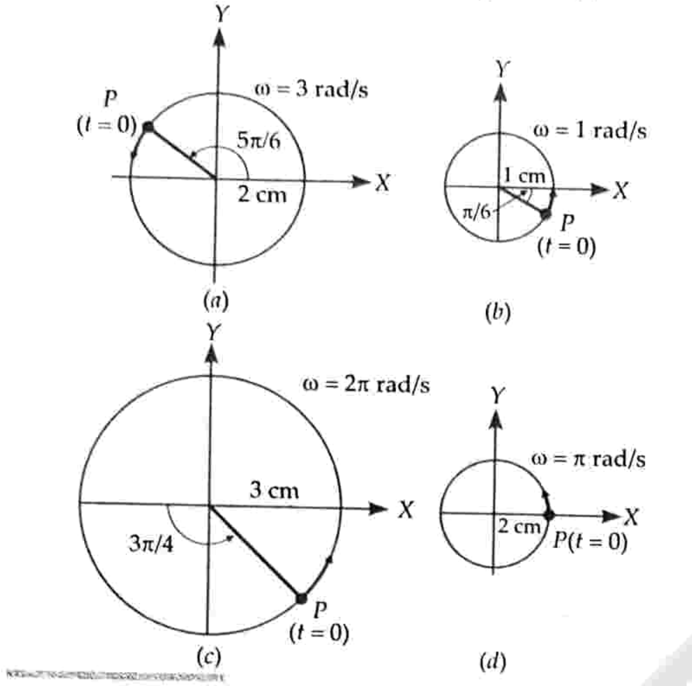


Fig. 14.49

(*ii*) $x = \cos(\pi/6 - t) = \cos[-(t - \pi/6)]$

or $\quad x = \cos(t - \pi/6) \quad [\because \cos(-\theta) = \cos\theta]$

Comparing with $x = A\cos(\omega t + \phi_0)$, it follows that

$$A = 1\text{cm},\quad \omega = 1\text{ rad s}^{-1},\quad \phi_0 = -\pi/6\text{ rad}.$$

The reference circle is shown in Fig. 14.49(*b*).

(*iii*) $x = 3\sin(2\pi t + \pi/4) = -3\cos\left(2\pi t + \frac{\pi}{4} + \frac{\pi}{2}\right)$

or $\quad x = -3\cos(2\pi t + 3\pi/4)$

The negative sign shows that the motion starts on the negative side of *x*-axis.

Here $A = 3$ cm, $\omega = 2\pi$ rad s^{-1}, $\phi_0 = 3\pi/4$ rad

The reference circle is shown in Fig. 14.49(*c*).

(*iv*) $x = 2\cos\pi t$

Comparing with $x = A\cos(\omega t + \phi_0)$, it follows that

$$A = 2\text{ cm},\ \omega = \pi\text{ rad s}^{-1},\ \phi_0 = 0.$$

The reference circle is shown in Fig. 14.49(*d*).

14.13. *Fig. 14.50(a) shows a spring of force constant k clamped rigidly at one end and a mass m attached to its free end. The spring is stretched by a force F at its free end. Fig. 14.50(b) shows the same spring with both ends free and attached to a mass m at either end. Each end of the spring in Fig. 14.50(b) is stretched by the same force F.*

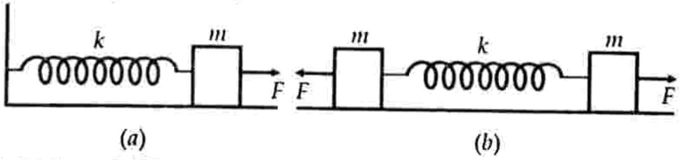


Fig. 14.50

(i) What is the maximum extension of the spring in the two cases ? (ii) If the mass in (a) and the two masses in (b) are released free, what is the period of oscillation in each case ?

Ans. (*i*) *Maximum extension of the spring.* In case (*b*), the force at either end of the spring is *F* and they act in opposite directions. In case (*a*), the force of reaction at the clamped end is also *F*, so both systems are identical. The maximum extension in each case is given by

$$y = \frac{F}{k}.$$

(*ii*) In case (*a*), the period of oscillation is given by

$$T = 2\pi\sqrt{\frac{m}{k}}$$

In case (*b*), the spring can be considered to be divided into two equal halves and its centre can be regarded to be fixed as it does not move. Let *k′* be the force constant of each half and *x′* be the extension produced in each half. Then

$$x' = \frac{F}{k'}$$

Total extension, $x = 2x'$

or $\quad \frac{F}{k} = 2 \cdot \frac{F}{k'}$

$\therefore \quad k' = 2k$

Hence the period of oscillation in case (*b*) is

$$T = 2\pi\sqrt{\frac{m}{k'}} = 2\pi\sqrt{\frac{m}{2k}}.$$

14.14. *The piston in the cylinder head of a locomotive has a stroke (twice the amplitude) of 1.0 m. If the piston moves with simple harmonic motion with an angular frequency of 200 rev/min, what is its maximum speed ?*

Ans. Here $A = \frac{1}{2}$ m, $\omega = 200$ rev / min

$$v_{\max} = \omega A = 200 \times \frac{1}{2} = \mathbf{100\ m\ /\ min.}$$

14.15. *The acceleration due to gravity on the surface of the moon is 1.7 ms^{-2}. What is the time period of a simple pendulum on the moon if its time period on the earth is 3.5 s ? Given g on earth = 9.8 ms^{-2}.* **[Delhi 06]**

Ans. *For the moon* : $g_m = 1.7\text{ ms}^{-2}$, $T_m = ?$

For the earth : $g_e = 9.8\text{ ms}^{-2}$, $T_e = 3.5\text{ s}$

But $\quad T_e = 2\pi\sqrt{\frac{l}{g_e}}$ and $T_m = 2\pi\sqrt{\frac{l}{g_m}}$

$\therefore \quad \frac{T_m}{T_e} = \sqrt{\frac{g_e}{g_m}}$

or $\quad T_m = \sqrt{\frac{g_e}{g_m}} \times T_e$

$$= \sqrt{\frac{9.8}{1.7}} \times 3.5 = \mathbf{8.4\ s.}$$

14.16. *Answer the following questions :*

(*a*) *Time period of a particle in SHM depends on the force constant k and mass m of the particle :*

$T = 2\pi\sqrt{\frac{m}{k}}$ *. A simple pendulum executes SHM approximately. Why then is the time period of a pendulum independent of the mass of the pendulum ?* **[Delhi 12]**

(*b*) *The motion of a simple pendulum is approximately simple harmonic for small angle oscillations. For larger angles of oscillation, a more involved analysis shows that T is greater than* $2\pi\sqrt{\frac{l}{g}}$ *. Think of a qualitative argument to appreciate this result.*

(*c*) *A man with a wristwatch on his hand falls from the top of a tower. Does the watch give correct time during the free fall ?*

(*d*) *What is the frequency of oscillation of a simple pendulum mounted in a cabin that is freely falling under gravity ?*

Ans. (*a*) For simple pendulum, force constant

$$k = \frac{mg}{l} \quad i.e., \quad k \propto m$$

$$\therefore \quad T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{m}{mg/l}} = 2\pi\sqrt{\frac{l}{g}}$$

Thus m cancels out. Hence time period of a simple pendulum is independent of mass.

(*b*) The acceleration of the bob of a simple pendulum is given by

$$a = -g\sin\theta.$$

If θ is small, then

$$\sin\theta \simeq \theta \quad \text{and} \quad a = -g\theta$$

If θ is large, then $\sin\theta < \theta$, so that there is effective decrease in the value of g for large angles. Hence the time period, $T = 2\pi\sqrt{l/g}$ increases.

(*c*) Yes, the wrist watch will give correct time because the working of a wrist watch depends on its spring action (*i.e.*, the P.E. stored in the wound spring) and is independent of the gravity.

(*d*) Inside a cabin falling freely under gravity, $g = 0$. Hence the frequency, $\nu = \frac{1}{2\pi}\sqrt{\frac{g}{l}}$ of a simple pendulum mounted in the cabin will be zero.

14.17. *A simple pendulum of length l and having a bob of mass M is suspended in a car. The car is moving on a circular track of radius R with a uniform speed v. If the pendulum makes small oscillations in a radial direction about its equilibrium position, what will be its time period ?*

Ans. The bob of the pendulum has two accelerations :

(*i*) Centripetal acceleration, $a_c = \frac{v^2}{R}$, acting horizontally

(*ii*) Acceleration due to gravity $= g$, acting vertically downwards.

The effective acceleration due to gravity,

$$g' = \sqrt{g^2 + a_c^2} = \sqrt{g^2 + \frac{v^4}{R^2}}$$

$\therefore$ Time period,

$$T = 2\pi\sqrt{\frac{l}{g'}} = 2\pi\sqrt{\frac{l}{\sqrt{g^2 + v^4/R^2}}}.$$

14.18. *A cylindrical piece of cork of base area A and height h floats in a liquid of density* ρ_1*. The cork is depressed slightly and then released. Show that the cork oscillates up and down simple harmonically with a period* $T = 2\pi\sqrt{\frac{h\rho}{\rho_1 g}}$*, where* ρ *is the density of cork. (Ignore damping due to viscosity of the liquid).*

Ans. Refer answer to Q. 22 on page 14.31.

14.19. *One end of a U-tube containing mercury is connected to a suction pump and the other end is connected to the atmosphere. A small pressure difference is maintained between the two columns. Show that when the suction pump is removed, the liquid in the U-tube executes SHM.*

Ans. Refer answer to Q. 20 on page 14.30.

14.20. *An air chamber of volume V has a neck of area of cross-section A into which a ball of mass m can move without friction. Show that when the ball is pressed down through some distance and released, the ball executes SHM. Obtain the formula for the time period of this SHM, assuming pressure- volume variations of the air to be (i) isothermal and (ii) adiabatic.*

Ans. Refer answer to Q. 23 on page 14.31.

14.21. *You are riding in an automobile of mass 3000 kg. Assuming that you are examining the oscillation characteristics of its suspension system. The suspension sags 15 cm when the entire automobile is placed on it. Also, the amplitude of oscillation decreases by 50% during one complete oscillation. Estimate the values of (a) the spring constant and (b) the damping constant b for the spring and shock absorber system of one wheel, assuming that each wheel supports 750 kg.*

Ans. (*a*) Here $m = 3000$ kg, $x = 0.15$ m

If k is the spring constant of each spring, then the spring constant of the four springs connected in parallel will be $4k$.

$$\therefore \quad 4kx = mg$$

or
$$k = \frac{mg}{4x} = \frac{3000 \times 10}{4 \times 0.15} = \mathbf{5 \times 10^4\ Nm^{-1}}.$$

(*b*) As $A' = Ae^{-bt/2m}$

$$\therefore \quad \frac{A}{2} = Ae^{-bt/2m} \quad \text{or} \quad 2 = e^{bt/2m}$$

or
$$\log_e 2 = \frac{bt}{2m}\log_e e = \frac{bt}{2m} \quad \text{or} \quad b = \frac{2m\log_e 2}{t}$$

But
$$t = 2\pi\sqrt{\frac{m}{4k}} = 2 \times \frac{22}{7} \times \sqrt{\frac{3000}{4 \times 5 \times 10^4}} = \frac{44}{70}\sqrt{\frac{3}{2}}\ \text{s}$$

Hence
$$b = \frac{2 \times 750 \times 0.693}{\frac{44}{70}\sqrt{\frac{3}{2}}} = \mathbf{1350.4\ kg\ s^{-1}}.$$

14.22. *Show that for a particle in linear SHM, the average kinetic energy over a period of oscillation equals the average potential energy over the same period.*

Ans. Suppose a particle of mass m executes SHM of period T. The displacement of the particle at any instant t is given by $\quad y = A \sin \omega t$

$$\therefore \text{Velocity,} \quad v = \frac{dy}{dt} = \omega A \cos \omega t.$$

$$\text{Kinetic energy,} \quad E_k = \frac{1}{2} mv^2 = \frac{1}{2} m \omega^2 A^2 \cos^2 \omega t.$$

$$\text{Potential energy,} \quad E_p = \frac{1}{2} m \omega^2 y^2 = \frac{1}{2} m \omega^2 A^2 \sin^2 \omega t.$$

∴ Average K.E. over a period of oscillation,

$$E_{k_{av}} = \frac{1}{T}\int_0^T E_k \, dt = \frac{1}{T}\int_0^T \frac{1}{2} m \omega^2 A^2 \cos^2 \omega t \, dt$$

$$= \frac{1}{2T} m \omega^2 A^2 \int_0^T \frac{(1 + \cos 2\omega t)}{2} dt$$

$$= \frac{1}{4T} m \omega^2 A^2 \left[t + \frac{\sin 2\omega t}{2\omega} \right]_0^T$$

$$= \frac{1}{4T} m \omega^2 A^2 (T) = \frac{1}{4} m \omega^2 A^2 \qquad \text{...(1)}$$

Average P.E. over a period of oscillation,

$$E_{p_{av}} = \frac{1}{T}\int_0^T E_p \, dt = \frac{1}{T}\int_0^T \frac{1}{2} m \omega^2 A^2 \sin^2 \omega t \, dt$$

$$= \frac{1}{2T} m \omega^2 A^2 \int_0^T \frac{(1 - \cos 2\omega t)}{2} dt$$

$$= \frac{1}{4T} m \omega^2 A^2 \left[t - \frac{\sin 2\omega t}{2\omega} \right]_0^T$$

$$= \frac{1}{4T} m \omega^2 A^2 (T) = \frac{1}{4} m \omega^2 A^2 \qquad \text{...(2)}$$

Clearly, from equations (1) and (2), $E_{k_{av}} = E_{p_{av}}$.

14.23. *A circular disc of mass 10 kg is suspended by a wire attached to its centre. The wire is twisted by rotating the disc and released. The period of torsional oscillation is found to be 1.5 s. The radius of the disc is 15 cm. Determine the torsional spring constant of the wire.*

Ans. Period of torsional oscillations is given by

$$T = 2\pi \sqrt{\frac{I}{C}} \quad \text{or} \quad T^2 = \frac{4\pi^2 I}{C}$$

$$\therefore \text{Torsional spring constant,} \quad C = \frac{4\pi^2 I}{T^2}$$

But $I = \frac{1}{2} MR^2$, $M = 10$ kg, $R = 15 \text{ cm} = 0.15$ m,

$T = 1.5$ s

$$\therefore \quad C = \frac{4\pi^2 \times \frac{1}{2} MR^2}{T^2}$$

$$= \frac{2 \times (3.14)^2 \times 10 \times (0.15)^2}{(1.5)^2} = \mathbf{2.0 \text{ Nm rad}^{-1}}$$

14.24. *A body describes simple harmonic motion with an amplitude of 5 cm and a period of 0.2 s. Find the acceleration and velocity of the body when the displacement is (a) 5 cm, (b) 3 cm, (c) 0 cm.*

Ans. Here $A = 5$ cm, $T = 0.2$ s

Velocity and acceleration at any displacement x are given by

$$v = \omega \sqrt{A^2 - x^2} = \frac{2\pi}{T} \sqrt{A^2 - x^2}$$

$$a = -\omega^2 x = -\frac{4\pi^2}{T^2} x$$

(*a*) When $x = 5$ cm,

$$v = \frac{2\pi}{0.2} \sqrt{5^2 - 5^2} = \mathbf{0.}$$

or

$$a = -\frac{4\pi^2}{(0.2)^2} \times 5 \text{ cm s}^{-2}$$

$$= -500\,\pi^2 \text{ cm s}^{-2} = \mathbf{-5\pi^2 \text{ ms}^{-2}}.$$

(*b*) When $x = 3$ cm,

$$v = \frac{2\pi}{0.2} \sqrt{5^2 - 3^2} \text{ cm s}^{-2}$$

$$= 40\,\pi \text{ cm s}^{-2} = \mathbf{0.40\,\pi \text{ ms}^{-1}}.$$

$$a = -\frac{4\pi^2}{(0.2)^2} \times 3 \text{ cm s}^{-2}$$

$$= -300\,\pi^2 \text{ cm s}^{-2} = \mathbf{-3\pi^2 \text{ ms}^{-2}}.$$

(*c*) When $x = 0$ cm,

$$v = \frac{2\pi}{0.2} \sqrt{5^2 - 0^2} \text{ cm s}^{-1} = \mathbf{50\,\pi \text{ cm s}^{-1}}$$

$$= \mathbf{0.50\,\pi \text{ m s}^{-1}}.$$

$$a = -\frac{4\pi^2}{(0.2)^2} \times 0 = \mathbf{0.}$$

14.25. *A mass attached to a spring is free to oscillate, with angular velocity ω, in a horizontal plane without friction or damping. It is pulled to a distance x_0 and pushed towards the centre with a velocity v_0 at time $t = 0$. Determine the amplitude of the resulting oscillations in terms of the parameters ω, x_0 and v_0.*

Ans. By conservation of energy,

(K.E. + P.E.) at distance x_0

= Total energy at the extreme position

or

$$\frac{1}{2} mv_0^2 + \frac{1}{2} kx_o^2 = \frac{1}{2} kA^2$$

or

$$\frac{m}{k} v_o^2 + x_o^2 = A^2$$

or

$$\frac{v_o^2}{\omega^2} + x_o^2 = A^2 \qquad \left[\because \omega^2 = \frac{k}{m}\right]$$

$$\therefore \qquad A = \sqrt{\frac{v_o^2}{\omega^2} + x_o^2}.$$

Text Based Exercises

Type A : Very Short Answer Questions · 1 Mark Each

1. What is periodic motion ? **[Himachal 04]**
2. What is oscillatory motion ? **[Himachal 04, 05C]**
3. What are harmonic functions ?
4. What is the period of each of the functions sec ωt and cosec ωt ?
5. Justify that sin θ and cos θ are periodic functions.
6. Define force constant. Give its SI unit.
7. Write the values of oscillation—amplitude and frequency from the equation $y = A \sin \omega t$ of S.H.M.
8. Write the relation between acceleration, displacement and frequency of a particle executing S.H.M.
9. The equation of motion of a particle executing S.H.M. is $a = -bx$, where a is the acceleration of the particle, x is the displacement from the mean position and b is a constant. What is the time period of the particle ?
10. Write the relation between time period T, displacement x and acceleration a of a particle in S.H.M.
11. Is spring constant a dimensional or non-dimensional constant ?
12. What is meant by phase of an oscillating particle ?
13. What is initial phase or epoch. Give a unit for its measurement.
14. Two simple pendulums of same length are crossing at their mean positions, what is phase difference between them ?
15. What is the phase relationship between particle displacement, velocity and acceleration in S.H.M. ?
16. What is phase difference between the displacement and acceleration of a particle executing S.H.M. ?
17. What is a second's pendulum ? What is its length ? **[Himachal 95C]**
18. A simple pendulum moves from one end to the other in 1/4 second. What is its frequency ?
19. Write the values of amplitude and angular frequency for the following simple harmonic motion.

$$y = 0.2 \sin(99\,t + 0.36)$$

20. Write the displacement equation representing the following conditions obtained in a simple harmonic motion :

Amplitude = 0.01 m,
Frequency = 600 Hz,
Initial phase = $\pi / 6$. **[Delhi 06]**

21. How will the time period of a simple pendulum change if its length is doubled ? **[Delhi 98]**
22. What would be the effect on the time period, if the amplitude of a simple pendulum increases ?
23. How will a simple pendulum behave if it is taken to the moon ?
24. A pendulum clock is thrown out of an aeroplane. How will it behave during its free fall in air ?
25. If on going up a hill, the value of g decreases by 10%, then what change must be made in the length of a pendulum clock in order to obtain accurate time ?
26. Which quantity is conserved during the oscillation of a simple pendulum ?
27. A girl is sitting in a swing. Another girl sits by her side. What will be the effect on the periodic time of the swing ?
28. What is the frequency of a second pendulum in an elevator rising up with an acceleration equal to $g / 2$?
29. Two identical springs of force constant k each are connected in series. What will be the equivalent spring constant ?
30. Two identical springs of force constant k each are connected in parallel. What will be the equivalent spring constant ?
31. The time period of a body executing S.H.M. is 0.05 s and the amplitude of vibration is 4 cm. What is the maximum velocity of the body ?
32. A particle executes S.H.M. of 2 cm. At the extreme position, the force is 4 N. What is the force at a point midway between mean and extreme positions ?
33. The potential energy of a particle in S.H.M. varies periodically. If ν is the frequency of oscillation of the particle, then what is the frequency of variation of potential energy ?

34. When will the motion of a simple pendulum be simple harmonic ?

35. When is the potential energy and kinetic energy of a harmonic oscillator maximum ? What are these maximum values ?

36. On what factors does the energy of a harmonic oscillator depend ?

37. What would be the time period of a simple pendulum at the centre of the earth ?

38. Can an ideal simple pendulum be realised in practice ? Is the motion of a simple pendulum-linear simple harmonic or angular simple harmonic ?

39. A simple harmonic motion of acceleration a and displacement x is represented by

$$a + 4\pi^2 x = 0.$$

What is the time period of S.H.M ?

40. State force law for a simple harmonic motion. **[Delhi 03]**

41. Give the general expression for displacement of a particle undergoing S.H.M. **[Central Schools 03]**

42. What are the two basic characteristics of an oscillating system ? **[Delhi 97]**

43. What is the frequency of oscillation of a simple pendulum mounted in a cabin that is freely falling under gravity ? **[Delhi 97]**

44. The amplitudes of oscillations of two simple pendulums similar in all respects are 2 cm and 5 cm respectively. Find the ratio of their energies of oscillations. **[Delhi 96]**

45. Define periodic time. Give its SI unit. **[Delhi 96]**

46. What is the main difference between forced oscillations and resonance ? **[Delhi 02]**

47. What is meant by SHM ? **[Himachal 05]**

48. What is meant by the displacement of a particle executing SHM ? **[Himachal 05]**

49. Define amplitude of SHM. **[Himachal 05]**

50. Define force constant and give its dimensional formula. **[Himachal 03]**

51. List any two characteristics of simple harmonic motion. **[Delhi 04]**

52. What is the time period of second's pendulum ? **[Himachal 03, 04]**

53. A pendulum is making one oscillation in every two seconds. What is the frequency of oscillation ? **[Delhi 04]**

54. A simple pendulum is inside a space craft. What should be its time period of vibration ? **[Central Schools 05]**

55. What is the condition to be satisfied by a mathematical relation between time and displacement to describe a periodic motion ? **[Central Schools 08]**

Answers

1. The motion which repeats itself over and over again after a fixed interval of time is called a periodic motion.

2. The motion which repeats itself over and over again about a mean position such that it remains confined within well defined limits (known as extreme positions) on either side of the mean position is called oscillatory motion.

3. The functions which can be represented by a sine or cosine curve are called harmonic functions.

4. Period of sec ωt or cosec $\omega t = 2\pi / \omega$.

5. Both sin θ and cos θ are the periodic functions of θ because,

$$\sin(\theta + 2\pi n) = \sin\theta \text{ and } \cos(\theta + 2\pi n) = \cos\theta,$$

where $n = 1, 2, 3....$

6. The restoring force produced per unit displacement of an oscillating body is called force constant or spring factor (k). Its SI unit is Nm^{-1}.

7. Amplitude = A,

frequency $= \omega / 2\pi$.

8. Acceleration,

$$a = -\omega^2 y = -4\pi^2 \nu^2 y.$$

9. Here $a = -bx = -\omega^2 x$,

where $\omega = \sqrt{b}$.

$$\therefore \quad T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{b}}.$$

10. $T = 2\pi\sqrt{\frac{x}{a}}$.

11. Spring constant,

$$k = \frac{F}{x} = \frac{\text{Restoring force}}{\text{Displacement}}$$

$$= \frac{[\text{MLT}^{-2}]}{[\text{L}]} = [\text{ML}^0\text{T}^{-2}]$$

Hence spring constant is a *dimensional constant*.

12. The phase of an oscillating particle at any instant gives the state of the particle as regards to its position and the direction of motion at that instant.

13. The phase of a vibrating particle corresponding to the time $t = 0$ is called initial phase or epoch. It is measured in radian.

14. 180° or π radian.

15. In S.H.M., the particle velocity leads the displacement in phase by $\pi / 2$ rad and acceleration leads the velocity in phase by $\pi / 2$ rad.

16. 180° or π radian.

17. A simple pendulum whose time period is 2 seconds is called a second's pendulum. Its length is 99.3 cm.

18. 2 Hz

19. Amplitude $A = 0.2$ m, angular frequency $\omega = 99$ Hz.

20. $y = a \sin(2\pi v t + \phi_0) = 0.01 \sin\left(1200\,\pi\, t + \frac{\pi}{6}\right)$.

21. As $T = 2\pi\sqrt{\frac{l}{g}}$, so when the length is doubled, the time period will be increased by $\sqrt{2}$ times.

22. The time period of the simple pendulum will remain the same, because time period is independent of its amplitude.

23. On the moon, the simple pendulum will oscillate $\sqrt{6}$ times slower than that it does on the surface of the earth because the value of g on the moon is $1/6$th of that on the earth.

24. During its free fall in air, the pendulum clock is in a state of weightlessness *i.e.*, $g = 0$. Hence

$$T = 2\pi\sqrt{\frac{l}{g}} = \infty$$

The pendulum clock will not oscillate at all.

25. The length of the pendulum clock should be decreased by 10%.

26. Total mechanical energy of the bob is conserved.

27. The periodic time remains unchanged because the length of the pendulum does not change when the second girl sits besides the first girl and T is independent of the mass of oscillating bob.

28. Frequency of a second pendulum, $v = \frac{1}{2}\,s^{-1}$

The effective value of g in the elevator,

$$g' = g + a = g + g/2 = 3g/2$$

As $\quad v = \frac{1}{2\pi}\sqrt{\frac{g}{l}}\quad$ *i.e.*, $\quad v \propto \sqrt{g}$

$$\therefore \quad \frac{v'}{v} = \sqrt{\frac{g'}{g}} = \sqrt{\frac{3}{2}} = 1.225$$

Hence $v' = 1.225v = 1.225 \times \frac{1}{2}$

$= \mathbf{0.612\ Hz.}$

29. $k_s = \frac{k \times k}{k + k} = \frac{k}{2}$.

30. $k_p = k + k = 2k$.

31. $v_{\max} = \frac{2\pi}{T}A = \frac{2\pi}{0.05} \times \frac{4}{100}$

$= \mathbf{1.6\pi\ ms^{-1}}$.

32. 2 N, because $F \propto x$.

33. $2v$.

34. When the displacement of the bob from the mean position is so small that $\sin\theta \simeq \theta$, the oscillations of the pendulum will be simple harmonic.

35. Potential energy of a harmonic oscillator is maximum at extreme position and minimum at extreme position, while kinetic energy is maximum at mean position.

Max. value of K.E. = Max. value of P.E. = $\frac{1}{2}m\omega^2 A^2$.

36. The energy of a harmonic oscillator depends on its (*i*) mass m (*ii*) frequency v and (*iii*) amplitude A.

$$E = 2\pi^2 m v^2 A^2$$

37. At the centre of the earth, $g = 0$, so

$$T = 2\pi\sqrt{l/g} = \infty.$$

38. No. The motion of simple pendulum is angular simple harmonic.

39. $a = -4\pi^2 x = -\omega^2 x$, where $\omega = 2\pi$

$$\therefore \quad T = \frac{2\pi}{\omega} = \frac{2\pi}{2\pi} = \mathbf{1\,s}$$

40. The force acting in S.H.M. is proportional to the displacement and is always directed towards the mean position. Hence the force law for S.H.M. is

$$F = -kx.$$

41. $x(t) = A\cos(\omega t + \phi_0)$ or $x(t) = a\cos\omega t + b\sin\omega t$.

42. The oscillations of a system result from its two basic characteristics, namely, *elasticity* and *inertia*.

43. In a freely falling cabin, $g = 0$, therefore

$$v = \frac{1}{2\pi}\sqrt{\frac{g}{l}} = 0.$$

44. $\frac{E_1}{E_2} = \left(\frac{A_1}{A_2}\right)^2 = \left(\frac{2}{5}\right)^2 = \mathbf{4 : 25.}$

45. The smallest interval of time after which a motion repeats itself over and over again is called its periodic time. The SI unit of periodic time is second (s).

46. In forced oscillations the frequency of the external periodic force is different from the natural frequency of the oscillator while these two frequencies are equal in resonant oscillations.

47. Refer to point 5 of Glimpses.

48. Refer to point 7 of Glimpses.

49. Refer to point 8 of Glimpses.

50. The restoring force produced per unit displacement of an oscillating body is called force constant. Its dimensional formula is $[ML^0T^{-2}]$

51. **Characteristics of SHM :**

(*i*) It is the simplest kind of oscillatory motion of constant amplitude and fixed frequency.

(*ii*) Restoring force is proportional to the displacement of the particle from its mean position.

52. 2 s.

53. $\nu = 1/2$ cps.

54. Inside a spacecraft, $g = 0$.

Therefore,

$$T = 2\pi\sqrt{\frac{l}{g}} = 2\pi\sqrt{\frac{l}{0}} = \infty.$$

55. A periodic motion repeats after a definite time interval T. So

$$y(t) = y(t+T) = y(t+2T), \text{ etc.}$$

Type B : Short Answer Questions **2 or 3 Marks Each**

1. Giving examples of each type, distinguish between periodic, harmonic and non-harmonic functions.

2. What is simple harmonic motion ? State its characteristics. **[Delhi 98]**

3. Write down the differential equation for S.H.M. Give its solution. Hence obtain expression for the time period of S.H.M.

4. Prove that the displacement equation

$$x(t) = a\cos\omega t + b\sin\omega t$$

represents a simple harmonic motion. Determine its amplitude and phase constant. **[Central Schools 05]**

5. Write expression for the particle velocity and acceleration during simple harmonic motion as function of time. **[Delhi 03C]**

6. Derive an expression for the instantaneous velocity and acceleration of a particle executing S.H.M. **[Himachal 01, 04, 05]**

7. Obtain an expression for the velocity of a particle executing S.H.M. When is this velocity (*i*) maximum and (*ii*) minimum ?

8. What is S.H.M. ? Show that the acceleration of a particle in S.H.M. is proportional to its displacement. Also write expression for the time-period in terms of acceleration. **[Central Schools 04]**

9. Show that in simple harmonic motion (S.H.M.), the acceleration is directly proportional to its displacement at the given instant. **[Delhi 08]**

10. The relation between the acceleration a and displacement x of a particle executing SHM is

$$a = -\left(\frac{p}{q}\right)y; \quad \text{where } p \text{ and } q \text{ are constants.}$$

What will be the time period T of the particle ?

11. Find an expression for the total energy of a particle executing S.H.M. **[Delhi 02 ; Himachal 05 ; Central Schools 05]**

12. Show that the total energy of a body executing S.H.M. is constant. **[Central Schools 07]**

13. Show that the total energy of a particle executing simple harmonic motion is directly proportional to the square of amplitude and frequency. **[Himachal 05C]**

14. A body is executing simple harmonic motion. At what distance from its mean position, its energy is half kinetic and half potential ? **[Delhi 96]**

15. Show that the horizontal oscillations of a massless loaded spring are simple harmonic. Deduce an expression for its time period.

16. Show that when a body is suspended from a spring and is pulled down a little and released, it executes S.H.M. Also find an expression for its time period. Does it depend on acceleration due to gravity ? **[Himachal 05 C]**

17. What is an ideal simple pendulum ? Derive an expression for its time period. **[Himachal 05C ; Chandigarh 07]**

18. What is a simple pendulum ? Show that motion executed by the bob of the pendulum is S.H.M. Derive an expression for its time period. **[Himachal 06 ; Chandigarh 08 ; Central Schools 12]**

19. Show that for small oscillations the motion of a simple pendulum is simple harmonic. Derive an expression for its time period. Does it depend on the mass of the bob ? **[Himachal 04 ; Delhi 08, 11]**

20. Prove that if a liquid taken in a U-tube is disturbed from the state of equilibrium, it will oscillate harmonically. Find expressions for the angular frequency and time period.

21. A ball of mass m fits smoothly in the cylindrical neck of an air chamber of volume V. The neck area is A. Show that the oscillations of the ball in the neck of the air chamber are simple harmonic. Calculate the time-period.

22. Show that the angular oscillations of the balance-wheel of a watch are simple harmonic. Hence deduce an expression for the time-period of its oscillations.

23. A cylindrical piece of cork of base area A and height h floats in a liquid of density ρ_1. The cork is depressed slightly and then released. Show that the cork oscillates up and down simple harmonically with a time period $T = 2\pi\sqrt{\dfrac{h\rho}{\rho_1 g}}$, where ρ is the density of the cork. **[Delhi 03]**

24. What are free, damped and maintained oscillations ? Give examples.

25. With the help of examples, differentiate between free oscillations and forced oscillations. **[Delhi 03]**

26. Briefly explain the principle underlying the tuning of a radio receiver.

27. What are coupled oscillations ? Give examples.

28. Show that in a S.H.M. the phase difference between displacement and velocity is $\pi/2$, and between displacement and acceleration it is π. **[Delhi 06]**

29. Draw the graphical representation of simple harmonic motion, showing the
    (*a*) displacement-time curve.
    (*b*) velocity-time curve and
    (*c*) acceleration-time curve. **[Chandigarh 07]**

Answers

1. Refer answer to Q. 5 on page 14.2.
2. Refer answer to Q. 6 and Q. 7 on page 14.4.
3. Refer answer to Q. 8 on page 14.4.
4. Refer answer to Q. 11 on page 14.7.
5. Refer answer to Q. 14 on page 14.8.
6. Refer to solution of Q. 12 on page 14.7 and Q. 13 on page 14.8.
7. Refer answer to Q. 12 on page 14.7.
8. Displacement, $x = A\cos\omega t$

   Velocity, $v = \dfrac{dx}{dt} = -\,\omega A\sin\omega t$

   Acceleration,
   $$a = \frac{dv}{dt} = -\,\omega^2 A\cos\omega t = -\,\omega^2 x$$

   $\therefore$ Therefore, acceleration $\propto$ displacement.

   Magnitude of acceleration in S.H.M. is
   $$a = \omega^2 x$$
   or $$\omega^2 = a/x$$
   $$\therefore \quad T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{a/x}}$$
   $$= 2\pi\sqrt{\frac{x}{a}} = 2\pi\sqrt{\frac{\text{Displacement}}{\text{Acceleration}}}$$
9. Refer answer to the above question.
10. $a = -\left(\dfrac{p}{q}\right)y = -\omega^2 y$
    $$\therefore \quad T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{p/q}} = 2\pi\sqrt{\frac{q}{p}}$$
11. Refer answer to Q. 15 on page 14.16.
12. Refer answer to Q. 15. on page 14.16.
13. Refer answer to Q. 15 on page 14.16.
14. Refer to the solution of Example 28 on page 14.18.
15. Refer answer to Q. 16 on page 14.20.
16. Refer answer to Q. 17 on page 14.20.
17. Refer answer to Q. 19 on page 14.27.
18. Refer answer to Q. 19 on page 14.27.
19. Refer answer to Q. 19 on page 19.27.
20. Refer answer to Q. 20 on page 14.30.
21. Refer answer to Q. 23 on page 14.31.
22. Refer answer to Q. 24 on page 14.32.
23. Refer answer to Q. 22 on page 14.31.
24. Refer answer to Q. 25 on page 14.34.
25. Refer answer to Q. 26 on page 14.36.
26. Refer answer to Q. 27 on page 14.37.
27. Refer answer to Q. 28 on page 14.37.
28. Refer answer to Q. 14 on page 14.8.
29. Refer answer to Q. 14 on page 14.8.

Type C : Long Answer Questions

5 Marks Each

1. With suitable examples, explain the meaning of a periodic function. Construct two infinite sets of periodic functions with period *T*. Hence state Fourier theorem.
2. Define the terms harmonic oscillator, displacement, amplitude, cycle, time period, frequency, angular frequency, phase and epoch with reference to an oscillatory system.
3. Show that simple harmonic motion may be regarded as the projection of uniform circular motion along a diameter of the circle. Hence derive an expression for the displacement of a particle in S.H.M.
4. Explain the relation in phase between displacement, velocity and acceleration in SHM, graphically as well as theoretically. **[Chandigarh 04]**
5. Derive expressions for the kinetic and potential energies of a harmonic oscillator. Hence show that total energy is conserved in S.H.M. **[Delhi 12]**
6. Find the total energy of the particle executing S.H.M. and show graphically the variation of P.E. and K.E. with time in S.H.M. What is the frequency of these energies with respect to the frequency of the particle executing S.H.M ? **[Delhi 05]**
7. Show that for a particle in linear S.H.M., the average kinetic energy over a period of oscillation is equal to the average potential energy over the same period.

   At what distance from the mean position is the kinetic energy in simple harmonic oscillator equal potential energy ? **[Delhi 06]**
8. What is a spring factor ? Derive the expression for resultant spring constant when two springs having constants k_1 and k_2 are connected in

   (*i*) parallel, and (*ii*) in series.

   [Chandigarh 04 ; Central Schools 05]

Answers

1. Refer answer to Q. 4 on page 14.2.
2. Refer answer to Q. 9 on page 14.4.
3. Refer answer to Q. 10 on page 14.6.
4. Refer answer to Q. 14 on page 14.8.
5. Refer answer to Q. 15 on page 14.16.
6. Refer answer to Q. 15 on page 14.16.
7. Refer to the solution on NCERT Exercise 14.22 on page 14.54 and Example 28 on page 14.18.
8. Refer answer to Q. 18 on page 14.21.

Competition Section

Oscillations

GLIMPSES

1. **Periodic motion.** A motion which repeats itself over and over again after a regular interval of time is called a periodic motion.
2. **Oscillatory motion.** A motion in which a body moves back and forth repeatedly about a fixed point (called mean position) is called oscillatory or vibratory motion.
3. **Periodic function.** Any function that repeats its value at regular intervals of its argument is called a periodic function. The following sine and cosine functions are periodic with period T.

$$f(t) = \sin \frac{2\pi t}{T} \quad \text{and} \quad g(t) = \cos \frac{2\pi t}{T}$$

The periodic functions which can be represented by a sine or cosine curve are called *harmonic functions*. All harmonic functions are necessarily periodic but all periodic functions are not harmonic.

The periodic functions which cannot be represented by single sine or cosine function are called *non-harmonic functions.*

4. **Fourier theorem.** Two infinite sets of periodic functions with period T are

$$f_n(t) = \sin \frac{2\pi nt}{T}, \quad n = 1, 2, 3, 4, \ldots$$

$$g_n(t) = \cos \frac{2\pi nt}{T}, \quad n = 0, 1, 2, 3, \ldots$$

Fourier theorem states that any periodic function $F(t)$ with period T can be expressed as the unique combination of sine and cosine functions $f_n(t)$ and $g_n(t)$ with suitable coefficients. Mathematically,

$$F(t) = b_0 + \sum_n b_n \cos n\omega t + \sum_n a_n \sin n\,\omega t$$

where $\omega = 2\pi / T$. The coefficients $b_0, b_1, b_2, \ldots$; $a_1, a_2, a_3, \ldots$ are called *Fourier coefficients*. The special case of Fourier theorem in which only a_1 and b_1 are non-zero represents *simple harmonic motion* (S.H.M.).

$$F(t) = a_1 \sin \frac{2\pi t}{T} + b_1 \cos \frac{2\pi t}{T}.$$

5. **Simple harmonic motion.** A particle is said to execute simple harmonic motion if it moves to and fro about a mean position under the action of a restoring force which is directly proportional to its displacement from the mean position and is always directed towards the mean position. If the displacement of the oscillating particle from the mean position is small, then

Restoring force $\propto$ Displacement

or $\quad F \propto x$

or $\quad F = -k\,x$

where k is a positive constant called *force constant* or *spring factor* and is defined as the restoring force produced per unit displacement. The negative sign shows that the restoring force always acts in the opposite direction of displacement x. The above equation defines SHM.

6. **Oscillation or cycle.** One complete back and forth motion of a particle is called cycle or vibration or oscillation.
7. **Displacement.** It is the distance of the oscillating particle from the mean position at any instant. It is denoted by x.
8. **Amplitude (A).** The maximum displacement of the oscillating particle on either side of its mean position is called its amplitude. Thus $x_{\max} = \pm A$.
9. **Time period.** It is the time taken by a particle to complete one oscillation about its mean position. It is denoted by T.
10. **Frequency.** It is the number of oscillations completed per second by a particle about its mean position. It is denoted by ν and is equal to the reciprocal of time period. Thus $\nu = \frac{1}{T}$

Frequency is measured in hertz.

1 hertz = 1 Hz = 1 oscillation per second = $1\,\text{s}^{-1}$.

11. **Angular frequency.** It is the quantity obtained by multiplying frequency ν by a factor of 2π. It is denoted by ω.

Thus $\omega = 2\pi\nu = \dfrac{2\pi}{T}$.

SI unit of $\omega = \text{rad s}^{-1}$.

12. **Phase.** The phase of vibrating particle at any instant gives the state of the particle as regards its position and the direction of motion at that instant. It is denoted by ϕ.

13. **Initial phase or epoch.** The phase of a vibrating particle corresponding to time $t = 0$ is called initial phase or epoch. It is denoted by ϕ_0.

14. **Phase difference.** The phase difference between two vibrating particles tells the lack of harmony in the vibrating states of the two particles at any instant.

15. **Relation between SHM and uniform circular motion.** Simple harmonic motion is the projection of uniform circular motion upon a diameter of a circle. This circle is called the *reference circle* and the particle which revolves along it is called *reference particle* or *generating particle*.

16. **Displacement in SHM.** In a simple harmonic motion, the displacement of a particle from its equilibrium position at any instant t is given by

$$x(t) = A\cos(\omega t + \phi_0)$$

Here A is *amplitude* of the displacement, the quantity $(\omega t + \phi_0)$ is the *phase* of the motion and ϕ_0 is the initial phase.

When the time is measured from the mean position,

$$x(t) = A\sin\omega t$$

When the time is measured from the extreme position,

$$x(t) = A\cos\omega t$$

The angular frequency ω, frequency ν and time period T of the motion are given by

$$\omega = \sqrt{\frac{a}{x}} \quad \text{or} \quad \omega = \sqrt{\frac{k}{m}}$$

$$\nu = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{a}{x}} \quad \text{or} \quad \nu = \frac{1}{2\pi}\sqrt{\frac{k}{m}}.$$

$$T = 2\pi\sqrt{\frac{\text{Displacement}}{\text{Acceleration}}} = 2\pi\sqrt{\frac{x}{a}}$$

$$\text{or} \quad T = 2\pi\sqrt{\frac{\text{Inertia factor}}{\text{Spring factor}}} = 2\pi\sqrt{\frac{m}{k}}.$$

17. **Velocity in SHM.** It is the rate of change of displacement of the particle at any instant. It is given by

$$v = \frac{dx}{dt} = \frac{d}{dt}[A\cos(\omega t + \phi_0)]$$
$$= -\omega A\sin(\omega t + \phi_0) = -\omega\sqrt{A^2 - x^2}$$

The maximum value of velocity is called *velocity amplitude* v_m of the motion.

Thus $v_m = \omega A = \dfrac{2\pi}{T}A$

At the mean position, particle velocity $= v_m = \omega A$

At the extreme position, particle velocity $= 0$.

18. **Acceleration in SHM.** It is the rate of change of velocity of the particle at any instant. It is given by

$$a = \frac{dv}{dt} = \frac{d}{dt}[-\omega A\sin(\omega t + \phi_0)]$$
$$= -\omega^2 A\cos(\omega t + \phi_0) = -\omega^2 x$$

i.e., $a \propto x$

The maximum value of acceleration of particle is called acceleration amplitude a_m. Thus

$$a_m = \omega^2 A$$

At the mean position, particle acceleration = 0

At the extreme position, particle acceleration,

$$a_m = \omega^2 A.$$

19. **Phase relationship between displacement, velocity and acceleration.** In SHM, the particle velocity is ahead of displacement by $\pi/2$ rad while acceleration is ahead of displacement by π rad.

20. **Energy of SHM.** If a particle of mass m executes SHM, then at a displacement x from mean position, the particle possesses potential and kinetic energy. At any displacement x,

Potential energy, $U = \dfrac{1}{2}m\omega^2 x^2 = \dfrac{1}{2}kx^2$

Kinetic energy, $K = \dfrac{1}{2}m\omega^2(A^2 - x^2)$

$= \dfrac{1}{2}k(A^2 - x^2)$

Total energy,

$$E = U + K = \frac{1}{2}m\omega^2 A^2 = 2\pi^2 m\nu^2 A^2.$$

If there is no friction, the total mechanical energy, $E = K + U$, of the system always remains constant even though K and U change.

21. **Motion of a massless loaded spring.** When a mass m is attached to a massless spring and pulled downwards, it executes SHM. If l is extension in the spring on attaching mass m and k is its force constant, then time period of SHM executed by the spring

$$T = 2\pi\sqrt{\frac{l}{g}} = 2\pi\sqrt{\frac{m}{k}}$$

22. **Spring cut into parts.** If we divide the spring of spring constant k into n equal parts, the spring constant of each part becomes $n\,k$. Hence the time period when the same mass m is suspended from each part is

$$T = 2\pi\sqrt{\frac{m}{n\,k}}$$

23. **Springs connected in series.** If two springs of spring constants k_1 and k_2 are connected in series, then the spring constant k of the combination is given by

$$\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2} \quad \text{or} \quad k = \frac{k_1 k_2}{k_1 + k_2}$$

$$\therefore \quad T = 2\pi\sqrt{\frac{m\,(k_1 + k_2)}{k_1\,k_2}}$$

24. **Springs connected in parallel.** If two springs of spring constants k_1 and k_2 are connected in parallel, then the spring constant k of combination is

$$k = k_1 + k_2 \qquad \therefore \quad T = 2\pi\sqrt{\frac{m}{k_1 + k_2}}$$

25. **Simple pendulum.** A simple pendulum is a heavy point mass suspended by a weightless, inextensible and a perfectly flexible string from a rigid support about which it can vibrate freely. The distance between the point of suspension and the point of oscillation is called *length of the pendulum* (l). When the metallic bob is displaced from mean position, it executes SHM.

$$\text{Time period,} \quad T = 2\pi\sqrt{\frac{l}{g}}$$

26. **Second's pendulum.** A second's pendulum is a pendulum whose time period is two seconds. Its length is 99.3 cm.

27. **Motion of a liquid in a U-tube.** When a liquid of density ρ and contained in a U-tube upto height h is depressed, it executes SHM of time period,

$$T = 2\pi\sqrt{\frac{h}{g}}$$

28. **Motion of a body dropped in a tunnel dug along the diameter of earth.** When a body is dropped in a tunnel dug along the diameter of the earth, it executes SHM. If R is radius of the earth, then its time period is

$$T = 2\pi\sqrt{\frac{R}{g}}$$

29. **Motion of a body floating in a liquid.** When a body made of material of density ρ and total vertical length L floats in a liquid of density ρ, such that its length h is submerged in the liquid, it executes SHM on being pushed into the liquid.

$$T = 2\pi\sqrt{\frac{\rho\,L}{\sigma\,g}} = 2\pi\sqrt{\frac{h}{g}}$$

30. **Free oscillations.** If a body, capable of oscillation, is slightly displaced from its position of equilibrium and then released, it starts oscillating with a frequency of its own. Such oscillations are called free oscillations. The frequency with which a body oscillates is called *natural frequency* and is given by

$$\nu_0 = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

Here a body continues to oscillate with constant amplitude and fixed frequency.

31. **Damped oscillations.** The oscillations in which amplitude decreases gradually with the passage of time are called damped oscillations.

The energy of a real oscillator decreases because a part of its mechanical energy is used in doing work against the frictional forces and is lost as heat. If the *damping force* is given by $F_d = -bv$, where v is the velocity of the oscillator and b is a *damping constant*, then the displacement of the oscillator is given by,

$$x\,(t) = Ae^{-bt/2m}\cos\,(\omega' t + \phi)$$

where ω', the angular frequency of the damped oscillator, is given by

$$\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$

If the damping constant is small then $\omega' \simeq \omega$, where ω is the angular frequency of the undamped oscillator. The mechanical energy E of the oscillator is given by

$$E\,(t) = \frac{1}{2}\,kA^2\,e^{-bt/m}.$$

32. **Forced oscillations.** When a body oscillates under the influence of an external periodic force, not with its own natural frequency but the frequency of the external periodic force, its oscillations are said to be forced oscillations.

33. **Resonant oscillations.** It is a particular case of forced oscillations in which the frequency of the driving force is equal to the natural frequency of the oscillator itself and the amplitude of oscillations is greatest. Such oscillations are called resonant oscillations and phenomenon is called *resonance*.

34. **Coupled oscillations.** A system of two or more oscillators linked together in such a way that there is mutual exchange of energy between them is called a coupled oscillator. The oscillations of such a system are called coupled oscillations.

IIT Entrance Exam

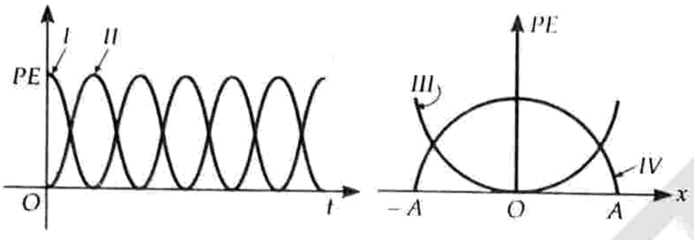
MULTIPLE CHOICE QUESTIONS WITH ONE CORRECT ANSWER

1. A particle executes simple harmonic motion between $x = -A$ and $x = +A$. The time taken for it to go from 0 to $A/2$ is T_1 and to go from $A/2$ to A is T_2. Then

(*a*) $T_1 < T_2$ (*b*) $T_1 > T_2$

(*c*) $T_1 = T_2$ (*d*) $T_1 = 2T_2$ [IIT 01]

2. For a particle executing SHM the displacement x is given by $x = A\cos\omega t$. Identify the graph which represents the variation of potential energy (PE) as a function of time t and displacement x



(*i*) (*ii*)

(*a*) I, III (*b*) II, IV

(*c*) II, III (*d*) I, IV [IIT 03]

3. A particle free to move along the x-axis has potential energy given by $U(x) = k[1 - \exp(-x)^2]$ for $-\infty \le x \le +\infty$, where k is a positive constant of appropriate dimensions. Then

(*a*) at points away from the origin, the particle is in unstable equilibrium

(*b*) for any finite nonzero value of x, there is a force directed away from the origin

(*c*) if its total mechanical energy is $k/2$, it has its minimum kinetic energy at the origin

(*d*) for small displacements from $x = 0$, the motion is simple harmonic. [IIT 99]

4. A spring of force constant k is cut into two pieces, such that one piece is double the length of the other. Then, the long piece will have a force constant of

(*a*) $\frac{2}{3}k$ (*b*) $\frac{3}{2}k$

(*c*) $3k$ (*d*) $6k$ [IIT 99]

5. Two bodies M and N of equal masses are suspended from two separate massless springs of spring constants k_1 and k_2 respectively. If the two bodies oscillate vertically such that their maximum velocities are equal, the ratio of the amplitude of vibration of M to that of N is

(*a*) $\frac{k_1}{k_2}$ (*b*) $\sqrt{k_1/k_2}$

(*c*) $\frac{k_2}{k_1}$ (*d*) $\sqrt{\frac{k_2}{k_1}}$ [IIT 98]

6. An object of mass 0.2 kg executes simple harmonic motion along the x-axis with a frequency of $(25/\pi)$ Hz. At the position $x = 0.04$, the object has kinetic energy of 0.5 J and potential energy 0.4 J. The amplitude of oscillations is

(*a*) 6 cm (*b*) 4 cm

(*c*) 8 cm (*d*) 2 cm [IIT 94]

7. A simple pendulum has a time period T_1, when on the earth's surface ; and T_2, when taken to a height R above the earth's surface (R is the radius of the earth). The value of T_2/T_1 is

(*a*) 1 (*b*) $\sqrt{2}$

(*c*) 4 (*d*) 2 [IIT 01]

8. The period of oscillation of a simple pendulum of length L suspended from the roof of a vehicle which moves without friction down an inclined plane of inclination α, is given by

(*a*) $2\pi\sqrt{\frac{L}{g\cos\alpha}}$ (*b*) $2\pi\sqrt{\frac{L}{g\sin\alpha}}$

(*c*) $2\pi\sqrt{\frac{L}{g}}$ (*d*) $2\pi\sqrt{\frac{L}{g\tan\alpha}}$ [IIT 2K]

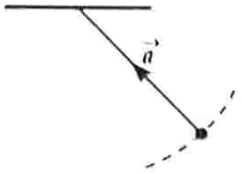
9. A simple pendulum has time period T_1. The point of suspension is now moved upward according to the relation $y = Kt^2$, $(K = 1\,\text{m}/\text{s}^2)$ where y is the vertical displacement. The time period now becomes T_2. The ratio of $\frac{T_1^2}{T_2^2}$ $(g = 10\,\text{m}/\text{s}^2)$ is

(*a*) $\frac{5}{6}$ (*b*) $\frac{6}{5}$

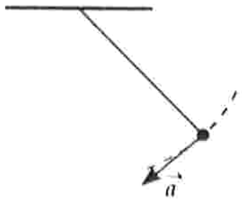
(*c*) 1 (*d*) $\frac{4}{5}$ [IIT 05]

10. A simple pendulum is oscillating without damping. When the displacement of the bob is less than maximum, its acceleration vector is $\vec{a}$ correctly shown in

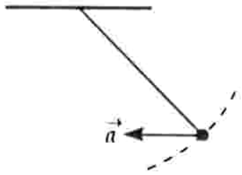
(*a*)



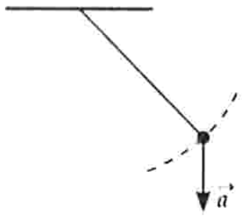
(*b*)



(*c*)

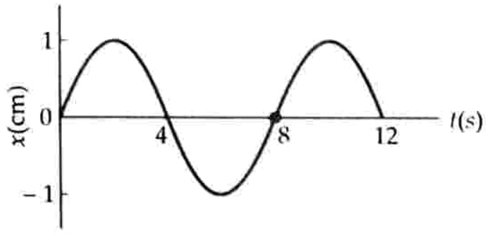


(*d*)



[IIT 02]

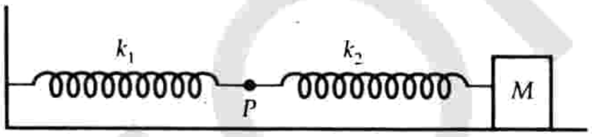
11. The x-t graph of a particle undergoing simple harmonic motion is shown below. The acceleration of the particle at $t = 4/3$ is



(*a*) $\frac{\sqrt{3}}{32}\pi^2$ cm/s^2 (*b*) $\frac{-\pi^2}{32}$ cm/s^2

(*c*) $\frac{\pi^2}{32}$ cm/s^2 (*d*) $-\frac{\sqrt{3}}{32}\pi^2$ cm/s^2 [IIT 09]

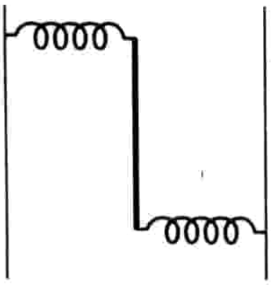
12. The mass M shown in the figure oscillates in simple harmonic motion with amplitude A. The amplitude of the point P is



(*a*) $\frac{k_1 A}{k_2}$ (*b*) $\frac{k_2 A}{k_1}$

(*c*) $\frac{k_1 A}{k_1 + k_2}$ (*d*) $\frac{k_2 A}{k_1 + k_2}$ [IIT 09]

13. A uniform rod of length L and mass M is pivoted at the centre. Its two ends are attached to two springs of equal constants k. The springs are fixed to rigid supports as shown in the figure, and the rod is free to oscillate in the horizontal plane. The rod is gently pushed through a small angle θ in one direction and released. The frequency of oscillation is



(*a*) $\frac{1}{2\pi}\sqrt{\frac{2k}{M}}$ (*b*) $\frac{1}{2\pi}\sqrt{\frac{k}{M}}$

(*c*) $\frac{1}{2\pi}\sqrt{\frac{6k}{M}}$ (*d*) $\frac{1}{2\pi}\sqrt{\frac{24k}{M}}$ [IIT 09]

14. A uniform cylinder of length L and mass M having cross-sectional area A is suspended, with its length vertical, from a fixed point by a massless spring, such that it is half-submerged in a liquid of density ρ at equilibrium position. When the cylinder is given a small downward push and released it starts oscillating vertically with small amplitude. If the force constant of the spring is k, the frequency of oscillation of the cylinder is

(*a*) $\frac{1}{2\pi}\left(\frac{k - A\rho g}{M}\right)^{1/2}$ (*b*) $\frac{1}{2\pi}\left(\frac{k + A\rho g}{M}\right)^{1/2}$

(*c*) $\frac{1}{2\pi}\left(\frac{k + \rho g L}{M}\right)^{1/2}$ (*d*) $\frac{1}{2\pi}\left(\frac{k + \rho g L}{M}\right)^{1/2}$ [IIT 90]

15. One end of a long metallic wire of length L is tied to the ceiling. The other end is tied to a massless spring of spring constant k. A mass m hangs freely from the free end of the spring. The area of cross-section and the Young's modulus of the wire are A and Y respectively. If the mass is slightly pulled down and released, it will oscillate with a time period T equal to

(*a*) $2\pi(m/k)^{1/2}$ (*b*) $2\pi\sqrt{\frac{m(YA + kL)}{YAk}}$

(*c*) $2\pi(mYA/kL)^{1/2}$ (*d*) $2\pi(mL/YA)^{1/2}$ [IIT 93]

16. A highly rigid cubical block A of small mass M and side L is fixed rigidly onto another cubical block B of the same dimensions and of low modulus of rigidity η such that the lower face of A completely covers the upper face of B. The lower face of B is rigidly held on a horizontal surface. A small force F is applied perpendicular to one of the side faces of A. After the force is withdrawn, block A executes small oscillations, the time period of which is given by

(*a*) $2\pi\sqrt{M\eta L}$ (*b*) $2\pi\sqrt{\frac{M\eta}{L}}$

(*c*) $2\pi\sqrt{\frac{ML}{\eta}}$ (*d*) $2\pi\sqrt{\frac{M}{\eta L}}$. [IIT 92]

17. A point mass is subjected to two simultaneous sinusoidal displacements in x-direction, $x_1(t) = A\sin\omega t$ and $x_2(t) = A\sin\left(\omega t + \frac{2\pi}{3}\right)$. Adding a third sinusoidal displacement $x_3(t) = B\sin(\omega t + \phi)$ brings the mass to a complete rest. The values of B and ϕ are

(*a*) $\sqrt{2}A, \frac{3\pi}{4}$ (*b*) $A, \frac{4\pi}{3}$

(*c*) $\sqrt{3}A, \frac{5\pi}{6}$ (*d*) $A, \frac{\pi}{3}$ [IIT 2011]

MULTIPLE CHOICE QUESTIONS WITH ONE OR MORE THAN ONE CORRECT ANSWER

18. The function $x = A\sin^2\omega t + B\cos^2\omega t + C\sin\omega t\cos\omega t$ represents simple harmonic motion for which of the option (s) ?

(*a*) for all values of A, B and C $(C \neq 0)$
(*b*) $A = B, C = 2B$
(*c*) $A = -B, C = 2B$
(*d*) $A = B, C = 0$. [IIT 06]

19. Three simple harmonic motions in the same direction having the same amplitude a and same period are suspended. If each differs in phase from the next by 45°, then

(*a*) the resultant amplitude is $(1+\sqrt{2})a$
(*b*) the phase of the resultant relative to the first is 90°
(*c*) the energy associated with the resulting motion is $(3+2\sqrt{2})$ times the energy associated with any single motion
(*d*) the resulting motion is not simple harmonic. [IIT 99]

20. A particle executes simple harmonic motion with a frequency f. The frequency with which its kinetic energy oscillates is

(*a*) $\frac{f}{2}$ (*b*) f

(*c*) $2f$ (*d*) $4f$ [IIT 87]

21. A linear harmonic oscillator of force constant 2×10^6 N/m and amplitude 0.01 m has a total mechanical energy of 160 J. Its

(*a*) maximum potential energy is 100 J
(*b*) maximum kinetic energy is 100 J
(*c*) maximum potential energy is 160 J
(*d*) maximum potential energy is zero. [IIT 89]

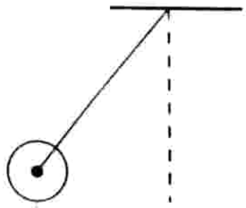
22. A particle of mass m is executing oscillations about the origin on the x-axis. Its potential energy is $U(x) = k|x|^3$, where k is a positive constant. If the amplitude of oscillation is a, then its time period T is

(*a*) proportional to $1/\sqrt{a}$
(*b*) independent of a
(*c*) proportional to $\sqrt{a}$
(*d*) proportional to $a^{3/2}$. [IIT 98]

23. A simple pendulum of length L and mass (bob) M is oscillating in a plane about a vertical line between angular limits $-\phi$ and $+\phi$. For an angular displacement $\theta(|\theta| < \phi)$, the tension in the string and the velocity of the bob are T and v respectively. The following relations hold good under the above conditions :

(*a*) $T\cos\theta = Mg$
(*b*) $T - Mg\cos\theta = \frac{Mv^2}{L}$
(*c*) The magnitude of the tangenial acceleration of the bob $|a_T| = g\sin\theta$
(*d*) $T = Mg\cos\theta$. [IIT 86]

24. A metal rod of length L and mass m is pivoted at one end. A thin disc of mass M and radius $R(< L)$ is attached at its centre to the free end of the rod. Consider two ways the disc is attached : (case A). The disc is not free to rotate about its centre and (case B) the disc is free to rotate about its centre. The rod-disc system performs SHM in vertical plane after being released from the same displaced position. Which of the following statement (s) is/are true ?

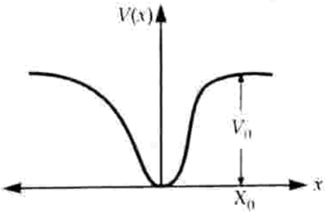


(*a*) restoring torque in case A = restoring torque in case B
(*b*) restoring torque in case $A <$ restoring torque in case B
(*c*) angular frequency for case $A >$ angular frequency for case B
(*d*) angular frequency for case $A <$ angular frequency for case B. [IIT 2011]

COMPREHENSION BASED QUESTIONS

PARAGRAPH FOR QUESTIONS 25 TO 27

When a particle of mass m moves on the x-axis in a potential of the form $V(x) = kx^2$, it performs simple harmonic motion. The corresponding time period is proportional to $\sqrt{\frac{m}{k}}$, as can be seen easily using dimensional analysis. However, the motion of a particle can be periodic even when its potential energy increases on both sides of $x = 0$ in a way different from kx^2 and its total energy is such that the particle does not escape to infinity. Consider a particle of mass m moving on the x-axis. Its potential energy is $V(x) = \alpha x^4 (\alpha > 0)$ for $|x|$ near the origin and becomes a constant equal to V_0 for $|x| \geq X_0$ (see figure).



25. If the total energy of the particle is E, it will perform periodic motion only if

(a) $E < 0$ (b) $E > 0$

(c) $V_0 > E > 0$ (d) $E > V_0$ [IIT 2010]

26. For periodic motion of small amplitude A, the time period T of this particle is proportional to

(a) $A\sqrt{\frac{m}{\alpha}}$ (b) $\frac{1}{A}\sqrt{\frac{m}{\alpha}}$

(c) $A\sqrt{\frac{\alpha}{m}}$ (d) $\frac{1}{A}\sqrt{\frac{\alpha}{m}}$ [IIT 2010]

27. The acceleration of this particle for $|x| > X_0$ is

(a) proportional to V_0

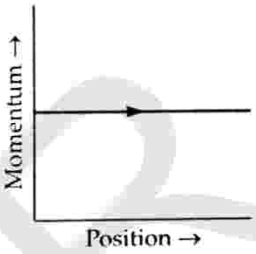
(b) proportional to $\frac{V_0}{mX_0}$

(c) proportional to $\sqrt{\frac{V_0}{mX_0}}$

(d) zero [IIT 2010]

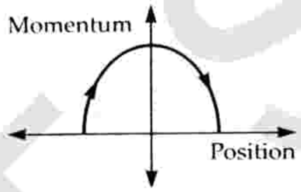
PARAGRAPH FOR QUESTION 28 TO 30

Phase space diagrams are useful tools in analyzing all kinds of dynamical problems. They are especially useful in studying the changes in motion as initial position and momentum are changed. Here we consider some simple dynamical systems in one-dimension. For such system, phase space is a plane in which position is plotted along horizontal axis and momentum is plotted along vertical axis. The phase space diagram is $x(t)$ vs $p(t)$ curve in this plane. The arrow on the curve indicates the time flow. For example, the phase space diagram for a particle moving with constant velocity is a straight line as shown in the figure. We use the sign convention in which position or momentum upwards (or to right) is positive and downwards (or to left) is negative.

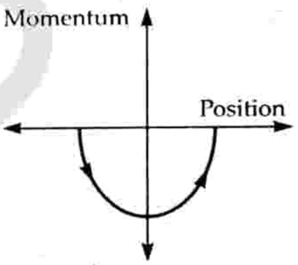


28. The phase space diagram for a ball thrown vertically up from ground is

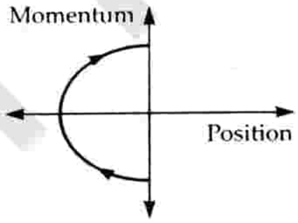
(a)



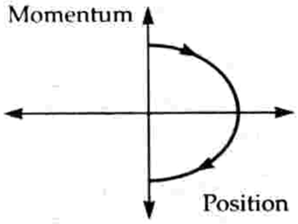
(b)



(c)

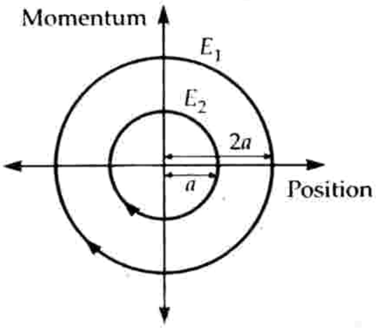


(d)



[IIT 2011]

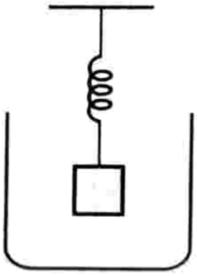
29. The phase space diagram for simple harmonic motion is a circle centered at the origin. In the figure, the two circles represent the same oscillator but for different initial conditions, and E_1 and E_2 are the total mechanical energies respectively. Then



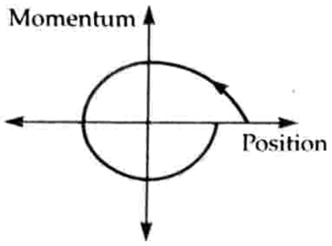
(a) $E_1 = \sqrt{2}\ E_2$ (b) $E_1 = 2\ E_2$

(c) $E_1 = 4\ E_2$ (d) $E_1 = 16\ E_2$ [IIT 2011]

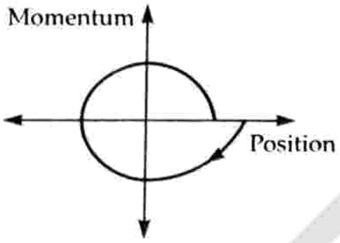
30. Consider the spring mass system, with the mass submerged in water, as shown in the figure. The phase space diagram for one cycle of this system is



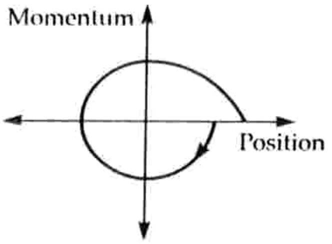
(*a*)



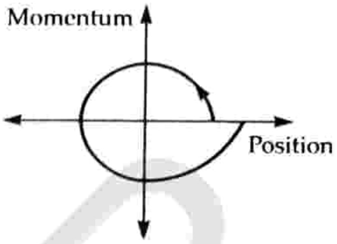
(*b*)



(*c*)



(*d*)



[IIT 2010]

Integer Type Answer

31. A 0.1 kg mass is suspended from a wire of negligible mass. The length of the wire is 1 m and its cross-sectional area is $4.9 \times 10^{-7}\,\text{m}^2$. If the mass is pulled a little in the vertically downward direction and released, it performs simple harmonic motion of angular frequency 140 rad s^{-1}. If the Young's modulus of the material of the wire is $n \times 10^9\,\text{Nm}^{-2}$, find the value of n. [IIT 2010]

Match – Matrix Type

32. Column I describes some situations in which a small object moves. Column II describes some characteristics of these motions. Match the situations in column I with the characteristics in column II. [IIT 07]

Column I	Column II
(*a*) The object moves on the x-axis under a conservative force in such a way that its speed and position satisfy $v = c_1\sqrt{c_2 - x^2}$, where c_1 and c_2 are positive constants.	(*p*) The object executes a simple harmonic motion.
(*b*) The object moves on the x-axis in such a way that its velocity and its displacement from the origin satisfy $v = -kx$, where k is a positive constant.	(*q*) The object does not change its direction.
(*c*) The object is attached to one end of a massless spring of a given spring constant. The other end of the spring is attached to the ceiling of an elevator. Initially everything is at rest. The elevator starts going upwards with a constant acceleration a. The motion of the object is observed from the elevator during the period it maintains this acceleration.	(*r*) The kinetic energy of the objects keeps on decreasing.
(*d*) The object is projected from the earth's surface vertically upwards with a speed $2\sqrt{GM_e / R_e}$, where M_e is the mass of the earth and R_e is the radius of the earth. Neglect forces from objects other than the earth.	(*s*) The object can change its direction only once.

Answers and Explanations

1. (*a*) $x = A\sin\omega t$

For $x = \frac{A}{2}$, $\sin\omega T_1 = \frac{1}{2}$

or $\omega T_1 = \frac{\pi}{6}$ or $T_1 = \frac{\pi}{6\omega}$

For $x = A$, $\sin\omega(T_1 + T_2) = 1$

or $\omega(T_1 + T_2) = \frac{\pi}{2}$

or $T_1 + T_2 = \frac{\pi}{2\omega}$

$$\therefore \quad T_2 = \frac{\pi}{2\omega} - \frac{\pi}{6\omega} = \frac{\pi}{3\omega} = \mathbf{2T_1}.$$

2. (*a*) At $t = 0$, $x = A\cos 0 = A$. The particle is at extreme position and its P.E. must be maximum. Hence the correct options are I and III.

3. (*d*) $U(x) = k(1 - e^{-x^2})$

$$F = -\frac{dU}{dx} = -2kxe^{-x^2} = -2kx(1 - x^2 + ...)$$

For small x, $F \simeq -2\,kx$.

This shows that the force is directed towards the origin and for smaller x, $F \propto x$. Hence the motion is simple harmonic.

4. (*b*) Force constant, $k = \frac{F}{x}$

The length of the long piece is $2x/3$.

So, its force constant is

$$k' = \frac{F}{2x/3} = \frac{3}{2}\frac{F}{x} = \mathbf{\frac{3}{2}k}.$$

5. (*d*) $v_{\max}(A) = v_{\max}(B)$

$$\omega_1 A_1 = \omega_2 A_2$$

$$\sqrt{\frac{k_1}{m}}\,A_1 = \sqrt{\frac{k_2}{m}}\,A_2$$

$$\frac{A_1}{A_2} = \sqrt{\frac{\mathbf{k_2}}{\mathbf{k_1}}}.$$

6. (*a*) Total energy,

$$E = 2\pi^2 m v^2 A^2$$

$$0.5 + 0.4 = 2\pi^2 \times 0.2 \times \left(\frac{25}{\pi}\right)^2 A^2$$

$$A^2 = \frac{0.9}{0.4 \times (25)^2}$$

$$A = \frac{3}{2 \times 25} = \frac{3}{50}\text{ m} = \mathbf{6\ cm.}$$

7. (*d*) $\frac{g'}{g} = \left(\frac{R}{R+R}\right)^2 = \frac{1}{4}$

As $T \propto \frac{1}{\sqrt{g}}$

$$\frac{T_2}{T_1} = \sqrt{\frac{g}{g'}} = \sqrt{\frac{4}{1}} = \mathbf{2.}$$

8. (*a*) The effective value of g will be equal to the component of g normal to the inclined plane which is

$$g' = g\cos\alpha$$

$$\therefore \quad T = 2\pi\sqrt{\frac{L}{g'}} = \mathbf{2\pi\sqrt{\frac{L}{g\cos\alpha}}}$$

9. (*b*) $y = kt^2$

Velocity $= \frac{dy}{dt} = 2kt$

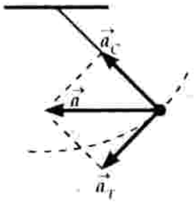
Acceleration $= \frac{d^2y}{dt^2} = 2k = 2 \times 1 = 2\text{ ms}^{-2}$

$\therefore \quad g_2 = g_1 + 2 = 10 + 2 = 12\text{ ms}^{-2}$

As $T = 2\pi\sqrt{\frac{l}{g}}$ or $T \propto \frac{1}{\sqrt{g}}$

$$\therefore \quad \frac{T_1^2}{T_2} = \frac{g_2}{g_1} = \frac{12}{10} = \mathbf{\frac{6}{5}}.$$

10. (*c*) When the displacement of the bob is less than maximum, there will two component accelerations of the bob :



Transverse component $= \vec{a_T}$

Centripetal or radial component $= \vec{a_C}$

The resultant acceleration $\vec{a}$ will be along the diagonal of the parallelogram.

11. (*d*) From the x-t graph,

$$T = 8\text{ s}$$

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{8} = \frac{\pi}{4}\text{ rad}, \quad t = \frac{4}{3}\text{s}$$

$$\omega t = \frac{\pi}{4} \times \frac{4}{3} = \frac{\pi}{3}\text{ rad}$$

$$a = -\omega^2 A\sin\omega t$$

$$= -\left(\frac{\pi}{4}\right)^2 \times 1 \times \sin\frac{\pi}{3}$$

$$= -\frac{\mathbf{\sqrt{3}}}{\mathbf{32}}\pi^2\ \mathbf{cm/s^2}.$$

12. (*d*) Here $x_1 + x_2 = A$

As internal forces in the two springs are the same,

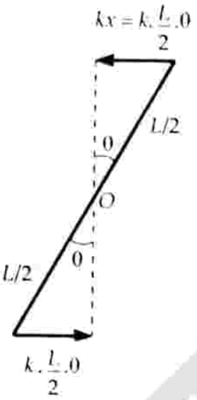
$$k_1 x_1 = k_2 x_2$$

or $$k_1 x_1 = k_2 (A - x_1)$$

or $$x_1 = \frac{k_2 A}{k_1 + k_2}.$$

13. (*c*) Restoring torque about *O*,

$$\tau = -2\left(k \cdot \frac{L}{2} \cdot \theta\right)\frac{L}{2} = -\frac{kL^2\theta}{2}$$



$$I = \frac{ML^2}{12}$$

Angular acceleration,

$$\alpha = \frac{\tau}{I} = \frac{-\frac{kL^2\theta}{2}}{\frac{ML^2}{12}}$$

or $$\alpha = -\frac{6k}{M}\theta = -\omega^2\theta$$

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{6k}{M}}.$$

14. (*b*) Suppose the cylinder is given a downward push through a small distance *y*. Then

Restoring upward force set up in the spring $= -ky$

Additional upward force of buoyancy $= -Ay\rho g$

Total upward restoring force,

$$F = -(ky + Ay\rho g) = -(k + A\rho g)y$$

Clearly, $F \propto y$ and it acts towards equilibrium position. Hence motion of the cylinder is simple harmonic. Here spring factor $= k + A\rho g$

Inertia factor = Mass of cylinder = *M*

∴ Frequency of oscillation of the cylinder,

$$\nu = \frac{1}{2\pi}\sqrt{\frac{\text{Spring factor}}{\text{Inertia factor}}}$$

$$= \frac{1}{2\pi}\sqrt{\frac{M}{k + A\rho g}}.$$

15. (*b*) Young's modulus of wire,

$$Y = \frac{F}{A} \cdot \frac{L}{\Delta L}$$

∴ Stretching force, $$F = YA\frac{\Delta L}{L}$$

Force constant of wire, $$k_1 = \frac{F}{\Delta L} = \frac{YA}{L}$$

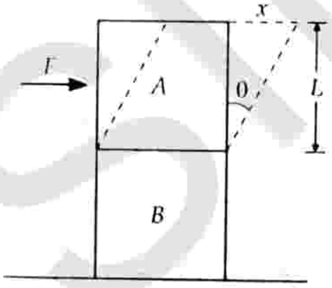
As the wire and the spring are connected in series, their effective force constant is

$$K = \frac{kk_1}{k + k_1} = \frac{k(YA/L)}{k + (YA/L)} = \frac{kYA}{kL + YA}$$

∴ Time period,

$$T = 2\pi\sqrt{\frac{m}{K}} = 2\pi\sqrt{\frac{m(kL + YA)}{kYA}}.$$

16. (*d*) When the force *F* is applied, the upper face of block *A* gets displaced through distance *x*.



Modulus of rigidity,

$$\eta = -\frac{F/A}{\theta}$$

$$= -\frac{F}{A\theta} = -\frac{F}{L^2\left(\frac{x}{L}\right)} = -\frac{F}{Lx}$$

Restoring force,

$$F = -\eta Lx \quad i.e., \quad F \propto x$$

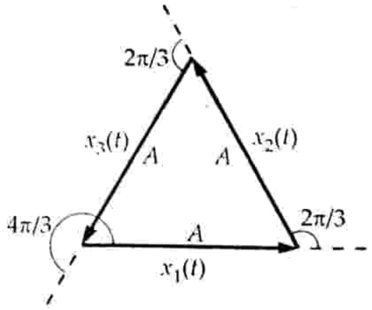
Hence the motion of *A* is simple harmonic with

$$k = \eta L$$

Time period of oscillation,

$$T = 2\pi\sqrt{\frac{M}{k}} = 2\pi\sqrt{\frac{M}{\eta L}}.$$

17. (*b*) Displacements $x_1(t)$ and $x_2(t)$ have amplitude *A* each, and phase difference $\frac{2\pi}{3}$. The third displacement $x_3(t)$ brings the mass to complete rest. For this, $x_3(t)$ must have amplitude *A* and phase difference $4\pi/3$ with $x_1(t)$ as shown in the figure.



18. (*a*), (*b*), (*c*)

$$x = A\sin^2\omega t + B\cos^2\omega t + C\sin\omega t\cos\omega t$$

or $$x = A\left(\frac{1-\cos 2\omega t}{2}\right) + B\left(\frac{1+\cos 2\omega t}{2}\right) + \frac{C\sin 2\omega t}{2}$$

(*a*) For $A = 0$, $B = 0$, $x = \frac{C}{2}\sin 2\omega t$

This represents SHM. Hence option (*a*) is correct.

(*b*) For $A = B$, $C = 2B$, $x = B + B\sin 2\omega t$

This represents SHM of amplitude B.

Hence option (*b*) is correct.

(*c*) For $A = -B$, $C = 2B$,

$$x = B\cos 2\omega t + B\sin 2\omega t$$

$$= \sqrt{2}\,B\sin\left(2\omega t + \frac{\pi}{4}\right)$$

This represents SHM. Hence option (*c*) is correct.

(*d*) For $A = B$, $C = 0$, $x = A$.

This does not represent SHM.

19. (*a*), (*c*) Using the principle of superposition,

$$y = y_1 + y_2 + y_3$$

$$= a\sin(\omega t + 45^\circ) + a\sin\omega t + a\sin(\omega t - 45^\circ)$$

$$= a[\sin(\omega t + 45^\circ) + \sin(\omega t - 45^\circ)] + a\sin\omega t$$

$$= 2a\sin\omega t\cos 45^\circ + a\sin\omega t$$

$$= \sqrt{2}a\sin\omega t + a\sin\omega t$$

$$y = (1+\sqrt{2})a\sin\omega t$$

(*a*) Amplitude of resultant motion $= (1+\sqrt{2})a$.

Hence option (*a*) is correct.

(*b*) Option (*b*) is incorrect because the phase of resultant motion relative to the first is 45° not 90°.

(*c*) $E \propto (\text{amplitude})^2$

$$\therefore \quad \frac{E_{\text{resultant}}}{E_{\text{single}}} = \frac{(1+\sqrt{2})^2 a^2}{a^2}$$

$$= 3 + 2\sqrt{2}$$

$$E_{\text{resultant}} = (3 + 2\sqrt{2})E_{\text{single}}$$

Hence option (*c*) is correct.

(*d*) Option (*d*) is incorrect as the resultant motion is simple harmonic.

20. (*c*) In one oscillation, the energy of an oscillator becomes twice kinetic and twice potential.

$\therefore$ Frequency of oscillation of K.E. $= \mathbf{2}\,f$.

21. (*b*), (*c*) Energy of oscillation of the particle,

$$= \frac{1}{2}kA^2 = \frac{1}{2} \times 2 \times 10^6 \times (0.01)^2$$

$$= 100 \text{ J}$$

Different energies at mean and extreme positions are shown below :

$x = 0$	$x = \pm A$
$K = 100$ J = Maximum	$K = 0$ J
$U = 60$ J = Minimum	$U = 160$ J = Maximum
$E = 160$ J = Constant	$E = 160$ J = Constant.

22. (*a*) $U(x) = k\,|x|^3$

$$\therefore \quad [k] = \frac{[U]}{[x^3]} = \frac{[\mathrm{ML^2T^{-2}}]}{[\mathrm{L^3}]} = [\mathrm{ML^{-1}T^{-2}}]$$

$$T \propto (\text{mass})^x\,(\text{amplitude})^y\,(k)^z$$

$$\therefore \quad [\mathrm{M^0L^0T}] = [\mathrm{M}]^x\,[\mathrm{L}]^y\,[\mathrm{ML^{-1}T^{-2}}]^z$$

$$= [\mathrm{M}^{x+y}\mathrm{L}^{y-z}\mathrm{T}^{-2z}]$$

Equating the powers, we get

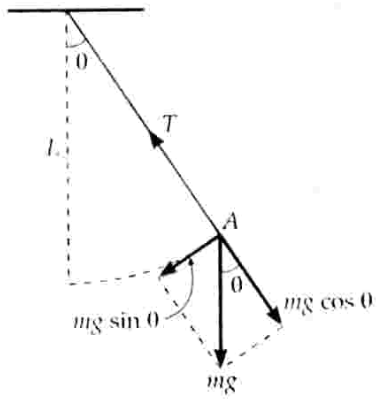
$$-2z = 1 \quad \text{or} \quad z = -1/2$$

$$y - z = 0 \quad \text{or} \quad y = z = -1/2$$

$$\therefore \quad T \propto (\text{amplitude})^{-1/2}$$

or $$T \propto \frac{1}{\sqrt{a}}.$$

23. (*b*), (*c*) As shown in the figure, $T - Mg\sin\theta$ provides the centripetal force.



$$T - Mg\sin\theta = \frac{Mv^2}{L}$$

Also, $$A_T = \frac{Mg\sin\theta}{M}$$

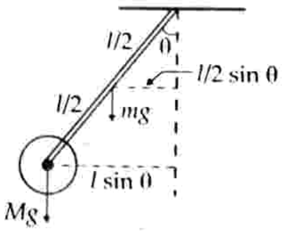
$$= g\sin\theta$$

Hence options (*b*) and (*c*) are correct.

24. (*a*), (*d*) We use $\tau = I\alpha$

For case A : $\tau_A = I_A \alpha_A$

or $$mg\left(\frac{l}{2}\sin\theta\right) + Mg(l\sin\theta) = \left(\frac{ml^2}{3} + \frac{MR^2}{2} + Ml^2\right)\alpha_A$$



For case B : $\tau_B = I_B \alpha_B$

or $$mg\left(\frac{l}{2}\sin\theta\right) + Mg(l\sin\theta) = \left(\frac{ml^2}{3} + Ml^3\right)\alpha_B$$

As $\tau_A = \tau_B$ and $I_A > I_B$, so

$$\alpha_A < \alpha_B \Rightarrow \omega_A < \omega_B$$

25. (*c*) For motion to be periodic, it must reverse its path *i.e.*, K.E. should become zero for a finite value of x.

Now $E = K + U \Rightarrow K = E - U$

Given $U_{max} = V_0$ $\therefore$ $K_{min} = E - V_0$

The particle will escape if $K_{min} > 0$

$\therefore$ $E - V_0 < 0 \Rightarrow E < V_0$

Also, $E = K + U > 0$

26. (*b*) As $V = \alpha x^4$

$$\Rightarrow \quad [\alpha] = \frac{ML^2T^{-2}}{L^4} = ML^{-2}T^{-2}$$

Hence, $$\left[\frac{1}{A}\sqrt{\frac{m}{\alpha}}\right] = \frac{1}{L}\left[\frac{M}{ML^{-2}T^{-2}}\right]^{1/2}$$

$$= M^\circ L^\circ T$$

27. (*d*) For $|x| > X_0$, potential energy $= V_0$ (constant)

$$\therefore \quad F = -\frac{dV_0}{dx} = 0$$

Hence acceleration is zero for $|x| > X_0$

28. (*d*) The momentum is initially positive as the ball moves up, becomes zero at the highest position and then becomes negative as the ball moves down.

29. (*c*) $\dfrac{E_1}{E_2} = \left(\dfrac{2a}{a}\right)^2 = 4 \Rightarrow E_1 = 4E_2$

30. (*b*) Amplitude of the mass oscillating in water should decrease with time. So options (*c*) and (*d*) are ruled out.

When the position of the mass is at one extreme end in the positive side (the topmost point), the momentum is zero. As the mass moves towards the mean position the momentum increases in the negative direction.

$\therefore$ Only option (*b*) is correct.

31.

0	0	0	4

Young's modulus,

$$Y = \frac{F}{A} \cdot \frac{l}{\Delta l}$$

Elongation, $$\Delta l = \frac{Fl}{AY} = \frac{F}{\frac{AY}{L}} = \frac{F}{k}$$

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{YA}{ml}}$$

$$\therefore \quad Y = \frac{\omega^2 ml}{A} = \frac{140 \times 140 \times 0.1 \times 1}{4.9 \times 10^{-7}} = 4 \times 10^9$$

$$= n \times 10^9\,\text{Nm}^{-2} \Rightarrow n = 4$$

32. $a \to p$; $b \to q, r$; $c \to p$; $d \to r, q$

AIEEE

1. The function $\sin^2 \omega t$ represents

(*a*) a periodic but not simple harmonic motion with a period $2\pi/\omega$

(*b*) a periodic, but not simple harmonic motion with a period π/ω

(*c*) a simple harmonic motion with a period $2\pi/\omega$

(*d*) a simple harmonic motion with a period π/ω

[AIEEE 05]

2. Two simple harmonic motions are represented by the equations

$$y_1 = 0.1\sin(100\pi t + \pi/3)$$

and $$y_2 = 0.1\cos \pi t$$

The phase difference of the velocity of particle 1 with respect to the velocity of particle 2 is

(*a*) $-\pi/6$ (*b*) $\pi/3$

(*c*) $-\pi/3$ (*d*) $\pi/6$.

[AIEEE 05]

3. The displacement of an object attached to a spring and executing S.H.M. is given by

$$x = 2 \times 10^{-2} \cos \pi t \text{ (in m)}$$

The time at which the maximum speed first occurs is

(*a*) 0.5 s (*b*) 0.75 s

(*c*) 0.125 s (*d*) 0.25 s [AIEEE 07]

4. A point mass oscillates along the *X*-axis according to the relation

$$x = x_0 \cos(\omega t - \pi / 4)$$

If acceleration of the particle is written as

$$a = a_0 \cos(\omega t + \delta),$$

then

(*a*) $a_0 = x_0\omega^2$; $\delta = -\pi / 4$

(*b*) $a_0 = x_0\omega^2$; $\delta = \pi / 4$

(*c*) $a_0 = x_0\omega^2$; $\delta = -\pi / 4$

(*d*) $a_0 = x_0\omega^2$; $\delta = 3\pi / 4$. [AIEEE 07]

5. The maximum velocity of a particle, executing simple harmonic motion with an amplitude 7 mm, is 4.4 m s^{-1}. The period of oscillation is

(*a*) 0.01 s (*b*) 0.1 s

(*c*) 10 s (*d*) 100 s [AIEEE 06]

6. If a simple harmonic motion is represented by

$$\frac{d^2x}{dt^2} + \alpha x = 0,$$

its time period is

(*a*) $2\pi / \alpha$ (*b*) $2\pi / \sqrt{\alpha}$

(*c*) $2\pi\alpha$ (*d*) $2\pi\sqrt{\alpha}$ [AIEEE 05]

7. If *x*, *v* and *a* denote the displacement, the velocity and the acceleration of a particle executing simple harmonic motion of time period *T*, then which of the following does not change with time ?

(*a*) $a^2T^2 + 4\pi^2v^2$ (*b*) aT / x

(*c*) $aT + 2\pi v$ (*d*) aT / v [AIEEE 09]

8. A coin is placed on a horizontal platform, which undergoes vertical simple harmonic motion of angular frequency ω The amplitude of oscillation is gradually increased. The coin will leave contact with the platform for the first time

(*a*) for an amplitude of g^2 / ω^2.

(*b*) for an amplitude of g / ω^2

(*c*) at the highest position of the platform

(*d*) at the mean position of the platform.

[AIEEE 06]

9. If a spring has time period *T* and is cut into *n* equal parts, then the time period of each part will be

(*a*) $T\sqrt{n}$ (*b*) $T / \sqrt{n}$

(*c*) nT (*d*) T [AIEEE 02]

10. A mass *M* is suspended from a spring of negligible mass. The spring is pulled a little and then released so that the mass executes S.H.M. of time period *T*. If the mass is increased by *m*, the time becomes $5T / 3$. Then the ratio of m / M is

(*a*) $3 / 5$ (*b*) $25 / 9$

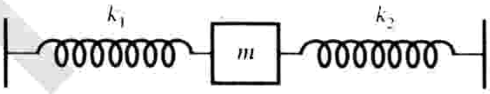
(*c*) $16 / 9$ (*d*) $5 / 3$. [AIEEE 03]

11. A particle at the end of a spring executes simple harmonic motion with a period t_1, while the corresponding period for another spring is t_2. If the period of oscillation with the two springs in series is *T*, then

(*a*) $T = t_1 + t_2$ (*b*) $T^2 = t_1^2 + t_2^2$

(*c*) $T^{-1} = t_1^{-1} + t_2^{-1}$ (*d*) $T^{-2} = t_1^{-2} + t_2^{-2}$ [AIEEE 04]

12. Two springs of force constant k_1 and k_2 are connected to a mass *m* as shown in the figure.



The frequency of oscillation of the mass is ν. If both k_1 and k_2 are made four times their original values, the frequency of oscillation becomes

(*a*) $\nu / 2$ (*b*) $\nu / 4$

(*c*) 4ν (*d*) 2ν [AIEEE 07]

13. Two bodies *M* and *N* of equal masses are suspended from two separate massless springs of spring constants k_1 and k_2 respectively. If the two bodies oscillate vertically, such that their maximum velocities are equal, then the ratio of the amplitude of *M* to that of *N* is

(*a*) k_1 / k_2 (*b*) $\sqrt{k_1 / k_2}$

(*c*) k_2 / k_1 (*d*) $\sqrt{k_2 / k_1}$

[AIEEE 03 ; IIT 88]

14. A child swinging on a swing in sitting position stands up. Then the time period of the swing will

(*a*) increase

(*b*) decrease

(*c*) remain the same

(*d*) increase, if the child is long and decrease, if the child is short. [DPMT 04 ; AIEEE 02]

15. The length of a simple pendulum executing simple harmonic motion is increased by 21%. The percentage increase in the time period of the pendulum of increased length is

(*a*) 50% (*b*) 21%

(*c*) 30% (*d*) 10%.

[AIEEE 03 ; AFMC 01 ; AIIMS 01]

16. The bob of a simple pendulum is a spherical hollow ball filled with water. A plugged hole near the bottom of the oscillating bob gets suddenly unplugged. During observation, till water is coming out, the time period of oscillation would

(*a*) first increase and then decrease to the original value

(*b*) first decreases and then increase to the original value

(*c*) remain unchanged

(*d*) increase towards a saturation value. [AIEEE 05]

17. The bob of a simple pendulum executes simple harmonic motion in water with a period t, while the period of oscillation of the bob is t_0 in air. Neglecting frictional force of water and given that the density of the bob is 4000/3 kg m^{-3}, what relationship between t and t_0 is true ?

(*a*) $t = t_0$ (*b*) $t = t_0/2$

(*c*) $t = 2t_0$ (*d*) $t = 4t_0$ [AIEEE 04]

18. The total energy of a particle executing simple harmonic motion is

(*a*) $\propto x$ (*b*) $\propto x^2$

(*c*) independent of x (*d*) $\propto x^{1/2}$ [AIEEE 04]

19. In a simple harmonic oscillator, at the mean position

(*a*) kinetic energy is minimum, potential energy is maximum

(*b*) both kinetic and potential energies are maximum

(*c*) kinetic energy is maximum, potential energy is minimum

(*d*) both kinetic and potential energies are minimum. [AIEEE 02]

20. A body executes simple harmonic motion. The potential energy (P.E.), the kinetic energy (K.E.) and total energy (T.E.) are measured as function of displacement x. Which of the following statements is true ?

(*a*) K.E. is maximum, when $x = 0$

(*b*) T.E. is zero, when $x = 0$

(*c*) K.E. is maximum, when x is maximum

(*d*) P.E. is maximum, when $x = 0$. [AIEEE 03]

21. A spring of force constant 800 N m^{-1} has an extension of 5 cm. The work done in extending it from 5 cm to 15 cm is

(*a*) 8 J (*b*) 16 J

(*c*) 24 J (*d*) 32 J. [AIEEE 02]

22. A spring of spring constant 5×10^3 Nm^{-1} is stretched initially by 5 cm from the unstretched position. Then the work done to stretch it further by another 5 cm is

(*a*) 6.25 N m (*b*) 12.50 N m

(*c*) 18.75 N m (*d*) 25.00 N m [AIEEE 03]

23. A particle of mass m executes S.H.M. with amplitude a and frequency ν. The average kinetic energy during its motion from the position of equilibrium to the end is

(*a*) $\pi^2 m a^2 \nu^2$ (*b*) $\frac{1}{4}\pi^2 m a^2 \nu^2$

(*c*) $4\pi^2 m a^2 \nu^2$ (*d*) $2\pi^2 m a^2 \nu^2$ [AIEEE 07]

24. Starting from the origin, a body oscillates simple harmonically with a period of 2 s. After what time will its kinetic energy be 75% of the total energy ?

(*a*) 1/12 s (*b*) 1/6 s

(*c*) 1/4 s (*d*) 1/3 s. [AIEEE 06]

25. A particle of mass m is attached to a spring (of spring constant k) and has a natural angular frequency ω_0. An external force $F(t)$ proportional to $\cos \omega t (\omega \neq \omega_0)$ is applied to the oscillator. The time displacement of the oscillator will be proportional to

(*a*) $\frac{m}{\omega_0^2 - \omega^2}$ (*b*) $\frac{1}{m(\omega_0^2 - \omega^2)}$

(*c*) $\frac{1}{m(\omega_0^2 + \omega^2)}$ (*d*) $\frac{m}{\omega_0^2 + \omega^2}$ [AIEEE 04]

26. In forced oscillation of a particle, the amplitude is maximum for a frequency ω_1 of the force, while the energy is maximum for a frequency ω_2 of the force. Then

(*a*) $\omega_1 = \omega_2$

(*b*) $\omega_1 > \omega_2$

(*c*) $\omega_1 < \omega_2$, when damping is small and $\omega_1 > \omega_2$, when damping is large

(*d*) $\omega_1 < \omega_2$. [AIEEE 04]

27. Two particles are executing simple harmonic motion of the same amplitude A and frequency ω along the x-axis. Their mean position is separated by distance

$X_0(X_0 > A)$. If the maximum separation between them is $(X_0 + A)$, the phase difference between their motions is

(*a*) $\frac{\pi}{2}$ (*b*) $\frac{\pi}{3}$

(*c*) $\frac{\pi}{4}$ (*d*) $\frac{\pi}{6}$ [AIEEE 2011]

28. A mass M, attached to a horizontal spring, executes SHM with amplitude A_1. When the mass M passes through its mean position then a smaller mass m is placed over it and both of them move together with amplitude A_2. The ratio of $\left(\frac{A_1}{A_2}\right)$ is

(*a*) $\frac{M}{M+m}$ (*b*) $\frac{M+m}{M}$

(*c*) $\left(\frac{M}{M+m}\right)^{1/2}$ (*d*) $\left(\frac{M+m}{M}\right)^{1/2}$ [AIEEE 2011]

Answers and Explanations

1. (*b*) $\sin^2 \omega t = \frac{1}{2} - \left(\frac{1}{2}\right)\cos 2\omega t$

The function does not represent SHM but it is periodic with

$$T = \frac{2\pi}{2\omega} = \frac{\pi}{\omega}$$

Refer to the solution of Problem 14 on page 14.43.

2. (*a*) Refer to the solution of Problem 4 on page 14.44.

3. (*a*) Displacement, $x = 2 \times 10^{-2} \cos \pi t$

Velocity, $v = \frac{dx}{dt} = -2 \times 10^{-2} \pi \sin \pi t$

Velocity becomes maximum when

$$\sin \pi t = 1$$

or $$\pi t = \frac{\pi}{2}$$

or $$t = 0.5 \text{ s.}$$

4. (*d*) $x = x_0 \cos(\omega t - \pi/4)$

$$v = \frac{dx}{dt} = -x_0 \omega \sin(\omega t - \pi/4)$$

$$a = \frac{dv}{dt} = -x_0 \omega^2 \cos(\omega t - \pi/4)$$

$$= x_0 \omega^2 \cos[\pi + (\omega t - \pi/4)]$$

or $$a = x_0 \omega^2 \cos(\omega t + 3\pi/4)$$

Given : $a = a_0 \cos(\omega t + \delta)$

On comparing,

$$a_0 = x_0 \omega^2, \ \delta = \frac{3\pi}{4}.$$

5. (*a*) $v_{\max} = \omega A = \frac{2\pi}{T} A$

$$\therefore \quad T = \frac{2\pi A}{v_{\max}} = \frac{2 \times 22 \times 7 \times 10^{-3}}{7 \times 4.4} = 0.01 \text{ s.}$$

6. (*b*) Refer to the solution of Problem 13 on page 14.43.

7. (*b*) $\frac{aT}{x} = \frac{\omega^2 x T}{x} = \frac{4\pi^2}{T^2} \times T$

$$= \frac{4\pi^2}{T} = \text{constant.}$$

8. (*b*) The coin will remain in contact with the platform if $a_{\max}$ does not exceed g *i.e.*, $a_{\max}$ is at the most equal to g.

$$\therefore \quad a_{\max} = g$$

or $$a\omega^2 = g$$

or $$a = g / \omega^2$$

9. (*b*) $T' = T / \sqrt{n}$.

Refer to the solution of Problem 3 on page 14.44.

10. (*c*) Refer to the solution of Problem 8 on Page 14.45.

11. (*b*) Refer to the solution of Problem 10 on Page 14.45.

12. (*d*) Initial frequency,

$$\nu = \frac{1}{2\pi}\sqrt{\frac{k_1 + k_2}{M}}$$

When both k_1 and k_2 are made four times their original values,

$$\nu = \frac{1}{2\pi}\sqrt{\frac{4(k_1 + k_2)}{M}} = 2\nu.$$

13. (*d*) $\frac{A_1}{A_2} = \sqrt{\frac{k_2}{k_1}}$.

Refer to the solution of Problem 9 on Page 14.45.

14. (*b*) Refer to the solution of Problem 21 on Page 14.39.

15. (*d*) Refer to the solution of Problem 15 on Page 14.43.

16. (*a*) Refer to the solution of Problem 2 on Page 14.44.

17. (*c*) $t = 2t_0$. Refer to the solution of Problem 7 on Page 14.45.

18. (*c*) $E = 2\pi^2 m\nu^2 A^2$

Clearly, the total energy is independent of displacement x.

19. (*c*) At the mean position, the kinetic energy is maximum and potential energy is minimum.

20. (*a*) The K.E. of a simple harmonic oscillator is maximum when $x = 0$.

21. (*a*) At $x_1 = 5$ cm,

$$U_1 = \frac{1}{2}kx_1^2$$

$$= \frac{1}{2} \times 800 \times (0.05)^2 = 1 \text{ J}$$

At $x_2 = 15$ cm,

$$U_2 = \frac{1}{2}kx_2^2 = \frac{1}{2} \times 800 \times (0.15)^2 = 9 \text{ J}.$$

$$\therefore \quad W = U_2 - U_1 = 9 - 1$$

$$= \mathbf{8\ J.}$$

22. (*c*) $$W = \frac{1}{2}k(x_2^2 - x_1^2)$$

$$= \frac{1}{2} \times 5 \times 10^3 \times [(0.10)^2 - (0.05)^2]$$

$$= \mathbf{18.75\ Nm.}$$

23. (*a*) K.E. of a simple harmonic oscillator at any instant t,

$$K = \frac{1}{2}ma^2\omega^2 \sin^2 \omega t$$

The average value of $\sin^2 \omega t$ over a cycle is $\frac{1}{2}$.

$$\therefore \quad K_{av} = \frac{1}{2}ma^2\omega^2 \times \frac{1}{2}$$

$$= \frac{1}{4}ma^2(2\pi\nu)^2 = \pi^2 ma^2\nu^2.$$

24. (*b*) K.E. = 75% of total energy

$$\frac{1}{2}k(a^2 - y^2) = \frac{75}{100} \times \frac{1}{2}ka^2$$

$$a^2 - y^2 = \frac{3}{4}a^2$$

$$y^2 = \frac{1}{4}a^2$$

$$\text{or} \quad y = \frac{a}{2}$$

For a body starting from mean position,

$$y = a \sin \omega t$$

$$\text{or} \quad \frac{a}{2} = a \sin \omega t$$

$$-\sin \omega t = \frac{1}{2}$$

$$\text{or} \quad \omega t = \frac{\pi}{6}$$

$$\text{or} \quad \frac{2\pi}{T}t = \frac{\pi}{6}$$

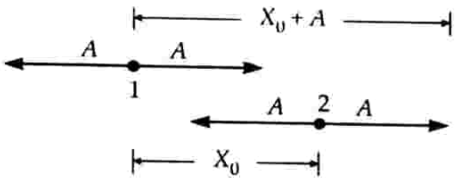
$$\text{or} \quad t = \frac{T}{12} = \frac{2}{12} = \frac{1}{6} \text{ s.}$$

25. (*b*) $$y \propto \frac{1}{m(\omega_0^2 - \omega^2)}.$$

Refer to the solution of Problem 5 on Page 14.44.

26. (*a*) Only in case of resonance, both the amplitude and energy of oscillation are maximum. Hence, $\omega_1 = \omega_2$.

27. (*a*) When the maximum separation is $X_0 + A$, one particle is at mean position and the other is at the extreme position. So phase difference = $\pi/2$



28. (*d*) Energy of the simple harmonic oscillator is constant.

$$\therefore \quad \frac{1}{2}M\omega A_1^2 = \frac{1}{2}(M+m)\omega^2 A_2^2$$

$$\Rightarrow \quad \frac{A_1^2}{A_2^2} = \frac{M+m}{M}$$

$$\Rightarrow \quad \frac{A_1}{A_2} = \left(\frac{M+m}{M}\right)^{1/2}$$