CHAPTER 4

MOTION IN A PLANE

4.1 NEED FOR VECTORS

1. *Briefly describe the necessity for introducing the concept of vectors.*

Need for vectors. In one-dimensional motion, only two directions are possible. So the directional aspect of the quantities like position, displacement, velocity and acceleration can be taken care of by using + and – signs. But in case of motion in two-dimensions (plane) or three dimensions (space), an object can have a large number of directions. In order to deal with such situations effectively, we need to introduce the concept of new physical quantities, called vectors, in which we take care of both magnitude and direction.

4.2 SCALARS AND VECTORS

2. *What are scalar and vector quantities ? Give examples.*

Scalar quantities. *The physical quantities which have only magnitude and no direction are called scalar quantities or scalars.* A scalar quantity can be specified by a single number, along with the proper unit.

Examples : Mass, volume, density, time, temperature, electric current, etc.

Vector quantities. *The physical quantities which have both magnitude and direction and obey the laws of vector addition are called vector quantities or vectors.* A vector quantity is specified by a number with a unit and its direction.

Examples : Displacement, velocity, force, momentum, etc.

3. *Give some points of difference between scalars and vectors.*

Scalars	Vectors
1. Scalars have only magnitude.	Vectors have both magnitude and direction.
2. They change if their magnitude changes.	They change if either their magnitude, direction or both change.
3. They can be added according to ordinary laws of algebra.	They can be added only by using special laws of vector addition.

4.3 REPRESENTATION OF A VECTOR

4. *With the help of a suitable example, explain how is a vector quantity represented.*

Representation of a vector. *A vector quantity is represented by a straight line with an arrowhead over it.* The length of the line gives the magnitude and the arrowhead gives the direction. Suppose a body has a velocity of 40 kmh^{-1} due east. If 1 cm is chosen to

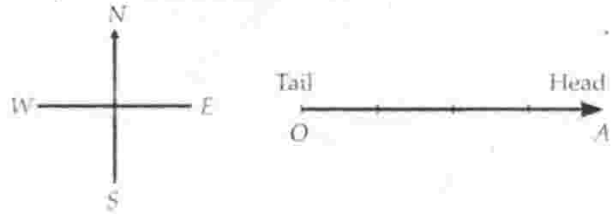


Fig. 4.1 Representation of a vector.

represent a velocity of 10 kmh^{-1}, a line OA, 4 cm in length and drawn towards east with arrowhead at A will completely represent the velocity of the body. The point A is called head or terminal point and point O is called tail or initial point of the vector $\vec{OA}$ (Fig. 4.1).

In a simpler notation, a vector is represented by single letter of alphabet either in bold face or with an arrow over it. For example, a force vector can be represented as $\vec{F}$ or **F**.

4.4 POSITION AND DISPLACEMENT VECTORS

5. *Distinguish between position vector and displacement vector.*

Position vector. *A vector which gives position of an object with reference to the origin of a co-ordinate system is called position vector.* Consider the motion of an object in X-Y plane with origin at O. Suppose an object is at point P at any instant t, as shown in Fig. 4.2. Then $\vec{OP}$ is the position vector of the object at point P. The position vector provides two informations :

(*i*) It tells the straight line distance of the object from the origin O.

(*ii*) It tells the direction of the object with respect to the origin.

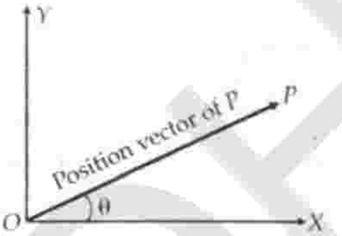


Fig. 4.2 Position vector.

Displacement vector. *It is that vector which tells how much and in which direction an object has changed its position in a given time interval.* Consider an object moving in the XY-plane. Suppose it is at point P at any instant t and at point Q at any later instant t', as shown in Fig. 4.3. Then vector $\vec{PQ}$ is the displacement vector of the object in time t to t'.

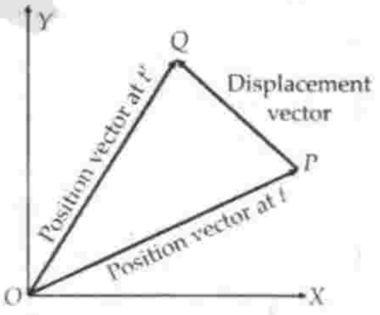


Fig. 4.3. Displacement vector.

4.5 POLAR AND AXIAL VECTORS

6. *What is meant by polar and axial vectors ? Give examples.*

Broadly speaking, vectors are of *two* types :

Polar vectors. *The vectors which have a starting point or a point of application are called polar vectors.*

Examples : Displacement, velocity, force, etc., are polar vectors.

Axial vectors. *The vectors which represent rotational effect and act along the axis of rotation in accordance with right hand screw rule are called axial vectors.*

Examples : Angular velocity, torque, angular momentum, etc. are axial vectors.

As shown in Fig. 4.4, axial vector will have its direction along its axis of rotation depending on its anticlockwise or clockwise rotational effect.

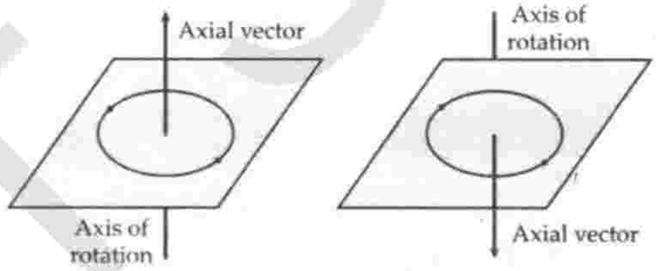


Fig. 4.4

For Your Knowledge

- The Latin word *vector* means *carrier.*
- The physical quantities which have no specified direction and have different values in different directions are called *tensors*. For example, moment of inertia.

4.6 SOME DEFINITIONS IN VECTOR ALGEBRA

7. *Define and illustrate the following terms :*

(*i*) *Equal vectors* (*ii*) *Negative of a vector*
(*iii*) *Modulus of a vector* (*iv*) *Unit vector*
(*v*) *Fixed vector* (*vi*) *Free vector*
(*vii*) *Collinear vectors* (*viii*) *Coplanar vectors*
(*ix*) *Co-initial vectors* (*x*) *Co-terminus vectors.*

(*i*) **Equal vectors.** *Two vectors are said to be equal if they have the same magnitude and same direction.* In Fig. 4.5, $\vec{A}$ and $\vec{B}$ are two equal vectors.

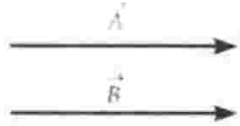


Fig. 4.5 Equal Vectors.

Note If a vector is displaced parallel to itself, it remains equal to itself.

(*ii*) **Negative of a vector.** *The negative of a vector is defined as another vector having the same magnitude but having an opposite direction.* In Fig. 4.6, vector $\vec{A}$ is the negative of vector $\vec{B}$ or vice versa.

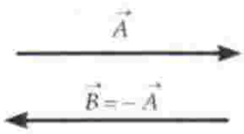


Fig. 4.6 Negative of a vector.

(*iii*) **Modulus of a vector.** *The modulus of a vector means the length* or *the magnitude of that vector.* It is a scalar quantity.

Modulus of vector $\vec{A} = |\vec{A}| = A$

(*iv*) **Unit vector.** *A unit vector is a vector of unit magnitude drawn in the direction of a given vector.* A unit vector in the direction of a given vector is found by dividing the given vector by its modulus. Thus a unit vector in the direction of vector $\vec{A}$ is given by

$$\hat{A} = \frac{\vec{A}}{|\vec{A}|} = \frac{\vec{A}}{A}$$

A unit vector in the direction of a given vector $\vec{A}$ is written as $\hat{A}$ and is pronounced as 'A carat' or 'A hat' or 'A cap'.

Any vector can be expressed as the magnitude times the unit vector along its own direction.

$$\vec{A} = |\vec{A}|\,\hat{A}$$

Note The magnitude of a unit vector is unity. It just gives the direction of a vector. A unit vector has no units or dimensions.

(*v*) **Fixed vector.** *The vector whose initial point is fixed is called a fixed vector or a localised vector.* For example, the position vector of a particle is a fixed vector because its initial point lies at the origin.

(*vi*) **Free vector.** *A vector whose initial point is not fixed is called a free vector or a non-localised vector.* For example, the velocity vector of a particle moving along a straight line is a free vector.

(*vii*) **Collinear vectors.** *The vectors which either act along the same line or along parallel lines are called collinear vectors.* Two collinear vectors having the same direction ($\theta = 0^\circ$) are called *like* or *parallel vectors*. Two collinear vectors having the opposite directions ($\theta = 180^\circ$) are called *unlike* or *antiparallel vectors*.

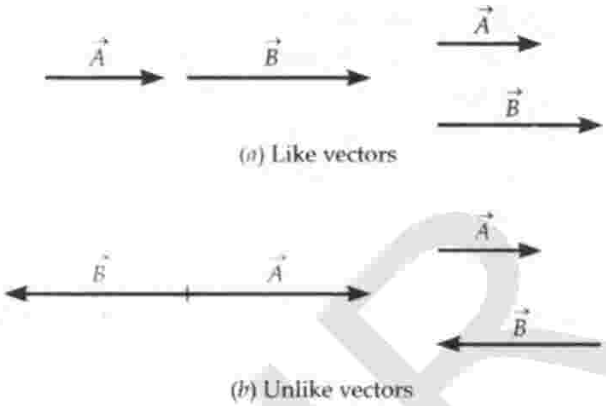


Fig. 4.7 Collinear vectors

(*viii*) **Coplanar vectors.** *The vectors which act in the same plane are called coplanar vectors.*

(*ix*) **Co-initial vectors.** *The vectors which have the same initial point are called co-initial vectors.* In Fig. 4.8, $\vec{A}$, $\vec{B}$ and $\vec{C}$ are co-initial vectors.

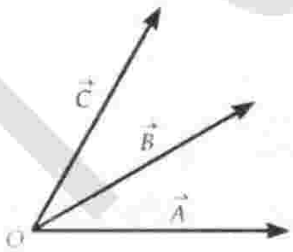


Fig. 4.8 Co-initial vectors.

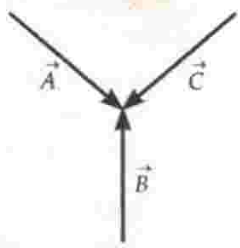


Fig. 4.9 Co-terminus vectors.

(*ix*) **Co-terminus vectors.** *The vectors which have the common terminal point are called co-terminus vectors.* In Fig. 4.9, $\vec{A}$, $\vec{B}$ and $\vec{C}$ are co-terminus vectors.

4.7 ZERO VECTOR AND ITS PROPERTIES

8. *What is a zero vector ? Explain the need of a zero vector. Give the important properties and physical examples of zero vectors.*

Zero vector. *A zero or null vector is a vector that has zero magnitude and an arbitrary direction.* It is represented by $\vec{0}$ (arrow over the number 0).

Need of a zero vector. The need of a zero vector arises due to the following situations :

(*i*) If $\vec{A} = \vec{B}$, then what is $\vec{A} - \vec{B}$?

(*ii*) If $\mu = -\lambda$, then what is $(\lambda + \mu)\vec{A}$?

In all these cases the resultant has to be a vanishingly vector and not a scalar. Hence there is need for introducing the concept of zero vector.

Properties of zero vectors :

(*i*) When a vector is added to zero vector, we get the same vector

$$\vec{A} + \vec{0} = \vec{A}$$

(*ii*) When a real number is multiplied by a zero vector, we get a zero vector

$$\lambda\, \vec{0} = \vec{0}$$

(*iii*) When a vector is multiplied by zero, we get zero vector

$$0\, \vec{A} = \vec{0}$$

(*iv*) If λ and μ are two different non-zero real numbers, then the relation

$$\lambda\, \vec{A} = \mu\, \vec{B}$$

can hold only if both $\vec{A}$ and $\vec{B}$ are zero vectors.

Physical examples of zero vector :

(*i*) The position vector of a particle lying at the origin is a zero vector.

(*ii*) The velocity vector of a stationary object is a zero vector.

(*iii*) The acceleration vector of an object moving with uniform velocity is a zero vector.

4.8 MULTIPLICATION OF A VECTOR BY A REAL NUMBER

9. *What do you mean by multiplication of a vector by a real number ? Illustrate it by some examples.*

Multiplication of a vector by a real number. When a vector $\vec{A}$ is multiplied by a real number λ, we get another vector $\lambda\, \vec{A}$. The magnitude of $\lambda\, \vec{A}$ is λ times the magnitude of $\vec{A}$. If λ is positive, then the direction of $\lambda\, \vec{A}$ is same as that of $\vec{A}$. If λ is negative, then the direction of $\lambda\, \vec{A}$ is opposite to that of $\vec{A}$. Fig. 4.10 shows multiplication of vector $\vec{A}$ by different real numbers.

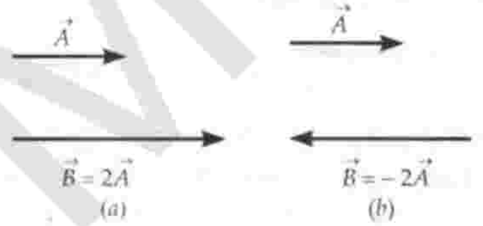


Fig. 4.10 Multiplication of a vector by a real number when (*a*) $\lambda = 2$ and (*b*) $\lambda = -2$

Examples :

(*i*) If a vector $\vec{A}$ is multiplied by real number $\lambda = 2$, then we get another vector $\vec{B}$ such that $\vec{B} = 2\, \vec{A}$. As shown in Fig. 4.10(*a*), the magnitude of $\vec{B}$ is twice the magnitude of $\vec{A}$ while the direction of $\vec{B}$ is the same as that of $\vec{A}$.

(*ii*) If a vector $\vec{A}$ is multiplied by real number $\lambda = -2$, then the new vector is such that $\vec{B} = -2\, \vec{A}$. As shown in Fig. 4.10(*b*), the magnitude of $\vec{B}$ is again twice the magnitude of $\vec{A}$ but direction is opposite to that of $\vec{A}$.

10. *How are the units of a vector affected when it is multiplied by a scalar having units or dimensions ? Give example.*

Multiplication of a vector by scalar. If λ is a pure number having no units or dimensions, then the units of $\lambda\, \vec{A}$ are the same as that of $\vec{A}$. However, when a vector $\vec{A}$ is multiplied by a scalar λ which has certain units, the units of resultant $\lambda\, \vec{A}$ are obtained by multiplying the units of $\vec{A}$ by the units of λ. For example, when velocity vector is multiplied by mass (a scalar), we get momentum. The units of momentum are obtained by multiplying the units of velocity by units of mass.

4.9 ADDITION OR COMPOSITION OF VECTORS

11. *What do you mean by resultant of two or more vectors ? Mention the various laws of vector addition.*

Composition of vectors. *The resultant of two or more vectors is that single vector which produces the same effect as the individual vectors together would produce. The process of adding two or more vectors is called composition of vectors.*

As the vectors have both magnitude and direction, so they cannot be added by using ordinary rules of algebra. Vectors can be added *geometrically*. The following three *laws of vector addition* can be used to add two or more vectors having any inclination to each other.

(*i*) *Triangle law of vector addition* for adding two vectors.

(*ii*) *Parallelogram law of vector addition* for adding two vectors.

(*iii*) *Polygon law of vector addition* for adding more than two vectors.

12. *State and illustrate triangle law of vector addition.*

Triangle law of vector addition. *If two vectors can be represented both in magnitude and direction by the two sides of a triangle taken in the same order, then their resultant is represented completely, both in magnitude and direction, by the third side of the triangle taken in the opposite order.*

Illustration. Suppose we wish to add two vectors $\vec{A}$ and $\vec{B}$ as shown in Fig. 4.11(*a*). Draw a vector $\vec{OP}$ equal and parallel to vector $\vec{A}$, as shown in Fig. 4.11(*b*). From head P of $\vec{OP}$, draw a vector $\vec{PQ}$ equal and parallel to vector $\vec{B}$. Then the resultant vector is given by $\vec{OQ}$ which joins the tail of $\vec{A}$ and head of $\vec{B}$.

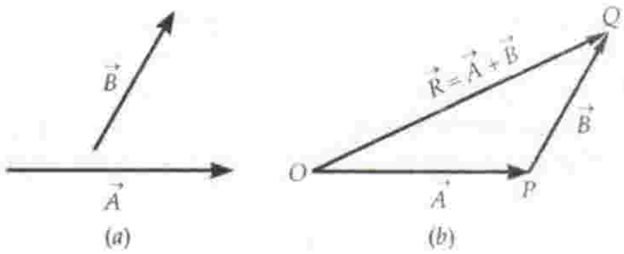


Fig. 4.11 Triangle law of vector addition.

According to triangle law of vector addition,

$$\vec{OQ} = \vec{OP} + \vec{PQ}$$

or $$\vec{R} = \vec{A} + \vec{B}$$

13. *State and illustrate parallelogram law of vector addition.*

Parallelogram law of vector addition. *If two vectors can be represented both in magnitude and direction by the two adjacent sides of a parallelogram drawn from a common point, then their resultant is completely represented, both in magnitude and direction, by the diagonal of the parallelogram passing through that point.*

Illustration. Suppose we wish to add two vectors $\vec{A}$ and $\vec{B}$, as shown in Fig. 4.12(*a*). From a common point O, draw a vector $\vec{OP}$ equal and parallel to $\vec{A}$ and vector $\vec{OQ}$ equal and parallel to $\vec{B}$. Complete the parallelogram $OPSQ$. According to the parallelogram law of vector addition, the diagonal $\vec{OS}$ gives the resultant vector $\vec{R}$.

Thus $\quad \vec{OS} = \vec{OP} + \vec{OQ} \quad$ or $\quad \vec{R} = \vec{A} + \vec{B}$

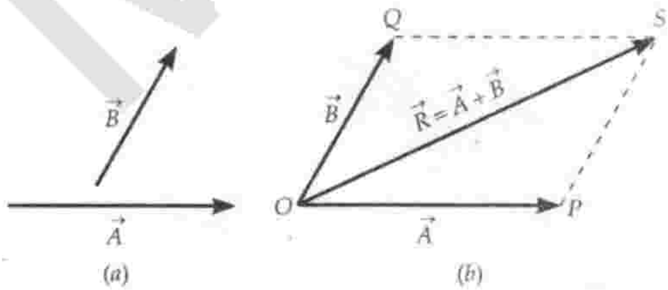


Fig. 4.12 Parallelogram law of vector addition.

14. *State and prove the polygon law of vector addition.*

Polygon law of vector addition. *If a number of vectors are represented both in magnitude and direction by the sides of an open polygon taken in the same order, then their resultant is represented both in magnitude and direction by the closing side of the polygon taken in opposite order.*

Illustration. Suppose we wish to add four vectors $\vec{A}, \vec{B}, \vec{C}$ and $\vec{D}$, as shown in Fig. 4.13(*a*). Draw vector $\vec{OP} = \vec{A}$. Move vectors $\vec{B}$, $\vec{C}$ and $\vec{D}$ parallel to themselves so that the tail of $\vec{B}$ touches the head of $\vec{A}$, the tail of $\vec{C}$ touches the head of $\vec{B}$ and the tail of $\vec{D}$ touches the head of $\vec{C}$, as shown in Fig. 4.13(*b*). According to the polygon law, the closing side OT of the polygon taken in the reverse order represents the resultant $\vec{R}$. Thus

$$\vec{R} = \vec{A} + \vec{B} + \vec{C} + \vec{D}$$

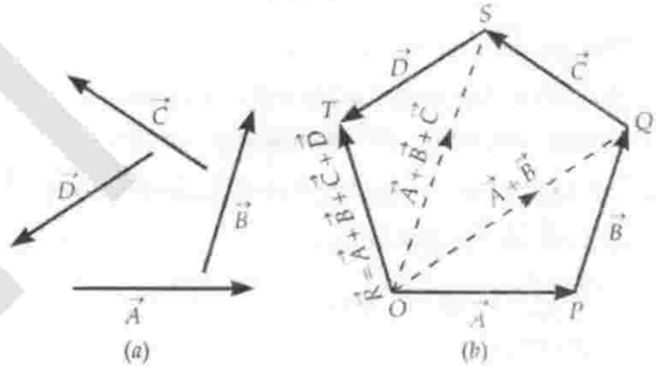


Fig. 4.13 Polygon law of vector addition.

Proof. We apply triangle law of vector addition to different triangles of the polygon shown in Fig. 4.13(*b*),

In $\Delta\, OPQ$, $\quad \vec{OQ} = \vec{OP} + \vec{PQ} = \vec{A} + \vec{B}$

In $\Delta\, OQS$, $\quad \vec{OS} = \vec{OQ} + \vec{QS} = \vec{A} + \vec{B} + \vec{C}$

In $\Delta\, OST$, $\quad \vec{OT} = \vec{OS} + \vec{ST} = \vec{A} + \vec{B} + \vec{C} + \vec{D}$

or $$\vec{R} = \vec{A} + \vec{B} + \vec{C} + \vec{D}.$$

This proves the polygon law of vector addition.

15. *Prove that the vector addition is commutative.*

Vector addition is commutative. In Fig. 4.14, the sides OP and OQ of a parallelogram $OPSQ$ represent vectors $\vec{A}$ and $\vec{B}$ respectively. According to parallelogram law of vector addition, diagonal OS gives the resultant. Thus

$$\vec{A} + \vec{B} = \vec{OS}$$

or $$\vec{OP} + \vec{OQ} = \vec{OS} \qquad ...(1)$$

As $OP = QS$ and $OP \parallel QS$, therefore

$$\vec{QS} = \vec{OP} = \vec{A}$$

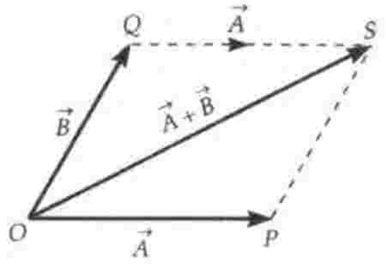


Fig. 4.14 Vector addition is commutative.

Using triangle law of vector addition in $\Delta\, OQS$, we get

$$\vec{OQ} + \vec{QS} = \vec{OS}$$

or $$\vec{B} + \vec{A} = \vec{OS} \quad \text{...(2)}$$

From equations (1) and (2), we find that

$$\vec{A} + \vec{B} = \vec{B} + \vec{A}$$

This proves that the vector addition is commutative.

16. *Prove that vector addition is associative.*

Vector addition is associative. Suppose three vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ are represented by the sides $\vec{OP}$, $\vec{PQ}$ and $\vec{QS}$ of a polygon $OPQS$. According to polygon law, $\vec{OS}$ represents the resultant $\vec{R}$ both in magnitude and direction. Join O to Q and P to S.

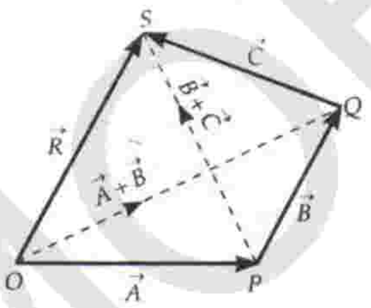


Fig. 4.15. Vector addition is associative.

Using triangle law of vector addition,

In $\Delta\, OPQ$, $\quad \vec{OQ} = \vec{OP} + \vec{PQ} = \vec{A} + \vec{B}$

In $\Delta\, OQS$, $\quad \vec{OS} = \vec{OQ} + \vec{QS} = (\vec{A} + \vec{B}) + \vec{C} \quad$ ...(1)

In $\Delta\, PQS$, $\quad \vec{PS} = \vec{PQ} + \vec{QS} = \vec{B} + \vec{C}$

In $\Delta\, OPS$, $\quad \vec{OS} = \vec{OP} + \vec{PS} = \vec{A} + (\vec{B} + \vec{C}) \quad$ ...(2)

From equations (1) and (2), we find that

$$(\vec{A} + \vec{B}) + \vec{C} = \vec{A} + (\vec{B} + \vec{C})$$

This proves that vector addition is associative.

17(*a*) *Is the flying of a bird an example of composition of vectors ? Explain.*

Flight of a bird. When a bird flies, it pushes the air with forces F_1 and F_2 in the downward direction with its wings W_1 and W_2. The lines of action of these two forces meet at point O. In accordance with Newton's third law of motion, the air exerts equal and opposite reactions R_1 and R_2. According to the parallelogram law, the resultant R of the reactions R_1 and R_2 acts on the bird in the upward direction and helps the bird to fly upward.

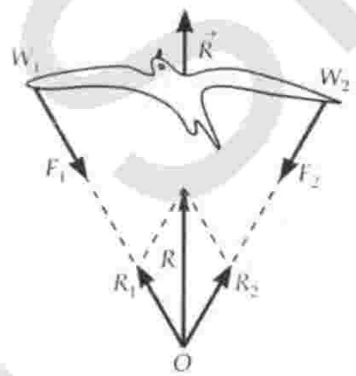


Fig. 4.16 Flight of a bird.

17(*b*) *Is the working of a sling based on the parallelogram law of vector addition ? Explain.*

Working of a sling. A sling consists of a Y-shaped wooden or metallic frame, to which a rubber band is attached, as shown in Fig. 4.17. When a stone held at the point O on the rubber band is pulled, the tensions T_1 and T_2 are produced along OA and OB in the two segments of the rubber band. According to the parallelogram law of forces, the resultant T of the tensions T_1 and T_2 acts on the stone along OC. As the stone is released, it moves under the action of the resultant tension T in forward direction with a high speed.

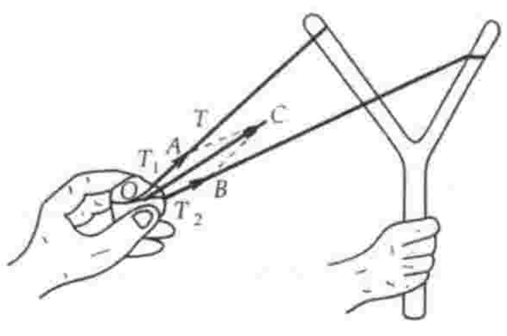


Fig. 4.17

4.10 ANALYTICAL METHOD OF VECTOR ADDITION

18. *Two vectors $\vec{A}$ and $\vec{B}$ are inclined to each other at an angle θ. Using triangle law of vector addition, find the magnitude and direction of their resultant.*

Analytical treatment of the triangle law of vector addition. Let the two vectors $\vec{A}$ and $\vec{B}$ be represented both in magnitude and direction by the sides $\vec{OP}$ and $\vec{PQ}$ of ΔOPQ taken in the same order. Then according to the triangle law of vector addition, the resultant $\vec{R}$ is given by the closing side OQ taken in the reverse order, as shown in Fig. 4.18.

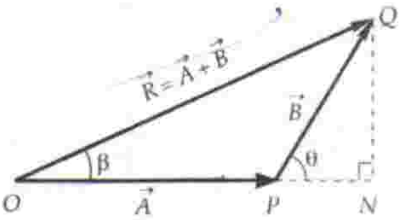


Fig. 4.18 Analytical treatment of Δ law of vector addition.

Magnitude of the resultant $\vec{R}$. From Q, draw QN perpendicular to OP produced.

Then $\angle QPN = \theta$, $OP = A$, $PQ = B$, $OQ = R$.

From right angled $\Delta\, QNP$, we have

$$\frac{QN}{PQ} = \sin\theta \quad \text{or} \quad QN = PQ\sin\theta = B\sin\theta$$

and
$$\frac{PN}{PQ} = \cos\theta \quad \text{or} \quad PN = PQ\cos\theta = B\cos\theta$$

Using Pythagoras theorem in right angled ΔONQ, we get

$$OQ^2 = ON^2 + QN^2$$
$$= (OP + PN)^2 + QN^2$$

or
$$R^2 = (A + B\cos\theta)^2 + (B\sin\theta)^2$$
$$= A^2 + B^2\cos^2\theta + 2AB\cos\theta + B^2\sin^2\theta$$
$$= A^2 + B^2(\cos^2\theta + \sin^2\theta) + 2AB\cos\theta$$
$$= A^2 + B^2 + 2\,AB\cos\theta$$

or
$$R = \sqrt{A^2 + B^2 + 2\,AB\cos\theta}$$

Direction of the resultant $\vec{R}$. Let the resultant $\vec{R}$ make an angle β with the direction of $\vec{A}$. Then from right angled ΔONQ, we get

$$\tan\beta = \frac{QN}{ON} = \frac{QN}{OP + PN}$$

or
$$\tan\beta = \frac{B\sin\theta}{A + B\cos\theta}$$

19. *Two vectors $\vec{A}$ and $\vec{B}$ are inclined to each other at an angle θ. Using parallelogram law of vector addition, find the magnitude and direction of their resultant. Discuss the special cases, when (i) θ = 0°, (ii) θ = 180° and (iii) θ = 90°.*

Analytical treatment of the parallelogram law of vector addition. Let the two vectors $\vec{A}$ and $\vec{B}$ inclined to each other at an angle θ be represented both in magnitude and direction by the adjacent sides $\vec{OP}$ and $\vec{OQ}$ of the parallelogram $OPSQ$. Then according to the parallelogram law of vector addition, the resultant of $\vec{A}$ and $\vec{B}$ is represented both in magnitude and direction by the diagonal $\vec{OS}$ of the parallelogram.

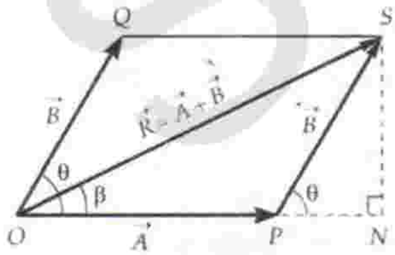


Fig. 4.19 Analytical treatment of parallelogram law.

Magnitude of resultant $\vec{R}$. Draw SN perpendicular to OP produced.

Then $\angle SPN = \angle QOP = \theta$, $OP = A$, $PS = OQ = B$, $OS = R$

From right angled $\Delta\, SNP$, we have

$$\frac{SN}{PS} = \sin\theta \quad \text{or} \quad SN = PS\sin\theta = B\sin\theta$$

$$\frac{PN}{PS} = \cos\theta \quad \text{or} \quad PN = PS\cos\theta = B\cos\theta$$

Using Pythagoras theorem in right-angled ΔONS, we get

$$OS^2 = ON^2 + SN^2 = (OP + PN)^2 + SN^2$$

or
$$R^2 = (A + B\cos\theta)^2 + (B\sin\theta)^2$$
$$= A^2 + B^2(\cos^2\theta + \sin^2\theta) + 2AB\cos\theta$$
$$= A^2 + B^2 + 2\,AB\cos\theta$$

or
$$R = \sqrt{A^2 + B^2 + 2\,AB\cos\theta}$$

Direction of resultant $\vec{R}$. Let the resultant $\vec{R}$ make angle β with the direction of $\vec{A}$. Then from right angled $\Delta\, ONS$, we get

$$\tan\beta = \frac{SN}{ON} = \frac{SN}{OP + PN} \quad \text{or} \quad \tan\beta = \frac{B\sin\theta}{A + B\cos\theta}.$$

Special cases. (*i*) If the two vectors $\vec{A}$ and $\vec{B}$ are acting along the same direction, $\theta = 0°$. Therefore the magnitude of the resultant is given by

$$R = \sqrt{A^2 + B^2 + 2\ AB \cdot \cos 0°}$$

$$= \sqrt{A^2 + B^2 + 2\ AB} \qquad [\because\ \cos 0° = 1]$$

$$= \sqrt{(A+B)^2}$$

or $$R = A + B$$

$\therefore$ Magnitude of the resultant vector is equal to the sum of magnitudes of two vectors acting along the same direction and the resultant vector also acts along that direction.

(*ii*) If the vectors $\vec{A}$ and $\vec{B}$ are acting along mutually opposite directions, $\theta = 180°$ and therefore,

$$R = \sqrt{A^2 + B^2 + 2\ AB \cos 180°}$$

$$= \sqrt{A^2 + B^2 - 2\ AB} \qquad [\because\ \cos 180° = -1]$$

or $$R = (A \sim B)$$

$\therefore$ Resultant is equal to the positive difference between magnitudes of two vectors and acts along the direction of bigger vector.

(*iii*) When the two vectors are acting at right angle to each other, $\theta = 90°$ and therefore,

$$R = \sqrt{A^2 + B^2 + 2\ AB \cos 90°}$$

$$= \sqrt{A^2 + B^2} \qquad [\because\ \cos 90° = 0]$$

Also $$\tan\beta = \frac{B \sin 90°}{A + B\cos 90°} = \frac{B}{A}$$

$\therefore$ $$\beta = \tan^{-1}\left(\frac{B}{A}\right)$$

Examples based on Composition of Vectors

FORMULAE USED

1. By triangle law or parallelogram law of vector addition, the magnitude of resultant $\vec{R}$ of two vectors $\vec{P}$ and $\vec{Q}$ inclined to each other at angle θ, is given by

$$R = \sqrt{P^2 + Q^2 + 2PQ\cos\theta}$$

2. If resultant $\vec{R}$ makes an angle β with $\vec{P}$, then

$$\tan\beta = \frac{Q\sin\theta}{P + Q\cos\theta}.$$

EXAMPLE 1. *ABCD is a parallelogram and $\vec{AC}$ and $\vec{BD}$ are its diagonals. Prove that :*

(i) $\vec{AC} + \vec{BD} = 2\ \vec{BC}$ *and (ii)* $\vec{AC} - \vec{BD} = 2\ \vec{AB}$.

Solution. Using triangle law of vector addition in Fig. 4.20,

$$\vec{AC} = \vec{AB} + \vec{BC}$$

$$\vec{BD} = \vec{BC} + \vec{CD}$$

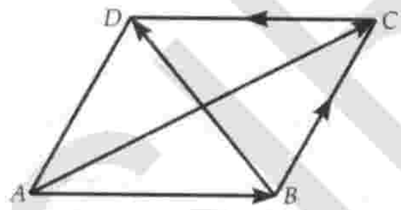


Fig. 4.20

(*i*) $\vec{AC} + \vec{BD} = \vec{AB} + \vec{BC} + \vec{BC} + \vec{CD}$

But $\vec{AB} = -\vec{CD}$

$\therefore$ $\vec{AC} + \vec{BD} = -\vec{CD} + 2\ \vec{BC} + \vec{CD} = 2\ \vec{BC}$.

(*ii*) $\vec{AC} - \vec{BD} = \vec{AB} + \vec{BC} - \vec{BC} - \vec{CD}$

$$= \vec{AB} - \vec{CD} = \vec{AB} - (-\vec{AB}) = 2\ \vec{AB}$$

EXAMPLE 2. *A body is simultaneously given two velocities, one 30 ms^{-1} due east and other 40 ms^{-1} due north. Find the resultant velocity.*

Solution. Let the body start moving from O, as shown in Fig. 4.21.

$$\vec{v}_A = \vec{OA} = 30\ \text{ms}^{-1}, \text{ due east}$$

$$\vec{v}_B = \vec{OB} = 40\ \text{ms}^{-1}, \text{ due north}$$

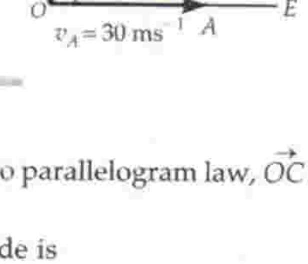


Fig. 4.21

According to parallelogram law, $\vec{OC}$ is the resultant velocity.

Its magnitude is

$$v = \sqrt{v_A^2 + v_B^2} = \sqrt{30^2 + 40^2} = \mathbf{50\ ms^{-1}}.$$

Suppose the resultant velocity $\vec{v}$ makes angle β with the east direction. Then

$$\tan\beta = \frac{CA}{OA} = \frac{40}{30} = 1.3333$$

$$\therefore \quad \beta = \tan^{-1}(1.3333) = \mathbf{53°8'}.$$

EXAMPLE 3. *A particle has a displacement of 12 m towards east and 5 m towards the north and then 6 m vertically upward. Find the magnitude of the sum of these displacements.*

Solution. As shown in Fig. 4.22, suppose initially the particle is at origin O. Then its displacement vectors are

$$\vec{OA} = 12 \text{ m, due east}$$

$$\vec{AB} = 5 \text{ m, due north}$$

$$\vec{CD} = 6 \text{ m, vertically upwards.}$$

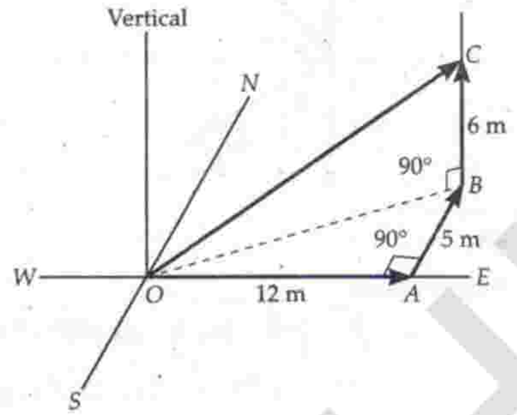


Fig. 4.22

According to polygon law, $\vec{OC}$ is the resultant displacement. From right ΔOAB,

$$OB = \sqrt{OA^2 + AB^2} = \sqrt{12^2 + 5^2}$$

$$= \sqrt{144 + 25} = 13 \text{ m}$$

From right ΔOBC,

$$OC = \sqrt{OB^2 + BC^2} = \sqrt{13^2 + 6^2}$$

$$= \sqrt{169 + 36} = \sqrt{205} = \mathbf{14.32 \text{ m}}.$$

EXAMPLE 4. *Two forces of 5 N and 7 N act on a particle with an angle of 60° between them. Find the resultant force.*

Solution. Here $P = 5$ N, $Q = 7$ N, $\theta = 60°$

The magnitude of the resultant force is

$$R = \sqrt{P^2 + Q^2 + 2PQ\cos\theta}$$

$$= \sqrt{5^2 + 7^2 + 2 \times 5 \times 7 \times \cos 60°}$$

$$= \sqrt{109} = \mathbf{10.44 \text{ N}}.$$

If $\vec{R}$ makes angle β with the force $\vec{P}$, then

$$\tan\beta = \frac{Q\sin\theta}{P + Q\cos\theta} = \frac{7\sin 60°}{5 + 7\cos 60°} = 0.7132$$

$$\text{or} \quad \beta = \tan^{-1} 0.7132 = \mathbf{35°29'}.$$

EXAMPLE 5. *Two vectors, both equal in magnitude, have their resultant equal in magnitude of the either. Find the angle between the two vectors.* **[Central Schools 04]**

Or

Two equal forces have their resultant equal to either. What is the inclination between them ? **[Delhi 02]**

Solution. Here $P = Q = R$

$$\text{As} \quad R = \sqrt{P^2 + Q^2 + 2PQ\cos\theta}$$

$$\therefore \quad P = \sqrt{P^2 + P^2 + 2P.P\cos\theta}$$

$$\text{or} \quad P^2 = 2P^2(1 + \cos\theta)$$

$$\text{or} \quad 1 + \cos\theta = \frac{1}{2}$$

$$\text{or} \quad \cos\theta = -\frac{1}{2} = \cos 120° \quad \text{or} \quad \theta = \mathbf{120°}.$$

EXAMPLE 6. *Two forces whose magnitudes are in the ratio of 3:5 give a resultant of 35 N. If the angle of inclination be 60°, calculate the magnitude of each force.* **[Himachal 04]**

Solution. Let $P = 3x$ newton, $Q = 5x$ newton,

$$R = 35 \text{ N}, \quad \theta = 60°$$

$$R = \sqrt{P^2 + Q^2 + 2PQ\cos\theta}$$

$$\text{or} \quad 35 = \sqrt{(3x)^2 + (5x)^2 + 2 \times 3x \times 5x\cos 60°}$$

$$\text{or} \quad 35 = 7x \quad \text{or} \quad x = \frac{35}{7} = 5$$

$$\therefore \quad P = 3 \times 5 = \mathbf{15 \text{ N}} \text{ and } Q = 5 \times 5 = \mathbf{25 \text{ N}}.$$

EXAMPLE 7. *Two forces equal to P and 2 P newton act on a particle. If the first be doubled and the second be increased by 20 newton, the direction of the resultant is unaltered. Find the value of P.*

Solution. Let the resultant make angle β with the force P.

$$\therefore \text{In first cast,} \quad \tan\beta = \frac{2P\sin\theta}{P + 2P\cos\theta}.$$

$$\text{In second case,} \quad \tan\beta = \frac{(2P + 20)\sin\theta}{2P + (2P + 20)\cos\theta}$$

$$\text{Hence} \quad \frac{(2P + 20)\sin\theta}{2P + (2P + 20)\cos\theta} = \frac{2P\sin\theta}{P + 2P\cos\theta}$$

$$\text{or} \quad \frac{2P\sin\theta}{P + 2P\cos\theta} = \frac{20\sin\theta}{P + 20\cos\theta} \quad \left[\because \frac{a}{b} = \frac{c}{d} = \frac{a-c}{b-d}\right]$$

From the above equation,

$$2P = 20$$

$$\text{or} \quad P = \mathbf{10 \text{ N}}.$$

Example 8. *The greatest and the least resultant of two forces acting at a point are 29 N and 5 N respectively. If each force is increased by 3 N, find the resultant of two new forces acting at right angle to each other.*

Solution. Let P and Q be the two forces. Then

Greatest resultant $= P + Q = 29$ N ...(*i*)

Least resultant $= P - Q = 5$ N ...(*ii*)

On solving (*i*) and (*ii*), we get

$$P = 17 \text{ N}, \quad Q = 12 \text{ N}$$

When each force is increased by 3 N, new forces are

$$p = P + 3 = 17 + 3 = 20 \text{ N}$$

$$q = Q + 3 = 12 + 3 = 15 \text{ N}$$

As the new forces act at right angle to each other, their resultant is

$$R = \sqrt{p^2 + q^2} = \sqrt{20^2 + 15^2} = \sqrt{625} = 25 \text{ N}$$

If the resultant R makes angle β with the force p, then

$$\tan\beta = \frac{q}{p} = \frac{15}{20} = 0.75$$

or $\beta = \tan^{-1}(0.75) = \mathbf{36°52'}$.

Example 9. *The sum of the magnitudes of two forces acting at a point is 18 N and the magnitude of their resultant is 12 N. If the resultant makes an angle of 90° with the force of smaller magnitude, what are the magnitudes of the two forces?* **[AIEEE 02]**

Solution. Let the two individual forces be $\vec{P}$ and $\vec{Q}$ and θ be the angle between them. Let $P < Q$. If the resultant $\vec{R}$ makes angle β with the force $\vec{P}$, then

$$\tan\beta = \frac{Q\sin\theta}{P + Q\cos\theta}$$

But $\beta = 90°$

$$\therefore \quad \frac{Q\sin\theta}{P + Q\cos\theta} = \tan 90° = \infty \quad \text{or} \quad P + Q\cos\theta = 0$$

Also $P + Q = 18$ N

As $R = \sqrt{P^2 + Q^2 + 2PQ\cos\theta} = 12$

$$\therefore \quad P^2 + Q^2 + 2PQ\cos\theta = 144$$

or $P^2 + (18 - P)^2 + 2P(-P) = 144$

$[\because \; Q = 18 - P, \; Q\cos\theta = -P]$

or $P^2 + (324 + P^2 - 36P) - 2P^2 = 144$

or $6P = 180$

or $P = \mathbf{5\ N}$ and $Q = 18 - 5 = \mathbf{13\ N}$.

Example 10. *A motorboat is racing towards north at 25 km h^{-1} and the water current in that region is 10 km h^{-1} in the direction of 60° east of south. Find the resultant velocity of the boat.* **[NCERT]**

Solution. Let the motorboat start moving from O, as shown in Fig. 4.23.

$\vec{v}_b$ = velocity of motorboat

$= 25$ km h^{-1}, due north

$\vec{v}_c$ = velocity of water current

$= 10$ km h^{-1}, 60° east of south

By parallelogram law, the resultant velocity $\vec{v}$ is equal to the diagonal $\vec{OC}$.

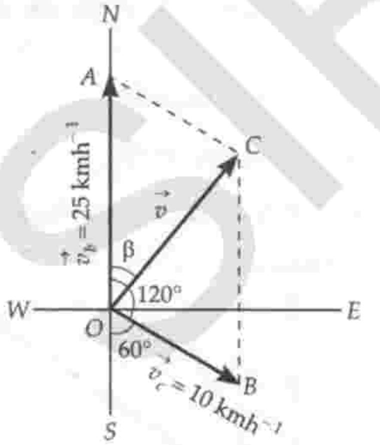


Fig. 4.23

Its magnitude is

$$v = \sqrt{v_b^2 + v_c^2 + 2v_b v_c \cos 120°}$$

$$= \sqrt{25^2 + 10^2 + 2 \times 25 \times 10(-1/2)}$$

$$= \mathbf{21.8\ km\ h^{-1}}.$$

Suppose the resultant velocity $\vec{v}$ makes angle β with the north direction. Then

$$\tan\beta = \frac{v_c \sin 120°}{v_b + v_c \cos 120°} = \frac{10 \times (\sqrt{3}/2)}{25 + 10 \times (-1/2)}$$

$$= \frac{\sqrt{3}}{4} = 0.433$$

$\therefore \quad \beta = \tan^{-1}(0.433) = \mathbf{23.4°}$

Example 11. *On a certain day, rain was falling vertically with a speed of 35 ms^{-1}. A wind started blowing after some time with a speed of 12 ms^{-1} in east to west direction. In which direction should a boy waiting at a bus stop hold his umbrella?* **[NCERT ; Delhi 02]**

Solution. In Fig. 4.24,

Velocity of rain,

$$\vec{v}_R = \vec{OA} = 35 \text{ ms}^{-1}, \quad \text{(vertically downward)}$$

Velocity of wind,

$$\vec{v}_W = \vec{OB} = 12 \text{ ms}^{-1}, \text{ east to west.}$$

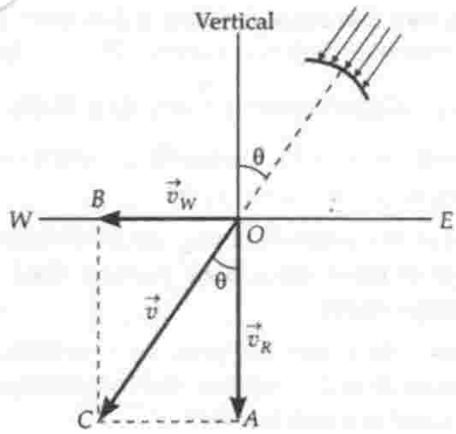


Fig. 4.24

The magnitude of the resultant velocity is

$$v = \sqrt{v_R^2 + v_W^2} = \sqrt{35^2 + 12^2} = 37\,\text{ms}^{-1}$$

Let the resultant velocity $\vec{v}\ (= \vec{OC})$ make an angle θ with the vertical. Then

$$\tan\theta = \frac{AC}{OA} = \frac{v_W}{v_R} = \frac{12}{35} = 0.343$$

$$\therefore \qquad \theta = \tan^{-1}(0.343) \simeq \mathbf{19°}.$$

Hence the boy should hold umbrella bending it towards east making an angle of about 19° with the vertical.

Example 12. *A river 800 m wide flows at the rate of 5 kmh^{-1}. A swimmer who can swim at 10 kmh^{-1} in still water, wishes to cross the river straight. (i) Along what direction must he strike ? (ii) What should be his resultant velocity ? (iii) How much time he would take ?*

Solution. (*i*) The situation is shown in Fig. 4.25.

$$\vec{OA} = \vec{v}_1 = \text{Velocity of river} = 5\ \text{kmh}^{-1}$$

$$\vec{OB} = \vec{v}_2 = \text{Velocity of swimmer in still water.}$$

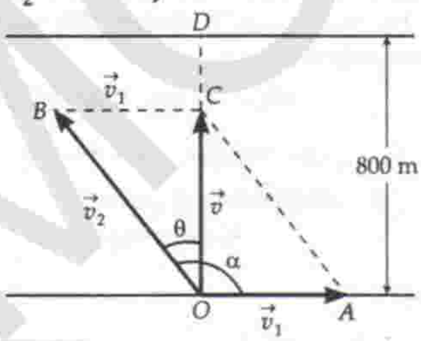


Fig. 4.25

The swimmer will cross the river straight if the resultant velocity $\vec{v}$ is perpendicular to the bank of the river. This will be possible if the swimmer moves making an angle θ with the upstream of the river.

In right ΔOCB,

$$\sin\theta = \frac{BC}{OB} = \frac{v_1}{v_2} = \frac{5}{10} = 0.5$$

$$\theta = \sin^{-1}(0.5) = \mathbf{30°}.$$

(*ii*) Resultant velocity of the swimmer,

$$v = \sqrt{v_2^2 - v_1^2} = \sqrt{10^2 - 5^2} = \sqrt{75} = 8.66\ \text{kmh}^{-1}$$

$$= \frac{8.66 \times 5}{18} = \mathbf{2.4\ ms^{-1}}.$$

(*iii*) Time taken to cross the river,

$$t = \frac{\text{width of river}}{v} = \frac{800\ \text{m}}{2.4\ \text{ms}^{-1}} = \mathbf{333.3\ s}.$$

EXAMPLE 13. *A boatman can row with a speed for 10 kmh^{-1} in still water. If the river flows steadily at 5 kmh^{-1}, in which direction should the boatman row in order to reach a point on the other bank directly opposite to the point from where he started ? The width of the river is 2 km.*

Solution. As shown in Fig. 4.26, the boatman starts from *S*. He should reach *Q*. Since the river flows along *PQ* with a velocity 5 kmh^{-1}, he should travel along *SP*.

$$\vec{PQ} = 5\ \text{kmh}^{-1}$$

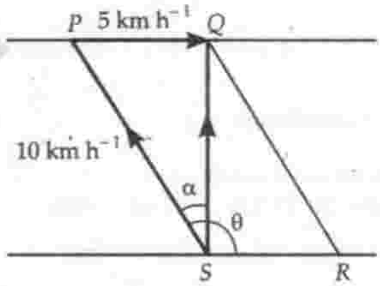


Fig. 4.26

Speed of boatman is shown by vector $\vec{SP}$.

$$\vec{SP} = 10\ \text{kmh}^{-1}$$

$\vec{SQ}$ is the resultant of $\vec{SP}$ and $\vec{PQ}$

$$\angle QSP = \alpha, \quad \angle PQS = 90°$$

Since *Q* and *S* are directly opposite.

$$\sin\alpha = \frac{PQ}{SP} = \frac{5}{10} = \frac{1}{2} \qquad i.e., \qquad \alpha = 30°$$

$$\therefore \qquad \theta = 90° + \alpha = \mathbf{120°}$$

Thus the boatman must row the boat in a direction at an angle of 120° with the direction of river flow. The direction does not depend on width of the river.

EXAMPLE 14. *A car travelling at a speed of 20 ms^{-1} due north along the highway makes a right turn on to a side road that heads due east. It takes 50 s for the car to complete the* *turn. At the end of 50 seconds, the car has a speed of 15 ms^{-1} along the side road. Determine the magnitude of average acceleration over the 50 second interval.*

Solution. In Fig. 4.27,

Initial velocity $\vec{v}_1 = \vec{OA} = 20 \text{ ms}^{-1}$, due north

Final velocity $\vec{v}_2 = \vec{OB} = 15 \text{ ms}^{-1}$, due east

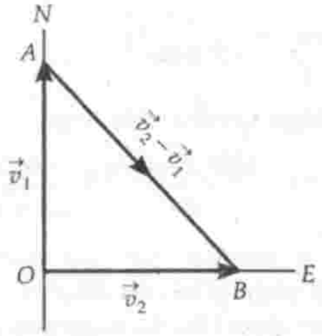


Fig. 4.27

As $\vec{OA} + \vec{AB} = \vec{OB}$

$$\therefore \quad \vec{AB} = \vec{OB} - \vec{OA} = \vec{v}_2 - \vec{v}_1$$

$$= \text{change in velocity}$$

$$\therefore \quad |\vec{v}_2 - \vec{v}_1| = AB = \sqrt{OA^2 + OB^2}$$

$$= \sqrt{20^2 + 15^2} = \sqrt{625} = 15 \text{ ms}^{-1}$$

$$\text{Average acceleration} = \frac{|\vec{v}_2 - \vec{v}_1|}{t} = \frac{15}{50} = \mathbf{0.3\ ms^{-2}}.$$

✖ PROBLEMS FOR PRACTICE

1. *ABCDE* is a pentagon. Prove that
$$\vec{AB} + \vec{BC} + \vec{CD} + \vec{DE} + \vec{EA} = \vec{0}.$$

2. In Fig. 4.28, *ABCDEF* is a regular hexagon. Prove that $\vec{AB} + \vec{AC} + \vec{AD} + \vec{AE} + \vec{AF} = 6\,\vec{AO}$.

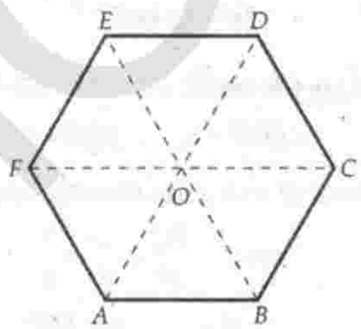


Fig. 4.28

3. A boy travels 10 m due north and then 7 m due east. Find the displacement of the boy. **(Ans.** 12.21 m, 35° east of north)

4. Find the resultant of two forces, one 6 N due east and other 8 N due north. **(Ans.** 10 N, 53°8′ with 6 N force)

5. Calculate the angle between a 2 N force and a 3 N force so that their resultant is 4 N. **(Ans.** 75°31′)

6. The resultant vector of $\vec{P}$ and $\vec{Q}$ is $\vec{R}$. On reversing the direction of $\vec{Q}$, the resultant vector becomes $\vec{S}$. Show that : $R^2 + S^2 = 2(P^2 + Q^2)$.

7. Two equal forces have the square of their resultant equal to three times their product. Find the angle between them. **(Ans.** 60°)

8. When the angle between two vectors of equal magnitude is $2\pi/3$, prove that the magnitude of the resultant is equal to either.

9. At what angle do the two forces $(P + Q)$ and $(P - Q)$ act so that the resultant is $\sqrt{3P^2 + Q^2}$. **(Ans.** 60°) **[Himachal 04]**

10. The resultant of two equal forces acting at right angles to each other is 1414 dyne. Find the magnitude of either force. **[Himachal 04]** **(Ans.** 1000 dyne)

11. A particle is acted upon by four forces simultaneously :
    (*i*) 30 N due east
    (*ii*) 20 N due north
    (*iii*) 50 N due west and
    (*iv*) 40 N due south. Find the resultant force on the particle. **(Ans.** $20\sqrt{2}$ N, 45° south of west)

12. Two boys raising a load pull at an angle to each other. If they exert forces of 30 N and 60 N respectively and their effective pull is at right angles to the direction of the pull of the first boy, what is the angle between their arms ? What is the effective pull ? **(Ans.** 120°, $30\sqrt{3}$ N)

13. A ship is steaming due east at 12 ms^{-1}. A woman runs across the deck at 5 ms^{-1} in a direction at right angles to the direction of motion of the ship and then towards north. Calculate the velocity of the woman relative to sea. **(Ans.** 13 ms^{-1}, 22°37′ north of east)

14. Find the angle between two vectors $\vec{P}$ and $\vec{Q}$ if resultant of the vectors is given by $R^2 = P^2 + Q^2$. **[Central Schools 07]** **(Ans.** 90°)

✖ HINTS

1. Refer to Fig. 4.29. Applying triangle law of vector addition,

$$\text{L.H.S.} = (\vec{AB} + \vec{BC}) + \vec{CD} + \vec{DE} + \vec{EA}$$

$$= \vec{AC} + \vec{CD} + \vec{DE} + \vec{EA}$$

$$= (\vec{AC} + \vec{CD}) + \vec{DE} + \vec{EA}$$

$= \vec{AD} + \vec{DE} + \vec{EA} = (\vec{AD} + \vec{DE}) + \vec{EA}$

$= \vec{AE} + \vec{EA} = -\vec{EA} + \vec{EA} = \vec{0} =$ R.H.S.

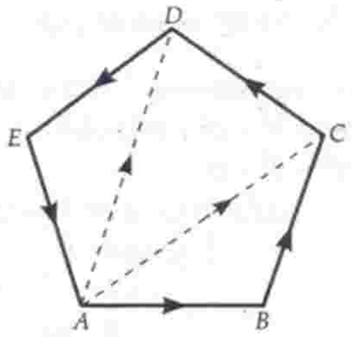


Fig. 4.29

2. We use triangle law of vector addition.

$\vec{AB} + \vec{AC} + \vec{AD} + \vec{AE} + \vec{AF}$

$= \vec{AB} + (\vec{AD} + \vec{DC}) + \vec{AD} + (\vec{AD} + \vec{DE}) + \vec{AF}$

$= 3\vec{AD} + (\vec{AB} + \vec{DE}) + (\vec{DC} + \vec{AF})$

$= 3 \times 2\,\vec{AO} + \vec{O} + \vec{O} = 6\,\vec{AO}$

$[\because\ \vec{AD} = 2\,\vec{AO},\ \vec{DE} = -\vec{AB},\ \vec{AF} = -\vec{DC}]$

3. Let the boy start moving from point O as shown in Fig. 4.30.

$\vec{OA} = 10$ m, due north ; $\vec{AB} = 7$ m, due east

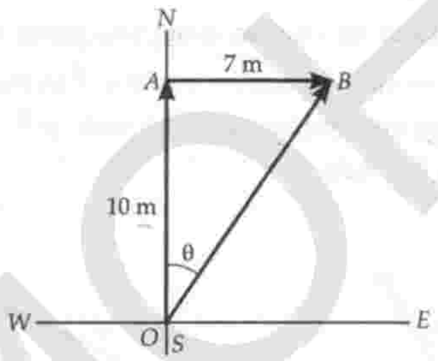


Fig. 4.30

By triangle law of vector addition, the resultant displacement is

$\vec{OA} + \vec{AB} = \vec{OB}$

$|\vec{OB}| = OB = \sqrt{OA^2 + AB^2}$

$= \sqrt{10^2 + 7^2} = 12.21$ m

$\tan\theta = \dfrac{AB}{OA} = \dfrac{7}{10} = 0.7$

$\theta = \tan^{-1}(0.7) \simeq 35°$

$\therefore$ Displacement = **12.21 m, along N 35° E.**

4. As shown in Fig. 4.31,

$P = 6$ N, $Q = 8$ N, $R = ?$

$R = \sqrt{P^2 + Q^2} = \sqrt{6^2 + 8^2} = \mathbf{10\ N}.$

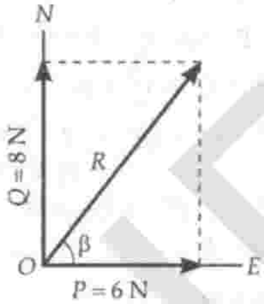


Fig. 4.31

If the resultant R makes angle β with the force of 6 N, then

$\tan\beta = \dfrac{Q}{P} = \dfrac{8}{6} = 1.3333$

$\therefore\ \beta = \tan^{-1}(1.3333) = \mathbf{53°8'}.$

5. Here $P = 2$ N, $Q = 3$ N, $R = 4$ N, $\theta = ?$

As $R^2 = P^2 + Q^2 + 2PQ\cos\theta$

$\therefore\ 4^2 = 2^2 + 3^2 + 2 \times 2 \times 3\cos\theta$

or $16 = 4 + 9 + 12\cos\theta$

or $\cos\theta = \dfrac{3}{12} = \dfrac{1}{4} = 0.25$

$\theta = \cos^{-1}(0.25) = \mathbf{75°31'}.$

6. Let θ be the angle between $\vec{P}$ and $\vec{Q}$.

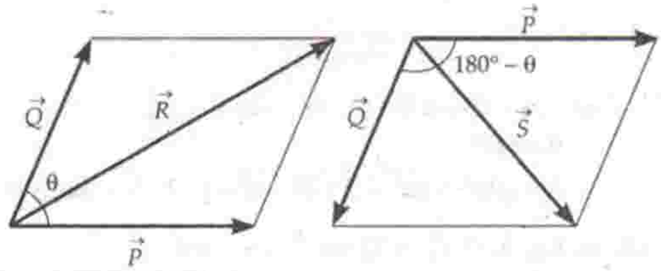


Fig. 4.32

As the resultant of $\vec{P}$ and $\vec{Q}$ is $\vec{R}$, therefore

$$R^2 = P^2 + Q^2 + 2PQ\cos\theta \quad ...(i)$$

When the direction of $\vec{Q}$ is reversed, the resultant becomes $\vec{S}$, therefore

$$S^2 = P^2 + Q^2 + 2PQ\cos(180° - \theta)$$
$$= P^2 + Q^2 - 2PQ\cos\theta \quad ...(ii)$$

Adding (*i*) and (*ii*), we get :

$$R^2 + S^2 = 2P^2 + 2Q^2 = 2(P^2 + Q^2).$$

8. Here $P = Q$,

$\theta = \dfrac{2\pi}{3} = 120°$

$R = \sqrt{P^2 + Q^2 + 2PQ\cos\theta}$

or $R = \sqrt{P^2 + P^2 + 2PP\cos 120°}$

$= \sqrt{2P^2 + 2P^2 \times (-1/2)} = P$

9. Here $F_1 = P + Q,\ F_2 = P - Q,\ R = \sqrt{3P^2 + Q^2}$

As $R^2 = F_1^2 + F_2^2 + 2F_1F_2\cos\theta$

$\therefore\ 3P^2 + Q^2 = (P+Q)^2 + (P-Q)^2 + 2(P+Q)(P-Q)\cos\theta$

or $P^2 - Q^2 = 2(P^2 - Q^2)\cos\theta$

or $\cos\theta = \frac{1}{2}$

$\therefore$ $\theta = 60°$

10. Here $P = Q,\ \theta = 90°,\ R = 1414$ dyne

As $R = \sqrt{P^2 + Q^2 + 2PQ\cos\theta}$

$\therefore$ $1411 = \sqrt{P^2 + P^2 + 2P^2\cos 90°} = \sqrt{2}\,P$

or $P = \frac{1414}{\sqrt{2}} = \frac{1414}{1.414} = \mathbf{1000\ dyne.}$

11. Net force due west, $P = 50 - 30 = 20$ N

Net force due south, $Q = 40 - 20 = 20$ N

$\therefore$ $R = \sqrt{P^2 + Q^2} = \sqrt{20^2 + 20^2} = 20\sqrt{2}$ N

$\tan\beta = \frac{Q}{P} = \frac{20}{20} = 1$ or $\beta = \mathbf{45°}$.

12. Here $P = 30$ N, $Q = 60°,\ \beta = 90°,\ R = ?,\ \theta = ?$

As $\tan\beta = \frac{Q\sin\theta}{P + Q\cos\theta}$

$\therefore\ \tan 90° = \frac{60\sin\theta}{30 + 60\cos\theta} = \infty$

or $30 + 60\cos\theta = 0$ or $\cos\theta = -\frac{1}{2}$

$\therefore\ \theta = \mathbf{120°}$

$R = \sqrt{30^2 + 60^2 + 2\times 30\times 60\times(-1/2)} = \mathbf{30\sqrt{3}\ N.}$

13. In Fig. 4.33, $\vec{v}_S = 12\ \text{ms}^{-1}$, due east

$\vec{v}_W = 5\ \text{ms}^{-1}$, due north

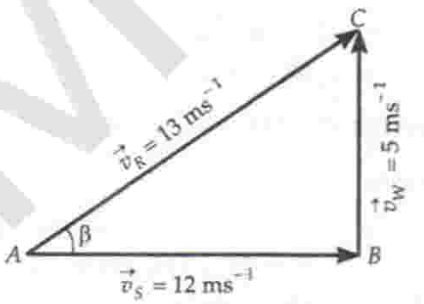


Fig. 4.33

Resultant velocity of woman is

$\vec{v}_R = \vec{v}_S + \vec{v}_W$

$\therefore$ $v_R = \sqrt{12^2 + 5^2} = \mathbf{13\ ms^{-1}}$.

Also $\tan\beta = \frac{5}{12} = 0.4167$

$\therefore$ $\beta = \mathbf{22°37'}$**, north of east.**

14. If θ is the angle between $\vec{P}$ and $\vec{Q}$, then

$R^2 = P^2 + Q^2 + 2PQ\cos\theta$

Given $R^2 = P^2 + Q^2$

$\therefore$ $P^2 + Q^2 + 2PQ\cos\theta = P^2 + Q^2$

or $2PQ\cos\theta = 0$

or $\cos\theta = 0$

or $\theta = 90°$

4.11 RESOLUTION OF A VECTOR

20. *What is meant by resolution of a vector ? Prove that a vector can be resolved along two given directions in one and only one way.*

Resolution of a vector. *It is the process of splitting a vector into two or more vectors in such a way that their combined effect is same as that of the given vector. The vectors into which the given vector is splitted are called* **component vectors.**

A component of a vector in any direction gives a measure of the effect of the given vector in that direction. The resolution of a vector is just opposite to the process of vector addition.

Resolution of a vector along two given directions. Suppose we wish to resolve a vector $\vec{R}$ in the direction of two coplanar and non-parallel vectors $\vec{A}$ and $\vec{B}$, as shown in Fig. 4.34.

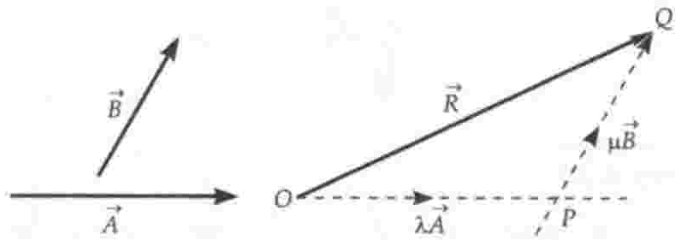


Fig. 4.34 Resolving a vector $\vec{R}$ in the directions of $\vec{A}$ and $\vec{B}$.

Suppose $\vec{OQ}$ represents vector $\vec{R}$. Through O and Q, draw lines parallel to vectors $\vec{A}$ and $\vec{B}$ respectively to meet at point P. From triangle law of vector addition,

$$\vec{OQ} = \vec{OP} + \vec{PQ}$$

As $\overrightarrow{OP} \parallel \vec{A}$ therefore, $\overrightarrow{OP} = \lambda \vec{A}$

As $\overrightarrow{PQ} \parallel \vec{B}$ therefore, $\overrightarrow{PQ} = \mu \vec{B}$

Here λ and μ are scalars. Hence

$$\vec{R} = \lambda \vec{A} + \mu \vec{B} \quad ...(1)$$

Thus the vector $\vec{R}$ has been resolved in the directions of $\vec{A}$ and $\vec{B}$. Here $\lambda \vec{A}$ is the component of $\vec{R}$ in the direction $\vec{A}$ and $\mu \vec{B}$ is the component in the direction of $\vec{B}$.

Uniqueness of resolution. Let us assume that $\vec{R}$ can be resolved in the directions of $\vec{A}$ and $\vec{B}$ in another way.

Then $$\vec{R} = \lambda' \vec{A} + \mu' \vec{B} \quad ...(2)$$

From equations (1) and (2), we have

$$\lambda \vec{A} + \mu \vec{B} = \lambda' \vec{A} + \mu' \vec{B}$$

or $$(\lambda - \lambda') \vec{A} = (\mu' - \mu) \vec{B}$$

But $\vec{A}$ and $\vec{B}$ are non-zero vectors acting along different directions. The above equation is possible only if

$$\lambda - \lambda' = 0 \quad \text{and} \quad \mu' - \mu = 0$$

or $$\lambda' = \lambda \quad \text{and} \quad \mu' = \mu.$$

Hence *there is one and only one way in which a vector $\vec{R}$ can be resolved in the directions of vectors $\vec{A}$ and $\vec{B}$*.

4.12 ORTHOGONAL TRIAD OF UNIT VECTORS : BASE VECTORS

21. *What do you mean by orthogonal triad of unit vectors or base vectors ? Show them in a diagram.*

Orthogonal triad of unit vectors or base vectors. In a right-handed Cartesian coordinate system, three unit vectors $\hat{i}, \hat{j}, \hat{k}$ are used to represent the positive directions of X-axis, Y-axis and Z-axis respectively. These three mutually perpendicular unit vectors $\hat{i}, \hat{j}, \hat{k}$ are collectively known as *orthogonal triad of unit vectors* or *base vectors*. Thus : $|\hat{i}| = |\hat{j}| = |\hat{k}| = 1$

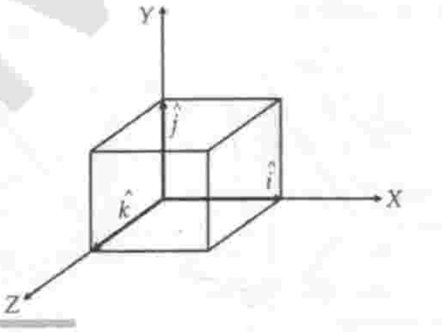


Fig. 4.35 Orthogonal triad of unit vectors.

4.13 RECTANGULAR COMPONENTS OF A VECTOR

22. *What do you mean by rectangular components of a vector ? Explain how a vector can be resolved into two rectangular components in a plane.*

Rectangular components. *When a vector is resolved along two mutually perpendicular directions, the components so obtained are called rectangular components of the given vector.*

Rectangular components of a vector in a plane. Suppose we wish to resolve vector $\vec{A}$ along X- and Y-axes. Taking the initial point of $\vec{A}$ as the origin O, draw two axes OX and OY perpendicular to each other. From the head P of $\vec{A}$, draw $PM \perp OX$ and $PN \perp OY$, as shown in Fig. 4.36 (*a*).

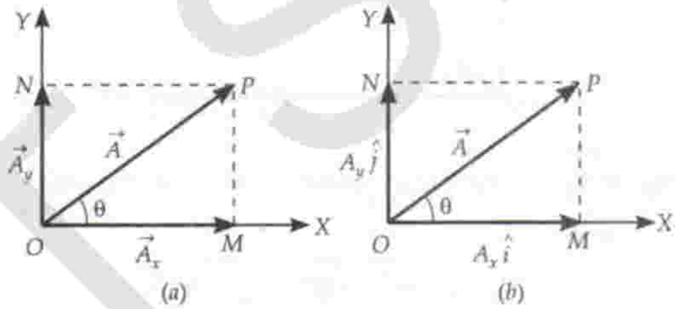


Fig. 4.36. Rectangular components of $\vec{A}$.

According to parallelogram law of vector addition,

$$\overrightarrow{OP} = \overrightarrow{OM} + \overrightarrow{ON} \quad \text{or} \quad \vec{A} = \vec{A}_x + \vec{A}_y$$

Thus $\vec{A}_x$ is the *horizontal* or *X-component* of $\vec{A}$ and $\vec{A}_y$ is the *vertical* or *Y-component* of $\vec{A}$.

Now, let $\hat{i}$ and $\hat{j}$ be the unit vectors along X- and Y-axes respectively, and A_x and A_y be the scalar magnitudes of $\vec{A}_x$ and $\vec{A}_y$ respectively. Then, we can write

$$\vec{A}_x = A_x \hat{i} \quad \text{and} \quad \vec{A}_y = A_y \hat{j}$$

$$\therefore \quad \vec{A} = A_x \hat{i} + A_y \hat{j}$$

This is the equation for vector $\vec{A}$ in terms of its rectangular components.

If vector $\vec{A}$ makes an angle θ with X-axis, then

$$A_x = A \cos\theta \quad \text{and} \quad A_y = A \sin\theta$$

Conversely, if A_x and A_y are given, we can find A and θ as follows :

$$A_x^2 + A_y^2 = A^2 (\cos^2\theta + \sin^2\theta) = A^2$$

or $$A = \sqrt{A_x^2 + A_y^2} \quad \text{and} \quad \tan\theta = \frac{A_y}{A_x} \quad \text{or} \quad \theta = \tan^{-1}\frac{A_y}{A_x}.$$

23. *How can the position vector $\vec{r}$ of any point $P(x, y)$ be expressed in terms of rectangular components?*

Resolution of a position vector into rectangular components. As shown in Fig. 4.37, draw $PM \perp X$-axis and $PN \perp Y$-axis. Then $OM = x$ and $ON = y$. According to parallelogram law of vector addition,

$$\vec{OP} = \vec{OM} + \vec{ON} \quad \text{or} \quad \vec{r} = x\hat{i} + y\hat{j}$$

This equation expresses position vector $\vec{r}$ in terms of its rectangular components along X- and Y-axes.

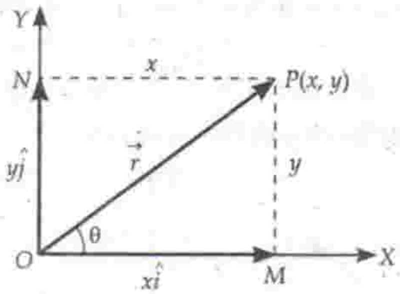


Fig. 4.37 Rectangular resolution of a position vector.

Clearly,

$$|\vec{r}| = \sqrt{OM^2 + ON^2} = \sqrt{x^2 + y^2}.$$

If $\vec{r}$ makes angle θ with X-axis, then

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta.$$

24. *Explain how a vector can be resolved into its rectangular components in three dimensions.*

Rectangular components of a vector in three dimensions. Suppose vector $\vec{A}$ is represented by $\vec{OP}$, as shown in Fig. 4.38. Taking O as origin, construct a rectangular parallelopiped with its three edges along the three rectangular axes *i.e.*, X-, Y- and Z-axes. Clearly, $\vec{A}$ represents the diagonal of the parallelopiped whose intercepts along X-, Y- and Z-axes are $\vec{A}_x$, $\vec{A}_y$ and $\vec{A}_z$ respectively.

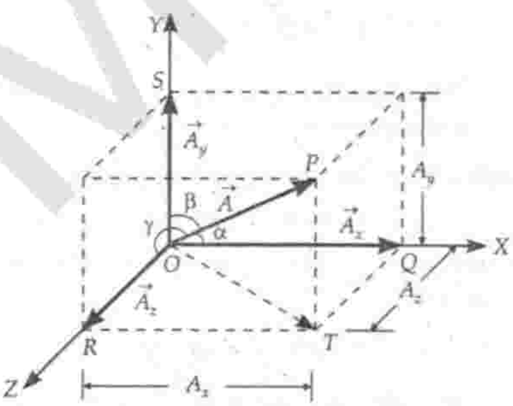


Fig. 4.38

Thus $\vec{A}_x$, $\vec{A}_y$ and $\vec{A}_z$ are the three rectangular components of $\vec{A}$.

Applying triangle law of vectors,

$$\vec{OP} = \vec{OT} + \vec{TP}$$

Applying parallelogram law of vectors,

$$\vec{OT} = \vec{OR} + \vec{OQ}$$

$$\therefore \quad \vec{OP} = \vec{OR} + \vec{OQ} + \vec{TP}$$

But $\quad \vec{TP} = \vec{OS}$

Hence $\quad \vec{OP} = \vec{OR} + \vec{OQ} + \vec{OS}$

or $\quad \vec{A} = \vec{A}_x + \vec{A}_y + \vec{A}_z = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$

If α, β and γ are the angles between $\vec{A}$ and X-, Y- and Z-axes respectively, then

$$A_x = A\cos\alpha, \quad A_y = A\cos\beta, \quad A_z = A\cos\gamma$$

Magnitude of $\vec{A}$. We note that

$$OP^2 = OT^2 + TP^2 = OQ^2 + QT^2 + TP^2$$

or $\quad A^2 = A_x^2 + A_y^2 + A_z^2$

or $\quad A = \sqrt{A_x^2 + A_y^2 + A_z^2}.$

Note A position vector $\vec{r}$ in three dimensions can be expressed as

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

where x, y and z are the components of $\vec{r}$ along X-, Y-, and Z-axes respectively.

25. *Can the walk of a man be an example of resolution of vectors? If yes, how?*

Walking of a man is an example of resolution of forces. While walking, a person presses the ground with his feet slightly slanted in the backward direction, as shown in Fig. 4.39. The ground exerts upon him an equal and opposite reaction R. Its horizontal component $H = R\cos\theta$ enables the person to move forward while the vertical component $V = R\sin\theta$ balances his weight.

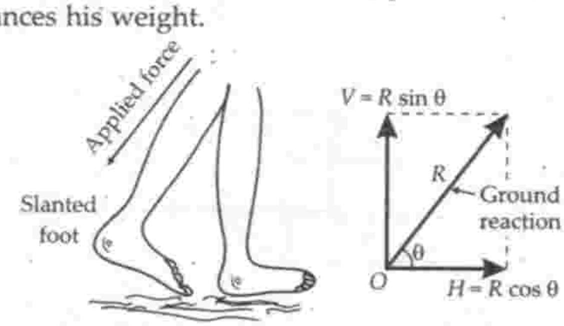


Fig. 4.39 Example of resolution of a vector.

Examples based on Expressing the Vectors in terms of Base Vectors and Rectangular Components of Vectors

FORMULAE USED

1. If A_x, A_y, A_z are the rectangular components of $\vec{A}$ and $\hat{i}$, $\hat{j}$, $\hat{k}$ are the unit vectors along X-, Y- and Z-axes respectively, then $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$
2. $|\vec{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$
3. $\hat{A} = \dfrac{\vec{A}}{|\vec{A}|} = \dfrac{A_x\hat{i} + A_y\hat{j} + A_z\hat{k}}{\sqrt{A_x^2 + A_y^2 + A_z^2}}$
4. If vector $\vec{A}$ makes angle θ with the horizontal, then horizontal component of $\vec{A} = A_x = A\cos\theta$

   vertical component of $\vec{A} = A_y = A\sin\theta$

   and $\quad A = \sqrt{A_x^2 + A_y^2}$

UNITS USED

Units of A_x, A_y and A_z are same as that of A and angle θ is in radians.

EXAMPLE 15. *Find the vector $\vec{AB}$ and its magnitude if it has initial point $A(1, 2, -1)$ and final point $B(3, 2, 2)$.*

Solution. Here $\vec{OA} = \hat{i} + 2\hat{j} - \hat{k}$

$$\vec{OB} = 3\hat{i} + 2\hat{j} + 2\hat{k}$$

$$\therefore\ \vec{AB} = \vec{OB} - \vec{OA}$$

$$= (3-1)\hat{i} + (2-2)\hat{j} + [2-(-1)]\hat{k}$$

$$= 2\hat{i} + 3\hat{k}.$$

EXAMPLE 16. *Find unit vector parallel to the resultant of the vectors $\vec{A} = \hat{i} + 4\hat{j} - 2\hat{k}$ and $\vec{B} = 3\hat{i} - 5\hat{j} + \hat{k}$.*

Solution. The resultant of $\vec{A}$ and $\vec{B}$ is

$$\vec{R} = \vec{A} + \vec{B} = (\hat{i} + 4\hat{j} - 2\hat{k}) + (3\hat{i} - 5\hat{j} + \hat{k})$$

$$= (1+3)\hat{i} + (4-5)\hat{j} + (-2+1)\hat{k} = 4\hat{i} - \hat{j} - \hat{k}$$

$$|\vec{R}| = \sqrt{4^2 + (-1)^2 + (-1)^2} = \sqrt{16+1+1} = 3\sqrt{2}$$

The unit vector parallel to $\vec{R}$ is

$$\hat{R} = \frac{\vec{R}}{|\vec{R}|} = \frac{1}{3\sqrt{2}}(4\hat{i} - \hat{j} - \hat{k})$$

EXAMPLE 17. *A Vector $\vec{X}$, when added to the resultant of the vectors $\vec{A} = 3\hat{i} - 5\hat{j} + 7\hat{k}$ and $\vec{B} = 2\hat{i} + 4\hat{j} - 3\hat{k}$ gives a unit vector along Y-axis. Find the vector $\vec{X}$.*

Solution. Resultant of $\vec{A}$ and $\vec{B}$ is

$$\vec{R} = \vec{A} + \vec{B} = (3\hat{i} - 5\hat{j} + 7\hat{k}) + (2\hat{i} + 4\hat{j} - 3\hat{k})$$

$$= 5\hat{i} - \hat{j} + 4\hat{k}$$

Unit vector along Y-axis $= \hat{j}$

$\therefore$ Required vector,

$$\vec{X} = \hat{j} - \vec{R} = \hat{j} - (5\hat{i} - \hat{j} + 4\hat{k})$$

$$= -5\hat{i} + 2\hat{j} - 4\hat{k}.$$

EXAMPLE 18. *Two forces $\vec{F_1} = 3\hat{i} + 4\hat{j}$ and $\vec{F_2} = 3\hat{j} + 4\hat{k}$ are acting simultaneously at a point. What is the magnitude of the resultant force ?*

Solution. The resultant force is

$$\vec{R} = \vec{F_1} + \vec{F_2} = (3\hat{i} + 4\hat{j}) + (3\hat{j} + 4\hat{k})$$

$$= 3\hat{i} + 7\hat{j} + 4\hat{k}$$

Magnitude of the resultant force is

$$|\vec{R}| = \sqrt{3^2 + 7^2 + 4^2} = \sqrt{9 + 49 + 16}$$

$$= \sqrt{74} \textbf{ units of force.}$$

EXAMPLE 19. *If $\vec{A} = 3\hat{i} + 4\hat{j}$ and $\vec{B} = 7\hat{i} + 24\hat{j}$, find a vector having the same magnitude as $\vec{B}$ and parallel to $\vec{A}$.* **[EAMCET, 89]**

Solution. $\quad |\vec{A}| = \sqrt{3^2 + 4^2} = 5$

Unit vector in the direction of $\vec{A}$ is

$$\hat{A} = \frac{\vec{A}}{|\vec{A}|} = \frac{1}{5}(3\hat{i} + 4\hat{j})$$

Also, $\quad |\vec{B}| = \sqrt{7^2 + 24^2} = 25$

The vector having the same magnitude as $\vec{B}$ and parallel to $\vec{A}$

$$= |\vec{B}|\hat{A} = 25 \times \tfrac{1}{5}(3\hat{i} + 4\hat{j}) = 15\hat{i} + 20\hat{j}.$$

EXAMPLE 20. *A bird moves with velocity 20 ms^{-1} in a direction making an angle of 60° with the eastern line and 60° with vertical upward. Represent the velocity vector in rectangular form.*

Solution. Let eastern line be taken as X-axis, northern as Y-axis and vertical upward as Z-axis. Let the velocity vector $\vec{v}$ make angles α, β and γ with X-, Y- and Z-axis respectively. Then $\alpha = 60°$, $\gamma = 60°$.

As $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$\therefore \quad \cos^2 60° + \cos^2\beta + \cos^2 60° = 1$

or $\quad \left(\frac{1}{2}\right)^2 + \cos^2\beta + \left(\frac{1}{2}\right)^2 = 1$

or $\quad \cos^2\beta = 1 - \frac{1}{2} = \frac{1}{2}$

or $\quad \cos\beta = \frac{1}{\sqrt{2}}$

$$\therefore \quad \vec{v} = v\cos\alpha\,\hat{i} + v\cos\beta\,\hat{j} + v\cos\gamma\,\hat{k}$$

$$= 20 \times \frac{1}{2}\,\hat{i} + 20 \times \frac{1}{\sqrt{2}}\,\hat{j} + 20 \times \frac{1}{2}\,\hat{k}$$

$$= 10\,\hat{i} + 10\sqrt{2}\,\hat{j} + 10\,\hat{k}.$$

EXAMPLE 21. *One of the rectangular components of a velocity of $80\ kmh^{-1}$ is $40\ kmh^{-1}$. Find the other component.*

Solution. Let $v = 80$ kmh^{-1}, $v_x = 40$ kmh^{-1}, then $v_y = ?$

As $v = \sqrt{v_x^2 + v_y^2}$

$$\therefore \quad v_y = \sqrt{v^2 - v_x^2} = \sqrt{80^2 - 40^2} = \sqrt{6400 - 1600}$$

$$= \sqrt{4800} = \mathbf{69.28\ kmh^{-1}}.$$

EXAMPLE 22. *A force is inclined at 50° to the horizontal. If its rectangular component in the horizontal direction be 50 N, find the magnitude of the force and its vertical component.*

Solution. Here $F_x = 50$ N, $\theta = 50°$

But $\quad F_x = F\cos\theta$

$$\therefore \quad F = \frac{F_x}{\cos\theta} = \frac{50}{\cos 50°} = \frac{50}{0.6428} = \mathbf{77.78\ N}.$$

Also, $\quad F_y = F\sin\theta = 50 \times \sin 50°$

$$= 77.78 \times 0.7660 = \mathbf{59.58\ N}.$$

EXAMPLE 23. *An aeroplane takes off at angle of 30° to the horizontal. If the component of its velocity along the horizontal is $250\ kmh^{-1}$, what is the actual velocity ? Find also the vertical component of the velocity.*

Solution. Let v_x and v_y be the horizontal and vertical components of actual velocity v (Fig. 4.40).

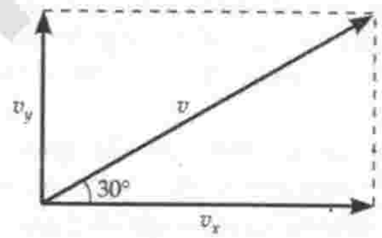


Fig. 4.40

Then $\quad v_x = v\cos 30° = 250$ kmh^{-1}

$$\therefore \quad v = \frac{250}{\cos 30°} = \frac{250 \times 2}{\sqrt{3}} = \mathbf{288.67\ kmh^{-1}}.$$

$$v_y = v\sin 30° = 288.67 \times 0.5 = \mathbf{144.33\ kmh^{-1}}.$$

EXAMPLE 24. *A man rows a boat with a speed of $18\ kmh^{-1}$ in the north-west direction. The shoreline makes an angle of 15° south of west. Obtain the component of the velocity of the boat along the shoreline and perpendicular to the shoreline.*

Solution. As shown in Fig. 4.41, the boat makes an angle of 45° with the west direction while the shorline makes an angle of 15° with it. Thus the boat makes angle of $45° + 15° = 60°$ with the shoreline.

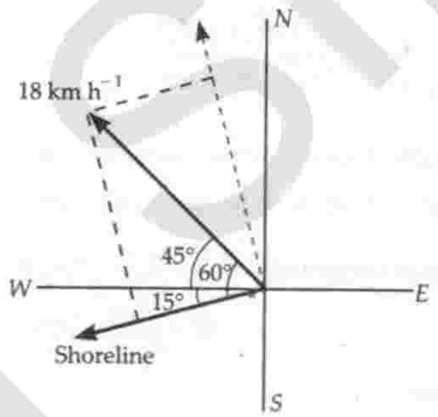


Fig. 4.41

$\therefore$ Component of the velocity of the boat along the shoreline

$$= 18\cos 60° = 18 \times \tfrac{1}{2} = \mathbf{9\ kmh^{-1}}.$$

Component of the velocity of the boat perpendicular to the shoreline

$$= 18\sin 60° = \frac{18 \times \sqrt{3}}{2} = \mathbf{15.6\ kmh^{-1}}.$$

EXAMPLE 25. *Two billiard balls are rolling on a flat table. One has the velocity components $v_x = 1\ ms^{-1}$, $v_y = \sqrt{3}\ ms^{-1}$ and the other has components $v'_x = 2\ ms^{-1}$ and $v'_y = 2$ ms^{-1}. If both the balls start moving from the same point, what is the angle between their paths ?*

Solution. *For first ball :*

$$OC = v_x = 1\ \text{ms}^{-1}, \quad OE = v_y = \sqrt{3}\ \text{ms}^{-1}.$$

Let θ be the angle which the resultant OA of v_x and v_y makes with the X-axis. Then

$$\tan\theta = \frac{v_y}{v_x} = \frac{\sqrt{3}}{1} = \sqrt{3}$$

$$\therefore \quad \theta = 60°$$

For second ball :

$$OD = v'_x = 2\ \text{ms}^{-1}, \quad OF = v'_y = 2\ \text{ms}^{-1}$$

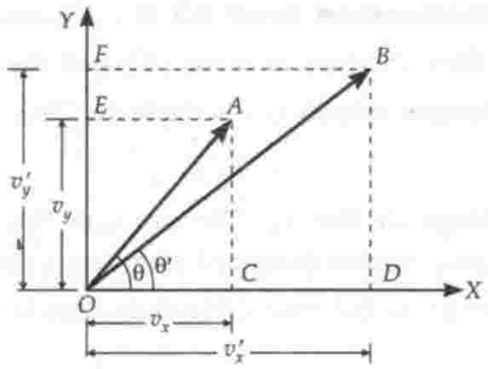


Fig. 4.42

Let θ' be the angle which the resultant OB of v'_x and v'_y makes with the X-axis. Then

$$\tan \theta' = \frac{v'_y}{v'_x} = \frac{2}{2} = 1$$

$$\therefore \quad \theta' = 45°$$

$\therefore$ Angles between the paths of two balls

$$= \theta - \theta' = 60° - 45° = \mathbf{15°}.$$

✖ PROBLEMS FOR PRACTICE

1. If $\vec{A} = 3\hat{i} + 2\hat{j}$ and $\vec{B} = \hat{i} - 2\hat{j} + 3\hat{k}$, find the magnitudes of $\vec{A} + \vec{B}$ and $\vec{A} - \vec{B}$. **[Ans.** 5, $\sqrt{29}$**]**
2. Find the unit vector parallel to the resultant of the vectors $\vec{A} = 2\hat{i} - 6\hat{j} - 3\hat{k}$ and $\vec{B} = 4\hat{i} + 3\hat{j} - \hat{k}$.
   [Ans. $1/\sqrt{61}\,(6\hat{i} - 3\hat{j} - 4\hat{k})$**]**
3. Determine the vector which when added to the resultant of $\vec{A} = 2\hat{i} - 4\hat{j} - 6\hat{k}$ and $\vec{B} = 4\hat{i} + 3\hat{j} + 3\hat{k}$ gives the unit vector along z-axis. **[Ans.** $-6\hat{i} + \hat{j} + 4\hat{k}$**]**
4. Find the value of λ in the unit vector $0.4\hat{i} + 0.8\hat{j} + \lambda\hat{k}$. **[Ans.** $\sqrt{0.2}$**]**
5. Given three coplanar vectors $\vec{a} = 4\hat{i} - \hat{j}$, $\vec{b} = -3\hat{i} + 2\hat{j}$ and $\vec{c} = -3\hat{j}$. Find the magnitude of the sum of the three vectors. **[Ans.** $\sqrt{5}$**]**
6. A force is inclined at 30° to the horizontal. If its rectangular component in the horizontal direction is 50 N, find the magnitude of the force and its vertical component. **[Ans.** 57.74 N, 28.87 N**]**
7. A velocity of 10 ms^{-1} has its Y-component $5\sqrt{2}$ ms^{-1}. Calculate its X-component. **[Ans.** $5\sqrt{2}$ ms^{-1}**]**
8. An aeroplane takes off at an angle of 30° to the horizontal. If the component of its velocity along the horizontal is 200 km h^{-1}, what is its actual velocity? Also find the vertical component of its velocity.
   [Ans. 230.94 km h^{-1}, 115.47 km h^{-1}**]**
9. A child pulls a rope attached to a stone with a force of 60 N. The rope makes an angle of 40° to the ground. (*i*) Calculate the effective value of the pull tending to move the stone along the ground. (*ii*) Calculate the force tending to lift the stone.
   [Ans. (*i*) 45.96 N (*ii*) 38.57 N**]**

✖ HINTS

4. As the given vector is a unit vector, so

$$|0.4\hat{i} + 0.8\hat{j} + \lambda\hat{k}| = 1$$

or $$\sqrt{(0.4)^2 + (0.8)^2 + \lambda^2} = 1$$

or $$\lambda^2 = 1 - (0.4)^2 - (0.8)^2 = 1 - 0.16 - 0.64 = 0.2$$

or $$\lambda = \sqrt{\mathbf{0.2}}.$$

9. From Fig. 4.43,

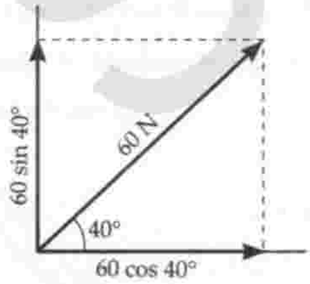


Fig. 4.43

(*i*) Pull along the ground

$$= 60 \cos 40° = \mathbf{45.96\ N.}$$

(*ii*) Force tending to lift the stone vertically up

$$= 60 \sin 40° = \mathbf{38.57\ N.}$$

4.14 SCALAR PRODUCT OF TWO VECTORS

26. *What are the different ways in which a vector can be multiplied by another vector ?*

These are *two* ways in which a vector can be multiplied by another vector :

(*i*) One way produces a scalar and is known as **scalar product**.

(*ii*) Another way produces a new vector and is known as **vector product**.

27. *Define scalar product of two vectors. Give its geometrical interpretation.*

Scalar or dot product. *The scalar or dot product of two vectors $\vec{A}$ and $\vec{B}$ is defined as the product of the magnitudes of $\vec{A}$ and $\vec{B}$ and cosine of the angle θ between them.* Thus

$$\vec{A} \cdot \vec{B} = |\vec{A}|\,|\vec{B}| \cos \theta = AB \cos \theta$$

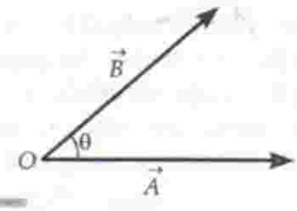


Fig. 4.44 Scalar product of $\vec{A}$ and $\vec{B}$ is a scalar : $\vec{A}\cdot\vec{B} = AB\cos\theta$.

As A, B and $\cos\theta$ are all scalars, so the dot product of $\vec{A}$ and $\vec{B}$ is a scalar quantity. Both $\vec{A}$ and $\vec{B}$ have directions, but their dot product has no direction.

Geometrical interpretation of scalar product. As shown in Fig. 4.45(*a*), suppose two vectors $\vec{A}$ and $\vec{B}$ are represented by $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ and $\angle POQ = \theta$.

$$\vec{A}\cdot\vec{B} = AB\cos\theta = A(B\cos\theta) = A(OR)$$

$= A\times$ Magnitude of component of $\vec{B}$ in the direction of $\vec{A}$

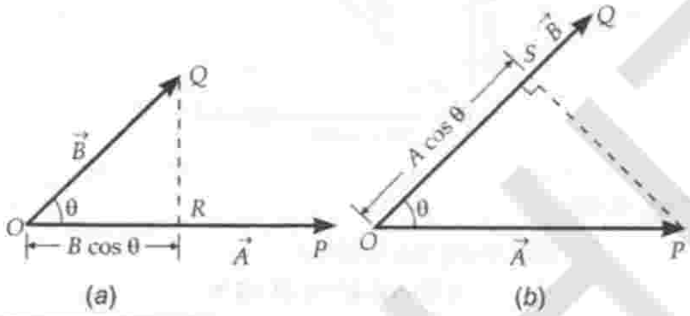


Fig. 4.45 (*a*) $B\cos\theta$ is the projection of $\vec{B}$ onto $\vec{A}$.
(*b*) $A\cos\theta$ is the projection of $\vec{A}$ onto $\vec{B}$.

From Fig. 4.45(*b*), we have

$$\vec{A}\cdot\vec{B} = AB\cos\theta = B(A\cos\theta)$$
$$= B(OS)$$

$= B\times$ Magnitude of component of $\vec{A}$ in the direction of $\vec{B}$.

Thus *the scalar product of two vectors is equal to the product of magnitude of one vector and the magnitude of component of other vector in the direction of first vector.*

28. *Give some examples of physical quantities that may be expressed as the scalar product of two vectors.*

Physical examples of scalar product of two vectors :

(*i*) **Work done (W).** *It is defined on the scalar product of the force ($\vec{F}$) acting on the body and the displacement ($\vec{s}$) produced.* Thus

$$W = \vec{F}\cdot\vec{s}$$

(*ii*) **Instantaneous power (P).** It is defined as the scalar product of force ($\vec{F}$) and the instantaneous velocity ($\vec{v}$) of the body. Thus

$$P = \vec{F}\cdot\vec{v}$$

(*iii*) **Magnetic flux** (ϕ). The magnetic flux linked with a surface is defined as the scalar product of magnetic induction ($\vec{B}$) and the area vector ($\vec{A}$). Thus

$$\phi = \vec{B}\cdot\vec{A}$$

Note As the scalar product of two vectors is a scalar quantity, so *work, power and magnetic flux are all scalar quantities.*

4.15 PROPERTIES OF SCALAR PRODUCT OF TWO VECTORS

29. *Mention important properties of the scalar product of vectors.*

Properties of scalar product :

(*i*) The scalar product is commutative *i.e.,*

$$\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$$

(*ii*) The scalar product is distributive over addition *i.e.,*

$$\vec{A}\cdot(\vec{B}+\vec{C}) = \vec{A}\cdot\vec{B} + \vec{A}\cdot\vec{C}$$

(*iii*) If $\vec{A}$ and $\vec{B}$ are two vectors perpendicular to each other, then their scalar product is zero.

$$\vec{A}\cdot\vec{B} = AB\cos 90° = 0.$$

(*iv*) If $\vec{A}$ and $\vec{B}$ are two parallel vectors having same direction, then their scalar product has the maximum positive magnitude.

$$\vec{A}\cdot\vec{B} = AB\cos 0° = AB$$

(*v*) If $\vec{A}$ and $\vec{B}$ are two parallel vectors having opposite directions, then their scalar product has the maximum negative magnitude.

$$\vec{A}\cdot\vec{B} = AB\cos 180° = -AB$$

(*vi*) The scalar product of a vector with itself is equal to the square of its magnitude.

$$\vec{A}\cdot\vec{A} = A\cdot A\cos 0° = A\cdot A = A^2 = |\vec{A}|^2$$

(*vii*) Scalar product of two similar base vectors is unity and that of two different base vectors is zero.

$$\hat{i}\cdot\hat{i} = (1)(1)\cos 0° = 1$$

$\therefore \quad \hat{i}.\hat{i} = \hat{j}.\hat{j} = \hat{k}.\hat{k} = 1$

$\hat{i}.\hat{j} = (1)(1)\cos 90° = 0$

$\therefore \quad \hat{i}.\hat{j} = \hat{j}.\hat{k} = \hat{k}.\hat{i} = 0$

(*viii*) Scalar product of two vectors $\vec{A}$ and $\vec{B}$ is equal to the sum of the products of their corresponding rectangular components.

$$\vec{A}\cdot\vec{B} = A_x B_x + A_y B_y + A_z B_z$$

(*ix*) The cosine of the angle θ between $\vec{A}$ and $\vec{B}$ is given by

$$\cos\theta = \frac{\vec{A}\cdot\vec{B}}{|\vec{A}||\vec{B}|}$$

$$= \frac{A_x B_x + A_y B_y + A_z B_z}{\sqrt{A_x^2 + A_y^2 + A_z^2}\sqrt{B_x^2 + B_y^2 + B_z^2}}$$

30. *Show that the scalar product of two vectors $\vec{A}$ and $\vec{B}$ is commutative.*

Scalar product of two vectors is commutative. We know that

$$\vec{A}\cdot\vec{B} = AB\cos\theta \qquad ...(i)$$

where θ is the angle between $\vec{A}$ and $\vec{B}$ measured anticlockwise, as shown in Fig. 4.46.

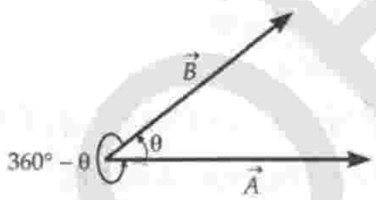


Fig. 4.46

Also, $\vec{B}\cdot\vec{A} = BA\cos(360° - \theta)$

$$= BA\cos\theta = AB\cos\theta \qquad ...(ii)$$

where $(360° - \theta)$ is the angle between $\vec{B}$ and $\vec{A}$ measured anticlockwise.

From equations (*i*) and (*ii*), we get

$$\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$$

Hence *the scalar product of two vectors is commutative.*

31. *Prove that the scalar product of vectors is distributive over addition.*

Scalar product of vectors is distributive over addition. In Fig. 4.47, OP, PQ and OQ are the projections or components of $\vec{B}$, $\vec{C}$ and $(\vec{B}+\vec{C})$ in the direction of vector $\vec{A}$. By definition of scalar product,

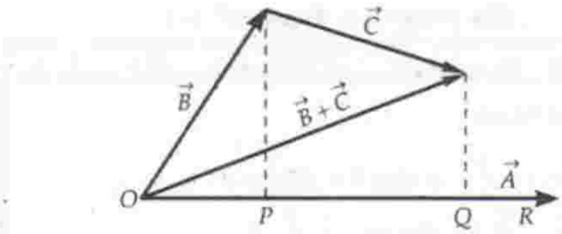


Fig. 4.47

$\vec{A}\cdot(\vec{B}+\vec{C})$

= Magnitude of $\vec{A}$ × Magnitude of component of $(\vec{B}+\vec{C})$ in the direction of $\vec{A}$

$= (OR)(OQ) = (OR)(OP + PQ)$

$= (OR)(OP) + (OR)(PQ)$

= Magnitude of $\vec{A}$ × Magnitude of component of $\vec{B}$ in the direction of $\vec{A}$

+ Magnitude of $\vec{A}$ × Magnitude of component of $\vec{C}$ in the direction of $\vec{A}$

$= \vec{A}\cdot\vec{B} + \vec{A}\cdot\vec{C}$ [By definition of scalar product]

Hence *the scalar product is distributive over addition.*

32. *Show that the scalar product of two vectors is equal to the sum of the products of their corresponding rectangular components.*

Scalar product in terms of rectangular components. We can express $\vec{A}$ and $\vec{B}$ in terms of their rectangular components as

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$

and $\quad \vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$

$$\vec{A}\cdot\vec{B} = (A_x\hat{i} + A_y\hat{j} + A_z\hat{k}).(B_x\hat{i} + B_y\hat{j} + B_z\hat{k})$$

$$= A_x\hat{i}\cdot(B_x\hat{i} + B_y\hat{j} + B_z\hat{k}) + A_y\hat{j}\cdot(B_x\hat{i} + B_y\hat{j} + B_z\hat{k}) + A_z\hat{k}\cdot(B_x\hat{i} + B_y\hat{j} + B_z\hat{k})$$

$$= A_xB_x\,\hat{i}.\hat{i} + A_xB_y\,\hat{i}.\hat{j} + A_xB_z\,\hat{i}.\hat{k} + A_yB_x\,\hat{j}.\hat{i} + A_yB_y\,\hat{j}.\hat{j} + A_yB_z\,\hat{j}.\hat{k} + A_zB_x\,\hat{k}.\hat{i} + A_zB_y\,\hat{k}.\hat{j} + A_zB_z\,\hat{k}.\hat{k}$$

$$= A_xB_x(1) + A_xB_y(0) + A_xB_z(0) + A_yB_x(0) + A_yB_y(1) + A_yB_z(0) + A_zB_x(0) + A_zB_y(0) + A_zB_z(1)$$

or $\quad \vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z$

Examples based on Scalar or Dot Product of two Vectors

FORMULAE USED

1. $\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos\theta = AB\cos\theta$
2. If $\vec{A}\perp\vec{B}$, $\theta = 90°$ and $\vec{A}\cdot\vec{B} = 0$
3. Angle θ between $\vec{A}$ and $\vec{B}$ is given by
$$\cos\theta = \frac{\vec{A}\cdot\vec{B}}{|\vec{A}||\vec{B}|}$$
4. In terms of rectangular components,
$$\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z$$
5. Work done, $W = \vec{F}\cdot\vec{S}$
6. Power, $P = \vec{F}\cdot\vec{v}$

EXAMPLE 26. *Find the angle between the vectors $\vec{A} = \hat{i} + 2\hat{j} - \hat{k}$ and $\vec{B} = -\hat{i} + \hat{j} - 2\hat{k}$.* **[Delhi 1995]**

Solution.
$$|\vec{A}| = \sqrt{1^2 + 2^2 + (-1)^2} = \sqrt{6}$$
$$|\vec{B}| = \sqrt{(-1)^2 + 1^2 + (-2)^2} = \sqrt{6}$$
$$\vec{A}\cdot\vec{B} = 1\times(-1) + 2\times 1 + (-1)\times(-2)$$
$$= -1 + 2 + 2 = 3$$
$$\therefore \quad \cos\theta = \frac{\vec{A}\cdot\vec{B}}{|\vec{A}||\vec{B}|} = \frac{3}{\sqrt{6}\times\sqrt{6}} = \frac{3}{6} = \frac{1}{2}$$

Hence $\theta = \mathbf{60°}$.

EXAMPLE 27. *Prove that the vectors $\vec{A} = \hat{i} + 2\hat{j} + 3\hat{k}$ and $\vec{B} = 2\hat{i} - \hat{j}$ are perpendicular to each other.*

Solution. $\vec{A}\cdot\vec{B} = (\hat{i} + 2\hat{j} + 3\hat{k})\cdot(2\hat{i} - \hat{j} + 0\hat{k})$
$$= 1\times 2 + 2\times(-1) + 3\times 0 = 0$$

Hence $\vec{A}\perp\vec{B}$.

EXAMPLE 28. *Find the value of λ so that the vectors $\vec{A} = 2\hat{i} + \lambda\hat{j} + \hat{k}$ and $\vec{B} = 4\hat{i} - 2\hat{j} - 2\hat{k}$ are perpendicular to each other.*

Solution. As $\vec{A}\perp\vec{B}$, so $\vec{A}\cdot\vec{B} = 0$

or $(2\hat{i} + \lambda\hat{j} + \hat{k})\cdot(4\hat{i} - 2\hat{j} - 2\hat{k}) = 0$

or $2\times 4 + \lambda\times(-2) + 1\times(-2) = 0$

or $\lambda = 3$.

EXAMPLE 29. *If the magnitudes of two vectors are 3 and 4 and their scalar product is 6, then find the angle between the two vectors.*

Solution. Here $|\vec{A}| = 3$, $|\vec{B}| = 4$, $\vec{A}\cdot\vec{B} = 6$
$$\therefore \quad \cos\theta = \frac{\vec{A}\cdot\vec{B}}{|\vec{A}||\vec{B}|} = \frac{6}{3\times 4} = \frac{1}{2}$$

Hence $\theta = \mathbf{60°}$.

EXAMPLE 30. *A body constrained to move along the z-axis of a co-ordinate system is subjected to a constant force $\vec{F}$ given by $\vec{F} = -\hat{i} + 2\hat{j} + 3\hat{k}$ newton where $\hat{i}$, $\hat{j}$ and $\hat{k}$ represent unit vectors along x-, y- and z-axis of the system respectively. Calculate the work done by the force in displacing the body through a distance of 4 m along the z-axis.* **[NCERT]**

Solution. As the body moves 4 m along the z-axis, so the displacement vector is
$$\vec{S} = 4\hat{k} = 0\hat{i} + 0\hat{j} + 4\hat{k} \text{ metre}$$
Also $\vec{F} = -\hat{i} + 2\hat{j} + 3\hat{k}$ newton
$$\therefore \quad W = \vec{F}\cdot\vec{S}$$
$$= (-\hat{i} + 2\hat{j} + 3\hat{k})\cdot(0\hat{i} + 0\hat{j} + 4\hat{k})$$
$$= -1\times 0 + 2\times 0 + 3\times 4 = \mathbf{12\ joule.}$$

EXAMPLE 31. *A force of $7\hat{i} + 6\hat{k}$ newton makes a body move on a rough plane with a velocity of $3\hat{j} + 4\hat{k}$ ms^{-1}. Calculate the power in watt.*

Solution. Power
$$P = \vec{F}\cdot\vec{v} = (7\hat{i} + 0\hat{j} + 6\hat{k})\cdot(0\hat{i} + 3\hat{j} + 4\hat{k})$$
$$= 7\times 0 + 0\times 3 + 6\times 4 = \mathbf{24\ W.}$$

EXAMPLE 32. *Three vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ are such that $\vec{A} = \vec{B} + \vec{C}$ and their magnitudes are 5, 4 and 3 respectively. Find the angle between $\vec{A}$ and $\vec{C}$.*

Solution. Given $\vec{A} = \vec{B} + \vec{C}$ or $\vec{B} = \vec{A} - \vec{C}$

Now $\vec{B}\cdot\vec{B} = (\vec{A} - \vec{C})\cdot(\vec{A} - \vec{C})$
$$= \vec{A}\cdot\vec{A} - \vec{A}\cdot\vec{C} - \vec{C}\cdot\vec{A} + \vec{C}\cdot\vec{C}$$
$$= \vec{A}\cdot\vec{A} + \vec{C}\cdot\vec{C} - 2\,\vec{A}\cdot\vec{C}$$
$$[\because \vec{C}\cdot\vec{A} = \vec{A}\cdot\vec{C}]$$

or $B^2 = A^2 + C^2 - 2AC\cos\theta$

where θ is the angle between $\vec{A}$ and $\vec{C}$.

Thus

$$\cos\theta = \frac{A^2 + C^2 - B^2}{2\,AC} = \frac{5^2 + 3^2 - 4^2}{2 \times 5 \times 3} = \frac{18}{30} = 0.6$$

$$\theta = \cos^{-1}(0.6) = \mathbf{53^\circ}.$$

EXAMPLE 33. *If $|\vec{A} + \vec{B}| = |\vec{A} - \vec{B}|$, find the angle between $\vec{A}$ and $\vec{B}$.* **[CBSE PMT 2K]**

Solution. Given :

$$|\vec{A} + \vec{B}| = |\vec{A} - \vec{B}|$$

or $$|\vec{A} + \vec{B}|^2 = |\vec{A} - \vec{B}|^2$$

or $$(\vec{A} + \vec{B}).(\vec{A} + \vec{B}) = (\vec{A} - \vec{B}).(\vec{A} - \vec{B})$$

$$[\because |\vec{A}|^2 = \vec{A} \cdot \vec{A}]$$

or $$\vec{A}.\vec{A} + \vec{A}.\vec{B} + \vec{B}.\vec{A} + \vec{B}.\vec{B} = \vec{A}.\vec{A} - \vec{A}.\vec{B} - \vec{B}.\vec{A} + \vec{B}.\vec{B}$$

or $$A^2 + 2\,\vec{A}.\vec{B} + B^2 = A^2 - 2\,\vec{A}.\vec{B} + B^2$$

$$[\because \vec{B}.\vec{A} = \vec{A}.\vec{B}]$$

or $$4\,\vec{A}.\vec{B} = 0 \quad \text{or} \quad 4AB\cos\theta = 0$$

As $\vec{A}$ and $\vec{B}$ are non-zero vectors, so

$$\cos\theta = 0 \quad \text{or} \quad \theta = \mathbf{90^\circ}$$

EXAMPLE 34. *If vectors $\vec{P}$, $\vec{Q}$ and $\vec{R}$ have magnitudes 5, 12 and 13 units and $\vec{P} + \vec{Q} = \vec{R}$, find the angle between $\vec{Q}$ and $\vec{R}$.*

Solution. As $\vec{P} + \vec{Q} = \vec{R}$.

$\therefore$ $$\vec{R} - \vec{Q} = \vec{P}$$

and $$(\vec{R} - \vec{Q}).(\vec{R} - \vec{Q}) = \vec{P}.\vec{P}$$

or $$\vec{R}.\vec{R} - \vec{R}.\vec{Q} - \vec{Q}.\vec{R} + \vec{Q}.\vec{Q} = \vec{P}.\vec{P}$$

or $$R^2 - 2\,\vec{R}.\vec{Q} + Q^2 = P^2$$

or $$\cos\theta = \frac{R^2 + Q^2 - P^2}{2\,RQ} = \frac{13^2 + 12^2 - 5^2}{2 \times 13 \times 12} = \frac{288}{2 \times 13 \times 12} = \frac{12}{13}$$

$\therefore$ $$\theta = \mathbf{\cos^{-1} 12/13.}$$

EXAMPLE 35. *Determine the angles which the vector $\vec{A} = 5\,\hat{i} + 0\,\hat{j} + 5\,\hat{k}$, makes with X-, Y- and Z-axes.*

Solution. Here $A = |\vec{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$

$$= \sqrt{5^2 + 0^2 + 5^2} = 5\sqrt{2}$$

If vector $\vec{A}$ makes angles α, β and γ with X-, Y- and Z-axis respectively, then

$$\cos\alpha = \frac{A_x}{A} = \frac{5}{5\sqrt{2}} = \frac{1}{\sqrt{2}} \qquad \therefore \qquad \alpha = \mathbf{45^\circ}.$$

$$\cos\beta = \frac{A_y}{A} = \frac{0}{5\sqrt{2}} = 0 \qquad \therefore \qquad \beta = \mathbf{90^\circ}$$

$$\cos\gamma = \frac{A_z}{A} = \frac{5}{5\sqrt{2}} = \frac{1}{\sqrt{2}} \qquad \therefore \qquad \gamma = \mathbf{45^\circ}.$$

EXAMPLE 36. *If unit vectors $\hat{a}$ and $\hat{b}$ are inclined at angle θ, then prove that*

$$|\hat{a} - \hat{b}| = 2\sin\frac{\theta}{2}.$$ **[NCERT]**

Solution. For any vector $\vec{a}$,

$$|\vec{a}|^2 = \vec{a}.\vec{a}$$

$$\therefore \; |\hat{a} - \hat{b}|^2 = (\hat{a} - \hat{b}).(\hat{a} - \hat{b})$$

$$= \hat{a}.\hat{a} - \hat{a}.\hat{b} - \hat{b}.\hat{a} + \hat{b}.\hat{b}$$

$$= 1 - 2\,\hat{a}.\hat{b} + 1$$

$$[\because \hat{a}.\hat{a} = 1 \times 1 \times \cos 0^\circ = 1]$$

$$= 2 - 2 \times 1 \times 1 \times \cos\theta = 2(1 - \cos\theta)$$

$$= 2.2\sin^2\frac{\theta}{2} = 4\sin^2\frac{\theta}{2}$$

$$[\because 1 - \cos 2\theta = 2\sin^2\theta]$$

Hence $$|\hat{a} - \hat{b}| = 2\sin\frac{\theta}{2}.$$

EXAMPLE 37. *If $\vec{A} + \vec{B} = \vec{C}$ and $A^2 + B^2 = C^2$, then prove that $\vec{A}$ and $\vec{B}$ are perpendicular to each other.*

Solution. Given $\vec{A} + \vec{B} = \vec{C}$

$$\therefore \; (\vec{A} + \vec{B}).(\vec{A} + \vec{B}) = \vec{C}.\vec{C}$$

or $$A^2 + 2\,\vec{A}.\vec{B} + B^2 = C^2$$

But $$A^2 + B^2 = C^2$$

$\therefore$ $$2\,\vec{A}.\vec{B} = 0$$

or $$2AB\cos\theta = 0$$

or $$\cos\theta = 0 \qquad [\because A \neq 0, B \neq 0]$$

$\therefore$ $$\theta = \mathbf{90^\circ}.$$

EXAMPLE 38. *A point P lies in the x-y plane. Its position can be specified by its x, y coordinates or by a radially directed vector $\vec{r} = (x\hat{i} + y\hat{j})$, making an angle θ with the x-axis. Find a vector $\hat{i}_r$ of unit magnitude in the direction of vector $\vec{r}$ and a vector $\hat{i}_\theta$ of unit magnitude normal to the vector $\hat{i}_r$ and lying in the x-y plane.* **[NCERT]**

Solution. The unit vector in the direction of vector $\vec{r}$ is given by

$$\hat{i}_r = \frac{\vec{r}}{r} = \frac{x\,\hat{i} + y\,\hat{j}}{r} = \hat{i}\,\frac{x}{r} + \hat{j}\,\frac{y}{r}$$

or $$\hat{i}_r = \hat{i}\,\cos\theta + \hat{j}\,\sin\theta$$

$[\because\ x = r\cos\theta,\ y = r\sin\theta]$

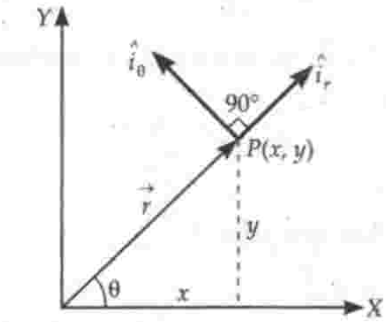


Fig. 4.48

Let the unit vector normal to the unit vector $\hat{i}_r$ be given by

$$\hat{i}_\theta = \hat{i}\,\alpha + \hat{j}\,\beta$$

where the coefficients α and β are to be determined. As $\hat{i}_r \perp \hat{i}_\theta$, so

$$\hat{i}_r \cdot \hat{i}_\theta = 0$$

or $(\hat{i}\cos\theta + \hat{j}\sin\theta) \cdot (\hat{i}\,\alpha + \hat{j}\,\beta) = 0$

or $$\alpha\cos\theta + \beta\sin\theta = 0$$

or $$\frac{\alpha}{\beta} = -\frac{\sin\theta}{\cos\theta}$$

and $$\alpha = -\beta\,\frac{\sin\theta}{\cos\theta}$$

Since $\hat{i}_\theta$ is a vector of unit magnitude, so

$$\alpha^2 + \beta^2 = 1$$

or $$\left(\frac{\alpha}{\beta}\right)^2 + 1 = \frac{1}{\beta^2}$$

or $$\left(-\frac{\sin\theta}{\cos\theta}\right)^2 + 1 = \frac{1}{\beta^2} \quad \text{or} \quad \frac{\sin^2\theta + \cos^2\theta}{\cos^2\theta} = \frac{1}{\beta^2}$$

or $$\beta^2 = \cos^2\theta$$

$\Rightarrow$ $$\beta = +\cos\theta \quad \text{and} \quad \alpha = -\sin\theta$$

The other solution, $\beta = -\cos\theta$ is rejected because we define a system of $\hat{i}_r$, $\hat{i}_\theta$ vectors in the increasing directions of $\vec{r}$ and θ.

Hence $\hat{i}_\theta = -\hat{i}\,\sin\theta + \hat{j}\,\cos\theta.$

PROBLEMS FOR PRACTICE

1. Find the angle between the vectors
$\vec{A} = 2\hat{i} - 4\hat{j} + 6\hat{k}$ and $\vec{B} = 3\hat{i} + \hat{j} + 2\hat{k}$.
(Ans. 60°)

2. Find the angles between the vectors $\vec{A} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{B} = -\hat{i} - \hat{j} + 2\hat{k}$. **[Delhi 95]** **(Ans.** 90°)

3. Find the value of m so that the vector $3\hat{i} - 2\hat{j} + \hat{k}$ is perpendicular to the vector $2\hat{i} + 6\hat{j} + m\hat{k}$.
(Ans. 6)

4. For what value of a are the vectors $\vec{A} = a\,\hat{i} - 2\hat{j} + \hat{k}$ and $\vec{B} = 2a\,\hat{i} + a\,\hat{j} - 4\,\hat{k}$ perpendicular to each other? **(Ans.** 2, – 1)

5. Find the angles between the following pairs of vectors :

   (i) $\vec{A} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{B} = -2\hat{i} - 2\hat{j} - 2\hat{k}$.
   (Ans. 180°)

   (ii) $\vec{A} = -2\hat{i} + 2\hat{j} - \hat{k}$ and $\vec{B} = 3\hat{i} + 6\hat{j} + 2\hat{k}$
   (Ans. 79°)

   (iii) $\vec{A} = 4\hat{i} + 6\hat{j} - 3\hat{k}$ and $\vec{B} = -2\hat{i} - 5\hat{j} + 7\hat{k}$.
   (Ans. 148.9°)

6. Calculate the values of (i) $\hat{j}\cdot(2\hat{i} - 3\hat{j} + \hat{k})$ and (ii) $(2\hat{i} - \hat{j})\cdot(3\hat{i} + \hat{k})$. **[Ans.** (i) – 3 (ii) 6]

7. A force $\vec{F} = 4\hat{i} + \hat{j} + 3\hat{k}$ newton acts on a particle and displaces it through displacement $\vec{S} = 11\hat{i} + 11\hat{j} + 15\hat{k}$ metre. Calculate the work done by the force. **(Ans.** 100 J)

8. Under a force of $10\hat{i} - 3\hat{j} + 6\hat{k}$ newton, a body of mass 5 kg is displaced from the position $6\hat{i} + 5\hat{j} - 3\hat{k}$ to the position $10\hat{i} - 2\hat{j} + 7\hat{k}$. Calculate the work done. **(Ans.** 121 J)

9. The sum and difference of two vectors $\vec{A}$ and $\vec{B}$ are $\vec{A} + \vec{B} = 2\hat{i} + 6\hat{j} + \hat{k}$ and $\vec{A} - \vec{B} = 4\hat{i} + 2\hat{j} - 11\hat{k}$. Find the magnitude of each vector and their scalar product $\vec{A}\cdot\vec{B}$. **(Ans.** $\sqrt{50}$, $\sqrt{41}$, – 25)

10. A force $\vec{F} = 5\,\hat{i} + 4\,\hat{j}$ newton displaces a body through $\vec{S} = 3\,\hat{i} + 4\,\hat{k}$ metre in 3 s. Find the power.
(Ans. 5 W)

11. If the resultant of the vectors $3\hat{i}+4\hat{j}+5\hat{k}$ and $5\hat{i}+3\hat{j}+4\hat{k}$ makes an angle θ with x-axis, then find $\cos\theta$. (**Ans.** 0.5744)
12. Show that the vectors $\vec{A}=3\hat{i}-2\hat{j}+\hat{k}$, $\vec{B}=\hat{i}-3\hat{j}+5\hat{k}$ and $\vec{C}=2\hat{i}+\hat{j}-4\hat{k}$ form a right angle triangle.
13. If vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ have magnitudes 8, 15 and 17 units and $\vec{A}+\vec{B}=\vec{C}$, find the angle between $\vec{A}$ and $\vec{B}$. (**Ans.** 90°)
14. If $\vec{A}=\vec{B}-\vec{C}$, then determine the angle between $\vec{A}$ and $\vec{B}$. $\left(\textbf{Ans. } \theta=\cos^{-1}\dfrac{A^2+B^2-C^2}{2AB}\right)$
15. For two vectors $\vec{A}$ and $\vec{B}$ if $\vec{A}+\vec{B}=\vec{C}$ and $A+B=C$, then prove that $\vec{A}$ and $\vec{B}$ are parallel to each other.
16. Prove that :
$(\vec{A}+2\vec{B}).(2\vec{A}-3\vec{B})=2A^2+AB\cos\theta-6B^2$.

4.16 VECTOR PRODUCT OF TWO VECTORS

33. *Define vector or cross product of two vectors. How is its direction determined ?*

Vector or cross product. *The vector or cross product of two vectors is defined as the vector whose magnitude is equal to the product of the magnitudes of two vectors and sine of the angle between them and whose direction is perpendicular to the plane of the two vectors and is given by right hand rule.* Mathematically, if θ is the angle between vectors $\vec{A}$ and $\vec{B}$, then

$$\vec{A}\times\vec{B}=AB\sin\theta\,\hat{n}$$

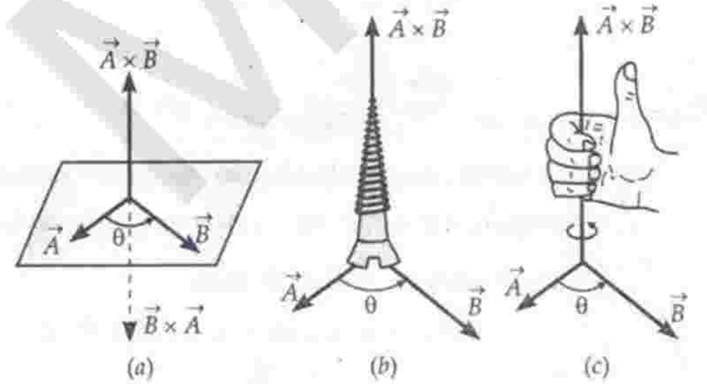


Fig. 4.49 Right hand rules for direction of vector product.

where $\hat{n}$ is a unit vector perpendicular to the plane of $\vec{A}$ and $\vec{B}$ and its direction is given by right hand rule. Thus, the direction of $\vec{A}\times\vec{B}$ is same as that of unit vector $\hat{n}$.

Rules for determining the direction of $\vec{A}\times\vec{B}$:

(*i*) **Right handed screw rule.** As shown in Fig. 4.49(*b*), *if a right handed screw is placed with its axis perpendicular to the plane of vectors $\vec{A}$ and $\vec{B}$ and is rotated from $\vec{A}$ to $\vec{B}$ through the smaller angle, then the direction in which the screw advances gives the direction of $\vec{A}\times\vec{B}$.*

(*ii*) **Right hand thumb rule.** As shown in Fig. 4.49(*c*), *curl the fingers of the right hand in such a way that they point in the direction of rotation from vector $\vec{A}$ to $\vec{B}$ through the smaller angle, then the stretched thumb points in the direction of $\vec{A}\times\vec{B}$.*

34. *Give geometrical interpretation of vector product of two vectors.*

Geometrical interpretation of vector product. Suppose two vectors $\vec{A}$ and $\vec{B}$ are represented by the sides OP and OQ of a parallelogram $OPRQ$, as shown in Fig. 4.50.

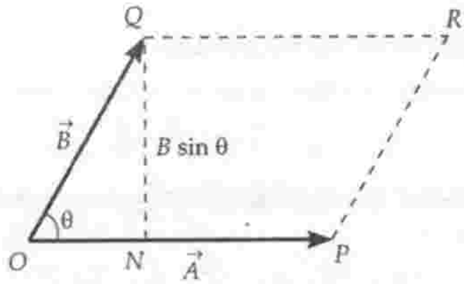


Fig. 4.50 Geometrical significance of vector product.

Let $\angle POQ=\theta$. Draw $QN\perp OP$. The magnitude of vector product $\vec{A}\times\vec{B}$ is

$$|\vec{A}\times\vec{B}|=AB\sin\theta=(OP)(OQ)\sin\theta$$
$$=(OP)(QN) \qquad [\because QN=OQ\sin\theta]$$
$$=\text{Area of parallelogram } OPRQ$$

Thus *the magnitude of the vector product of two vectors is equal to area of the parallelogram formed by the two vectors as its adjacent sides.* Moreover,

$$|\vec{A}\times\vec{B}|=2\times\tfrac{1}{2}(OP)(QN)=2\times\text{Area of }\Delta\, POQ$$

Thus *the magnitude of the vector product of two vectors is equal to twice the area of the triangle formed by the two vectors as its adjacent sides.*

35. *Give some examples of physical quantities which can be expressed as the vector product of two vectors.*

Physical examples of vector product :

(*i*) **Torque $\vec{\tau}$.** The torque acting on a particle is equal to the vector product of its position vector ($\vec{r}$) and force vector ($\vec{F}$). Thus

$$\vec{\tau} = \vec{r} \times \vec{F}$$

(*ii*) **Angular momentum $\vec{L}$.** The angular momentum of a particle is equal to the cross product of its position vector ($\vec{r}$) and linear momentum ($\vec{p}$). Thus

$$\vec{L} = \vec{r} \times \vec{p}$$

(*iii*) **Instantaneous velocity $\vec{v}$.** The instantaneous velocity of a particle is equal to the cross product of its angular velocity ($\vec{\omega}$) and the position vector ($\vec{r}$). Thus

$$\vec{v} = \vec{\omega} \times \vec{r}$$

4.17 PROPERTIES OF VECTOR PRODUCT

36. *Mention some important properties of vector product.*

Properties of vector product :

(*i*) **Vector product is anti-commutative** *i.e.,*

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$$

(*ii*) **Vector product is distributive over addition** *i.e.,*

$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$$

(*iii*) **Vector product of two parallel or antiparallel vectors is a null vector.** Thus

$$\vec{A} \times \vec{B} = AB \sin(0° \text{ or } 180°)\,\hat{n} = \vec{0}$$

(*iv*) **Vector product of a vector with itself is a null vector.**

$$\vec{A} \times \vec{A} = AA \sin 0°\,\hat{n} = \vec{0}$$

(*v*) **The magnitude of the vector product of two mutually perpendicular vectors is equal to the product of their magnitudes.**

$$|\vec{A} \times \vec{B}| = AB \sin 90° = AB$$

(*vi*) **Vector product of orthogonal unit vectors.** The magnitude of each of the vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$ is 1 and the angle between any of two of them is 90°.

$$\therefore \quad \hat{i} \times \hat{j} = (1)(1) \sin 90°\,\hat{n} = \hat{n}$$

As $\hat{n}$ is a unit vector perpendicular to the plane of $\hat{i}$ and $\hat{j}$, so it is just the third vector $\hat{k}$

$$\therefore \quad \hat{i} \times \hat{j} = \hat{k}$$

$$\hat{i} \times \hat{j} = \hat{k},\ \hat{j} \times \hat{k} = \hat{i},\ \hat{k} \times \hat{i} = \hat{j}$$

$$\hat{j} \times \hat{i} = -\hat{k},\ \hat{k} \times \hat{j} = -\hat{i},\ \hat{i} \times \hat{k} = -\hat{j}$$

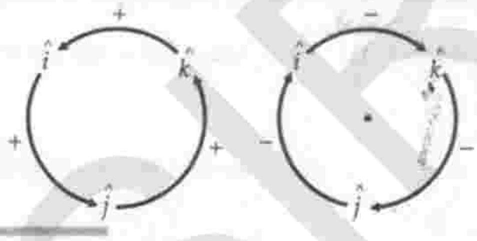


Fig. 4.51 Vector product of base vectors is cyclic (*a*) Anticlock-wise positive (*b*) Clockwise negative.

Aid to memory. Write $\hat{i}$, $\hat{j}$ and $\hat{k}$ cyclically round a circle, as shown in Fig. 4.51. Multiplying two unit vectors anticlockwise, we get positive value of third unit vector (*e.g.,* $\hat{i} \times \hat{j} = +\hat{k}$) ; and multiplying two unit vectors clockwise, we get negative value of third unit vector (*e.g.,* $\hat{j} \times \hat{i} = -\hat{k}$).

Also, $\hat{i} \times \hat{i} = (1)(1) \sin 0°\,\hat{n} = \vec{0}$

$$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \vec{0}$$

(*vii*) **The vector product of two vectors can be expressed in terms of their rectangular components as a determinant.**

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

(*viii*) **Sine of the angle between two vectors.** If θ is the angle between two vectors $\vec{A}$ and $\vec{B}$, then

$$|\vec{A} \times \vec{B}| = |\vec{A}|\,|\vec{B}| \sin\theta$$

or

$$\sin\theta = \frac{|\vec{A} \times \vec{B}|}{|\vec{A}|\,|\vec{B}|}$$

(*ix*) **Unit vector perpendicular to the plane of two vectors.** If $\hat{n}$ is a unit vector perpendicular to the plane of vectors $\vec{A}$ and $\vec{B}$, then

$$\vec{A} \times \vec{B} = AB \sin\theta\,\hat{n} = |\vec{A} \times \vec{B}|\,\hat{n}$$

or

$$\hat{n} = \frac{\vec{A} \times \vec{B}}{|\vec{A} \times \vec{B}|}$$

37. *Show that the cross product of two vectors is not commutative but anticommutative.*

Cross product is anticommutative. As shown in Figs. 4.52 (*a*) and (*b*), consider two vectors $\vec{A}$ and $\vec{B}$ such that the small angle between them is θ.

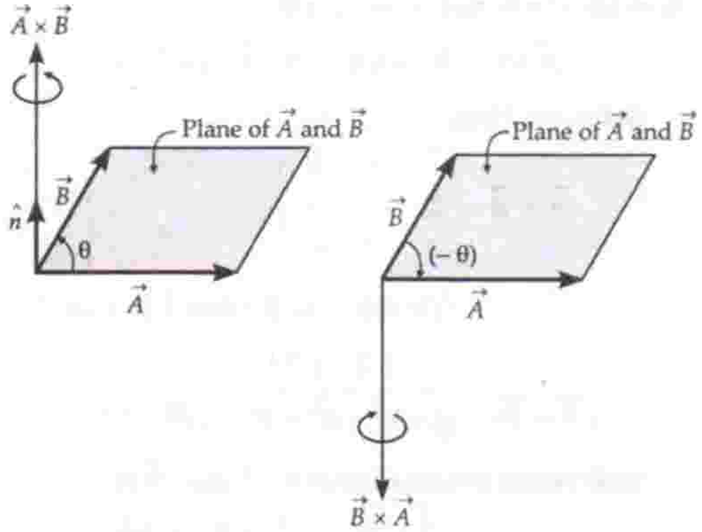


Fig. 4.52 Direction of $\vec{B} \times \vec{A}$ is opposite to that $\vec{A} \times \vec{B}$.

Now, $\vec{A} \times \vec{B} = AB \sin\theta\, \hat{n}$

and $\vec{B} \times \vec{A} = AB \sin(-\theta)\, \hat{n} = -AB\sin\theta\, \hat{n}$

Clearly, both $\vec{A} \times \vec{B}$ and $\vec{B} \times \vec{A}$ have equal magnitudes ($AB\sin\theta$) but they have opposite directions. Thus

$$\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A} \quad \text{and} \quad \vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$$

Hence *cross product of two vectors is not commutative, instead it is anticommutative.*

38. *Show how the vector product of two vectors can be expressed in terms of their rectangular components as a determinant.*

Vector product in terms of rectangular components. We can express $\vec{A}$ and $\vec{B}$ in terms of their rectangular components as

$$\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$$

and $\vec{B} = B_x \hat{i} + B_y \hat{j} + B_z \hat{k}$

$$\vec{A} \times \vec{B} = (A_x \hat{i} + A_y \hat{j} + A_z \hat{k}) \times (B_x \hat{i} + B_y \hat{j} + B_z \hat{k})$$
$$= A_x \hat{i} \times (B_x \hat{i} + B_y \hat{j} + B_z \hat{k})$$
$$+ A_y \hat{j} \times (B_x \hat{i} + B_y \hat{j} + B_z \hat{k})$$
$$+ A_z \hat{k} \times (B_x \hat{i} + B_y \hat{j} + B_z \hat{k})$$
$$= A_x B_x (\hat{i} \times \hat{i}) + A_x B_y (\hat{i} \times \hat{j}) + A_x B_z (\hat{i} \times \hat{k})$$
$$+ A_y B_x (\hat{j} \times \hat{i}) + A_y B_y (\hat{j} \times \hat{j}) + A_y B_z (\hat{j} \times \hat{k})$$
$$+ A_z B_x (\hat{k} \times \hat{i}) + A_z B_y (\hat{k} \times \hat{j}) + A_z B_z (\hat{k} \times \hat{k})$$
$$= A_x B_x (\vec{0}) + A_x B_y (\hat{k}) + A_x B_z (-\hat{j})$$
$$+ A_y B_x (-\hat{k}) + A_y B_y (\vec{0}) + A_y B_z (\hat{i})$$
$$+ A_z B_x (\hat{j}) + A_z B_y (-\hat{i}) + A_z B_z (\vec{0})$$

or
$$= \hat{i}(A_y B_z - A_z B_y) - \hat{j}(A_z B_x - A_x B_z)$$
$$+ \hat{k}(A_x B_y - A_y B_x)$$

or
$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

This equation expresses $\vec{A} \times \vec{B}$ in a determinant form.

Examples based on
Vector or Cross Product of two Vectors

FORMULAE USED

1. $\vec{A} \times \vec{B} = AB\sin\theta\, \hat{n}$
2. Unit vector $\hat{n}$ perpendicular to the plane of vectors $\vec{A}$ and $\vec{B}$ is given by $\hat{n} = \dfrac{\vec{A} \times \vec{B}}{|\vec{A} \times \vec{B}|}$
3. Angle θ between vectors $\vec{A}$ and $\vec{B}$ is given by
$$\sin\theta = \frac{|\vec{A} \times \vec{B}|}{|\vec{A}||\vec{B}|}$$
4. In terms of rectangular components, we have
$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$
or
$$\vec{A} \times \vec{B} = (A_y B_z - A_z B_y)\hat{i} + (A_z B_x - A_x B_z)\hat{j} + (A_x B_y - A_y B_x)\hat{k}$$
5. For parallel vectors, $\vec{A} \times \vec{B} = \vec{0}$
6. Moment of a force or torque, $\vec{\tau} = \vec{r} \times \vec{F}$.

EXAMPLE 39. *Prove that the vectors $\vec{A} = 2\hat{i} - 3\hat{j} - \hat{k}$ and $\vec{B} = -6\hat{i} + 9\hat{j} + 3\hat{k}$ are parallel.*

Solution. $\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -3 & -1 \\ -6 & 9 & 3 \end{vmatrix}$

$$= \hat{i} \begin{vmatrix} -3 & -1 \\ 9 & 3 \end{vmatrix} - \hat{j} \begin{vmatrix} 2 & -1 \\ -6 & 3 \end{vmatrix} + \hat{k} \begin{vmatrix} 2 & -3 \\ -6 & 9 \end{vmatrix}$$

$$= \hat{i}(-9+9) - \hat{j}(6-6) + \hat{k}(18-18) = \vec{0}$$

Hence $\vec{A} \parallel \vec{B}$.

EXAMPLE 40. *Calculate the area of the parallelogram whose two adjacent sides are formed by the vectors $\vec{A} = 3\hat{i} + 4\hat{j}$ and $\vec{B} = -3\hat{i} + 7\hat{j}$.*

Solution. $\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 4 & 0 \\ -3 & 7 & 0 \end{vmatrix}$

$$= \hat{i} \begin{vmatrix} 4 & 0 \\ 7 & 0 \end{vmatrix} - \hat{j} \begin{vmatrix} 3 & 0 \\ -3 & 0 \end{vmatrix} + \hat{k} \begin{vmatrix} 3 & 4 \\ -3 & 7 \end{vmatrix}$$

$$= \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(21+12) = 33\hat{k}$$

Area of parallelogram

$$= |\vec{A} \times \vec{B}| = \sqrt{0^2 + 0^2 + 33^2} = \mathbf{33 \text{ sq. units.}}$$

EXAMPLE 41. *If $\vec{A}$ and $\vec{B}$ denote the sides of a parallelogram and its area is $AB/2$, find the angle between $\vec{A}$ and $\vec{B}$.*

Solution. Area of parallelogram

$$= |\vec{A} \times \vec{B}| = AB \sin\theta = AB/2$$

$\therefore \quad \sin\theta = \frac{1}{2}$ or $\theta = \mathbf{30°}$

EXAMPLE 42. *Determine a unit vector perpendicular to both $\vec{A} = 2\hat{i} + \hat{j} + \hat{k}$ and $\vec{B} = \hat{i} - \hat{j} + 2\hat{k}$.* **[Chandigarh 04]**

Solution. The perpendicular unit vector $\hat{n}$ is given by

$$\vec{A} \times \vec{B} = AB \sin\theta\, \hat{n} = |\vec{A} \times \vec{B}|\, \hat{n}$$

$$\therefore \quad \hat{n} = \frac{\vec{A} \times \vec{B}}{|\vec{A} \times \vec{B}|}$$

Now $\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 1 \\ 1 & -1 & 2 \end{vmatrix}$

$$= \hat{i} \begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix} - \hat{j} \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \hat{k} \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix}$$

$$= \hat{i}(2+1) - \hat{j}(4-1) + \hat{k}(-2-1)$$

$$= 3\hat{i} - 3\hat{j} - 3\hat{k}$$

$$\therefore \quad |\vec{A} \times \vec{B}| = \sqrt{3^2 + (-3)^2 + (-3)^2} = \sqrt{27} = 3\sqrt{3}$$

Hence $\hat{n} = \dfrac{3\hat{i} - 3\hat{j} - 3\hat{k}}{3\sqrt{3}} = \dfrac{1}{\sqrt{3}}(\hat{i} - \hat{j} - \hat{k})$.

EXAMPLE 43. *Find a vector whose length is 7 and which is perpendicular to each of the vectors :*

$$\vec{A} = 2\hat{i} - 3\hat{j} + 6\hat{k} \text{ and } \vec{B} = \hat{i} + \hat{j} - \hat{k}$$

Solution. Here

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -3 & 6 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= \hat{i}(3-6) - \hat{j}(-2-6) + \hat{k}(2+3)$$

$$= -3\hat{i} + 8\hat{j} + 5\hat{k}$$

$$|\vec{A} \times \vec{B}| = \sqrt{(-3)^2 + 8^2 + 5^2} = \sqrt{98} = 7\sqrt{2}.$$

Unit vector perpendicular to $\vec{A}$ and $\vec{B}$ is

$$\hat{n} = \frac{\vec{A} \times \vec{B}}{|\vec{A} \times \vec{B}|} = \frac{-3\hat{i} + 8\hat{j} + 5\hat{k}}{7\sqrt{2}}$$

Required vector

$$= 7\hat{n} = 7\left(\frac{-3\hat{i} + 8\hat{j} + 5\hat{k}}{7\sqrt{2}}\right)$$

$$= \mathbf{\frac{1}{\sqrt{2}}(-3\hat{i} + 8\hat{j} + 5\hat{k}).}$$

EXAMPLE 44. *Determine the sine of the angle between the vectors $3\hat{i} + \hat{j} + 2\hat{k}$ and $2\hat{i} - 2\hat{j} + 4\hat{k}$.*

Solution. Let

$$\vec{A} = 3\hat{i} + \hat{j} + 2\hat{k} \text{ and } \vec{B} = 2\hat{i} - 2\hat{j} + 4\hat{k}$$

Then $\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & 2 \\ 2 & -2 & 4 \end{vmatrix}$

$$= \hat{i} \begin{vmatrix} 1 & 2 \\ -2 & 4 \end{vmatrix} - \hat{j} \begin{vmatrix} 3 & 2 \\ 2 & 4 \end{vmatrix} + \hat{k} \begin{vmatrix} 3 & 1 \\ 2 & -2 \end{vmatrix}$$

$$= \hat{i}(4+4) - \hat{j}(12-4) + \hat{k}(-6-2)$$

$$= 8\hat{i} - 8\hat{j} - 8\hat{k}$$

$$|\vec{A} \times \vec{B}| = \sqrt{8^2 + (-8)^2 + (-8)^2} = 8\sqrt{3}$$

$$|\vec{A}| = \sqrt{3^2 + 1^2 + 2^2} = \sqrt{14}$$

$$|\vec{B}| = \sqrt{2^2 + (-2)^2 + 4^2} = \sqrt{24}$$

$$\therefore \quad \sin\theta = \frac{|\vec{A} \times \vec{B}|}{|\vec{A}||\vec{B}|} = \frac{8\sqrt{3}}{\sqrt{14} \times \sqrt{24}} = \frac{2}{\sqrt{7}}.$$

Example 45. *Show that*

$$(\vec{A}-\vec{B})\times(\vec{A}+\vec{B})=2(\vec{A}\times\vec{B}).$$

Solution.

$$\text{L.H.S.}=(\vec{A}-\vec{B})\times(\vec{A}+\vec{B})$$
$$=\vec{A}\times\vec{A}+\vec{A}\times\vec{B}-\vec{B}\times\vec{A}-\vec{B}\times\vec{B}$$
$$=\vec{0}+\vec{A}\times\vec{B}+\vec{A}\times\vec{B}+\vec{0} \qquad [\because \vec{A}\times\vec{B}=-\vec{B}\times\vec{A}]$$
$$=2(\vec{A}\times\vec{B})=\text{R.H.S.}$$

Example 46. *For any three vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$, prove that*

$$\vec{A}\times(\vec{B}+\vec{C})+\vec{B}\times(\vec{C}+\vec{A})+\vec{C}\times(\vec{A}+\vec{B})=\vec{0}.$$

Solution. L.H.S.

$$=\vec{A}\times(\vec{B}+\vec{C})+\vec{B}\times(\vec{C}+\vec{A})+\vec{C}\times(\vec{A}+\vec{B})$$
$$=\vec{A}\times\vec{B}+\vec{A}\times\vec{C}+\vec{B}\times\vec{C}+\vec{B}\times\vec{A}+\vec{C}\times\vec{A}+\vec{C}\times\vec{B}$$
$$=\vec{A}\times\vec{B}+\vec{A}\times\vec{C}+\vec{B}\times\vec{C}-\vec{A}\times\vec{B}-\vec{A}\times\vec{C}-\vec{B}\times\vec{C} \qquad [\because \vec{B}\times\vec{A}=-\vec{A}\times\vec{B}]$$
$$=\vec{0}=\text{R.H.S.}$$

Example 47. *For any two vectors $\vec{A}$ and $\vec{B}$, prove that*

$$(\vec{A}\times\vec{B})^2=A^2B^2-(\vec{A}\cdot\vec{B})^2$$

[Himachal 07, 09C]

Solution. L.H.S. $=(\vec{A}\times\vec{B})^2$

$$=|\vec{A}\times\vec{B}|^2=(AB\sin\theta)^2$$
$$=A^2B^2(1-\cos^2\theta)$$
$$=A^2B^2-(AB\cos\theta)^2$$
$$=A^2B^2-(\vec{A}\cdot\vec{B})^2=\text{R.H.S.}$$

Example 48. *Find* $\vec{A}\cdot\vec{B}$ *if* $|\vec{A}|=2$, $|\vec{B}|=5$ *and* $|\vec{A}\times\vec{B}|=8$.

Solution. As

$$|\vec{A}\times\vec{B}|^2=|\vec{A}|^2|\vec{B}|^2-(\vec{A}\cdot\vec{B})^2$$
$$\therefore \quad 8^2=2^2\times5^2-(\vec{A}\cdot\vec{B})^2$$

or $\quad(\vec{A}\cdot\vec{B})^2=100-64=36$

$$\therefore \quad \vec{A}\cdot\vec{B}=\pm 6.$$

Example 49. *Find the area of the triangle formed by the tips of the vectors* $\vec{a}=\hat{i}-\hat{j}-3\hat{k}$, $\vec{b}=4\hat{i}-3\hat{j}+\hat{k}$ *and* $\vec{c}=3\hat{i}-\hat{j}+2\hat{k}$.

Solution. Let ABC the triangle formed by the tips of the given vectors.

Then

$$\vec{BA}=\vec{a}-\vec{b}=(\hat{i}-\hat{j}-3\hat{k})-(4\hat{i}-3\hat{j}+\hat{k})$$
$$=-3\hat{i}+2\hat{j}-4\hat{k}$$
$$\vec{BC}=\vec{c}-\vec{b}=(3\hat{i}-\hat{j}+2\hat{k})-(4\hat{i}-3\hat{j}+\hat{k})$$
$$=-\hat{i}+2\hat{j}+\hat{k}$$

Now

$$\vec{BA}\times\vec{BC}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -3 & 2 & -4\\ -1 & 2 & 1\end{vmatrix}$$
$$=\hat{i}(2+8)-\hat{j}(-3-4)+\hat{k}(-6+2)$$
$$=10\hat{i}+7\hat{j}-4\hat{k}$$
$$|\vec{BA}\times\vec{BC}|=\sqrt{(10)^2+7^2+(-4)^2}=\sqrt{165}=12.8$$

$$\text{Ar. of }\Delta ABC=\frac{1}{2}|\vec{BA}\times\vec{BC}|$$
$$=\frac{1}{2}\times12.8=\mathbf{6.4\ sq.\ units.}$$

Example 50. *Find the moment about the point* $(1,-1,-1)$ *of the force* $3\hat{i}+4\hat{j}-5\hat{k}$ *acting at the point* $(1,0,-2)$.

Solution. Here $\quad \vec{F}=3\hat{i}+4\hat{j}-5\hat{k}$

Let P be the point about which moment is to be obtained and A be the point at which force is applied. If O is the origin, then

$$\vec{OP}=\hat{i}-\hat{j}-\hat{k}, \quad \vec{OA}=\hat{i}+0\hat{j}-2\hat{k}$$
$$\therefore \quad \vec{PA}=\vec{OA}-\vec{OP}$$
$$=(\hat{i}-0\hat{j}-2\hat{k})-(\hat{i}-\hat{j}-\hat{k})$$
$$=\hat{j}-\hat{k}$$

Moment of force $\vec{F}$ about the point P is

$$\vec{\tau}=\vec{PA}\times\vec{F}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 0 & 1 & -1\\ 3 & 4 & -5\end{vmatrix}$$
$$=i(-5+4)-\hat{j}(0+3)+k(0-3)$$
$$=-\hat{i}-3\hat{j}-3\hat{k}.$$

Example 51. *The diagonals of a parallelogram are given by the vectors* $3\hat{i}+\hat{j}+2\hat{k}$ *and* $\hat{i}-3\hat{j}+4\hat{k}$. *Find the area of the parallelogram.*

Solution. If $\vec{A}$ and $\vec{B}$ are the adjacent sides of the parallelogram, then its diagonals will be

$$\vec{A}+\vec{B}=3\hat{i}+\hat{j}+2\hat{k}$$

and $\quad \vec{A}-\vec{B}=\hat{i}-3\hat{j}+4\hat{k}$

Now $\vec{A} \times \vec{B} = \frac{1}{2}(\vec{A} - \vec{B}) \times (\vec{A} + \vec{B})$

$$= \frac{1}{2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 4 \\ 3 & 1 & 2 \end{vmatrix}$$

$$= \frac{1}{2}[\hat{i}(-6-4) - \hat{j}(2-12) + \hat{k}(1+9)]$$

$$= \frac{1}{2}(-10\,\hat{i} + 10\,\hat{j} + 10\,\hat{k})$$

Area of parallelogram

$$= |\vec{A} \times \vec{B}| = \frac{1}{2}\sqrt{(-10)^2 + 10^2 + 10^2}$$

$$= \frac{1}{2} \times 17.32 = \mathbf{8.66\ sq.\ units.}$$

Example 52. *In any ΔABC, prove that*

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Solution. As shown in Fig. 4.53, the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are cyclic, therefore

$$\vec{a} + \vec{b} + \vec{c} = 0 \quad \text{or} \quad \vec{a} + \vec{b} = -\vec{c}$$

or $(\vec{a} + \vec{b}) \times \vec{c} = -\vec{c} \times \vec{c}$ or $\vec{a} \times \vec{c} + \vec{b} \times \vec{c} = \vec{0}$

or $\quad -\vec{c} \times \vec{a} + \vec{b} \times \vec{c} = \vec{0}$

or $\quad \vec{b} \times \vec{c} = \vec{c} \times \vec{a} \quad$ ...(*i*)

Similarly, $\quad \vec{a} \times \vec{b} = \vec{b} \times \vec{c} \quad$ ...(*ii*)

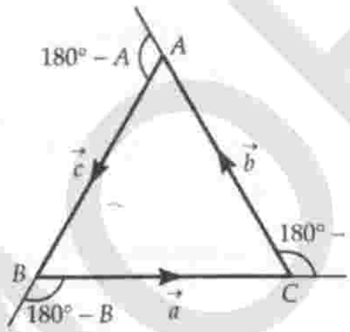


Fig. 4.53

From (*i*) and (*ii*), we get

$$\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$$

or $\quad |\vec{a} \times \vec{b}| = |\vec{b} \times \vec{c}| = |\vec{c} \times \vec{a}|$

or $\quad ab \sin(180° - C) = bc \sin(180° - A)$

$$= ca \sin(180° - B)$$

or $\quad ab \sin C = bc \sin A = ca \sin B$

Dividing throughout by abc, we get

$$\frac{\sin C}{c} = \frac{\sin A}{a} = \frac{\sin B}{b} \quad \text{or} \quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Example 53. *If $\vec{a}$, $\vec{b}$, $\vec{c}$ are three vectors such that*

$$\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c}, \quad \vec{a} \times \vec{b} = \vec{a} \times \vec{c}, \quad \vec{a} \neq \vec{0}$$

then prove that $\vec{b} = \vec{c}$.

Solution. Given : $\quad \vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c}$

or $\quad \vec{a} \cdot \vec{b} - \vec{a} \cdot \vec{c} = 0$

or $\quad \vec{a} \cdot (\vec{b} - \vec{c}) = 0$

But $\quad \vec{a} \neq \vec{0}$

$\therefore$ Either $\quad \vec{b} - \vec{c} = \vec{0}$ or $\vec{a} \perp (\vec{b} - \vec{c})$

$\Rightarrow$ Either $\quad \vec{b} = \vec{c}$

or $\quad \vec{a} \perp (\vec{b} - \vec{c}) \quad$ ...(*i*)

Also, $\quad \vec{a} \times \vec{b} = \vec{a} \times \vec{c}$

or $\quad \vec{a} \times \vec{b} - \vec{a} \times \vec{c} = \vec{0}$ or $\vec{a} \times (\vec{b} - \vec{c}) = \vec{0}$.

But $\quad \vec{a} \neq \vec{0}$

$\therefore$ Either $\quad \vec{b} - \vec{c} = \vec{0}$ or $\vec{a} \parallel (\vec{b} - \vec{c})$

$\Rightarrow$ Either $\quad \vec{b} = \vec{c}$

or $\quad \vec{a} \parallel (\vec{b} - \vec{c}) \quad$ ...(*ii*)

But $\vec{a}$ cannot be simultaneously perpendicular and parallel to $(\vec{b} - \vec{c})$, so equations (*i*) and (*ii*) will hold simultaneously if $\vec{b} = \vec{c}$.

Example 54. *If $\vec{a} = \hat{i} - 2\hat{j} - 3\hat{k}$, $\vec{b} = 2\hat{i} + \hat{j} - \hat{k}$ and $\vec{c} = \hat{i} + 3\hat{j} - 2\hat{k}$, then find $\vec{a} \times (\vec{b} \times \vec{c})$.*

Solution. $\vec{b} \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -1 \\ 1 & 3 & -2 \end{vmatrix}$

$$= \hat{i}(-2+3) - \hat{j}(-4+1) + \hat{k}(6-1)$$

$$= \hat{i} + 3\hat{j} + 5\hat{k}$$

Now

$$\vec{a} \times (\vec{b} \times \vec{c}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & -3 \\ 1 & 3 & 5 \end{vmatrix}$$

$$= \hat{i}(-10+9) - \hat{j}(5+3) + \hat{k}(3+2)$$

$$= \mathbf{-\hat{i} - 8\hat{j} + 5\hat{k}}.$$

Problems for Practice

1. If $\vec{A} = \hat{i} + 3\hat{j} + 2\hat{k}$ and $\vec{B} = 3\hat{i} + \hat{j} + 2\hat{k}$, then find the vector product $\vec{A} \times \vec{B}$. (**Ans.** $4\hat{i} + 4\hat{j} - 8\hat{k}$)

2. Prove that the vectors $\vec{A} = 4\hat{i} + 3\hat{j} + \hat{k}$ and $\vec{B} = 12\hat{i} + 9\hat{j} + 3\hat{k}$ are parallel to each other.

3. If $\vec{A} = 2\hat{i} + 3\hat{j} + \hat{k}$ and $\vec{B} = 3\hat{i} + 2\hat{j} + 4\hat{k}$, then find the value of $(\vec{A} + \vec{B}) \times (\vec{A} - \vec{B})$.

   (**Ans.** $-20\hat{i} + 10\hat{j} + 10\hat{k}$)

4. Find the value of a for which the vectors $3\hat{i} + 3\hat{j} + 9\hat{k}$ and $\hat{i} + a\hat{j} + 3\hat{k}$ are parallel.

   (**Ans.** $a = 1$)

5. Find a unit vector perpendicular to the vectors $\vec{A} = 4\hat{i} - \hat{j} + 3\hat{k}$ and $\vec{B} = -2\hat{i} + \hat{j} - 2\hat{k}$.

   [**Ans.** $\frac{1}{3}(-\hat{i} + 2\hat{j} + 2\hat{k})$]

6. Find the sine of the angle between the vectors $\vec{A} = 3\hat{i} - 4\hat{j} + 5\hat{k}$ and $\vec{B} = \hat{i} - \hat{j} + \hat{k}$. (**Ans.** 1/5)

7. Find a vector of magnitude 18 which is perpendicular to both the vectors $4\hat{i} - \hat{j} + 3\hat{k}$ and $-2\hat{i} + \hat{j} - 2\hat{k}$. (**Ans.** $-6\hat{i} + 12\hat{j} + 12\hat{k}$)

8. Determine the area of the parallelogram whose adjacent sides are formed by the vectors $\vec{A} = \hat{i} - 3\hat{j} + \hat{k}$ and $\vec{B} = \hat{i} + \hat{j} + \hat{k}$.

   (**Ans.** $4\sqrt{2}$ square units)

9. Find the area of the triangle formed by points O, A and B such that $\vec{OA} = \hat{i} + 2\hat{j} + 3\hat{k}$ and $\vec{OB} = -3\hat{i} - 2\hat{j} + \hat{k}$. (**Ans.** $3\sqrt{5}$ square units)

10. Find with the help of vectors, the area of the triangle with vertices $A(3, -1, 2)$, $B(1, -1, -3)$ and $C(4, -3, 1)$. (**Ans.** $\sqrt{165}/2$ sq. units)

11. If $\vec{A}$ and $\vec{B}$ are two such vectors that $|\vec{A}| = 2$, $|\vec{B}| = 7$ and $\vec{A} \times \vec{B} = 3\hat{i} + 2\hat{j} + 6\hat{k}$, find the angle between $\vec{A}$ and $\vec{B}$. (**Ans.** $\pi/6$)

12. Find the moment about the point $\hat{i} + 2\hat{j} - \hat{k}$ of a force represented by $3\hat{i} + \hat{k}$ acting through the point $2\hat{i} - \hat{j} + 3\hat{k}$. (**Ans.** $-3\hat{i} + 11\hat{j} + 9\hat{k}$)

13. Prove that $(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b}) = 2(\vec{b} \times \vec{a})$.

14. Prove that $|\vec{a} \times \vec{b}| = \sqrt{a^2b^2 - (\vec{a} \cdot \vec{b})^2}$.

15. Find $|\vec{A} \times \vec{B}|$ if $|\vec{A}| = 10$, $|\vec{B}| = 2$ and $\vec{A} \cdot \vec{B} = 12$.

    (**Ans.** 16)

16. If $\vec{a} = \hat{i} - 2\hat{j} - 3\hat{k}$, $\vec{b} = 2\hat{i} - \hat{j} - \hat{k}$ and $\vec{c} = \hat{i} + 3\hat{j} - 2\hat{k}$, find $(\vec{a} \times \vec{b}) \times \vec{c}$.

    (**Ans.** $\hat{i} + \hat{j} + 2\hat{k}$)

17. Calculate the area of the triangle determined by the two vectors : $\vec{A} = 3\hat{i} + 4\hat{j}$ and $\vec{B} = -3\hat{i} + 7\hat{j}$.

    [Central Schools 08, 09]

    (**Ans.** 16.5 square units)

18. Find the area of a parallelogram formed by vectors

    $$\vec{A} = 3\hat{i} + 2\hat{j}, \quad \vec{B} = -3\hat{i} + 7\hat{j}$$

    [Central Schools 07]

    (**Ans.** $3\sqrt{3}$ square units)

19. The diagonals of a parallelogram are represented by $\vec{R_1} = 3\hat{i} + 2\hat{j} - 7\hat{k}$ and $\vec{R_2} = 5\hat{i} + 6\hat{j} - 3\hat{k}$. Find the area of the parallelogram. **[Chandigarh 07]**

    (**Ans.** $2\sqrt{509}$ square units)

4.18 MOTION IN A PLANE

39. *Show the position vector for a particle in two dimensional motion. Write an expression for this position vector.*

Expression for position vector. Fig. 4.54 shows the position vector $\vec{OP}$ of a particle located at P with respect to the origin O. If (x, y) are the coordinates of point P, then

$$\vec{OP} = \vec{OA} + \vec{OB}$$

or $$\vec{r} = x\hat{i} + y\hat{j}$$

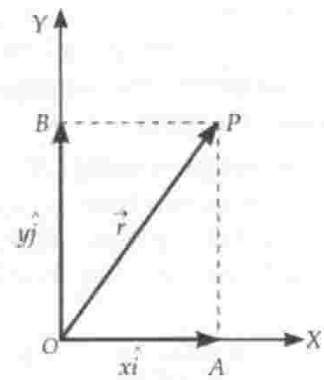


Fig. 4.54 Position vector.

This equation expresses position vector $\vec{r}$ in terms of its rectangular components x and y.

40. *Show graphically the displacement vector for a motion in two dimensions. Write an expression for displacement vector in terms of its rectangular components.*

Expression for displacement vector. Suppose a particle moves in the X-Y plane along the curved path shown in Fig. 4.55. The particle is at point $P(x, y)$ at time t and at $P'(x', y')$ at time t'.

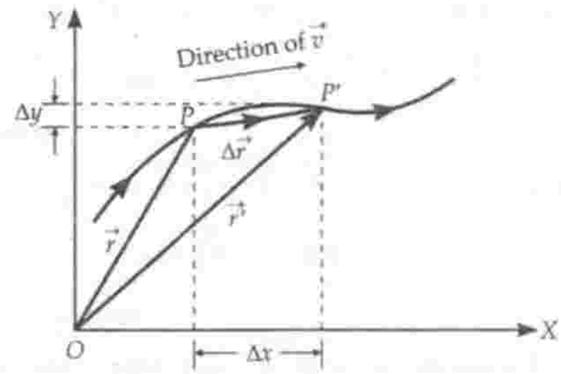


Fig. 4.55 Displacement $\Delta\vec{r}$ and average velocity $\vec{v}$.

Using triangle law of vectors addition,

$$\vec{OP'} = \vec{OP} + \vec{PP'}$$

So the displacement vector is

$$\vec{PP'} = \vec{OP'} - \vec{OP}$$

or $$\Delta\vec{r} = \vec{r}' - \vec{r}$$

The direction of $\Delta\vec{r}$ is from P to P'.

In terms of rectangular components,

$$\Delta\vec{r} = (x'\hat{i} + y'\hat{j}) - (x\hat{i} + y\hat{j})$$
$$= (x' - x)\hat{i} + (y' - y)\hat{j}$$

or $$\Delta\vec{r} = \Delta x\hat{i} + \Delta y\hat{j}$$

41. *Write an expression for average velocity in terms of its rectangular components. Hence write an expression for instantaneous velocity. Find the direction of instantaneous velocity.*

Average velocity. Refer to Fig. 4.55. Suppose the particle moves from point P to P' in time t to t'. *The average velocity of an object is the ratio of the displacement and the corresponding time interval.* So it is given by

$$\bar{v} = \frac{\Delta\vec{r}}{\Delta t} = \frac{\Delta x\hat{i} + \Delta y\hat{j}}{\Delta t} = \frac{\Delta x}{\Delta t}\hat{i} + \frac{\Delta y}{\Delta t}\hat{j}$$

[Here $\Delta t = t' - t$]

or $$\bar{v} = \bar{v}_x\hat{i} + \bar{v}_y\hat{j}.$$

This equation expresses average velocity $\bar{v}$ in terms of its rectangular components $\bar{v}_x$ and $\bar{v}_y$.

As $\bar{v} = \dfrac{\Delta\vec{r}}{\Delta t}$, so the direction of the average velocity is same as that of displacement vector $\Delta\vec{r}$, as shown in Fig. 4.55.

Instantaneous velocity. *The instantaneous velocity of a particle is equal to the limiting value of its average velocity when the time interval approaches zero.* It is given by

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta\vec{r}}{\Delta t} = \frac{d\vec{r}}{dt}$$

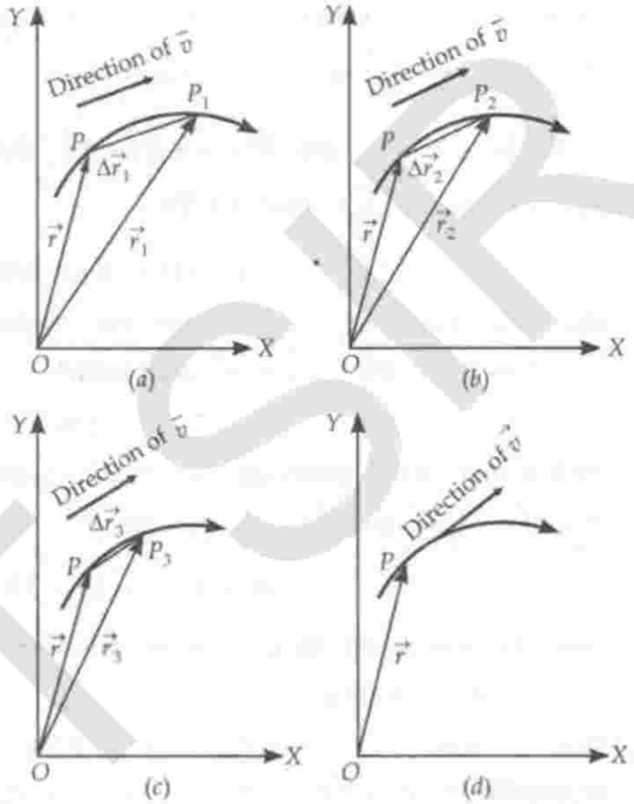


Fig. 4.56 As $\Delta t \to 0$, the average velocity approaches the velocity $\vec{v}$. The direction of $\vec{v}$ is parallel to the line tangent to the path.

Direction of instantaneous velocity. In Figs. 4.56(*a*) to (*d*), the thick line curve represents the path of an object. The object is at point P at time t. P_1, P_2 and P_3 are the positions of the object after time intervals Δt_1, Δt_2 and Δt_3; where $\Delta t_1 > \Delta t_2 > \Delta t_3$. The directions of average velocity $\bar{v}$ are shown parallel to the corresponding displacement vector $\Delta\vec{r}_1$, $\Delta\vec{r}_2$ and $\Delta\vec{r}_3$, for the decreasing values of Δt. As $\Delta t \to 0$, $\Delta\vec{r} \to 0$, and the direction of velocity is along the tangent to the path, as shown in Fig. 4.56(*d*). *Hence the direction of (instantaneous) velocity at any point on the path of an object is tangent to the path at that point and is in the direction of motion.*

42. *Obtain an expression for instantaneous velocity in terms of its rectangular components.*

Velocity in terms of rectangular components. The instantaneous velocity is given by

$$\vec{v} = \frac{d\vec{r}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta\vec{r}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta x\hat{i} + \Delta y\hat{j}}{\Delta t}$$

$$=\left(\lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}\right)\hat{i}+\left(\lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t}\right)\hat{j}=\frac{dx}{dt}\,\hat{i}+\frac{dy}{dt}\,\hat{z}$$

or $\qquad \vec{v}=v_x\,\hat{i}+v_y\,\hat{j}\,.$

If the coordinates x and y are known as functions of time t, then we can determine v_x and v_y. The magnitude of $\vec{v}$ will be

$$v=\sqrt{v_x^2+v_y^2}$$

Figure 4.57 shows the rectangular components v_x and v_y of velocity $\vec{v}$. If $\vec{v}$ makes angle θ with X-axis, then

$$\tan\theta=\frac{v_y}{v_x} \quad \text{or} \quad \theta=\tan^{-1}\left(\frac{v_y}{v_x}\right).$$

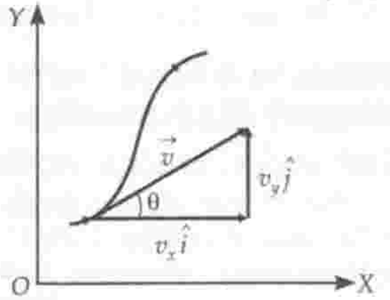


Fig. 4.57 Components v_x and v_y of velocity $\vec{v}$.

43. *Write an expression for average acceleration in terms of its rectangular components. Hence write an expression for instantaneous acceleration. Find the direction of instantaneous acceleration.*

Average acceleration. *The average acceleration of an object is the ratio of the change in velocity and the corresponding time interval.* If the velocity of an object changes from $\vec{v}$ to $\vec{v'}$ in time Δt, then the average acceleration is given by

$$\bar{a}=\frac{\vec{v'}-\vec{v}}{\Delta t}=\frac{\Delta\vec{v}}{\Delta t}=\frac{\Delta(v_x\,\hat{i}+v_y\,\hat{j})}{\Delta t}=\frac{\Delta v_x}{\Delta t}\,\hat{i}+\frac{\Delta v_y}{\Delta t}\,\hat{j}$$

or $\bar{a}=\bar{a}_x\,\hat{i}+\bar{a}_y\,\hat{j}$

This equation expresses average acceleration $\bar{a}$ in terms of its rectangular components $\bar{a}_x$ and $\bar{a}_y$. As $\bar{a}=\dfrac{\Delta\vec{v}}{\Delta t}$, so the direction of average acceleration is same as that of the change in velocity $\Delta\vec{v}$.

Instantaneous acceleration. *The instantaneous acceleration of an object is equal to the limiting value of its average acceleration when the time interval approaches zero.* It is given by

$$\vec{a}=\lim_{\Delta t\to 0}\frac{\Delta\vec{v}}{\Delta t}=\frac{d\vec{v}}{dt}\,.$$

Direction of instantaneous acceleration. In Figs. 4.58(*a*) to (*d*), the thick line curve represents the path of object's motion. The object is at point P at time t. P_1, P_2 and P_3 are the positions after time intervals Δt_1, Δt_2 and Δt_3; where $\Delta t_1 > \Delta t_2 > \Delta t_3$. The velocity vectors at points P, P_1, P_2 and P_3 are shown in the figures. In each case, the change in velocity $\Delta\vec{v}$ is obtained by using triangle law of vector addition. Again, in each case the direction of average acceleration $\bar{a}$ is shown parallel to $\Delta\vec{v}$.

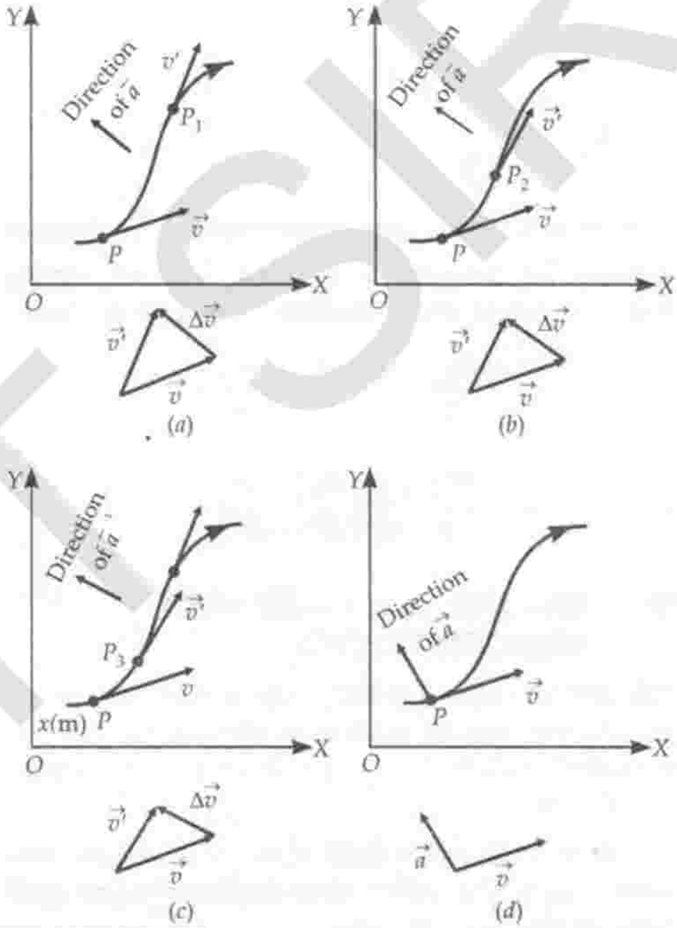


Fig. 4.58 The average acceleration for three time intervals (*a*) Δt_1, (*b*) Δt_2 and (*c*) Δt_3, $(\Delta t_1 > \Delta t_2 > \Delta t_3)$. (*d*) In the limit $\Delta t \to 0$, the average acceleration becomes the acceleration.

As Δt decreases from (*a*) to (*d*), *the direction of* $\Delta\vec{v}$ *and hence that of* $\bar{a}$ *changes. In Fig. 4.58(d)*, as $\Delta t \to 0$, the average acceleration becomes the instantaneous acceleration and its direction is as shown.

For Your Knowledge

- For motion in one dimension, the velocity and acceleration are always along the same line either in same direction (for accelerated motion) or in opposite direction (for decelerated motion).
- For motion in two or three dimensions, the angle between velocity and acceleration vectors may have any value between 0° and 180°.

44. *Obtain an expression for instantaneous acceleration in terms of its rectangular components.*

Acceleration in terms of rectangular components. The instantaneous acceleration is given by

$$\vec{a} = \frac{d\vec{v}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta(v_x \hat{i} + v_y \hat{j})}{\Delta t}$$

$$= \left(\lim_{\Delta t \to 0} \frac{\Delta v_x}{\Delta t}\right)\hat{i} + \left(\lim_{\Delta t \to 0} \frac{\Delta v_y}{\Delta t}\right)\hat{j}$$

$$= \frac{dv_x}{dt}\hat{i} + \frac{dv_y}{dt}\hat{j}$$

or $$\vec{a} = a_x \hat{i} + a_y \hat{j}$$

This equation expresses acceleration $\vec{a}$ in terms of its rectangular components a_x and a_y. We can express a_x and a_y in terms of coordinates x and y as follows :

$$a_x = \frac{dv_x}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2}$$

and $$a_y = \frac{dv_y}{dt} = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d^2y}{dt^2}.$$

4.19 MOTION IN A PLANE WITH UNIFORM VELCOITY

45. *Define uniform velocity. Show that two dimensional uniform velocity motion is equivalent to two one dimensional uniform velocity motions along two perpendicular directions.*

Uniform velocity. *A body is said to be moving with uniform velocity if it suffers equal displacements in equal intervals of time, however small.*

Position vector for uniform velocity. Consider an object moving with uniform velocity $\vec{v}$ in XY-plane. Let $\vec{r}(0)$ and $\vec{r}(t)$ be its position vectors at times $t = 0$ and $t = t$ respectively. Then

$$\vec{v} = \frac{\vec{r}(t) - \vec{r}(0)}{t - 0}$$

or $$\vec{r}(t) = \vec{r}(0) + \vec{v}\,t \qquad ...(1)$$

In terms of rectangular components, we can write

$$\vec{v} = v_x \hat{i} + v_y \hat{j}, \text{ where } v = \sqrt{v_x^2 + v_y^2}$$

$$\vec{r}(0) = x(0)\hat{i} + y(0)\hat{j}$$

$$\hat{r}(t) = x(t)\hat{i} + y(t)\hat{j}$$

Substituting these values in equation (1), we get

$$x(t)\hat{i} + y(t)\hat{j} = x(0)\hat{i} + y(0)\hat{j} + (v_x i + v_y \hat{j}).t$$

or $$x(t)\hat{i} + y(t)\hat{j} = [x(0) + v_x t]\hat{i} + [y(0) + v_y t]\hat{j} \qquad ...(2)$$

Equating the coefficients of $\hat{i}$ and $\hat{j}$ on both sides, we get

$$x(t) = x(0) + v_x t$$

$$y(t) = y(0) + v_y t$$

The above two equations represent uniform motions along X-axis and Y-axis respectively. Thus equation (2) shows that a *uniform motion in two-dimensions can be expressed as the sum of two uniform motions along two mutually perpendicular directions.*

4.20 MOTION IN A PLANE WITH CONSTANT ACCELERATION

46(*a*) *Define uniform acceleration. Show that in two-dimensional motion with uniform acceleration, each rectangular component of velocity is similar to that of uniformly accelerated motion along one dimension.*

Uniform acceleration. *A body is said to be moving with uniform acceleration if its velocity vector suffers the same change in the same interval of time, however small.*

Velocity vector for uniform acceleration. If $\vec{v}_0$ and $\vec{v}$ be the velocity vectors at times $t = 0$ and $t = t$ respectively, then the acceleration is given by

$$\vec{a} = \frac{\vec{v} - \vec{v}_0}{t - 0} = \frac{\vec{v} - \vec{v}_0}{t}$$

or $$\vec{v} = \vec{v}_0 + \vec{a}\,t$$

Writing the above equation in terms of rectangular components, we get

$$v_x \hat{i} + v_y \hat{j} = v_{0x}\hat{i} + v_{0y}\hat{j} + (a_x \hat{i} + a_y \hat{j})\,t$$

or $$v_x \hat{i} + v_y \hat{j} = (v_{0x} + a_x t)\hat{i} + (v_{0y} + a_y t)\hat{j}$$

Comparing the coefficients of $\hat{i}$ and $\hat{j}$ on both sides of the above equation, we get

$$v_x = v_{0x} + a_x t \quad \text{and} \quad v_y = v_{0y} + a_y t$$

The above two equations show that *each rectangular component of velocity of an object moving with uniform acceleration in a plane depends upon time as if it were the velocity vector of one- dimensional uniformly accelerated motion.*

46(b) *Write an expression for the instantaneous position vector of an object having two dimensional motion under uniform acceleration. Hence show that a two dimensional uniformly accelerated motion is a combination of two rectangular uniformly accelerated motions.*

Position vector for uniform acceleration. Consider a particle moving with uniform acceleration $\vec{a}$. Let $\vec{r_0}$ and $\vec{r}$ be its position vectors at times 0 and t and let the velocities at these instants be $\vec{v_0}$ and $\vec{v}$. Now

Displacement = Average velocity × time interval

or $$\vec{r} - \vec{r_0} = \frac{\vec{v_0} + \vec{v}}{2} \times t = \frac{\vec{v_0} + (\vec{v_0} + \vec{a}\,t)}{2} \times t$$

$$= \vec{v_0}\,t + \frac{1}{2}\vec{a}\,t^2$$

or $$\vec{r} = \vec{r_0} + \vec{v_0}\,t + \frac{1}{2}\vec{a}\,t^2$$

This equation gives position of a uniformly accelerated particle at time t. Writing the above equation in terms of rectangular components, we get

$$x\hat{i} + y\hat{j} = x_0\hat{i} + y_0\hat{j} + (v_{0x}\hat{i} + v_{0y}\hat{j})t + \frac{1}{2}(a_x\hat{i} + a_y\hat{j})t^2$$

Equating the coefficients of $\hat{i}$ and $\hat{j}$ on both sides, we get

$$x = x_0 + v_{0x}\,t + \frac{1}{2}a_x\,t^2$$

and $$y = y_0 + v_{0y}\,t + \frac{1}{2}a_y\,t^2$$

The above two equations show that the motions in x and y directions can be treated independently of each other. Thus, *the motion in a plane with uniform acceleration can be treated as the superposition of two separate simultaneous one-dimensional motions along two perpendicular directions.*

Note In uniform acceleration, the position vector at time t,

$$\vec{r} = \vec{r_0} + \vec{v_0}t + \frac{1}{2}\vec{a}\,t^2$$

Similarly, position vector at time t' is

$$\vec{r'} = \vec{r_0} + v_0 t' + \frac{1}{2}\vec{a}\,t'^2$$

$$\therefore \quad \vec{r'} - \vec{r} = v_0(t' - t) + \frac{1}{2}\vec{a}(t'^2 - t^2)$$

or $$\vec{r'} = \vec{r} + v_0(t' - t) + \frac{1}{2}\vec{a}(t'^2 - t^2)$$

Prove that $v^2 - v_0^2 = 2\vec{a}\,.(\vec{r} - \vec{r_0})$.

From first equation of motion,

$$\vec{v} = \vec{v_0} + \vec{a}\,t \quad \text{or} \quad \vec{v} - \vec{v_0} = \vec{a}\,t \quad ...(1)$$

Now,

displacement = Average velocity × time interval

or $$\vec{r} - \vec{r_0} = \frac{\vec{v} + \vec{v_0}}{2} \times t$$

or $$\vec{v} + \vec{v_0} = \frac{2(\vec{r} - \vec{r_0})}{t} \quad ...(2)$$

Taking the dot products of the corresponding sides of the equations (1) and (2), we get

$$(\vec{v} - \vec{v_0}).(\vec{v} + \vec{v_0}) = \vec{a}\,t\,.2\,\frac{(\vec{r} - \vec{r_0})}{t}$$

or $$v^2 - v_0^2 = 2\vec{a}\,.(\vec{r} - \vec{r_0}).$$

Examples based on Motion in a Plane

Concepts Used

1. Distance is the length of actual path traversed by a moving body between its initial and final positions.
2. Displacement is the shortest distance between the initial and final positions of a body.
3. Average speed = $\dfrac{\text{Distance travelled}}{\text{Time taken}}$
4. Average velocity = $\dfrac{\text{Displacement covered}}{\text{Time taken}}$
5. Instantaneous velocity, $\vec{v} = \dfrac{d\vec{r}}{dt}$
6. Instantaneous acceleration, $\vec{a} = \dfrac{d\vec{v}}{dt}$

Units Used

Displacement and distance are in metre, average speed and average velocity are in ms^{-1}.

Example 55. A cyclist moves along a circular path of radius 70 m. If he completes one round in 11 s, calculate (i) total length of path, (ii) magnitude of the displacement, (iii) average speed, and (iv) magnitude of average velocity.

Solution. Radius of the circular path, $r = 70$ m

Time taken to complete one round, $t = 11$ s

(*i*) Total length of path,

$$s = 2\pi r = 2 \times \frac{22}{7} \times 70 = \mathbf{440\ m}.$$

(*ii*) As the cyclist returns to the initial position after one round, magnitude of the displacement = **0**.

(*iii*) Average speed $= \frac{s}{t} = \frac{440\,\text{m}}{11\,\text{s}} = \mathbf{40\ ms^{-1}}$.

(*iv*) Average velocity $= \dfrac{\text{Displacement}}{\text{Time}} = \dfrac{0}{11} = \mathbf{0}$.

Example 56. A particle is moving eastwards with a velocity of 5 ms^{-1}. In 10 seconds, the velocity changes to 5 ms^{-1} northwards. Find the average acceleration of the particle in this time interval. **[AIEEE 05]**

Solution. In Fig. 4.59,

$\vec{OA}$ = Initial velocity $\vec{v}_1 = 5\ ms^{-1}$, due east

$\vec{OB}$ = Final velocity $\vec{v}_2 = 5\ ms^{-1}$, due north

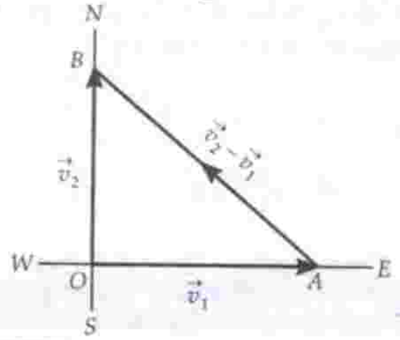


Fig. 4.59

By Δ law of vector addition,

$$\vec{OA} + \vec{AB} = \vec{OB}$$

$$\therefore \quad \vec{AB} = \vec{OB} - \vec{OA} = \vec{v}_2 - \vec{v}_1$$

$$= \text{Change in velocity}$$

$$|\vec{v}_2 - \vec{v}_1| = AB = \sqrt{OA^2 + OB^2}$$

$$= \sqrt{5^2 + 5^2} = 5\sqrt{2}\ \text{m}$$

Average acceleration

$$= \frac{|\vec{v}_2 - \vec{v}_1|}{t} = \frac{5\sqrt{2}}{10}$$

$$= \frac{1}{\sqrt{2}}\ \mathbf{ms^{-2}}, \textbf{ along N-W direction.}$$

Example 57. The position of a particle is given by

$$\vec{r} = 3.0\,t\,\hat{i} + 2.0\,t^2\,\hat{j} + 5.0\,\hat{k}$$

where t is in seconds and the coefficients have the proper units for r to be in metres. (a) Find $v(t)$ and $a(t)$ of the particle. (b) Find the magnitude and direction of $v(t)$ at $t = 3.0$ s. **[NCERT]**

Solution. Here $\vec{r} = 3.0t\,\hat{i} + 2.0t^2\,\hat{j} + 5.0\,\hat{k}$

$$\therefore \quad \vec{v}(t) = \frac{d\vec{r}}{dt} = \frac{d}{dt}(3.0\,t\,\hat{i} + 2.0\,t^2\,\hat{j} + 5.0\,\hat{k})$$

$$= \mathbf{3.0\,\hat{i} + 4.0\,t\,\hat{j}}$$

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = \mathbf{+\,4.0\,\hat{j}}$$

At $t = 3.0$ s, $\quad \vec{v} = 3.0\,\hat{i} + 12.0\,\hat{j}$

It's magnitude is $v = \sqrt{3^2 + 12^2} = \mathbf{12.4\ ms^{-1}}$ and direction is $\theta = \tan^{-1}\left(\frac{v_y}{v_x}\right) = \tan^{-1}\left(\frac{12}{3}\right) = \mathbf{76°}$.

Example 58. If the position vector of a particle is given by: $\vec{r} = (4\cos 2t)\,\hat{i} + (4\sin 2t)\,\hat{j} + (6t)\,\hat{k}$ m, calculate its acceleration at $t = x/4$. **[S.C.R.A. 91]**

Solution. Position,

$$\vec{r} = (4\cos 2t)\,\hat{i} + (4\sin 2t)\,\hat{j} + 6t\,\hat{k}.$$

Velocity, $\vec{v} = \frac{d\vec{r}}{dt} = [4(-\sin 2t).(2)]\,\hat{i}$

$$+ [4(\cos 2t).(2)]\,\hat{j} + 6\,\hat{k}$$

$$= (-8\sin 2t)\,\hat{i} + (8\cos 2t)\,\hat{j} + 6\,\hat{k}$$

Acceleration,

$$\vec{a} = \frac{d\vec{v}}{dt}$$

$$= [-8(\cos 2t)(2)]\,\hat{i} + [8(-\sin 2t)(2)]\,\hat{j}$$

$$= (-16\cos 2t)\,\hat{i} + (-16\sin 2t)\,\hat{j}$$

When $t = \pi/4$

$$\vec{a} = (-16\cos \pi/2)\,\hat{i} + (-16\sin \pi/2)\,\hat{j}$$

$$= (-16 \times 0)\,\hat{i} + (-16 \times 1)\,\hat{j} = \mathbf{-16\,\hat{j}\ ms^{-2}}.$$

PROBLEMS FOR PRACTICE

1. A body is moving with a uniform velocity of 10 ms^{-1} on a circular path of diameter 2.0 m. Calculate (*i*) the difference between the magnitude of the displacement of the body and the distance covered in half a round and (*ii*) the magnitude of the change in velocity of the body in half a round.

   (**Ans.** 1.14 m, 20 ms^{-1})

2. A particle starts from origin at $t = 0$ with a velocity 5.0 $\hat{i}$ m/s and moves in x-y plane under action of a force which produces a constant acceleration of $(3.0\,\hat{i} + 2.0\,\hat{j})\ m/s^2$. (*a*) What is the y-coordinate of the particle at the instant its x-coordinate is 84 m? (*b*) What is the speed of the particle at this time?

   [NCERT, Central Schools 03]

   [**Ans.** (*a*) 36.0 m (*b*) 25.9 ms^{-1}]

HINTS

1. (*i*) In Fig. 4.60, let A and B be the initial and final positions of the body after half around.

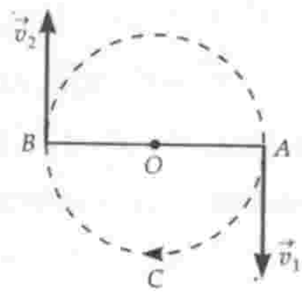


Fig. 4.60

Distance covered = Half circumference
$= \pi r = 3.14 \times 1.0 = 3.14$ m

Magnitude of displacement $= AB = 2.0$ m

Required difference $= 3.14 - 2.0 =$ **1.14 m.**

(ii) Let $\vec{v}_1$ and $\vec{v}_2$ be the velocity vectors at A and B.

Then $\Delta \vec{v} = \vec{v}_2 - \vec{v}_1 = 10 - (-10) =$ **20 ms^{-1}.**

2. The position of the particle is given by

$$\vec{r}(t) = \vec{v}_0 t + \frac{1}{2}\vec{a}\,t^2$$

$$= 5.0\,\hat{i}\,t + (1/2)(3.0\,\hat{i} + 2.0\,\hat{j})\,t^2$$

or $x(t)\,\hat{i} + y(t)\,\hat{j} = (5.0t + 1.5t^2)\,\hat{i} + 1.0t^2\,\hat{j}$

$\therefore \quad x(t) = 5.0t + 1.5t^2; \qquad y(t) = 1.0t^2$

Given $x(t) = 84$ m, $t = ?$

$5.0t + 1.5t^2 = 84 \quad \Rightarrow \quad t = 6$ s

At $t = 6$ s, $\quad y = 1.0(6)^2 =$ **36.0 m.**

Velocity, $\quad \vec{v} = \dfrac{d\vec{r}}{dt} = (5.0 + 3.0t)\,\hat{i} + 2.0t\,\hat{j}$

At $t = 6$ s, $\quad \vec{v} = 23.0\,\hat{i} + 12.0\,\hat{j}$

Speed $= |\vec{v}| = \sqrt{23^2 + 12^2} =$ **25.9 ms^{-1}.**

4.21 RELATIVE VELOCITY IN TWO DIMENSIONS

47. *Define the term relative velocity. How can it be obtained vectorially ?*

Relative velocity. *The relative velocity of an object A with respect to object B, when both are in motion, is the rate of change of position of object A with respect to object B.* Suppose two objects A and B are moving with velocities $\vec{v}_A$ and $\vec{v}_B$, with respect to ground or the earth. Then

Relative velocity of object A w.r.t. object B,

$$\vec{v}_{AB} = \vec{v}_A - \vec{v}_B$$

Relative velocity of object B w.r.t. object A,

$$\vec{v}_{BA} = \vec{v}_B - \vec{v}_A$$

Clearly,

$$\vec{v}_{AB} = -\vec{v}_{BA} \quad \text{and} \quad |\vec{v}_{AB}| = |\vec{v}_{BA}|$$

Now, relative velocity of object $A = \vec{v}_A + (-\vec{v}_B)$

= Velocity vector of A + Negative velocity vector of $\vec{B}$

Hence the relative velocity of object A with respect to object B is equal to the vector addition of velocity vector of A and the negative velocity vector of B.

48. *A man moving in rain holds his umbrella inclined to the vertical even though the rain drops are falling vertically downwards. Why ?*

Rain and man. The man experiences the velocity of rain relative to himself. To protect himself from the rain, the man should hold umbrella in the direction of relative velocity of rain w.r.t. the man.

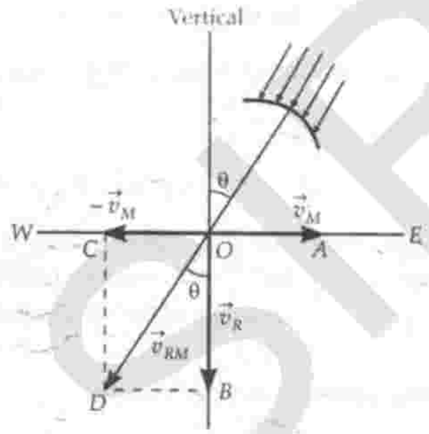


Fig. 4.61 Rain and man.

As shown in Fig. 4.61, consider a man moving due east with velocity $\vec{v}_M$. Suppose the rain falls vertically with velocity $\vec{v}_R$. The relative velocity of rain w.r.t. the man is

$$\vec{v}_{RM} = \vec{v}_R - \vec{v}_M = \vec{v}_R + (-\vec{v}_M) = \vec{OB} + \vec{OC} = \vec{OD}$$

If $\vec{OD}$ makes angle θ with the vertical, then

$$\tan\theta = \frac{DB}{OB} = \frac{v_M}{v_R}$$

So the man can protect himself from rain by holding his umbrella at an angle θ with the vertical in the direction of his motion.

Examples based on Relative Velocity of Two Inclined Motions

FORMULAE USED

1. The relative velocity of A w.r.t. B, $\vec{v}_{AB} = \vec{v}_A - \vec{v}_B$
2. The relative velocity of B w.r.t. A, $\vec{v}_{BA} = \vec{v}_B - \vec{v}_A$
3. For two objects moving with velocities v_A and v_B at an angle θ, the relative velocity of an object A w.r.t. B is given by
$$v_{AB} = \sqrt{v_A^2 + v_B^2 - 2v_A v_B \cos\theta}$$
4. If velocity v_{AB} makes angle β with v_A, then
$$\tan\beta = \frac{v_B \sin\theta}{v_A - v_B \cos\theta}$$

UNIT USED

Velocities v_A, v_B and v_{AB} are in ms^{-1}.

Example 59. *A boat is moving with a velocity $(3\hat{i}+4\hat{j})$ with respect to ground. The water in the river is moving with a velocity $-3\,i-4\,j$ with respect to ground. What is the relative velocity of boat with respect to river ?* **[EAMCET 91]**

Solution. Velocity of boat, $\vec{v}_B = 3\hat{i}+4\hat{j}$

Velocity of water, $\vec{v}_W = -3\hat{i}-4\hat{j}$

$\therefore$ Relative velocity of boat with respect to water is

$$\vec{v}_{BW} = \vec{v}_B - \vec{v}_W$$
$$= (3\hat{i}+4\hat{j})-(-3\hat{i}-4\hat{j}) = \mathbf{6\hat{i}+8\hat{j}}.$$

Example 60. *A particle P is moving along a straight line with a velocity of 3 ms^{-1} and another particle Q has a velocity of 4 ms^{-1} at an angle of 30° to the path of P. Find the speed of Q relative to P.*

Solution. The situation is shown in Fig. 4.62. Here

$$\vec{OA} = \vec{v}_P = 3\text{ ms}^{-1}$$
$$\vec{OB} = \vec{v}_Q = 4\text{ ms}^{-1},\ \text{inclined to } \vec{v}_P \text{ at } 30°$$
$$\vec{OC} = -\vec{v}_P = \text{opposite velocity of } P.$$

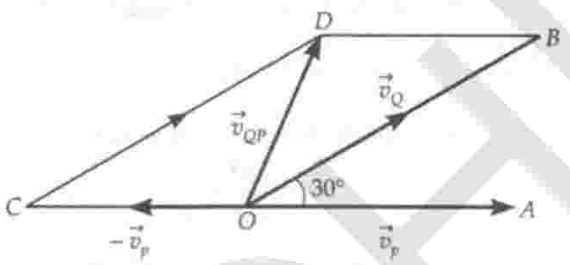


Fig. 4.62

Relative velocity of Q w.r.t. P is

$$\vec{v}_{QP} = \vec{v}_Q - \vec{v}_P = \vec{v}_Q + (-\vec{v}_P) = \vec{OB}+\vec{OC} = \vec{OD}$$

In parallelogram $OBDC$,

$$OC = 3\text{ ms}^{-1},\quad OB = 4\text{ ms}^{-1},\ \angle BOC = 150°$$

$$\therefore \quad \vec{v}_{QP} = OD$$
$$= \sqrt{OC^2+OB^2+2\,OC\,.\,OB\cos 150°}$$
$$= \sqrt{3^2+4^2+2\times 3\times 4\times(-\sqrt{3}/2)}$$
$$= \sqrt{9+16-12\times 1.732} = \sqrt{25-20.784}$$
$$= \sqrt{4.216} \simeq \mathbf{2.1\ ms^{-1}}.$$

Example 61. *To a driver going east in a car with a velocity of 40 km h^{-1}, a bus appears to move towards north with a velocity of $40\sqrt{3}$ km h^{-1}. What is the actual velocity and direction of motion of the bus ?*

Solution. In Fig. 4.63, the true velocity of the car is

$$\vec{v}_C = \vec{OA} = 40\text{ km h}^{-1},\ \text{due east}$$

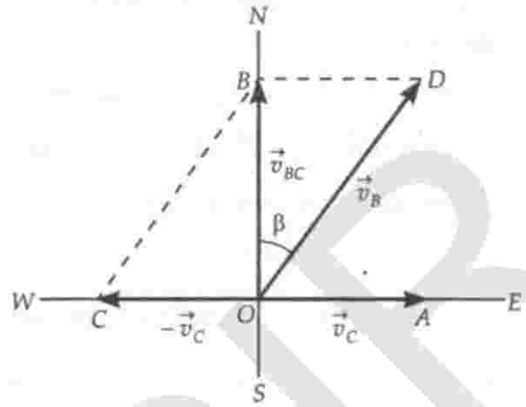


Fig. 4.63

Relative velocity of bus w.r.t. car is

$$\vec{v}_{BC} = \vec{OB} = 40\sqrt{3}\text{ km h}^{-1},\ \text{due north}$$

Let the true velocity of bus be along $\vec{OD}$ and $\angle BOD = \beta$. Then

$$v_B = OD = \sqrt{OA^2+OB^2}$$
$$= \sqrt{40^2+(40\sqrt{3})^2} = \mathbf{80\ km\ h^{-1}}$$
$$\tan\beta = \frac{BD}{OB} = \frac{40}{40\sqrt{3}} = \frac{1}{\sqrt{3}}$$
$$\therefore \quad \boldsymbol{\beta = 30°},\ \textbf{east of north.}$$

Example 62. *A man rows directly across a flowing river in time t_1 and rows an equal distance down the stream in time t_2. If u be the speed of man in still water and v that of stream, then show that :*

$$t_1 : t_2 = \sqrt{u+v} : \sqrt{u-v}$$

Solution. The situation is shown in Fig. 4.64. While rowing directly across the river, the resultant velocity is perpendicular to the velocity v of the stream.

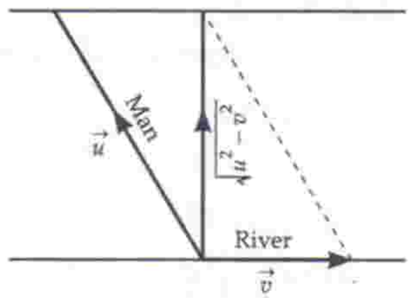


Fig. 4.64

$\therefore$ Resultant velocity across the river $= \sqrt{u^2-v^2}$

Resultant velocity down the stream $= u+v$

Let s be the distance covered in each case. Then

$$t_1 = \frac{s}{\sqrt{u^2 - v^2}} \quad \text{and} \quad t_2 = \frac{s}{u+v}$$

$$\therefore \quad \frac{t_1}{t_2} = \frac{u+v}{\sqrt{u^2 - v^2}} = \frac{\sqrt{u+v} \times \sqrt{u+v}}{\sqrt{u+v} \times \sqrt{u-v}} = \frac{\sqrt{u+v}}{\sqrt{u-v}}$$

or $t_1 : t_2 = \sqrt{u+v} : \sqrt{u-v}$.

PROBLEMS FOR PRACTICE

1. A train is moving with a velocity of 30 km h^{-1} due east and a car is moving with a velocity of 40 km h^{-1} due north. What is the velocity of car as appears to a passenger in the train ?

   (**Ans.** 50 km h^{-1}, 36°52′ west of north)

2. Rain is falling vertically with a speed of 35 ms^{-1}. A woman rides a bicycle with a speed of 12 ms^{-1} in east to west direction. What is the direction in which she should hold her umbrella ?

   (**Ans.** At an angle of 19° with the vertical towards the west)

3. To a person moving eastwards with a velocity of 4.8 km h^{-1}, rain appears to fall vertically downwards with a speed of 6.4 km h^{-1}. Find the actual speed and direction of the rain.

   (**Ans.** 8 kmh^{-1}, 53°7′33″ with the horizontal)

4. A ship is streaming towards east with a speed of 12 ms^{-1}. A woman runs across the deck at a speed of 5 ms^{-1} in the direction at right angles to the direction of motion of the ship *i.e.*, towards north. What is the velocity of the woman relative to the sea ? (**Ans.** 13 ms^{-1}, 22°37′ north of east)

5. A plane is travelling eastward at a speed of 500 km h^{-1}. But a 90 km h^{-1} wind is blowing southward. What is the direction and speed of the plane relative to the ground ?

   (**Ans.** 10.2° south of east, 508 km h^{-1})

6. A reckless drunk is playing with a gun in an airplane that is going directly east at 500 km h^{-1}. The drunk shoots the gun straight up at the ceiling of the plane. The bullet leaves the gun at a speed of 1000 km h^{-1}. Relative to an observer on earth, what angle does the bullet make with the vertical ?

   (**Ans.** 26.6°)

HINTS

1. In Fig. 4.65,

$$\vec{OA} = \vec{v}_T = 30 \text{ km h}^{-1}$$

$$\vec{OC} = \vec{v}_C = 40 \text{ km h}^{-1}$$

$$\vec{v}_{CT} = \vec{v}_C + (-\vec{v}_T) = \vec{OC} + \vec{OB} = \vec{OD}$$

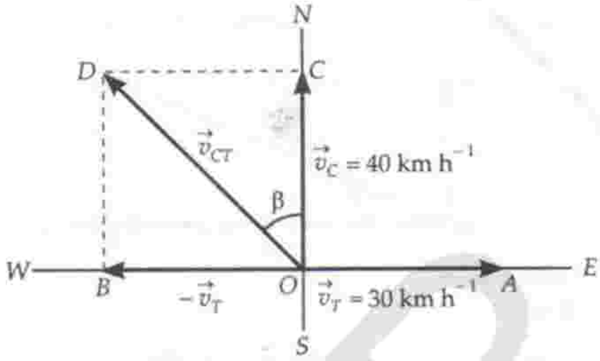


Fig. 4.65

3. The situation is shown in Fig. 4.66. Here

$$\vec{OA} = \text{Velocity of person} = 4.8 \text{ kmh}^{-1}, \text{ due east}$$

$$\vec{OB} = \text{Relative velocity of rain w.r.t. the person} = 6.4 \text{ kmh}^{-1}, \text{ vertically downwards}$$

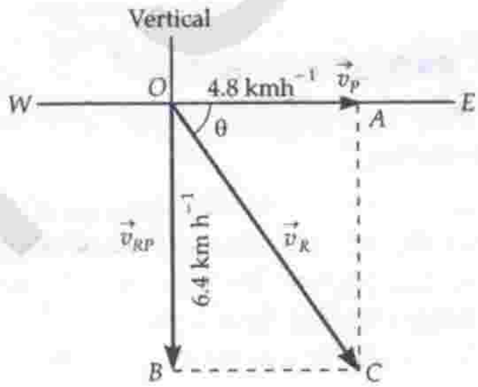


Fig. 4.66

If $\vec{v}_R$ is the actual velocity of rain, then

$$\vec{v}_{RP} = \vec{v}_R - \vec{v}_P$$

or $$\vec{v}_R = \vec{v}_{RP} + \vec{v}_P$$

Complete the parallelogram $OACB$. Then $\vec{OC}$ represents the rain velocity $\vec{v}_R$.

$$\therefore \quad v_R = OC = \sqrt{OA^2 + AC^2}$$

$$= \sqrt{v_P^2 + v_{RP}^2} = \sqrt{4.8^2 + 6.4^2} = 8 \text{ kmh}^{-1}.$$

If $\vec{v}_R$ makes angle θ with OE, then

$$\tan\theta = \frac{AC}{OA} = \frac{v_{RP}}{v_P} = \frac{6.4}{4.8} = \frac{4}{3} = 1.333$$

$$\therefore \quad \theta = 53°\ 7'33''.$$

4. As shown in Fig. 4.67, the woman has two velocities simultaneously : (*i*) a velocity equal to the velocity of ship which is 12 ms^{-1} due east and (*ii*) a velocity of 5 ms^{-1} due north.

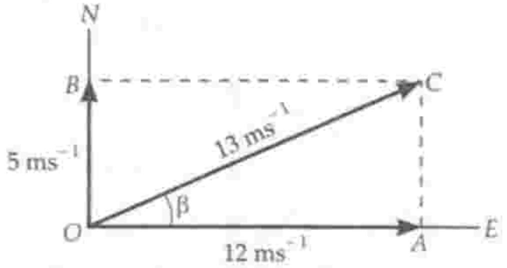


Fig. 4.67

Velocity of woman relative to the sea

= Resultant velocity of woman

$$= OC = \sqrt{OA^2 + OB^2} = \sqrt{12^2 + 5^2} = \mathbf{13\ ms^{-1}}.$$

If the resultant velocity makes angle β with the direction of the motion of the ship, then

$$\tan \beta = \frac{AC}{OA} = \frac{5}{12} = 0.4167$$

$\therefore$ $\beta =$ **22°37′, north of east**

4.22 PROJECTILE MOTION

49. *What is a projectile ? Give some examples of projectile motion.*

Projectile. *A projectile is the name given to any body which once thrown into space with some initial velocity, moves thereafter under the influence of gravity alone without being propelled by any engine or fuel. The path followed by a projectile is called its trajectory.*

Examples of projectile motion :

(*i*) A javelin thrown by an athlete.

(*ii*) An object dropped from an aeroplane.

(*iii*) A bullet fired from a rifle.

(*iv*) A jet of water coming out from the side hole of a vessel.

(*v*) A stone thrown horizontally from the top of a building.

50. *State the principle of physical independence of motions used in projectile motion. Give an illustration.*

Principle of physical independence of motions. In the absence of air resistance, the motion of a projectile is considered as the combination of the following two independent motions :

(*i*) Motion along horizontal direction with *uniform velocity.*

(*ii*) Motion along vertical direction under gravity *i.e., with uniform acceleration equal to g.*

The two motions of a projectile along horizontal and vertical directions are independent of each other. This is called the **principle of physical independence of motions**. For example, if a ball is dropped downward from the roof a building and simultaneously another ball is thrown in a horizontal direction, then both the balls will reach the ground at the same time but at different places. Clearly, the vertical motion is not being affected by the horizontal motion.

For Your Knowledge

- It was *Galileo* who first stated the principle of physical independence of the horizontal and vertical motions of a projectile in his book *"Dialogue on the great world systems"* in 1632.

51. *State the assumptions made in the study of projectile motion.*

Assumptions used in projectile motion. While studying the motion of a projectile, we use following assumptions :

(*i*) There is no air resistance on the projectile.

(*ii*) The effect due to curvature of the earth is negligible.

(*iii*) The effect due to rotation of the earth is negligible.

(*iv*) For all points of the trajectory, the acceleration due to gravity is constant both in magnitude and direction.

4.23 PROJECTILE GIVEN HORIZONTAL PROJECTION

52. *A projectile is fired horizontally with a velocity u. Show that its trajectory is a parabola. Also obtain expressions for its (i) time of flight (ii) horizontal range and (iii) velocity at any instant.*

Projectile fired parallel to horizontal. As shown in Fig. 4.68, suppose a body is projected horizontally with velocity u from a point O at a certain height h above the ground level. The body is under the influence of two simultaneous independent motions :

(*i*) Uniform horizontal velocity u.

(*ii*) Vertically downward accelerated motion with constant acceleration g.

Under the combined effect of the above two motions, the body moves along the path OPA.

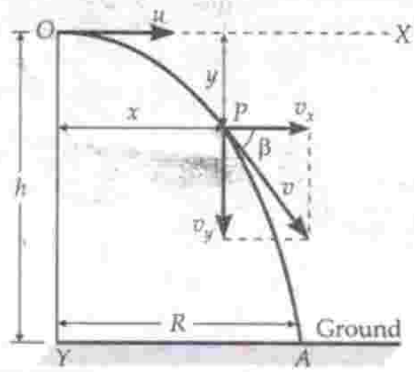


Fig. 4.68 Horizontal projection of a projectile.

Trajectory of the projectile. After the time t, suppose the body reaches the point $P(x, y)$.

The horizontal distance covered by the body in time t is

$$x = ut \qquad \therefore \quad t = \frac{x}{u}$$

The vertical distance travelled by the body in time t is given by

$$s = ut + \frac{1}{2}at^2$$

or $$y = 0 \times t + \frac{1}{2}gt^2 = \frac{1}{2}gt^2$$

[For vertical motion, $u = 0$]

or $$y = \frac{1}{2}g\left(\frac{x}{u}\right)^2 = \left(\frac{g}{2u^2}\right)x^2 \qquad \left[\because \; t = \frac{x}{u}\right]$$

or $$y = k\,x^2 \qquad \left[\text{Here } k = \frac{g}{2u^2} = \text{a constant}\right]$$

As y is a quadratic function of x, so *the trajectory of the projectile is a parabola.*

Time of flight. *It is the total time for which the projectile remains in its flight* (from O to A). Let T be its time of flight.

For the vertical downward motion of the body, we use

$$s = ut + \frac{1}{2}at^2 \quad \text{or} \quad h = 0 \times T + \frac{1}{2}gT^2$$

or $$T = \sqrt{\frac{2h}{g}}.$$

Horizontal range. *It is the horizontal distance covered by the projectile during its time of flight.* It is equal to $OA = R$. Thus

$$R = \text{Horizontal velocity} \times \text{time of flight} = u \times T$$

or $$R = u\sqrt{\frac{2h}{g}}.$$

Velocity of the projectile at any instant. At the instant t (when the body is at point P), let the velocity of the projectile be v. The velocity v has two rectangular components :

Horizontal component of velocity, $v_x = u$

Vertical component of velocity, $v_y = 0 + gt = gt$

$\therefore$ The resultant velocity at point P is

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{u^2 + g^2t^2}$$

If the velocity v makes an angle β with the horizontal, then

$$\tan\beta = \frac{v_y}{v_x} = \frac{gt}{u}$$

or $$\beta = \tan^{-1}\left(\frac{gt}{u}\right).$$

Examples based on Projectile Fired Horizontally

FORMULAE USED

1. Position of the projectile after time t :
   $x = ut\ ,\ y = \frac{1}{2}gt^2$
2. Equation of trajectory : $y = \frac{g}{2u^2} \cdot x^2$
3. Velocity after time t : $v = \sqrt{u^2 + g^2t^2}$
   $\beta = \tan^{-1}\frac{gt}{u}$
4. Time of flight : $T = \sqrt{\frac{2h}{g}}$
5. Horizontal range : $R = u \times T = u\sqrt{\frac{2h}{g}}$

UNITS USED

Distances x, y, h and R are in metres, velocities, u and v in ms^{-1}, acceleration due to gravity g in ms^{-2}, and times t and T in second.

EXAMPLE 63. *A hiker stands on the edge of a cliff 490 m above the ground and throws a stone horizontally with an initial speed of $15\,ms^{-1}$. Neglecting air resistance, find the time taken by the stone to reach the ground, and the speed with which it hits the ground. (Take $g = 9.8\ ms^{-2}$).* **[NCERT]**

Solution. Suppose the stone is thrown from the edge of a cliff with speed $u = 15\ ms^{-1}$ along the horizontal OX. It hits the ground at point P after time t.

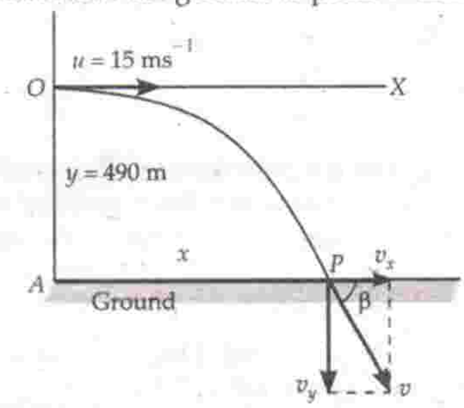


Fig. 4.69

Initial velocity in the downward direction $= 0$

Vertical distance, $\quad OA = y = 490$ m

As $\; y = \frac{1}{2}gt^2 \; \therefore \; 490 = \frac{1}{2} \times 9.8\,t^2$ or $t = \sqrt{100} = \mathbf{10\ s}$.

The horizontal and vertical components of speed v of the stone at point P are

$$v_x = u = 15\ ms^{-1}$$

$$v_y = u_y + gt = 0 + 9.8 \times 10 = 98\ ms^{-1}$$

$$\therefore \quad v = \sqrt{v_x^2 + v_y^2} = \sqrt{15^2 + 98^2} = \mathbf{99.1\ ms^{-1}}.$$

EXAMPLE 64. *A projectile is fired horizontally with a velocity of 98 ms^{-1} from the top of a hill 490 m high. Find (i) the time taken to reach the ground (ii) the distance of the target from the hill and (iii) the velocity with which the projectile hits the ground.*

Solution. (*i*) As shown in Fig. 4.69, the projectile is fired from the top O of a hill with velocity $u = 98\ \text{ms}^{-1}$ along the horizontal OX. It reaches the target P in time t.

Initial velocity in the downward direction = 0

Vertical distance,

$$OA = y = 490\ \text{m}$$

As $\quad y = \frac{1}{2}gt^2$

$\therefore \quad 490 = \frac{1}{2} \times 9.8\, t^2$

or $\quad t = \sqrt{100} = \mathbf{10\ s.}$

(*ii*) Distance of the target from the hill,

$$AP = x = \text{Horizontal velocity} \times \text{time}$$
$$= 98 \times 10 = \mathbf{980\ m.}$$

(*iii*) The horizontal and vertical components of velocity v of the projectile at point P are

$$v_x = u = 98\ \text{ms}^{-1}$$
$$v_y = u_x + gt = 0 + 9.8 \times 10 = 98\ \text{ms}^{-1}$$

$\therefore \quad v = \sqrt{v_x^2 + v_y^2}$

$$= \sqrt{98^2 + 98^2} = 98\sqrt{2} = \mathbf{138.59\ ms^{-1}}.$$

If the resultant velocity v makes angle β with the horizontal, then

$$\tan\beta = \frac{v_y}{v_x} = \frac{98}{98} = 1 \quad \therefore \quad \beta = \mathbf{45°}.$$

EXAMPLE 65. *A body is thrown horizontally from the top of a tower and strikes the ground after three seconds at an angle of 45° with the horizontal. Find the height of the tower and the speed with which the body was projected. Take $g = 9.8\ ms^{-2}$.*

Solution. As shown in Fig. 4.69, suppose the body is thrown horizontally from the top O of a tower of height y with velocity u. The body hits the ground after 3 s. Considering vertically downward motion of the body,

$$y = u_y t + \frac{1}{2}gt^2 = 0 \times 3 + \frac{1}{2} \times 9.8 \times (3)^2 = \mathbf{44.1\ m}$$

[$\because$ Initial vertical velocity, $u_y = 0$]

Final vertical velocity,

$$v_y = u_y + gt = 0 + 9.8 \times 3 = 29.4\ \text{ms}^{-1}$$

Final horizontal velocity, $\quad v_x = u$

As the resultant velocity v makes an angle of 45° with the horizontal, so

$$\tan 45° = \frac{v_y}{v_x} \quad \text{or} \quad 1 = \frac{29.4}{u} \quad \text{or} \quad u = \mathbf{29.4\ ms^{-1}}.$$

EXAMPLE 66. *A bomb is dropped from an aeroplane when it is directly above a target at a height of 1000 m. The aeroplane is moving horizontally with a speed of 500 kmh^{-1}. By how much distance will the bomb miss the target ?*

Solution. As the aeroplane is moving horizontally, the initial downward velocity of the bomb, $u_y = 0$.

Also $\quad y = 1000\ \text{m},\ g = 9.8\ \text{ms}^{-2},\ t = ?$

Now $\quad y = u_y t + \frac{1}{2}gt^2$

$\therefore \quad 1000 = 0 + \frac{1}{2} \times 9.8\, t^2$

or $\quad t = \sqrt{\frac{1000}{4.9}} = \frac{100}{7}\ \text{s}$

Horizontal velocity of the aeroplane

$$= 500\ \text{kmh}^{-1}$$
$$= 500 \times \frac{5}{18}\ \text{ms}^{-1} = \frac{1250}{9}\ \text{ms}^{-1}$$

Distance by which the bomb misses the target

= Horizontal distance covered by the bomb before it hits the ground

= Horizontal velocity × time

$$= \frac{1250}{9} \times \frac{100}{7} = \mathbf{1984.13\ m.}$$

EXAMPLE 67. *A body is projected horizontally from the top of a cliff with a velocity of 9.8 ms^{-1}. What time elapses before horizontal and vertical velocities become equal ? Take $g = 9.8\ ms^{-2}$.*

Solution. Horizontal velocity at any instant, $v_x = u = 9.8\ \text{ms}^{-1}$

Vertical velocity at any instant, $v_y = 0 + gt = 9.8\,t$

According to the question, $v_x = v_y$

$\therefore \quad 9.8 = 9.8\,t \quad \text{or} \quad t = \mathbf{1\ s.}$

EXAMPLE 68. *A marksman wishes to hit a target just in the same level as the line of sight. How high from the target he should aim, if the distance of the target is 1600 m and the muzzle velocity of the gun is 800 ms^{-1}? Take $g = 9.8\ ms^{-2}$.*

Solution. Let u be the speed of the bullet. In time t, it covers a horizontal distance,

$$x = 1600\ \text{m}$$

But $\quad x = ut$

$\therefore \quad 1600 = 800 \times t \quad \text{or} \quad t = 2\ \text{s}$

Distance through which the bullet is pulled down by the force of gravity in 2 s is

$$y = \frac{1}{2}gt^2 = \frac{1}{2} \times 9.8 \times (2)^2 = 19.6\ \text{m}$$

$\therefore$ Height of the gun from the target = **19.6 m.**

EXAMPLE 69. Two tall buildings face each other and are at a distance of 180 m from each other. With what velocity must a ball be thrown horizontally from a window 55 m above the ground in one building, so that it enters a window 10.9 m above the ground in the second building ?

Solution. In Fig. 4.70, A and B are two tall buildings which are 180 m apart. W_1 and W_2 are the two windows in A and B respectively.

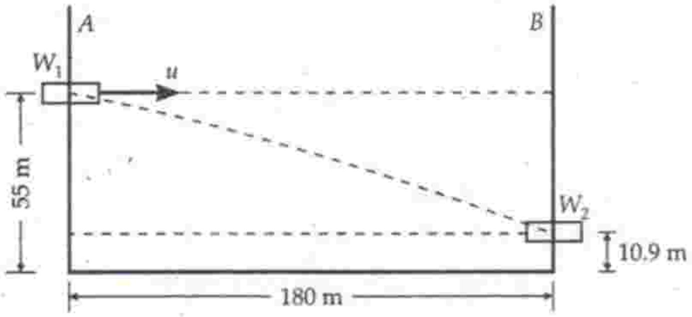


Fig. 4.70

Vertical downward distance to be covered by the ball

$$= \text{Height of } W_1 - \text{Height of } W_2$$
$$= 55 - 10.9 = 44.1 \text{ m}$$

Initial vertical velocity of ball,

$$u_y = 0$$

As $\quad y = u_y\, t + \frac{1}{2} g t^2$

$\therefore \quad 44.1 = 0 + \frac{1}{2} \times 9.8\, t^2$

or $\quad t^2 = \dfrac{44.1 \times 2}{9.8} = 9 \quad$ or $\quad t = 3$ s

Required horizontal velocity

$$= \frac{\text{Horizontal distance}}{\text{Time}} = \frac{180 \text{ m}}{3 \text{ s}} = \mathbf{60\ ms^{-1}}.$$

EXAMPLE 70. A particle is projected horizontally with a speed u from the top of plane inclined at an angle θ with the horizontal. How far from the point of projection will the particle strike the plane ?

Solution. The horizontal distance covered in time t,

$$x = ut \quad \text{or} \quad t = \frac{x}{u} \qquad ...(1)$$

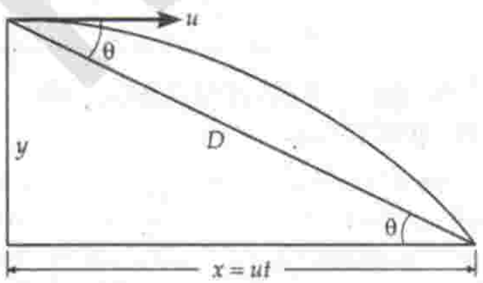


Fig. 4.71

The vertical distance covered in time t,

$$y = 0 + \tfrac{1}{2} g t^2 = \tfrac{1}{2} g \times \frac{x^2}{u^2} \qquad \text{[using (1)]}$$

Also $\quad \dfrac{y}{x} = \tan\theta \quad$ or $\quad y = x \tan\theta$

$\therefore \quad \dfrac{g x^2}{2 u^2} = x \tan\theta$

or $\quad x\left(\dfrac{g x}{2 u^2} - \tan\theta\right) = 0$

As $x = 0$ is not possible, so $x = \dfrac{2u^2 \tan\theta}{g}$

The distance of the point of strike from the point of projection is

$$D = \sqrt{x^2 + y^2} = \sqrt{x^2 + (x\tan\theta)^2}$$
$$= x\sqrt{1 + \tan^2\theta} = x \sec\theta$$

or $\quad D = \dfrac{\mathbf{2u^2}}{g}\ \mathbf{\tan\theta \sec\theta}.$

EXAMPLE 71. A helicopter on a flood relief mission flying horizontally with a speed u at an altitude h, has to drop a food packet for a victim standing on the ground. At what distance from the victim should the food packet be dropped ?

Solution. In Fig. 4.72, H represents position of the helicopter and V that of the victim. For vertical motion of the packet

$$h = 0 + \frac{1}{2} g t^2 \quad \text{or} \quad t = \sqrt{\frac{2h}{g}}$$

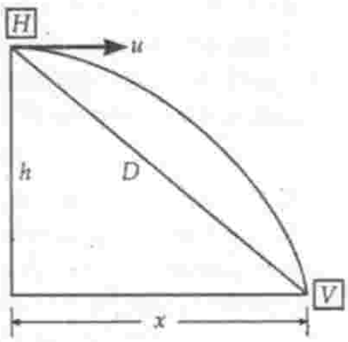


Fig. 4.72

Horizontal distance covered by the food packet in time t,

$$x = ut = u\sqrt{\frac{2h}{g}}$$

The distance of the point of projection from the food packet is

$$D = \sqrt{h^2 + x^2} = \sqrt{h^2 + \frac{2u^2 h}{g}}.$$

✖ PROBLEMS FOR PRACTICE

1. A plane is flying horizontally at a height of 1000 m with a velocity of 100 ms^{-1} when a bomb is released from it. Find (*i*) the time taken by it to reach the ground (*ii*) the velocity with which the bomb hits the target and (*iii*) the distance of the target. [**Ans.** (*i*) 14.28 s (*ii*) 172.1 ms^{-1}, $\beta = 54°28'$ (*iii*) 1428.5 m]
2. From the top of a building 19.6 m high, a ball is projected horizontally. After how long does it strike the ground ? If the line joining the point of projection to the point where it hits the ground makes an angle of 45° with the horizontal, what is the initial velocity of the ball ? (**Ans.** 2 s, 9.8 ms^{-1})
3. A body is thrown horizontally from the top of a tower and strikes the ground after two seconds at angle of 45° with the horizontal. Find the height of the tower and the speed with which the body was thrown. Take $g = 9.8\ ms^{-2}$. (**Ans.** 19.6 m, 19.6 ms^{-1})
4. Two tall buildings are situated 200 m apart. With what speed must a ball be thrown horizontally from the window 540 m above the ground in one building, so that it will enter a window 50 m above the ground in the other ? (**Ans.** 20 ms^{-1})
5. A stone is dropped from the window of a bus moving at 60 kmh^{-1}. If the window is 1.96 m high, find the distance along the track, which the stone moves before striking the ground. (**Ans.** 10.54 m)
6. An aeroplane is flying in a horizontal direction with a velocity of 600 kmh^{-1} and at a height of 1960 m. When it is vertically above a point *A* on the ground, a body is dropped from it. The body strikes the ground at a point *B*. Calculate the distance *AB*. (**Ans.** 3333.3 m)
7. A mailbag is to be dropped into a post office from an aeroplane flying horizontally with a velocity of 270 kmh^{-1} at a height of 176.4 m above the ground. How far must the aeroplane be from the post office at the time of dropping the bag so that it directly falls into the post office ? (**Ans.** 450 m)
8. In between two hills of heights 100 m and 92 m respectively, there is a valley of breadth 16 m. If a vehicle jumps from the first hill to the second, what must be its minimum horizontal velocity so that it may not fall into the valley ? Take $g = 9\ ms^{-2}$. (**Ans.** 12 ms^{-1})
9. A ball is projected horizontally from a tower with a velocity of 4 ms^{-1}. Find the velocity of the ball after 0.7 s. Take $g = 10\ ms^{-2}$. (**Ans.** 8.06 ms^{-1}, 60°15′)

✖ HINTS

1. (*i*) *For vertical motion :*

$$y = \frac{1}{2} g t^2 \qquad \therefore 1000 = \frac{1}{2} \times 9.8 t^2$$

or $$t^2 = \frac{10000}{49} \qquad \therefore \qquad t = \frac{100}{7}\ \text{s} = \mathbf{14.28\ s}$$

(*ii*) Velocity with which the bomb hits the target,

$$v = \sqrt{u^2 + g^2t^2} = \sqrt{(100)^2 + \left(9.8 \times \frac{100}{7}\right)^2}$$

$$= \sqrt{100^2 + 140^2} = \mathbf{172.05\ ms^{-1}}.$$

$$\tan\beta = \frac{gt}{u} = \frac{9.8 \times 100}{100 \times 7} = 1.4$$

$\therefore \qquad \beta = \mathbf{54°28'}$.

(*iii*) Distance of the target,

$$x = ut = 1000 \times \frac{100}{7} = \mathbf{1428.51\ m.}$$

7. For vertical motion :

$$y = \frac{1}{2} gt^2 \qquad \therefore \quad 176.4 = \frac{1}{2} \times 9.8 t^2$$

or $$t^2 = \frac{176.4}{4.9} = 36 \quad \therefore \quad t = 6\ \text{s}$$

Also $$u = 270\ kmh^{-1} = \frac{270 \times 5}{18} = 75\ ms^{-1}$$

$\therefore \qquad x = ut = 75 \times 6 = \mathbf{450\ m.}$

8. For vertical motion : $u = 0$,

y = Difference of heights = 8 m, $g = 9\ ms^{-2}$

As $$y = \frac{1}{2} gt^2 \quad \therefore t = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2 \times 8}{9}} = \frac{4}{3}\ \text{s}$$

For horizontal motion :

x = Distance between the two hills = 16 m

As $$x = ut \qquad \therefore \quad u = \frac{x}{t} = \frac{16}{4/3} = \mathbf{12\ ms^{-1}}.$$

9. Here $v_x = u_x = 4\ ms^{-1}$

$$v_y = u_y + a_y t = 0 + 10 \times 0.7 = 7\ ms^{-1}$$

Resultant velocity,

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{4^2 + 7^2} = \sqrt{65} = \mathbf{8.06\ ms^{-1}}.$$

If velocity v makes angle β with the horizontal, then

$$\tan\beta = \frac{v_y}{v_x} = \frac{7}{4} = 1.75 \quad \text{or} \quad \beta = \mathbf{60°15'}.$$

4.24 PROJECTILE GIVEN ANGULAR PROJECTION

53. *A projectile is fired with a velocity u making an angle θ with the horizontal. Show that its trajectory is a parabola. Derive expressions for (i) time of maximum height (ii) time of flight (iii) maximum height (iv) horizontal range.*

Projectile fired at an angle θ with the horizontal. As shown in Fig. 4.73, suppose a body is projected with initial velocity u, making an angle θ with the horizontal. The velocity u has two rectangular components :

(*i*) The horizontal component $u \cos \theta$, which remains constant throughout the motion.

(*ii*) The vertical component $u \sin \theta$, which changes with time under the effect of gravity. This component first decreases, becomes zero at the highest point A, after which it again increases, till the projectile hits the ground.

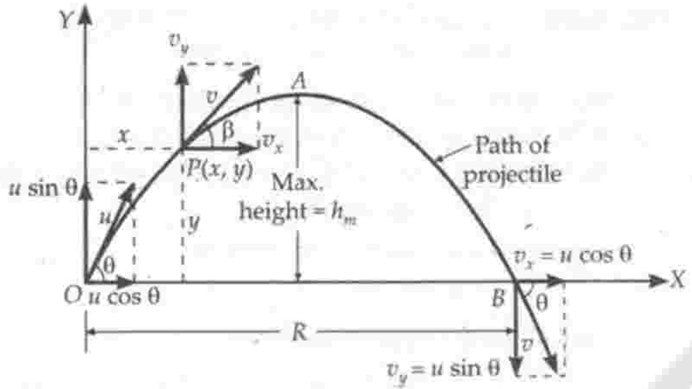


Fig. 4.73 Trajectory of a projectile fired at an angle θ with the horizontal.

Under the combined effect of the above two components, the body follows the parabolic path OAB as shown in the figure.

Equation of trajectory of a projectile. Suppose the body reaches the point $P(x, y)$ after time t.

∴ The horizontal distance covered by the body in time t,

$$x = \text{Horizontal velocity} \times \text{time} = u \cos \theta \,.\, t$$

or
$$t = \frac{x}{u \cos \theta}$$

For vertical motion : $u = u \sin \theta$, $a = -g$, so the vertical distance covered in time t is given by

$$s = ut + \frac{1}{2} at^2$$

or
$$y = u \sin \theta \,.\, \frac{x}{u \cos \theta} - \frac{1}{2} g \,.\, \frac{x^2}{u^2 \cos^2 \theta}$$

or
$$y = x \tan \theta - \frac{g}{2u^2 \cos^2 \theta} x^2$$

or
$$y = px - qx^2$$

where p and q are constants.

Thus y is a quadratic function of x. Hence the *trajectory of a projectile is a parabola.*

Time of maximum height. Let t_m be the time taken by the projectile to reach the maximum height h_m.

At the highest point, vertical component of velocity $= 0$

As $\quad v = u + at \quad \therefore \quad 0 = u \sin \theta - g\, t_m$

or
$$t_m = \frac{u \sin \theta}{g}$$

Time of flight. *It is the time taken by the projectile from the instant it is projected till it reaches a point in the horizontal plane of its projection.* The body reaches the point B after the time of flight T_f.

∴ Net vertical displacement covered during the time of flight $= 0$

As
$$s = ut + \frac{1}{2} at^2$$

∴
$$0 = u \sin \theta \,.\, T_f - \frac{1}{2} g T_f^2$$

or
$$T_f = \frac{2u \sin \theta}{g}$$

Obviously, $T_f = 2t_m$. This is expected because the time of ascent is equal to the time of descent for the symmetrical parabolic path.

Maximum height of a projectile. *It is the maximum vertical distance attained by the projectile above the horizontal plane of projection.* It is denoted by h_m.

At the highest point A, vertical component of velocity $= 0$

As
$$v^2 - u^2 = 2as$$

∴
$$0^2 - (u \sin \theta)^2 = 2(-g)\, h_m$$

or
$$h_m = \frac{u^2 \sin^2 \theta}{2g}$$

Horizontal range (R). *It is the horizontal distance travelled by the projectile during its time of flight.* So

Horizontal range = Horizontal velocity × time of flight

or
$$R = u \cos \theta \times \frac{2u \sin \theta}{g} = \frac{u^2}{g} . 2 \sin \theta \cos \theta,$$

or
$$R = \frac{u^2 \sin 2\theta}{g} \qquad [\because 2 \sin \theta \cos \theta = \sin 2\theta]$$

54. *A body is projected at an angle θ with the horizontal. Derive an expression for its horizontal range. Determine the condition for maximum horizontal range. Show that there are two angles of projections for the same horizontal range. Also calculate the velocity of the projectile at any instant t and at the end point of the flight.*

Horizontal range. Refer answer to the above question.

Condition for the maximum horizontal range. The horizontal range is given by

$$R = \frac{u^2 \sin 2\theta}{g}$$

Clearly, R will be maximum when

$$\sin 2\theta = 1 = \sin 90^\circ$$

or $$2\theta = 90^\circ \quad \text{or} \quad \theta = 45^\circ$$

Thus *the horizontal range of a projectile is maximum when it is projected an angle of 45° with the horizontal.*

The maximum horizontal range is given by

$$R_m = \frac{u^2 \sin 90^\circ}{g} = \frac{u^2 \times 1}{g}$$

or $$R_m = u^2 / g$$

Two angles of projection for the same horizontal range. The horizontal range of a projectile projected at an angle θ with the horizontal with velocity u is given by

$$R = \frac{u^2 \sin 2\theta}{g}$$

Replacing θ by $(90^\circ - \theta)$, we get

$$R' = \frac{u^2 \sin 2(90^\circ - \theta)}{g}$$

$$= \frac{u^2 \sin(180^\circ - 2\theta)}{g} = \frac{u^2 \sin 2\theta}{g}$$

i.e., $$R' = R$$

Hence *for a given velocity of projection, a projectile has the same horizontal range for the angles of projection* θ *and* $(90^\circ - \theta)$. As shown in Fig. 4.74, the horizontal range is maximum for 45°. Clearly, R is same for $\theta = 15^\circ$ and 75° but less than R_m. Again R is same for $\theta = 30^\circ$ and 60°.

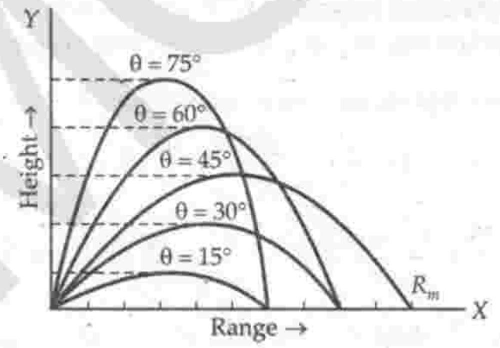


Fig. 4.74 R is same for θ and $(90^\circ - \theta)$. R is maximum for $\theta = 45^\circ$.

When the angle of projection is $(90^\circ - \theta)$ with the horizontal, the angle of projection with the vertical is θ. This indicates that the *horizontal range is same whether* θ *is the angle of projection with the horizontal or with the vertical, as shown in Fig. 4.75.*

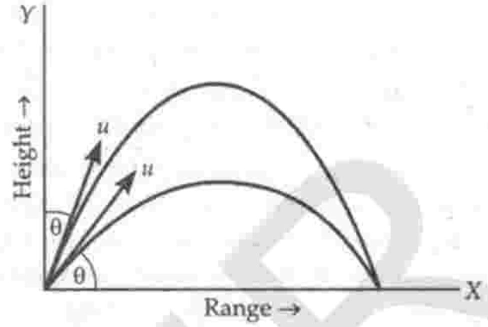


Fig. 4.75 R is same for angle of projection θ with horizontal or with vertical.

Velocity of projectile at any instant. As shown in Fig. 4.73, suppose the projectile has velocity v at the instant t when it is at point $P(x, y)$. The velocity v has two rectangular components :

Horizontal component of velocity,

$$v_x = u \cos \theta$$

Vertical component of velocity,

$$v_y = u \sin \theta - gt \qquad [\text{Using } v = u + at]$$

The resultant velocity at point P is

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(u \cos \theta)^2 + (u \sin \theta - gt)^2}$$

or $$v = \sqrt{u^2 + g^2 t^2 - 2u\, gt \sin \theta}$$

If the velocity v makes an angle β with the vertical, then

$$\tan \beta = \frac{v_y}{v_x} = \frac{u \sin \theta - gt}{u \cos \theta}$$

Velocity of projectile at the end point. At the end of flight,

$$t = \text{total time of flight} = \frac{2u \sin \theta}{g}$$

So the resultant velocity is

$$v' = \sqrt{u^2 + g^2 \cdot \frac{4u^2 \sin^2 \theta}{g^2} - 2ug \cdot \frac{2u \sin \theta}{g} \cdot \sin \theta}$$

$$= \sqrt{u^2} = u$$

Also, $$\tan \beta = \frac{u \sin \theta - g \cdot \dfrac{2u \sin \theta}{g}}{u \cos \theta}$$

$$= -\frac{u \sin \theta}{u \cos \theta} = -\tan \theta = \tan(-\theta)$$

or $$\beta = -\theta.$$

The negative sign shows that the projectile is moving downwards. Thus *in projectile motion, a body returns to the ground at the same angle and with the same speed at which it was projected.*

For Your Knowledge

- A body is said to be projectile if it is projected into space with some initial velocity and then it continues to move in a vertical plane such that its horizontal acceleration is zero and vertical downward acceleration is equal to g.
- In projectile motion, the horizontal motion and the vertical motion are independent of each other *i.e.*, neither motion affects the other.
- The horizontal range is maximum for $\theta = 45°$ and $R_m = u^2/g$.
- The horizontal range is same when the angle of projection is θ and $(90° - \theta)$.
- Again, the horizontal range is same for the angles of projection of $(45° + \theta)$ and $(45° - \theta)$.
- At the highest point of the parabolic path, the velocity and acceleration of a projectile are perpendicular each other.
- The velocity at the end of flight of an oblique projectile is the same in magnitude as at the beginning but the angle that it makes with the horizontal is negative of the angle of projection.
- In projectile motion, the kinetic energy is maximum at the point of projection or point of reaching the ground and is minimum at the highest point.
- There are two values of time for which the projectile is at the same height. The sum of these two times is equal to the time of flight.
- The maximum horizontal range is four times the maximum height attained by the projectile, when fired at $\theta = 45°$. Thus $h_m = R_m/4 = u^2/4g$.
- If a body is projected from a place above the surface of the earth, then for the maximum range the angle of projection should be slightly less than 45°. For javelin throw and discus throw, the athlete throws the projectile at an angle slightly less than 45° to the horizontal for achieving maximum range.
- The trajectory of a projectile is parabolic only when the acceleration of the projectile is constant and the direction of acceleration is different from the direction of projectile's initial velocity. The acceleration of a projectile thrown from the earth is equal to acceleration due to gravity (g) which remains constant if
  (*i*) the projectile does not go to a very large height.
  (*ii*) the range of the projectile is not very large.
  (*iii*) the initial velocity of the projectile is not large.
  Thus the trajectory of a bullet fired from a gun will be parabolic, but not so the trajectory of a missile.
- The shape of the trajectory of the motion of an object is not determined by position alone but also depends on its initial position and initial velocity. Under the same acceleration due to gravity, the trajectory of an object can be a straight line or a parabola depending on the initial conditions.

Examples based on Projectile Fired at an Angle with the Horizontal

FORMULAE USED

1. For a projectile fired with velocity u at an angle θ with the horizontal : $u_x = u\cos\theta$, $u_y = u\sin\theta$, $a_x = 0$, $a_y = -g$
2. Position after time t :
$$x = (u\cos\theta)\,t, \quad y = (u\sin\theta)\,t - \frac{1}{2}gt^2$$
3. Equation of trajectory :
$$y = x\tan\theta - \frac{g}{2u^2\cos^2\theta}\cdot x^2$$
4. Maximum height : $H = \dfrac{u^2\sin^2\theta}{2g}$
5. Time of flight, $T = \dfrac{2u\sin\theta}{g}$
6. Horizontal range, $R = \dfrac{u^2\sin 2\theta}{g}$
7. Maximum horizontal range is attained at $\theta = 45°$ and its value is $R_{max} = \dfrac{u^2}{g}$
8. Velocity after time t, $v_x = u\cos\theta$, $v_y = u\sin\theta - gt$
$$\therefore \quad v = \sqrt{v_x^2 + v_y^2} \quad \text{and} \quad \tan\beta = \frac{v_y}{v_x}$$

UNITS USED

Distances x, y, R and R_{max} are in metres, velocities u, v_x, v_y and v are in ms^{-1}, accelerations a_x, a_y and g are in ms^{-2} and times t and T in second.

EXAMPLE 72. *A cricket ball is thrown at a speed of 28 ms^{-1} in a direction 30° above the horizontal. Calculate (a) the maximum height; (b) the time taken by the ball to return to the same level, and (c) the horizontal distance from the thrower to the point where the ball returns to the same level.*

[NCERT ; Central Schools 08, 12]

Solution. Here $u = 28\ ms^{-1}$, $\theta = 30°$

(*a*) Maximum height,
$$H = \frac{u^2\sin^2\theta}{2g} = \frac{28^2\sin^2 30°}{2\times 9.8} = \mathbf{10.0\ m}.$$

(*b*) The time taken by the ball to return to the same level,
$$T = \frac{2u\sin\theta}{g} = \frac{2\times 28\times\sin 30°}{9.8} = \mathbf{2.9\ s}.$$

(*c*) The distance from the thrower to the point where the ball returns to the same level,
$$R = \frac{u^2\sin 2\theta}{g} = \frac{28\times 28\times\sin 60°}{9.8} = \mathbf{69.3\ m}.$$

Example 73. A body is projected with a velocity of 30 ms^{-1} at an angle of 30° with the vertical. Find the maximum height, time of flight and the horizontal range.

Solution. Here $u = 30\ \text{ms}^{-1}$,

Angle of projection, $\theta = 90° - 30° = 60°$

Maximum height,

$$H = \frac{u^2 \sin^2 \theta}{2g} = \frac{30^2 \sin^2 60°}{2 \times 9.8} = \mathbf{34.44\ m}$$

Time of flight,

$$T = \frac{2u \sin \theta}{g} = \frac{2 \times 30 \sin 60°}{9.8} = \mathbf{5.3\ s}$$

Horizontal range,

$$R = \frac{u^2 \sin 2\theta}{g} = \frac{30^2 \sin 120°}{9.8}$$

$$= \frac{30^2 \sin 60°}{9.8} = \mathbf{79.53\ m.}$$

Example 74. A projectile has a range of 50 m and reaches a maximum height of 10 m. Calculate the angle at which the projectile is fired.

Solution. Here $R = 50$ m, $H = 10$ m, $\theta = ?$

Horizontal range,

$$R = \frac{u^2 \sin 2\theta}{g} = \frac{2u^2 \sin \theta \cos \theta}{g} \quad ...(1)$$

Maximum height,

$$H = \frac{u^2 \sin^2 \theta}{2g} \quad ...(2)$$

Dividing (2) by (1), we get

$$\frac{H}{R} = \frac{u^2 \sin^2 \theta}{2g} \times \frac{g}{2u^2 \sin \theta \cos \theta} = \frac{1}{4} \tan \theta$$

or $\quad \tan \theta = \frac{4H}{R} = \frac{4 \times 10}{50} = 0.8$

or $\quad \theta = \tan^{-1}(0.8) = \mathbf{38.66°}$.

Example 75. A boy stands at 39.2 m from a building and throws a ball which just passes through a window 19.6 m above the ground. Calculate the velocity of projection of the ball.

Solution. Here $\quad H = 19.6$m

$\therefore \quad R = 39.2 + 39.2 = 78.4$ m

As proved in the above example,

$$\frac{H}{R} = \frac{1}{4} \tan \theta$$

or $\quad \tan \theta = \frac{4H}{R} = \frac{4 \times 19.6}{78.4} = 1$

$\therefore \quad \theta = \mathbf{45°}$

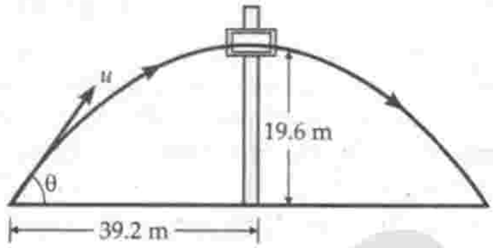


Fig. 4.76

As $\quad \frac{u^2 \sin 2\theta}{g} = R$

$\therefore \quad \frac{u^2 \sin 90°}{9.8} = 78.4$

or $\quad u = \sqrt{78.4 \times 9.8} = 19.6\sqrt{2} = \mathbf{27.72\ ms^{-1}}$.

Example 76. Find the angle of projection for which the horizontal range and the maximum height are equal.

Solution. Given

horizontal range = maximum height

or $\quad \frac{u^2 \sin 2\theta}{g} = \frac{u^2 \sin^2 \theta}{2g}$

or $\quad 2 \sin \theta \cos \theta = \frac{\sin^2 \theta}{2}$

or $\quad \frac{\sin \theta}{\cos \theta} = 4 \quad$ or $\quad \tan \theta = 4$

$\therefore \quad \theta = \mathbf{75°58'}$.

Example 77. Prove that the maximum horizontal range is four times the maximum height attained by the projectile, when fired at an inclination so as to have maximum horizontal range. **[Chandigarh 08]**

Solution. For $\theta = 45°$, the horizontal range is maximum and is given by

$$R_{max} = \frac{u^2}{g}$$

Maximum height attained,

$$H_{max} = \frac{u^2 \sin^2 45°}{2g} = \frac{u^2}{4g} = \frac{R_{max}}{4}$$

or $\quad R_{max} = 4H_{max}$.

Example 78. A ball is kicked at an angle of 30° with the vertical. If the horizontal component of its velocity is 19.6 ms^{-1}, find the maximum height and horizontal range.

Solution. Here $\theta = 90° - 30° = 60°$

Horizontal velocity $= u \cos 60° = 19.6\ \text{ms}^{-1}$

$\therefore \quad u = \frac{19.6}{\cos 60°} = \frac{19.6}{0.5} = 39.2\ \text{ms}^{-1}$

$\therefore$ Maximum height,

$$H = \frac{u^2 \sin^2 60°}{2g} = \frac{(39.2)^2}{2 \times 9.8} \times \left(\frac{\sqrt{3}}{2}\right)^2 = \mathbf{58.8\ m.}$$

Horizontal range,

$$R = \frac{u^2 \sin 2\theta}{g} = \frac{(39.2)^2 \times \sin 120°}{9.8}$$

$$= \frac{(39.2)^2}{9.8} \times \left(\frac{\sqrt{3}}{2}\right) = \mathbf{135.8\ m}$$

EXAMPLE 79. *Show that a given gun will shoot three times as high when elevated at an angle of 60° as when fired at angle of 30° but will carry the same distance on a horizontal plane.*

Solution. The vertical height attained by a projectile is given by

$$H = \frac{u^2 \sin^2 \theta}{2g}$$

When $\theta = 60°$,

$$H_1 = \frac{u^2 \sin^2 60°}{2g} = \frac{u^2}{2g}\left(\frac{\sqrt{3}}{2}\right)^2 = \frac{3u^2}{8g}$$

When $\theta = 30°$,

$$H_2 = \frac{u^2 \sin^2 30°}{2g} = \frac{u^2}{2g}\left(\frac{1}{2}\right)^2 = \frac{u^2}{8g}$$

$$\therefore \quad H_1 : H_2 = \frac{3u^2}{8g} \times \frac{8g}{u^2} = \mathbf{3:1}$$

Thus the gun will shoot three times as high when elevated at an angle of 60° as when fired at an angle of 30°.

Horizontal range of a projectile,

$$R = \frac{u^2 \sin 2\theta}{g}$$

When $\theta = 60°$, $\quad R_1 = \dfrac{u^2 \sin 120°}{g} = \dfrac{\sqrt{3}\, u^2}{2g}$

When $\theta = 30°$ $\quad R_2 = \dfrac{u^2 \sin 60°}{g} = \dfrac{\sqrt{3}\, u^2}{2g}$

Thus $R_1 = R_2$, *i.e.* the horizontal distance covered will be same in both cases.

EXAMPLE 80. *A ball is thrown at an angle θ and another ball is thrown at an angle (90° − θ) with the horizontal direction from the same point with velocity 39.2 ms⁻¹. The second ball reaches 50 m higher than the first ball. Find their individual heights. Take $g = 9.8\ ms^{-2}$.*

Solution. *For the first ball* : Angle of projection = θ, Velocity of projection, $u = 39.2\ \text{ms}^{-1}$

As maximum height,

$$H = \frac{u^2 \sin^2 \theta}{2g}$$

$$\therefore \quad H = \frac{(39.2)^2 \sin^2 \theta}{2 \times 9.8} \qquad ...(1)$$

For the second ball : Angle of projection $= 90° - \theta$, velocity of projection, $u = 39.2\ \text{ms}^{-1}$, maximum height reached $= (H + 50)$ m

$$\therefore \quad H + 50 = \frac{(39.2)^2 \sin^2 (90° - \theta)}{2 \times 9.8}$$

$$\text{or} \quad H + 50 = \frac{(39.2)^2 \cos^2 \theta}{2 \times 9.8} \qquad ...(2)$$

Adding (1) and (2), we get

$$2H + 50 = \frac{(39.2)^2}{2 \times 9.8}(\sin^2 \theta + \cos^2 \theta)$$

$$= \frac{(39.2)^2}{2 \times 9.8} = 78.4$$

or $\quad 2H = 78.4 - 50 = 28.4 \quad$ or $\quad H = 14.2$ m

$\therefore$ Height of first ball $= H = \mathbf{14.2\ m}$

Height of second ball

$$= H + 50 = 14.2 + 50 = \mathbf{64.2\ m.}$$

EXAMPLE 81. *If R is the horizontal range for θ inclination and h is the maximum height reached by the projectile, show that the maximum range is given by $\dfrac{R^2}{8h} + 2h$.*

Solution. Horizontal range, $\quad R = \dfrac{u^2 \sin 2\theta}{g}$

Maximum height, $\quad h = \dfrac{u^2 \sin^2 \theta}{2g}$

$$\therefore \quad \frac{R^2}{8h} + 2h = \frac{u^4 (\sin 2\theta)^2}{g^2} \times \frac{2g}{8u^2 \sin^2 \theta} + \frac{2u^2 \sin^2 \theta}{2g}$$

$$= \frac{u^2 (2 \sin \theta \cos \theta)^2}{4g \sin^2 \theta} + \frac{u^2 \sin^2 \theta}{g}$$

$$= \frac{u^2 \cos^2 \theta}{g} + \frac{u^2 \sin^2 \theta}{g}$$

$$= \frac{u^2}{g}(\cos^2 \theta + \sin^2 \theta) = \frac{u^2}{g}.$$

$$\text{or} \quad \frac{R^2}{8h} + 2h = R_{max}.$$

EXAMPLE 82. *Show that there are two angles of projection for which the horizontal range is the same. Also show that the sum of the maximum heights for these two angles is independent of the angle of projection.*

Solution. When a projectile is fired with velocity u at an angle θ with the horizontal, its horizontal range is

$$R = \frac{u^2 \sin 2\theta}{g}$$

Replacing θ by $(90° - \theta)$, we get

$$R' = \frac{u^2 \sin 2\,(90° - \theta)}{g} = \frac{u^2 \sin (180° - 2\theta)}{g} = \frac{u^2 \sin 2\theta}{g}$$

i.e., $R' = R$

Hence there are two angles of projection θ and $(90° - \theta)$ for which the horizontal range R is same.

Now $$H = \frac{u^2 \sin^2 \theta}{2g} \qquad ...(1)$$

and $$H' = \frac{u^2 \sin^2 (90° - \theta)}{2g} = \frac{u^2 \cos^2 \theta}{2g} \qquad ...(2)$$

Adding equations (1) and (2), we get

$$H + H' = \frac{u^2}{2g} (\sin^2 \theta + \cos^2 \theta)$$

or $$H + H' = \frac{u^2}{2g}$$

Clearly, the sum of the heights for the two angles of projection is independent of the angle of projection.

EXAMPLE 83. *Show that there are two values of time for which a projectile is at the same height. Also show that the sum of these two times is equal to the time of flight.*

Solution. For vertically upward motion of a projectile,

$$y = u \sin \theta \,.\, t - \tfrac{1}{2} g t^2$$

or $$\tfrac{1}{2} g t^2 - u \sin \theta \,.\, t + y = 0$$

This is a quadratic equation in t. Its roots are

$$t_1 = \frac{u \sin \theta - \sqrt{u^2 \sin^2 \theta - 2\,g\,y}}{g} \qquad \text{(Lower value)}$$

and $$t_2 = \frac{u \sin \theta + \sqrt{u^2 \sin^2 \theta - 2\,g\,y}}{g} \qquad \text{(Higher value)}$$

These are the two values of time for which the vertical height y is same, first while going up and second while going down.

Now $$t_1 + t_2 = \frac{u \sin \theta - \sqrt{u^2 \sin^2 \theta - 2\,g\,y}}{g} + \frac{u \sin \theta + \sqrt{u^2 \sin^2 \theta - 2\,g\,y}}{g}$$

or $$t_1 + t_2 = \frac{2u \sin \theta}{g} = T, \text{ the time of flight.}$$

EXAMPLE 84. *Two projectiles are thrown with different velocities and at different angles so as to cover the same maximum height. Show that the sum of the times taken by each to reach the highest point is equal to the total time taken by either of the projectiles.*

Solution. If the two projectiles are thrown with velocities u_1 and u_2 at angle θ_1 and θ_2 with horizontal, then their maximum heights will be

$$H_1 = \frac{u_1^2 \sin^2 \theta_1}{2g} \quad \text{and} \quad H_2 = \frac{u_2^2 \sin^2 \theta_2}{2g}$$

But $H_1 = H_2$

$$\therefore \quad \frac{u_1^2 \sin^2 \theta_1}{2g} = \frac{u_2^2 \sin^2 \theta_2}{2g}$$

or $$u_1 \sin \theta_1 = u_2 \sin \theta_2 \qquad ...(1)$$

Times of flight for the two projectiles are

$$T_1 = \frac{2u_1 \sin \theta_1}{g} \quad \text{and} \quad T_2 = \frac{2u_2 \sin \theta_2}{g}$$

Making use of equation (1), we get

$$T_1 = T_2 = \frac{2u_1 \sin \theta_1}{g} = \frac{2u_2 \sin \theta_2}{g}$$

Times taken to reach the highest point in the two cases will be

$$t_1 = \frac{u_1 \sin \theta_1}{g} \quad \text{and} \quad t_2 = \frac{u_2 \sin \theta_2}{g}$$

$$\therefore \quad t_1 + t_2 = \frac{u_1 \sin \theta_1}{g} + \frac{u_2 \sin \theta_2}{g}$$

$$= \frac{2u_1 \sin \theta_1}{g} \quad \text{or} \quad \frac{2u_2 \sin \theta_2}{g}$$

[using (1)]

or $t_1 + t_2$ = Time of flight of either projectile.

EXAMPLE 85. *A hunter aims his gun and fires a bullet directly at a monkey on a tree. At the instant the bullet leaves the barrel of the gun, the monkey drops. Will the bullet hit the monkey ? Substantiate your answer with proper reasoning.*

Solution. As shown in Fig. 4.77, the gun at O is directed towards the monkey at position M. Suppose the bullet leaves the barrel of the gun with velocity u at an angle θ with the horizontal. Let the bullet cross the vertical line MB at A after time t. Horizontal distance travelled,

$$OB = x = u \cos \theta \,.\, t$$

or $$t = \frac{x}{u \cos \theta} \qquad ...(1)$$

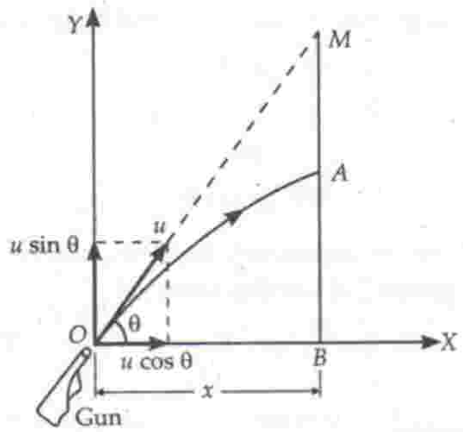


Fig. 4.77

For motion of the bullet from O to B, the vertical range is

$$AB = u\sin\theta \,.\, t - \tfrac{1}{2}gt^2 = u\sin\theta \,.\, \frac{x}{u\cos\theta} - \tfrac{1}{2}gt^2$$

$$= x\tan\theta - \tfrac{1}{2}gt^2 \qquad \text{[using (1)]}$$

Also $\quad MB = x\tan\theta$

$\therefore \quad MA = MB - AB$

$$= x\tan\theta - \left[x\tan\theta - \tfrac{1}{2}gt^2\right] = \frac{1}{2}gt^2$$

Thus in a time t the bullet passes through A a vertical distance $\frac{1}{2}gt^2$ below M.

The vertical distance through which the monkey falls in time t

$$= 0 + \tfrac{1}{2}gt^2 = \tfrac{1}{2}gt^2$$

Thus the bullet and the monkey will always reach the point A at the same time. Hence the bullet will always hit the monkey whatever be the velocity of the bullet.

Example 86. *A machine gun is mounted on the top of a tower 100 m high. At what angle should the gun be inclined to cover a maximum range of firing on the ground below? The muzzle speed of the bullet is 150 ms^{-1}, take $g = 10\ ms^{-2}$.*

Solution. Let u be the muzzle speed of the bullet fired from the gun (on the top of the tower) at an angle θ with the horizontal, as shown in Fig. 4.78.

Clearly, the total range of firing on the ground is

$$x = \frac{u^2 \sin 2\theta}{g} + 100\cot\theta$$

$$\therefore \quad \frac{dx}{d\theta} = \frac{u^2 \times 2\cos 2\theta}{g} + 100 \times (-\operatorname{cosec}^2\theta)$$

$$= \frac{2u^2}{g}(1 - 2\sin^2\theta) - \frac{100}{\sin^2\theta}$$

$$= 4500 - 9000\sin^2\theta - \frac{100}{\sin^2\theta}$$

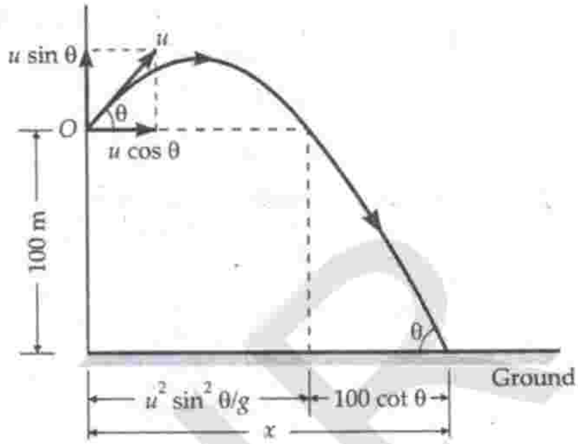


Fig. 4.78

For x to be maximum,

$$\frac{dx}{d\theta} = 0$$

or $\quad 4500 - 9000\sin^2\theta - \dfrac{100}{\sin^2\theta} = 0$

or $\quad 90\sin^4\theta - 45\sin^2\theta + 1 = 0$

or $\quad \sin^2\theta = \dfrac{45 \pm \sqrt{(-45)^2 - 4 \times 90 \times 1}}{2 \times 90}$

$$= \frac{45 \pm 40.80}{180}$$

Taking only positive sign,

$$\sin^2\theta = 0.4767$$

or $\quad \sin\theta = 0.6904$

or $\quad \theta = \mathbf{43.7°}$.

Example 87. *At what angle should a body be projected with a velocity 24 ms^{-1} just to pass over the obstacle 16 m high at a horizontal distance of 32 m? Take $g = 10\ ms^{-2}$.*

Solution. As shown in Fig. 4.79, if point of projection is taken as the origin of the coordinate system, the projected body must pass through a point having coordinates (32 m, 16 m). If u be the initial velocity of the projectile and θ the angle of projection, then

Horizontal component of initial velocity, $u_x = u\cos\theta$

Vertical component of initial velocity, $\quad u_y = u\sin\theta$

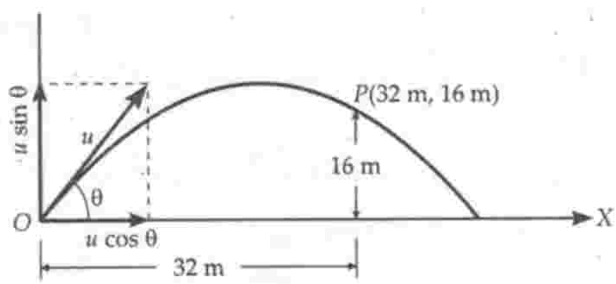


Fig. 4.79

If the body passes through point P after time t, then horizontal distance covered,

$$x = (u \cos\theta)\, t$$

or $$32 = (24 \cos\theta)\, t \qquad ...(1)$$

Similarly, vertical distance covered,

$$y = (u \sin\theta)\, t - \frac{1}{2} g t^2$$

or $$16 = (24 \sin\theta)\, t - \frac{1}{2} \times 10 \times t^2 \qquad ...(2)$$

From equation (1), $t = \dfrac{32}{24 \cos\theta}$

Putting this value of t in equation (2), we get

$$16 = (24 \sin\theta) \frac{32}{24 \cos\theta} - \frac{1}{2} \times 10 \times \left(\frac{32}{24 \cos\theta}\right)^2$$

or $$16 = 32 \tan\theta - 5 \times \frac{16}{9 \cos^2\theta}$$

or $$1 = 2 \tan\theta - \frac{5}{9} \sec^2\theta$$

or $$9 = 18 \tan\theta - 5(1 + \tan^2\theta)$$

or $$5 \tan^2\theta - 18 \tan\theta + 14 = 0$$

$$\therefore \quad \tan\theta = \frac{18 \pm \sqrt{(18)^2 - 4 \times 5 \times 14}}{10}$$

$$= 2.462 \text{ or } 1.137$$

Hence $\theta = \mathbf{67°54'}$ **or** $\mathbf{48°40'}$.

EXAMPLE 88. A target is fixed on the top of a pole 13 metre high. A person standing at a distance of 50 metre from the pole is capable of projecting a stone with a velocity $10\sqrt{g}$ ms^{-1}. If he wants to strike the target in shortest possible time, at what angle should he project the stone ?

Solution. The trajectory of a projectile is given by

$$y = x \tan\theta - \frac{g x^2}{2 u^2 \cos^2\theta}$$

As the target lies on the path of projectile, so

$$13 = 50 \tan\theta - \frac{g \times (50)^2}{2\,(10\sqrt{g})^2} \sec^2\theta$$

$$[\because u = 10\sqrt{g} \text{ ms}^{-1}]$$

or $$13 = 50 \tan\theta - \frac{25}{2} \sec^2\theta$$

$$= 50 \tan\theta - \frac{25}{2}(1 + \tan^2\theta)$$

or $$25 \tan^2\theta - 100 \tan\theta + 51 = 0$$

$$\therefore \quad \tan\theta = \frac{100 \pm \sqrt{4900}}{50} = \frac{17}{5} \text{ or } \frac{3}{5}$$

The horizontal distance covered in time t is given by

$$x = (u \cos\theta)\, t$$

or $$50 = (10\sqrt{g})(\cos\theta)\, t$$

or $$t = \frac{50}{10\sqrt{g} \cos\theta}$$

For t to be minimum, $\cos\theta$ should be maximum; $\cos\theta$ is greatest if $\tan\theta$ is least, *i.e.*

$$\tan\theta = \frac{3}{5} \quad \text{or} \quad \theta = \tan^{-1}\left(\frac{3}{5}\right) = \mathbf{30°58'}.$$

EXAMPLE 89. A particle is projected over a triangle from one end of a horizontal base and grazing the vertex falls on the other end of the base. If α and β be the base angles and θ the angle of projection, prove that $\tan\theta = \tan\alpha + \tan\beta$.

Solution. The situation is shown in Fig. 4.80.

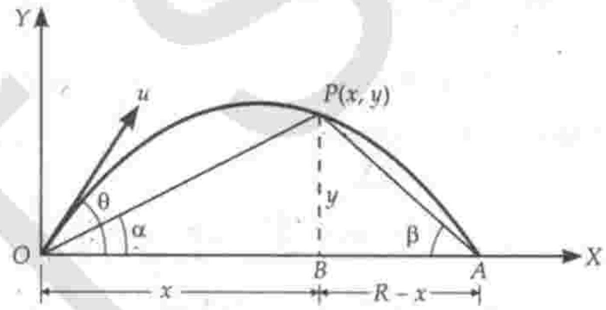


Fig. 4.80

If R is the range of the particle, then from the figure we have

$$\tan\alpha + \tan\beta = \frac{y}{x} + \frac{y}{R - x} = \frac{y(R - x) + xy}{x(R - x)}$$

or $$\tan\alpha + \tan\beta = \frac{y}{x} \times \frac{R}{(R - x)} \qquad ...(1)$$

Also, the trajectory of the particle is

$$y = x \tan\theta - \frac{1}{2} g \frac{x^2}{u^2 \cos^2\theta}$$

$$= x \tan\theta \left[1 - \frac{g x}{2 u^2 \cos^2\theta \tan\theta}\right]$$

$$= x \tan\theta \left[1 - \frac{g}{u^2 \sin 2\theta} x\right]$$

$$= x \tan\theta \left[1 - \frac{x}{R}\right]$$

or $$\tan\theta = \frac{y}{x} \times \frac{R}{(R - x)} \qquad ...(2)$$

From equations (1) and (2), we get

$$\tan\theta = \tan\alpha + \tan\beta.$$

EXAMPLE 90. A shell is fired from a gun from the bottom of a hill along its slope. The slope of the hill is $\alpha = 30°$, and the angle of the barrel to the horizontal $\beta = 60°$. The initial velocity u of the shell is 21 ms^{-1}. Find the distance from the gun to the point at which the shell falls.

Solution. The situation is shown in Fig. 4.81. The horizontal and vertical distances covered by the shell in time t are given by

$$x = u \cos \beta \,.\, t \qquad ...(1)$$

and
$$y = u \sin \beta \,.\, t - \frac{1}{2} g t^2 \qquad ...(2)$$

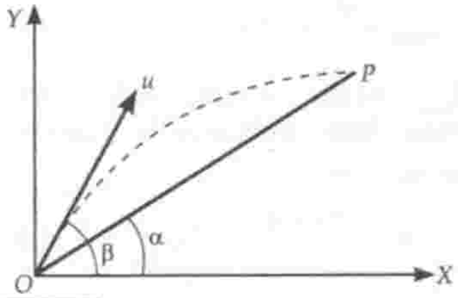


Fig. 4.81

If the shell falls on the inclined plane at distance $OP = s$, then

$$x = s \cos \alpha \qquad ...(3)$$

$$y = s \sin \alpha \qquad ...(4)$$

From (1) and (3),

$$s \cos \alpha = u \cos \beta \,.\, t \quad \text{or} \quad t = \frac{s \cos \alpha}{u \cos \beta}$$

From (2) and (4),

$$s \sin \alpha = u \sin \beta \,.\, t - \frac{1}{2} g t^2$$

$$= u \sin \beta \,.\, \frac{s \cos \alpha}{u \cos \beta} - \frac{1}{2} g \,.\, \frac{s^2 \cos^2 \alpha}{u^2 \cos^2 \beta}$$

$$\therefore \quad s = \frac{2u^2}{g} \left[\frac{\sin \beta \cos \alpha - \cos \beta \sin \alpha}{\cos^2 \alpha} \right] \cos \beta$$

$$= \frac{2u^2 \sin (\beta - \alpha) \cos \beta}{g \cos^2 \alpha}$$

$$= \frac{2 \times (21)^2 \sin (60° - 30°) \cos 60°}{9.8 \times \cos^2 30°} = \mathbf{30\ m.}$$

PROBLEMS FOR PRACTICE

1. A football player kicks a ball at an angle of 37° to the horizontal with an initial speed of 15 ms^{-1}. Assuming that the ball travels in a vertical plane, calculate (*i*) the time at which the ball reaches the highest point (*ii*) the maximum height reached (*iii*) the horizontal range of the projectile and (*iv*) the time for which the ball is in air.

   [**Ans.** (*i*) 0.92 s (*ii*) 4.16 m (*iii*) 21.2 m (*iv*) 1.84 s]

2. A body is projected with a velocity of 20 ms^{-1} in a direction making an angle of 60° with the horizontal. Calculate its (*i*) position after 0.5 s and (*ii*) velocity after 0.5 s. [**Ans.** (*i*) $x = 5$ m, $y = 7.43$ m

   (*ii*) 15.95 ms^{-1}, $\beta = 51.16°$]

3. The maximum vertical height of a projectile is 10 m. If the magnitude of the initial velocity is 28 ms^{-1}, what is the direction of the initial velocity ? Take $g = 9.8$ ms^{-2}. (**Ans.** 30°)

4. A bullet fired from a gun with a velocity of 140 ms^{-1} strikes the ground at the same level as the gun at a distance of 1 km. Find the angle of inclination with the horizontal at which the bullet is fired. Take $g = 9.8$ ms^{-2}. (**Ans.** 15°)

5. A bullet is fired at an angle of 15° with the horizontal and hits the ground 6 km away. Is it possible to hit a target 10 km away by adjusting the angle of projection assuming the initial speed to be the same ? (**Ans.** Yes)

6. A cricketer can throw a ball to maximum horizontal distance of 160 m. Calculate the maximum vertical height to which he can throw the ball. Given $g = 10$ ms^{-2}. (**Ans.** 80 m)

7. A football is kicked 20 ms^{-1} at a projection angle of 45°. A receiver on the goal line 25 metres away in the direction of the kick runs the same instant to meet the ball. What must be his speed, if he is to catch the ball before it hits the ground ?

   (**Ans.** 5.483 ms^{-1})

8. A bullet fired at an angle of 60° with the vertical hits the ground at a distance of 2 km. Calculate the distance at which the bullet will hit the ground when fired at an angle of 45°, assuming the speed to be the same. (**Ans.** 2.31 km)

9. A person observes a bird on a tree 39.6 m high and at a distance of 59.2 m. With what velocity the person should throw an arrow at an angle of 45° so that it may hit the bird ? (**Ans.** 41.86 ms^{-1})

10. A ball is thrown from the top of a tower with an initial velocity of 10 ms^{-1} at an angle of 30° with the horizontal. If it hits the ground at a distance of 17.3 m from the base of the tower, calculate the height of the tower. Given $g = 10$ ms^{-2}.(**Ans.** 10 m)

11. Prove that the time of flight T and the horizontal range R of a projectile are connected by the equation : $gT^2 = 2R \tan \theta$, where θ is the angle of projection.

12. Show that the range of a projectile for two angles of projection α and β is same, where $\alpha + \beta = 90°$.

13. A body is projected with velocity of 40 ms^{-1}. After 2 s it crosses a vertical pole of height 20.4 m. Calculate the angle of projection and horizontal range. (**Ans.** 30°, 141.39 m)

14. From the top of a tower 156.8 m high, a projectile is thrown up with velocity of 39.2 ms^{-1} making an angle of 30° with the horizontal direction. Find the distance from the foot of the tower where it strikes the ground and the time taken by it to do so.
(Ans. 271.6 m, 8 s)

15. As shown in Fig. 4.82, a body is projected with velocity u_1 from the point A. At the same time

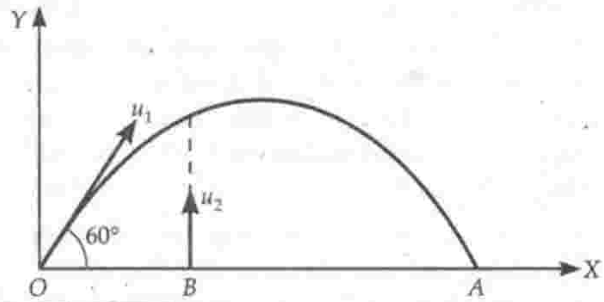


Fig. 4.82

another body is projected vertically upwards with the velocity u_2 from the point B. What should be the value of u_1 / u_2 for both the bodies to collide ?
(Ans. $2/\sqrt{3}$)

16. A body is projected such that its kinetic energy at the top is 3/4th of its initial kinetic energy. What is the initial angle of projection of the projectile with the horizontal ? **(Ans.** 30°)

17. A projectile is projected in the upward direction making an angle of 60° with the horizontal direction with a velocity of 147 ms^{-1}. After what time will its inclination with the horizontal be 45° ?
(Ans. 5.49 s)

Hints

1. (*i*) Time of ascent $= \dfrac{u \sin \theta}{g} = \dfrac{15 \times \sin 37°}{9.8} = \mathbf{0.92\ s}$

(*ii*) $H = \dfrac{u^2 \sin^2 \theta}{2g} = \dfrac{(15)^2 \sin^2 37°}{9.8} = \mathbf{4.16\ m}$

(*iii*) $R = \dfrac{u^2 \sin 2\theta}{g} = \dfrac{(15)^2 \sin 74°}{9.8} = \mathbf{21.2\ m}$

(*iv*) $T = \dfrac{2u \sin \theta}{g} = \dfrac{2 \times 15 \sin 37°}{9.8} = \mathbf{1.84\ s.}$

2. Here $u = 20\ ms^{-1}$, $\theta = 60°$, $t = 0.5\ s$

(*i*) $x = (u \cos \theta)\, t = (20 \cos 60°) \times 0.5 = \mathbf{5\ m}$

$y = (u \sin \theta)\, t - \frac{1}{2} g t^2$

$= (20 \times \sin 60°) \times 0.5 - \frac{1}{2} \times 9.8 \times (0.5)^2$

$= \mathbf{7.43\ m}$

(*ii*) $v_x = u \cos \theta = 20 \cos 60° = 10\ ms^{-1}$

$v_y = u \sin \theta - gt = 20 \sin 60° - 9.8 \times 0.5$

$= 12.42\ ms^{-1}$

$\therefore \quad v = \sqrt{v_x^2 + v_y^2} = \sqrt{(10)^2 + (12.42)^2} = \mathbf{15.95\ ms^{-1}}.$

$\tan \beta = \dfrac{v_y}{v_x} = \dfrac{12.42}{10} = 1.242$

$\therefore \quad \beta = \tan^{-1} 1.242 = \mathbf{51.16°}.$

5. As $\quad R = \dfrac{u^2 \sin 2\theta}{g} \quad \therefore \quad 6 = \dfrac{u^2 \sin 30°}{g}$

or $\quad \dfrac{u^2}{g} = \dfrac{6}{\sin 30°} = 12$

Also $R_{max} = \dfrac{u^2}{g} = 12$ km

As $R_{max} > 10$ km, the bullet can hit the target at a distance of 10 km by adjusting the angle of projection.

7. Horizontal range,

$$R = \frac{u^2 \sin 2\theta}{g} = \frac{(20)^2 \times \sin 90°}{9.8} = 40.82\ \text{m}$$

Distance required to be covered by the receiver to meet the football $= 40.82 - 25 = 15.82$ m. This distance must be covered during the time of flight of the ball (2.886 s).

$\therefore$ Velocity with which player should run to catch the ball

$$= \frac{15.82}{2.886} = \mathbf{5.483\ ms^{-1}}.$$

8. *In the first case :* $\theta = 90° - 60° = 30°$, $R = 2$ km

As $\quad R = \dfrac{u^2 \sin 2\theta}{g} \quad \therefore \; 2 = \dfrac{u^2 \sin 60°}{g} \qquad ...(1)$

In the second case : $\theta = 45°$

$\therefore \quad R' = \dfrac{u^2 \sin 90°}{g} \qquad ...(2)$

From (1) and (2),

$$\frac{R'}{2} = \frac{\sin 90°}{\sin 60°} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

$\therefore \quad R' = \dfrac{4 \times 1.732}{3} = \mathbf{2.31\ km.}$

9. For horizontal motion of the arrow :

$$u \cos 45° \,.\, t = 59.2 \text{ or } u = \frac{59.2\sqrt{2}}{t}$$

For vertical motion of the arrow :

$$39.6 = u \sin 45° \,.\, t - \frac{1}{2} \times 9.8 \times t^2$$

$\therefore \quad 39.6 = \dfrac{59.2\sqrt{2}}{t} \cdot \dfrac{1}{\sqrt{2}}\, t - \dfrac{1}{2} \times 9.8\, t^2$

or $\quad t^2 = \dfrac{59.2 - 39.6}{4.9} = \dfrac{19.6}{4.9} = 4$ or $t = 2$ s

Hence $\quad u = \dfrac{59.2\sqrt{2}}{2} = 29.6\sqrt{2} = \mathbf{41.86\ ms^{-1}}.$

10. Here $\theta = 30°$, $u = 10\ ms^{-1}$, $R = 17.3$ m, $g = 10\ ms^{-2}$

For horizontal motion,

$$R = u\cos\theta \,.\, t$$

or $$t = \frac{R}{u\cos\theta} = \frac{17.3}{10\cos 30°} = \frac{17.3 \times 2}{10 \times \sqrt{3}}$$

$$= \frac{17.3 \times 2}{10 \times 1.73} = 2\text{ s}$$

For vertical motion,

$$y = u\sin\theta \,.\, t - \tfrac{1}{2} gt^2$$

$$= 10\sin 30° \times 2 - \tfrac{1}{2} \times 10 \times 2^2$$

$$= 10 - 20 = -10\text{ m}$$

Height of tower = **10 m.**

11. As $R = \dfrac{u^2 \sin 2\theta}{g}$ and $T = \dfrac{2u\sin\theta}{g}$

$\therefore$ $$gT^2 = \frac{g \times 4u^2 \sin^2\theta}{g^2}$$

$$= \frac{2u^2 \times 2\sin\theta\cos\theta}{g} \times \frac{\sin\theta}{\cos\theta}$$

$$= \frac{2u^2 \sin 2\theta}{g} \times \tan\theta = 2R\tan\theta.$$

13. Here $u = 40\ ms^{-1}$, $t = 2$ s, $y = 20.4$ m

As $y = u\sin\theta \times t - \frac{1}{2} gt^2$

$\therefore$ $20.4 = 40 \times \sin\theta \times 2 - \frac{1}{2} 9.8 \times 2^2$

or $20.4 + 19.6 = 80\sin\theta$ or $40 = 80\sin\theta.$

$\therefore$ $\sin\theta = \dfrac{1}{2}$ and $\theta = 30°$

$$R = \frac{u^2 \sin 2\theta}{g} = \frac{(40)^2 \sin 60°}{9.8} = \mathbf{141.39\ m.}$$

14. As $y = u\sin\theta \times t - \frac{1}{2} gt^2$

$\therefore$ $-156.8 = 39.2 \times \frac{1}{2} \times t - 4.9t^2$

or $t^2 - 4t - 32 = 0$

$\therefore$ $t = 8, -4$

But $t = -4$ is meaningless, so $t =$ **8 s**

Hence $R = u\cos\theta \times t = 39.2\cos 30° \times 8$

$= \mathbf{271.6\ m.}$

15. For two bodies to collide in air, $y_1 = y_2$

$$u_1 \sin 60° \times t - \tfrac{1}{2} gt^2 = u_2 t - \tfrac{1}{2} gt^2$$

$$u_1 \sin 60° \times t = u_2 t$$

or $$\frac{u_1}{u_2} = \frac{1}{\sin 60°} = \frac{2}{\sqrt{3}}.$$

16. Initial K.E. $= \dfrac{1}{2} mu^2$

K.E. at the top $= \dfrac{1}{2} mu^2 \cos^2\theta$

$\therefore$ $$\frac{\frac{1}{2} mu^2 \cos^2\theta}{\frac{1}{2} mu^2} = \frac{3}{4} \quad \text{or} \quad \cos^2\theta = \frac{3}{4}$$

or $$\cos\theta = \frac{\sqrt{3}}{2} \quad \text{or} \quad \theta = \mathbf{30°}.$$

17. Suppose the body is projected with velocity u at an angle of 60° with the horizontal and after time t, its velocity becomes v making an angle of 45° with the horizontal. Equating the horizontal and vertical components of u and v,

$$u\cos 60° = v\cos 45° \quad \text{or} \quad v = \frac{u\cos 60°}{\cos 45°}$$

Also, $$v\sin 45° = u\sin 60° - gt$$

or $$\frac{u\cos 60°}{\cos 45°} \times \sin 45° = u\sin 60° - gt$$

or $$t = \frac{u}{g}[\sin 60° - \cos 60°] = \frac{147}{9.8}\left[\frac{\sqrt{3}}{2} - \frac{1}{2}\right]$$

$= \mathbf{5.49\ s.}$

4.25 UNIFORM CIRCULAR MOTION

55. *Define uniform circular motion. Give some examples.*

Uniform circular motion. *If a particle moves along a circular path with a constant speed (i.e., it covers equal distances along the circumference of the circle in equal intervals of time), then its motion is said to be a uniform circular motion.*

Examples :

(*i*) Motion of the tip of the second hand of a clock.

(*ii*) Motion of a point on the rim of a wheel rotating uniformly.

56. *Justify that a uniform circular motion is an accelerated motion.*

Uniform circular motion is an accelerated motion. In uniform circular motion, the speed of the body remains the same but the direction of motion changes at every point. Fig. 4.83 shows the different velocity vectors at different positions of the particle. At each position, the velocity vector $\vec{v}$ is perpendicular to the radius vector $\vec{r}$. Thus the velocity of the body changes continuously due to the continuous change in the direction of motion of the body. As the rate of change of velocity is acceleration, so a uniform circular motion is an accelerated motion.

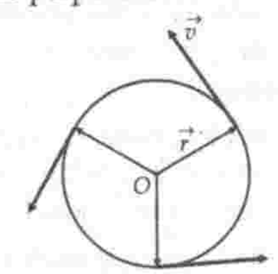


Fig. 4.83 Direction of velocity changes at every point.

57. *Define the terms : (i) angular displacement (ii) angular velocity (iii) time period and (iv) frequency, in connection with a circular motion.*

(*i*) **Angular displacement.** *The angular displacement of a particle moving along a circular path is defined as the angle swept out by its radius vector in the given time interval.*

As shown in Fig. 4.84, suppose a particles starts from the position P_0. Its angular position is θ_1 at time t_1 and θ_2 at time t_2. Let the particle cover a distance Δs in the time interval $t_2 - t_1 (= \Delta t)$. It revolves through angle $\theta_2 - \theta_1 (= \Delta\theta)$ in this interval. The angle of revolution $\Delta\theta$ is the angular displacement of the particle. If r is the radius of the circle, then

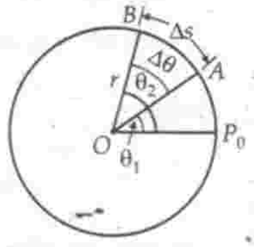


Fig. 4.84 Angular displacement.

$$\Delta\theta = \frac{\Delta s}{r} \qquad \left[\because \text{Angle} = \frac{\text{Arc}}{\text{Radius}}\right]$$

The unit of angular displacement is radian. It is a dimensionless quantity.

(*ii*) **Angular velocity.** *The time rate of change of angular displacement of a particle is called its angular velocity.* It is denoted by ω. It is measured in **radian per second** (rad s^{-1}) and its dimensional formula is $[M^0L^0T^{-1}]$.

As shown in Fig. 4.84, if $\Delta\theta$ is the angular displacement of a particle is time Δt, then its average angular velocity is

$$\bar{\omega} = \frac{\Delta\theta}{\Delta t}$$

When the time interval $\Delta t \to 0$, the limiting value of the average velocity is called the instantaneous angular velocity, which is given by

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$

(*iii*) **Time period.** *The time taken by a particle to complete one revolution along its circular path is called its period of revolution.* It is denoted by T and is measured in **second.**

(*iv*) **Frequency.** *The frequency of an object in circular motion is defined as the number of revolutions completed per unit time.* It is denoted by ν (nu).

If ν is the frequency of revolution of a particle, then time taken to complete ν revolutions = 1 second, time taken to complete 1 revolution $= \frac{1}{\nu}$ second

But time taken to complete 1 revolution is the time period T, so

$$T = \frac{1}{\nu} \quad \text{or} \quad \nu = \frac{1}{T}$$

58. *Deduce relations between angular velocity, frequency and time period.*

Relations between ω, ν and T. By definition of time period, a particle completes one revolution in time T *i.e.*, it traverses an angle of 2π radian in time T.

$\therefore$ When time $t = T$

angular displacement $\theta = 2\pi$ radian

$$\text{Angular velocity} = \frac{\text{Angular displacement}}{\text{Time}}$$

or
$$\omega = \frac{\theta}{t} = \frac{2\pi}{T} = 2\pi\nu \qquad \left[\because \frac{1}{T} = \nu\right]$$

59. *Derive the relation between linear velocity and angular velocity.*

Relation between linear velocity and angular velocity. Consider a particle moving along a circular path of radius r. As shown in Fig. 4.85, suppose the particle moves from A to B in time Δt covering distance Δs along the arc AB. Hence the angular displacement of the particle is

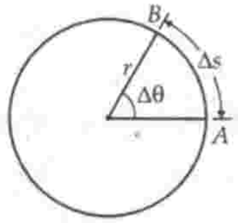


Fig. 4.85 Velocity in uniform circular motion

$$\Delta\theta = \frac{\Delta s}{r}$$

Dividing both sides by Δt, we get

$$\frac{\Delta\theta}{\Delta t} = \frac{1}{r}\frac{\Delta s}{\Delta t}$$

Taking the limit $\Delta t \to 0$ on both sides,

$$\lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{1}{r} \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t}$$

But
$$\lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt} = \omega,$$

is the instantaneous angular velocity

and
$$\lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt} = v,$$

is the instantaneous linear velocity.

$$\therefore \qquad \omega = \frac{1}{r} \cdot v$$

or
$$v = \omega r$$

Linear velocity = Angular velocity × radius

In vector notation, we have the relation

$$\vec{v} = \vec{\omega} \times \vec{r}$$

For a given angular velocity (ω), *the linear velocity v of a particle is directly proportional to its distance from the centre.*

60. *Define angular acceleration. Deduce its relation with linear acceleration.*

Angular acceleration. *The time rate of change of angular velocity of a particle is called its angular acceleration.* If $\Delta\omega$ is the change in angular velocity in time Δt, then the *average angular acceleration* is

$$\bar{\alpha} = \frac{\Delta\omega}{\Delta t}$$

The *instantaneous acceleration* is equal to the limiting value of the average acceleration $\Delta\omega/\Delta t$ when Δt approaches zero. It is given by

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt}$$

The angular acceleration is measured in *radian per second*^{2} (*rad* s^{-2}) and has the dimensions $[M^\circ L^\circ T^{-2}]$.

The relation between linear velocity v and angular velocity ω is $v = r\omega$

Differentiating both sides w.r.t. time t, we get

$$\frac{dv}{dt} = \frac{d}{dt}(r\omega)$$

or $$\frac{dv}{dt} = r\left(\frac{d\omega}{dt}\right) \qquad [\because r \text{ is constant}]$$

or $$a = \alpha r$$

Linear acceleration = Angular acceleration × radius

In vector rotation, we have the relation

$$\vec{a} = \vec{\alpha} \times \vec{r}$$

61. *Define centripetal acceleration. Derive an expression for the centripetal acceleration of a particle moving with uniform speed v along a circular path of radius r. Discuss the direction of this acceleration.*

Centripetal acceleration. When a body is in uniform circular motion, its speed remains constant but its velocity changes continuously due to the change in its direction. Hence the motion is accelerated. *A body undergoing uniform circular motion is acted upon by an acceleration which is directed along the radius towards the centre of the circular path. This acceleration is called centripetal (centre seeking) acceleration.*

Expression for centripetal acceleration. Consider a particle moving on a circular path of radius r and centre O, with a uniform speed v. As shown in Fig. 4.86(a), suppose at time t the particle is at P and at time $t + \Delta t$, the particle is at Q. Let $\vec{v}_1$ and $\vec{v}_2$ be the velocity vectors at P and Q, directed along the tangents at P and Q respectively.

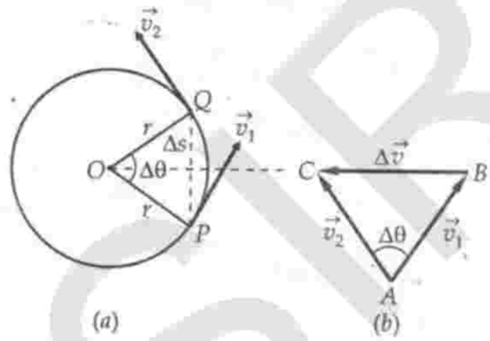


Fig. 4.86 Determination of centripetal acceleration.

To determine the change in velocity, take an external point A. Draw $\vec{AB}$ equal to and parallel to $\vec{v}_1$ and $\vec{AC}$ equal to and parallel to $\vec{v}_2$. Draw the vector $\vec{BC}$ to close the triangle, as shown in Fig. 4.86(b).

Applying triangle law of vector addition in $\Delta\, BAC$,

$$\vec{AB} + \vec{BC} = \vec{AC}$$

$$\therefore \qquad \vec{BC} = \vec{AC} - \vec{AB} = \vec{v}_2 - \vec{v}_1$$

Thus the change in velocity in time Δt is given by

$$\vec{BC} = \Delta \vec{v}$$

If Δt is small, the chord PQ becomes equal to arc PQ. Then OPQ can be considered as a triangle. $\angle POQ = \angle BAC = \Delta\theta$. This is because the angle between the radii PO and QO is same as the angle between the tangents at P and Q.

Also $OP = OQ = r$, radius of the circle.

$$|\vec{v}_1| = |\vec{v}_2| = v \qquad i.e., \quad AB = AC = v$$

And $\angle POQ = \Delta\theta$, $\angle BAC = \Delta\theta$

Thus the two triangles POQ and BAC are similar. Hence

$$\frac{PQ}{OP} = \frac{BC}{AB}$$

or $$\frac{\Delta s}{r} = \frac{\Delta v}{v}$$

or $$\Delta v = \frac{v}{r}\Delta s$$

Dividing both sides by Δt, we get

$$\frac{\Delta v}{\Delta t} = \frac{v}{r}\frac{\Delta s}{\Delta t}$$

Taking the limit $\Delta t \to 0$ on both sides, we get

$$\lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{v}{r} \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t}$$

But $$\lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt} = a,$$

is the instantaneous acceleration

and $$\lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt} = v,$$

is the instantaneous velocity

$$\therefore \qquad a = \frac{v}{r} \cdot v$$

or $$a = \frac{v^2}{r} = \omega^2 r \qquad [\because\ v = \omega r]$$

This gives the magnitude of the acceleration of a particle in uniform circular motion.

Direction of acceleration. As Δt tends to zero, the angle $\Delta\theta$ also approaches zero. In this limit, as $AB = AC$, so $\angle ABC = \angle ACB = 90°$. Thus the change in velocity $\Delta \vec{v}$ and hence the acceleration $\vec{a}$ is perpendicular to the velocity vector $\vec{v}_1$. But $\vec{v}_1$ is directed along tangent at point P, so acceleration $\vec{a}$ acts along the radius towards the centre of the circle. Such an acceleration is called *centripetal acceleration*. Its magnitude remains constant $(= v^2/r)$ but its direction continuously changes and remains perpendicular to the velocity vector at all positions.

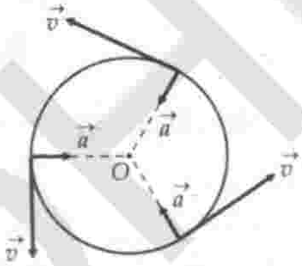


Fig. 4.87 Centripetal acceleration.

61. *Write an expression for the resultant acceleration of a particle having non-uniform circular motion.*

Circular motion with variable speed. Consider a particle moving along a circular path of radius r with a variable speed v.

As the speed of the particle changes, so acceleration has a tangential component,

$$a_T = \frac{dv}{dt} = r\alpha$$

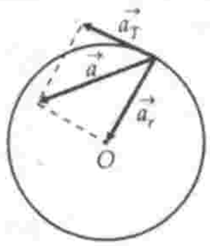


Fig. 4.88 Resultant acceleration in circular motion.

As the direction of motion changes continuously, so the acceleration has a radial component,

$$a_r \quad \text{or} \quad a_c = \frac{v^2}{r}$$

The resultant acceleration of the particle will be

$$a = \sqrt{a_T^2 + a_r^2}\,.$$

For Your Knowledge

- It was the Dutch scientist *Christiaan Huygens* who first gave a thorough analysis of centripetal acceleration in 1673.
- In uniform circular motion, the direction of velocity vector which acts along the tangent to the path, changes continuously but its magnitude always remains constant ($v = r\omega$). So circular motion is an accelerated motion.
- As v and r are constant, so the magnitude of the centripetal acceleration is a constant $(= v^2/r)$. But the direction of a_c changes continuously, pointing always towards the centre. So centripetal acceleration is not a constant vector.
- The resultant acceleration of an object in circular motion is towards the centre only if the speed is constant.
- For a body moving with a constant angular velocity, the angular acceleration is zero.
- In projectile motion, both the magnitude and direction of acceleration (g) remain constant, while in uniform circular motion the magnitude remains constant but the direction continuously changes. Hence the equations of motion $v = u + at$, etc., are not applicable to circular motion. These equations hold only when both the magnitude and direction of acceleration are constant.

Examples based on Uniform Circular Motion

FORMULAE USED

1. Angular displacement, $\theta = \dfrac{s}{r}$
2. Angular velocity, $\omega = \dfrac{\theta}{t}$
3. Also, $\omega = \dfrac{2\pi}{T} = 2\pi\nu$
4. Linear velocity, $v = r\,\omega$
5. Centripetal acceleration, $a = \dfrac{v^2}{r} = r\,\omega^2$
6. Linear acceleration, $a = r\,\alpha$

UNITS USED

Here θ is in radian, ω in rad s^{-1}, v in ms^{-1}, a in ms^{-2} and angular acceleration α in rad s^{-2}.

EXAMPLE 91. *Which is greater : the angular velocity of hour hand of a watch or angular velocity of earth around its own axis ? Give their ratio.* **[Delhi 12]**

Solution. The hour hand completes one rotation in 12 hours. Its angular speed is

$$\therefore\ \omega_1 = \frac{\theta}{t} = \frac{2\pi \text{ rad}}{12 \text{ h}} = \frac{2\pi \text{ rad}}{12 \times 60 \times 60 \text{ s}} = \frac{\pi}{\mathbf{21600}} \textbf{ rad s}^{-1}.$$

The earth completes one rotation about its axis in 24 hours. Its angular speed is

$$\therefore\ \omega_2 = \frac{\theta}{t} = \frac{2\pi \text{ rad}}{24 \text{ h}} = \frac{2\pi \text{ rad}}{24 \times 60 \times 60} = \frac{\pi}{\mathbf{43200}} \textbf{ rad s}^{-1}.$$

Clearly, $\omega_1 > \omega_2$ and $\omega_1 : \omega_2 = \mathbf{2 : 1}$

EXAMPLE 92. *Calculate the angular speed of flywheel making 420 revolutions per minute.*

Solution. Here $\nu = 420$ revolutions/min

$$= \frac{420}{60} \text{ revolutions/s}$$

$$\omega = 2\pi\nu = 2 \times \frac{22}{7} \times \frac{420}{60} = \mathbf{44 \text{ rad s}^{-1}}.$$

EXAMPLE 93. *A body of mass 10 kg revolves in a circle of diameter 0.40 m making 1000 revolutions per minute. Calculate its linear velocity and centripetal acceleration.*

Solution. Here $m = 10$ kg, $r = 0.20$ m, $\nu = \frac{1000}{60}\text{ s}^{-1}$

Angular speed,

$$\omega = 2\pi\nu = 2\pi \times \frac{1000}{60} = \frac{100\pi}{3} \text{ rad s}^{-1}$$

Linear velocity,

$$v = r\omega = 0.20 \times \frac{100\pi}{3} = \frac{\mathbf{20\pi}}{\mathbf{3}} \textbf{ ms}^{-1}.$$

Centripetal acceleration,

$$a = r\omega^2 = 0.20 \times \left(\frac{100\pi}{3}\right)^2 = \frac{\mathbf{2000\pi^2}}{\mathbf{9}} \textbf{ ms}^{-2}.$$

Example 94. *Find the magnitude of the centripetal acceleration of a particle on the tip of a fan blade, 0.30 metre in diameter, rotating at 1200 rev/minute.* **[BIT Ranchi 97]**

Solution. Here $\nu = \frac{1200}{60} = 20$ rps

$$\omega = 2\pi\nu = 2\pi \times 20 = 40\pi \text{ rad s}^{-1}$$

$$a_c = r\omega^2 = 0.15 \times (40\pi)^2 = \mathbf{2368.8 \text{ ms}^{-2}}.$$

EXAMPLE 95. *An insect trapped in a circular groove of radius 12 cm moves along the groove steadily and completes 7 revolutions in 100 s. (i) What is the angular speed and the linear speed of the motion ? (ii) Is the acceleration vector a constant vector ? What is the magnitude ? (iii) What is its linear displacement ?* **[NCERT, Central Schools 03]**

Solution. Here $r = 12$ cm, $\nu = \frac{7}{100}\text{ s}^{-1}$

(*i*) Angular speed,

$$\omega = 2\pi\nu = 2 \times \frac{22}{7} \times \frac{7}{100} = \mathbf{0.44 \text{ rads}^{-1}}.$$

Linear speed,

$$v = r\omega = 12 \times 0.44 = \mathbf{5.28 \text{ cms}^{-1}}.$$

(*ii*) Acceleration is always directed towards the centre of the circular groove. As the insect moves, the direction of the acceleration vector changes. So acceleration vector is not a constant vector.

Magnitude of acceleration,

$$a = r\omega^2 = 12 \times (0.44)^2 = \mathbf{2.32 \text{ cms}^{-2}}.$$

(*iii*) After completing 7 revolutions, the insect comes back to its initial position. So its linear displacement is zero.

EXAMPLE 96. *The radius of the earth's orbit around the sun is 1.5×10^{11} m. Calculate the angular and linear velocity of the earth. Through how much angle does the earth revolve in 2 days ?*

Solution. Here $r = 1.5 \times 10^{11}$ m,

Period of revolution of the earth,

$$T = 365 \text{ days} = 365 \times 24 \times 60 \times 60 \text{ s}$$

$\therefore$ Angular velocity,

$$\omega = \frac{2\pi}{T} = \frac{2 \times 3.14}{365 \times 24 \times 60 \times 60}$$

$$= \mathbf{1.99 \times 10^{-7} \text{ rad s}^{-1}}$$

Linear velocity,

$$v = r\omega = 1.5 \times 10^{11} \times 1.99 \times 10^{-7} = \mathbf{2.99 \times 10^4 \text{ ms}^{-1}}$$

In 365 days, the earth revolves through an angle of 2π radians.

$\therefore$ Angle through which the earth revolves in 2 days

$$= \frac{2\pi}{365} \times 2 = \frac{2 \times 3.14 \times 2}{365} = \mathbf{0.0344 \text{ rad.}}$$

EXAMPLE 97. *A particle moves in a circle of radius 4.0 cm clockwise at constant speed of 2 cms^{-1}. If $\hat{x}$ and $\hat{y}$ are unit acceleration vectors along X-axis and Y-axis respectively (in cms^{-2}), find the acceleration of the particle at the instant half way between P and Q. Refer to Fig. 4.89(a).*

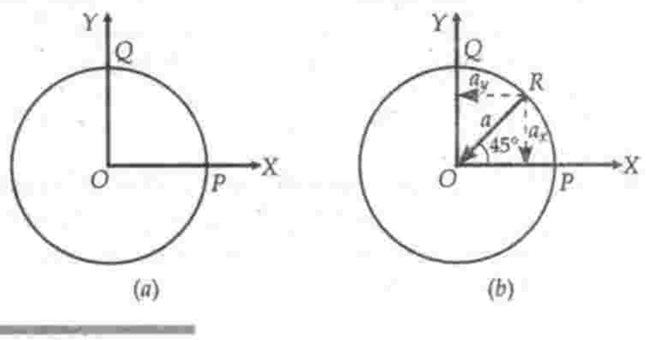


Fig. 4.89

Solution. As shown in Fig. 4.89(*b*), let R be the midpoint of arc PQ. Then $\angle POR = 45°$.

Magnitude of acceleration at R,

$$a = \frac{v^2}{r} = \frac{(2)^2}{4} = 1 \text{ cms}^{-2}$$

The acceleration a acts along RO.

Magnitude of component of a along X-axis,

$$a_x = a \cos 45° = 1 \times \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \text{ cms}^{-2}.$$

$$\therefore \quad \hat{a}_x = -\frac{1}{\sqrt{2}} \hat{x}$$

Magnitude of component of a along Y-axis,

$$a_y = 1 \times \sin 45° = \frac{1}{\sqrt{2}} \text{ cms}^{-2}.$$

$$\therefore \quad \hat{a}_y = -\frac{1}{\sqrt{2}} \hat{y}$$

Hence $\quad \hat{a} = \hat{a}_x + \hat{a}_y = -\frac{1}{\sqrt{2}} (\hat{x} + \hat{y}).$

PROBLEMS FOR PRACTICE

1. What is the angular velocity of a second hand and minute hand of a clock ? **[Himachal 06C]**
(**Ans.** 0.1047 rad s^{-1}, 0.0017 rad s^{-1})

2. A body of mass 0.4 kg is whirled in a horizontal circle of radius 2 m with a constant speed of 10 ms^{-1}. Calculate its (*i*) angular speed (*ii*) frequency of revolution (*iii*) time period and (*iv*) centripetal acceleration. **[Ans.** (*i*) 5 rad s^{-1} (*ii*) 0.8 Hz (*iii*) 1.25 s (*iv*) 50 ms^{-2}]

3. A circular wheel of 0.50 m radius is moving with a speed of 10 ms^{-1}. Find the angular speed.
(**Ans.** 20 rad s^{-1})

4. Assuming that the moon completes one revolution in a circular orbit around the earth in 27.3 days, calculate the acceleration of the moon towards the earth. The radius of the circular orbit can be taken as 3.85×10^5 km. (**Ans.** 2.73×10^{-3} ms^{-2})

5. The angular velocity of a particle moving along a circle of radius 50 cm is increased in 5 minutes from 100 revolutions per minute to 400 revolutions per minute. Find (*i*) angular acceleration and (*ii*) linear acceleration. $\left[\textbf{Ans. } (i) \frac{\pi}{30} \text{ rad s}^{-2} \ (ii) \frac{5\pi}{3} \text{ cms}^{-2}\right]$

6. Calculate the linear acceleration of a particle moving in a circle of radius 0.4 m at the instant when its angular velocity is 2 rad s^{-1} and its angular acceleration is 5 rad s^{-2}.
(**Ans.** 2.6 ms^{-2}, 38°40′ with a_T)

7. A threaded rod with 12 turns per cm and diameter 1.18 cm is mounted horizontally. A bar with a threaded hole to match the rod is screwed onto the rod. The bar spins at the rate of 216 rpm. How long will it take for the bar to move 1.50 cm along the rod ? (**Ans.** 5 s)

HINTS

1. The second hand of a clock completes one rotation in 60 s.

$$t = 60 \text{ s} \quad \text{and} \quad \theta = 2\pi \text{ rad}$$

Hence $\omega = \frac{\theta}{t} = \frac{2\pi}{60} = \frac{2 \times 3.14}{60} = \mathbf{0.1047 \text{ rad s}^{-1}}.$

(*ii*) The minute hand of a clock completes one rotation in 60 minutes.

$\therefore \quad t = 60 \text{ min} = 3600 \text{ s} \quad \text{and} \quad \theta = 2\pi \text{ rad}$

Hence $\omega = \frac{\theta}{t} = \frac{2\pi}{3600} = \frac{2 \times 3.14}{3600} = \mathbf{0.0017 \text{ rad s}^{-1}}.$

2. Here $m = 0.4$ kg, $r = 2$ m, $v = 10$ ms^{-1}

(*i*) Angular speed, $\omega = \frac{v}{r} = \frac{10}{2} = \mathbf{5 \text{ rad s}^{-1}}.$

(*ii*) Frequency, $\nu = \frac{\omega}{2\pi} = \frac{5 \times 7}{2 \times 22} = 0.795 \text{ Hz} \simeq \mathbf{0.8 \text{ Hz}}.$

(*iii*) Time period, $T = \frac{1}{\nu} = \frac{1}{0.8} = \mathbf{1.25 \text{ s}}.$

(*iv*) Centripetal acceleration,

$$a = r\omega^2 = 2 \times (5)^2 = \mathbf{50 \text{ ms}^{-2}}.$$

3. Here $r = 0.50$ m, $v = 10$ ms^{-1}

As $v = r\omega$

$$\therefore \quad \omega = \frac{v}{r} = \frac{10 \text{ ms}^{-1}}{0.50 \text{ m}} = \mathbf{20.0 \text{ rad s}^{-1}}.$$

4. $a = r\omega^2 = r\left(\frac{2\pi}{T}\right)^2$

$$= 3.85 \times 10^8 \times \left[\frac{2 \times 3.14}{27.3 \times 24 \times 60 \times 60}\right]^2$$

$$= \mathbf{2.73 \times 10^{-3} \text{ ms}^{-2}}.$$

5. Here $\nu_1 = \frac{100}{60}$ rps, $\nu_2 = \frac{400}{60}$ rps, $r = 50$ cm,

$$t = 5 \text{ min} = 300 \text{ s}$$

(*i*) Angular acceleration,

$$\alpha = \frac{\omega_2 - \omega_1}{t} = \frac{2\pi\nu_2 - 2\pi\nu_1}{2} = 2\pi\left(\frac{\nu_2 - \nu_1}{t}\right)$$

$$= 2\pi\left(\frac{400 - 100}{60 \times 300}\right) = \mathbf{\frac{\pi}{30} \text{ rad s}^{-1}}.$$

(*ii*) Linear acceleration,

$$a = r\,\alpha = 50 \times \frac{\pi}{30} = \mathbf{\frac{5\pi}{3} \text{ cms}^{-2}}.$$

6. Here $r = 0.4$ m, $\omega = 2$ rad s^{-1}, $\alpha = 5$ rad s^{-2}

Tangential acceleration,

$$a_T = r\alpha = 0.4 \times 5 = 2 \text{ ms}^{-2}$$

Centrepetal acceleration,

$$a_C = r\omega^2 = 0.4 \times (2)^2 = 1.6 \text{ ms}^{-2}$$

$\therefore$ Total linear acceleration,

$$a = \sqrt{a_T^2 + a_C^2} = \sqrt{2^2 + (1.6)^2} = \mathbf{2.6\ ms^{-2}}.$$

If $\vec{a}$ makes angle θ with a_T, then

$$\tan\theta = \frac{a_C}{a_T} = \frac{1.6}{2} = 0.8$$

$$\therefore \quad \theta = \mathbf{38°40'}.$$

7. Pitch of threaded screw $= \frac{1}{12}$ cm

Number of rotations required to move a distance of 1.5 cm,

$$n = \frac{\text{Distance}}{\text{Pitch}} = \frac{1.5}{1/12} = 18$$

$$\therefore \quad \theta = 2\pi n = 2\pi \times 18 = 36\,\pi \text{ rad}$$

Angular speed of the bar,

$$\omega = 2\pi\nu = 2\pi \times \frac{216}{60} = 7.2\,\pi \text{ rad s}^{-1}$$

$\therefore$ Required time,

$$t = \frac{\theta}{\omega} = \frac{36\,\pi}{7.2\,\pi} = \mathbf{5\ s}.$$

Very Short Answer Coceptual Problems

Problem 1. Why cannot be vectors added algebraically ?

Solution. Apart from magnitude, the vectors also have directions, so they cannot be added algebraically.

Problem 2. State the essential condition for the addition of vectors.

Solution. The essential condition for the addition of vectors is that they must represent the physical quantities of same nature.

Problem 3. Is time a vector quantity ? Give reason.

Solution. No, time flows from past to present and present to future. Thus the direction of time flow is unique and does not need to be specified. Hence time is not a vector, though it has a direction.

Problem 4. Is pressure a vector ? Give reason.

Solution. No. Pressure is always taken to be normal to the plane of the area on which it is acting. As this direction is unique, it does need any specification. So pressure is not a vector.

Problem 5. Can two vectors of different magnitudes be combined to give zero resultant ?

Solution. No, two vectors of different magnitudes cannot be combined to give zero resultant.

Problem 6. When is the magnitude of the resultant of two vectors equal to either of them ?

Solution. When two vectors of equal magnitude are inclined to each other at an angle of 120°, the magnitude of their resultant is equal to that of the either vector.

$$a^2 = a^2 + a^2 + 2a^2\cos^2\theta$$

$$\Rightarrow \quad \cos\theta = -\frac{1}{2} \quad \Rightarrow \quad \theta = 120°$$

Problem 7. Are the magnitude and direction of $(\vec{A} - \vec{B})$ same as that of $(\vec{B} - \vec{A})$? *[Delhi 10]*

Solution. The vectors $(\vec{A} - \vec{B})$ and $(\vec{B} - \vec{A})$ have equal magnitude but opposite directions.

Problem 8. Can $\vec{A} + \vec{B} = \vec{A} - \vec{B}$?

Solution. Yes. The equality holds when $\vec{B}$ is a null vector.

Problem 9. When is the magnitude of $(\vec{A} + \vec{B})$ equal to the magnitude of $(\vec{A} - \vec{B})$?

Solution. When the vectors $\vec{A}$ and $\vec{B}$ are perpendicular to each other, $|\vec{A} + \vec{B}| = |\vec{A} - \vec{B}|$.

Problem 10. Under what condition will the directions of sum and difference of two vectors be same ?

Solution. When the two vectors are unequal in magnitude and are in the same direction.

Problem 11. If $\vec{A} + \vec{B} = \vec{B} + \vec{A}$, what can you say about the angle between $\vec{A}$ and $\vec{B}$?

Solution. $\vec{A}$ and $\vec{B}$ can have any angle between them because commutative law holds good for any two coplanar vectors.

Problem 12. Can we add a velocity vector to a displacement vector ?

Solution. No, only vectors representing physical quantities of same nature can be added together.

Problem 13. What is the minimum number of coplanar vectors of different magnitudes which can give zero resultant ?

Solution. Three. If three vectors can be represented by the three sides of a triangle taken in the same order, then their resultant is a zero vector.

Problem 14. If $\vec{a} + \vec{b} = \vec{c}$ and $|\vec{a}| + |\vec{b}| = |\vec{c}|$, what can we say about the direction of these vectors ?

Solution. The three vectors have the same direction.

Problem 15. What is the resultant of vector $\vec{A}$ multiplied by real number m ?

Solution. The resultant vector $m\vec{A}$ has magnitude m times that of $\vec{A}$. It has same direction as that of $\vec{A}$ if m is positive. It has direction opposite to that of $\vec{A}$ if m is negative.

Problem 16. Can a vector be multiplied by both dimensional and non-dimensional scalars ?

Solution. Yes. When a vector is multiplied by a dimensional scalar, the resultant has different dimensions. When the vector is multiplied by a non-dimensional scalar, its dimensions remain unchanged.

Problem 17. What is the maximum number of components into which a vector can be resolved ?

Solution. Infinite.

Problem 18. Can the magnitude of the rectangular component of a vector be greater than the magnitude of that vector ?

Solution. No. For example, the rectangular components of vector $\vec{A}$ are $A_x = A\cos\theta$ and $A_y = A\sin\theta$. As both $\sin\theta$ and $\cos\theta$ can take values between -1 and $+1$, so the magnitudes of both A_x and A_y cannot be greater than A.

Problem 19. Can a vector be zero when one of the components is not zero while all the other components are zero ?

Solution. No. Any vector $\vec{A}$ in three dimensions can be written as

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$

where $A = \sqrt{A_x^2 + A_y^2 + A_z^2}$.

Clearly, if any of the components A_x, A_y or A_z is not zero, the vector $\vec{A}$ will not be a zero vector.

Problem 20. Can the increment Δa in the magnitude of vector $\vec{a}$ be greater than the modulus of the increment of the vector, that is, $|\Delta\vec{a}|$? Can they be equal ?

Solution. No, the increment Δa cannot be greater than the increment $|\Delta\vec{a}|$, as shown in Fig. 4.90. The two will be equal if $\vec{a}$ and $\Delta\vec{a}$ have the same direction.

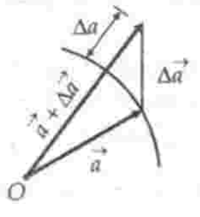


Fig. 4.90

Problem 21. The direction of vector $\vec{a}$ is reversed. Find $\Delta\vec{a}$, $|\Delta\vec{a}|$ and Δa.

Solution. $\Delta\vec{a} = -\vec{a} - \vec{a} = -2\vec{a}$

$|\Delta\vec{a}| = 2a$

Problem 22. If $\vec{a}$ and $\Delta\vec{a}$ are directed opposite to each other, what is the relation between Δa and $|\Delta\vec{a}|$?

Solution. $\Delta a = -|\Delta\vec{a}|$.

Problem 23. A vector $\vec{a}$ is turned through a small angle $d\theta$ without a change in its length. What are $|\Delta\vec{a}|$ and Δa ?

Solution. $|\Delta\vec{a}| = a\,d\theta$ and $\Delta a = 0$.

Problem 24. Give two conditions necessary for a given quantity to be a vector.

Solution. (*i*) The quantity must have both magnitude and direction.

(*ii*) It must obey the laws of vector addition.

Problem 25. Is finite rotation a vector ?

Solution. No. This is because the addition of two finite rotations about different axes does not obey commutative law.

Problem 26. Can the scalar product of two vectors be negative ?

Solution. Yes, the scalar product is negative when the angle between two vectors lies between 90° and 270°.

Problem 27. What is the dot product of two perpendicular vectors $\vec{A}$ and $\vec{B}$? *[Himachal 01, 02, 03, 04]*

Solution. Zero, as $\vec{A}\cdot\vec{B} = AB\cos 90° = 0$.

Problem 28. What is the dot product of two similar unit vectors ?

Solution. Unity. For example, $\hat{i}\cdot\hat{i} = (1)(1)\cos 0° = 1$.

Problem 29. What is the dot product of two dissimilar unit vectors ?

Or

Calculate the value of $\hat{i}\cdot\hat{j}$. *[Himachal 02]*

Solution. Zero.

For example, $\hat{i}\cdot\hat{j} = (1)(1)\cos 90° = 0$.

Problem 30. If $\vec{A}\cdot\vec{B} = \vec{A}\cdot\vec{C}$, is it correct to conclude that $\vec{B} = \vec{C}$?

Solution. No. We can write $\vec{A}\cdot(\vec{B} - \vec{C}) = 0$ which implies that $\vec{A}$ may be perpendicular to $(\vec{B} - \vec{C})$.

Problem 31. If $\vec{A}$, $\vec{B}$ and $\vec{C}$ are non-zero vectors and $\vec{A} \cdot \vec{B} = 0$ and $\vec{B} \cdot \vec{C} = 0$, then find out the value of $\vec{A} \cdot \vec{C}$.

Solution. $\vec{A} \cdot \vec{B} = 0 \quad \Rightarrow \quad \vec{A} \perp \vec{B}$

$$\vec{B} \cdot \vec{C} = 0 \quad \Rightarrow \quad \vec{B} \perp \vec{C}$$

$\therefore \quad \vec{A} \parallel \vec{C}$ and $\vec{A} \cdot \vec{C} = AC \cos 0° = AC.$

Problem 32. What is the magnitude and direction of $\hat{i} + \hat{j}$?

Solution. $|\hat{i} + \hat{j}| = \sqrt{1^2 + 1^2} = \sqrt{2}$

If the vector $\hat{i} + \hat{j}$ makes angle β with x-axis, then

$$\tan \beta = \frac{\text{coeff. of } \hat{i}}{\text{coeff. of } \hat{j}} = \frac{1}{1} = 1$$

$\therefore \qquad \beta = 45°.$

Problem 33. What is the angle made by vector $\vec{A} = 2\hat{i} + 2\hat{j}$ with x-axis ?

Solution. The angle θ between $\vec{A}$ and x-axis is given by

$$\cos \theta = \frac{\vec{A} \cdot \hat{i}}{|\vec{A}||\hat{i}|} = \frac{(2\hat{i} + 2\hat{j}) \cdot \hat{i}}{|2\hat{i} + 2\hat{j}||\hat{i}|} = \frac{2 \times 1 + 2 \times 0}{\sqrt{2^2 + 2^2}\sqrt{1^2}} = \frac{2}{2\sqrt{2}}$$

or $\qquad \cos \theta = \dfrac{1}{\sqrt{2}}$

$\therefore \qquad \theta = 45°.$

Problem 34. What should be the angle θ between two vectors $\vec{A}$ and $\vec{B}$ for their resultant $\vec{R}$ to be maximum ?

Solution. $R = \sqrt{A^2 + B^2 + 2AB \cos \theta}$

R will be maximum when $\cos \theta = +1$ or $\theta = 0°$.

$$R_{max} = A + B.$$

Problem 35. What should be the angle θ between two vectors $\vec{A}$ and $\vec{B}$ for their resultant $\vec{R}$ to be minimum ?

Solution. $R = \sqrt{A^2 + B^2 + 2AB \cos \theta}$

R will be minimum when $\cos \theta = -1$ or $\theta = 180°$.

$$R_{\min} = A \sim B.$$

Problem 36. What is the effect on the magnitude of the resultant of two vectors when the angle θ between them is increased from 0° to 180° ?

Solution. $R = \sqrt{A^2 + B^2 + 2AB \cos \theta}$

As the angle θ increases from 0° to 180°, the value of cos θ decreases, so the magnitude R of the resultant also decreases.

Problem 37. Two persons are pulling the ends of a string in such a way that the string is stretched horizontally. When a weight of 10 kg is suspended in the middle of the string, the string does not remain horizontal. Can the persons make it horizontal again by putting it with a greater force ?

Solution. No, the vertical weight cannot be balanced by the horizontal force, however large the two forces may be.

Problem 38. What is the vector sum of n coplanar forces, each of magnitude F, if each force makes an angle of $2\pi/n$ with the preceding force ?

Solution. Total angle between n coplanar forces $= (2\pi/n) \times n = 2\pi$. This shows that the n forces can be represented by the n sides of a closed polygon taken in the same order. So the resultant force is **zero**.

Problem 39. Two vectors $\vec{A}$ and $\vec{B}$ are in the same plane and angle between them is θ. What is the magnitude and direction of $\vec{A} \times \vec{B}$?

Solution. $|\vec{A} \times \vec{B}| = AB \sin \theta$

The direction of $\vec{A} \times \vec{B}$ is perpendicular to the plane of $\vec{A}$ and $\vec{B}$ and points in the same direction in which a right handed screw would advance when rotated from $\vec{A}$ to $\vec{B}$.

Problem 40. If $\vec{A}$ and $\vec{B}$ are two length vectors, then what is the geometrical significance of $|\vec{A} \times \vec{B}|$?

Solution. $|\vec{A} \times \vec{B}| = AB \sin \theta$, this gives area of the parallelogram with adjacent sides $\vec{A}$ and $\vec{B}$.

Problem 41. What is the unit vector perpendicular to the plane of vectors $\vec{A}$ and $\vec{B}$?

Solution. $\hat{n} = \dfrac{\vec{A} \times \vec{B}}{|\vec{A} \times \vec{B}|} = \dfrac{\vec{A} \times \vec{B}}{AB \sin \theta}$.

Problem 42. What is the value of $\vec{A} \times \vec{A}$?

[Himachal 02]

Solution. Zero, because $\vec{A} \times \vec{A} = AA \sin 0° = 0.$

Problem 43. What is the condition for two vectors to be collinear ?

Solution. For two given vectors to be collinear, their cross-product must be zero.

Problem 44. If $\vec{A} \times \vec{B} = \vec{C} \times \vec{B}$, show that $\vec{C}$ need not be equal to $\vec{A}$.

Solution. As $\quad \vec{A} \times \vec{B} = \vec{C} \times \vec{B}$

$\therefore \qquad \vec{A} \times \vec{B} - \vec{C} \times \vec{B} = 0$

or $\qquad (\vec{A} - \vec{C}) \times \vec{B} = 0$

This implies that if $\vec{C} \neq \vec{A}$, then either $\vec{B} = \vec{0}$

or $\quad (\vec{A} - \vec{C}) \parallel \vec{B}$.

Problem 45. Find the value of $\hat{i} \times \hat{j}$.

Solution. $\hat{i} \times \hat{j} = (1)(1) \sin 90° \hat{k} = \hat{k}$.

Problem 46. What is $\hat{i} . (\hat{j} \times \hat{k})$?

Solution. $\hat{i} . (\hat{j} \times \hat{k}) = \hat{i} . \hat{i} = (1)(1) \cos 0° = 1$.

Problem 47. Under what condition will the equality : $|\vec{A} \times \vec{B}| = \vec{A} . \vec{B}$ hold good ?

Solution. As $|\vec{A} \times \vec{B}| = \vec{A} . \vec{B}$

$\therefore \quad AB \sin \theta = AB \cos \theta$ or $\tan \theta = 1$

$\Rightarrow \quad \theta = 45°$.

Problem 48. What is the angle between $(\vec{A} + \vec{B})$ and $(\vec{A} \times \vec{B})$?

Solution. The resultant vector $(\vec{A} + \vec{B})$ lies in the plane of $\vec{A}$ and $\vec{B}$, while the cross product $(\vec{A} \times \vec{B})$ is perpendicular to this plane. So the angle between $(\vec{A} + \vec{B})$ and $(\vec{A} \times \vec{B})$ is 90°.

Problem 49. Can there be motion in two dimensions with an acceleration only in one dimension ?

Solution. Yes, in a projectile motion the acceleration acts vertically downwards while the projectile follows a parabolic path.

Problem 50. Can the direction of velocity of a body change when the acceleration is constant ?

Solution. Yes, in projectile motion the direction of velocity changes from point to point, but its acceleration is always constant and acts vertically downwards.

Problem 51. Is the rocket in flight an example of projectile ?

Solution. No, because it is propelled by combustion of fuel and does not move under the effect of gravity alone.

Problem 52. A stone is thrown vertically upwards and then it returns to the thrower. Is it a projectile ?

Solution. It is not a projectile, because a projectile should have two component velocities in two mutually perpendicular directions, but in this case the body has velocity only in one direction while going up or coming downwards.

Problem 53. Why does a projectile fired along the horizontal not follow a straight line path ?

Solution. Because the projectile fired horizontally is constantly acted upon by acceleration due to gravity acting vertically downwards.

Problem 54. A body projected horizontally moves with the same horizontal velocity although it is under the action of force of gravity. Why ?

Solution. The force of gravity acts in the vertically downward direction and has no effect on the horizontal component of velocity ; and this makes the body to move with constant horizontal velocity.

Problem 55. What is the angle between the direction of velocity and acceleration at the highest point of a projectile path ?

Solution. 90°. At the highest point the vertical component of velocity becomes zero, the projectile has only a horizontal velocity while the acceleration due to gravity acts vertically downwards.

Problem 56. A bullet is dropped from a certain height and at the same time, another bullet is fired horizontally from the same height. Which one will hit the ground earlier and why ?

Solution. Since the heights of both bullets from the ground is the same, so the time taken by both of them to reach the ground will be the same.

Problem 57. A stone dropped from the window of a stationary bus takes 5 seconds to reach the ground. In what time the stone will reach the ground when the bus is moving with (*a*) constant velocity of 80 kmh^{-1} (*b*) constant acceleration of 2 kmh^{-2} ?

Solution. (*a*) 5 seconds (*b*) 5 seconds. In both cases, initial vertical velocity is zero, downward acceleration is equal to '*g*' and also vertical distance covered is same.

Problem 58. A bomb thrown as projectile explodes in mid-air. What is the path traced by the centre of mass of the fragments assuming the friction to be negligible ?

Solution. The path traced by the centre of mass of the fragments is a parabola.

Problem 59. At what point in its trajectory does a projectile have its (*i*) minimum speed (*ii*) maximum speed ?

Solution. For a projectile given angular projection, the horizontal component of velocity remains constant throughout while the vertical component first decreases, becomes zero at the highest point and then increases again. Hence

(*i*) Projectile has *minimum speed at the highest point* of its trajectory.

(*ii*) Projectile has *maximum speed at the point of projection* and at the point where it returns to the horizontal plane of projection.

Problem 60. What will be the effect on horizontal range of a projectile when its initial velocity is doubled, keeping the angle of projection same ? *[Delhi 02]*

Solution. As $R = \dfrac{u^2 \sin 2\theta}{g}$ *i.e.*, $R \propto u^2$

When u is doubled, the horizontal range becomes four times the original horizontal range.

Problem 61. A projectile is fired at an angle of 15° to the horizontal with the speed v. If another projectile is projected with the same speed, then at what angle with the horizontal it must be projected so as to have the same range ?

Solution. For same R and v, the sum of the two angles of projection is 90°. As one angle is of 15°, other should be $90° - 15° = \mathbf{75°}$.

Problem 62. Is the maximum height attained by projectile is largest when its horizontal range is maximum ?

Solution. No. Horizontal range is maximum when $\theta = 45°$ and maximum height attained by projectile is largest when $\theta = 90°$.

Problem 63. What will be the effect on maximum height of a projectile when its angle of projection is changed from 30° to 60°, keeping the same initial velocity of projection ?

Solution. Maximum height, $H = \dfrac{u^2 \sin^2 \theta}{2g}$

$$\therefore \quad \frac{H_2}{H_1} = \frac{\sin^2 60°}{\sin^2 30°} = \frac{(\sqrt{3}/2)^2}{(1/2)^2} = 3$$

Thus the maximum height becomes three times the original maximum height.

Problem 64. A body is projected with speed u at an angle θ to the horizontal to have maximum range. What is the velocity at the highest point ?

Solution. For maximum horizontal range, $\theta = 45°$

Velocity at highest point = Horizontal component of velocity $= u \cos 45° = \mathbf{u / \sqrt{2}}$.

Problem 65. What is the angle of projection for a projectile motion whose range R is n times the maximum height H ?

Solution. Here $\quad R = nH$

$$\text{or} \quad \frac{u^2 \sin 2\theta}{g} = n \cdot \frac{u^2 \sin^2 \theta}{2g}$$

$$\text{or} \quad \frac{u^2 \times 2 \sin \theta \cos \theta}{g} = n \cdot \frac{u^2 \sin^2 \theta}{2g} \quad \text{or} \quad \tan \theta = \frac{4}{n}$$

$$\Rightarrow \quad \theta = \tan^{-1}\left(\frac{4}{n}\right).$$

Problem 66. The greatest height to which a man can throw a stone is h. What will be the greatest distance upto which he can throw the stone ?

Solution. Maximum height attained, $H = \dfrac{u^2 \sin^2 \theta}{2g}$

Clearly, H will be maximum when $\sin \theta = 1$

$$\therefore \quad H_{max} = \frac{u^2}{2g} = h$$

Maximum horizontal range, $R_{max} = \dfrac{u^2}{g} = \mathbf{2h}$.

Problem 67. A particle is projected at an angle θ from the horizontal with kinetic energy K. What is the kinetic energy of the particle at the highest point ?

Solution. If the particle is projected with velocity v, then its initial kinetic energy is

$$K = \frac{1}{2} mv^2$$

Velocity of the projectile at the highest point

$$= v \cos \theta$$

$\therefore$ K.E. of the particle at the highest point

$$= \frac{1}{2} m (v \cos \theta)^2 = (\tfrac{1}{2} mv^2) \cos^2 \theta = \mathbf{K \cos^2 \theta}.$$

Problem 68. A man can jump on the moon six times as high as on the earth. Why ?

Solution. Maximum height attained on earth,

$$H = \frac{u^2 \sin^2 \theta}{2g}$$

Acceleration due to gravity on moon, $g' = g/6$

$\therefore$ Maximum height attained on moon,

$$H' = \frac{u^2 \sin^2 \theta}{2g'} = \frac{6u^2 \sin^2 \theta}{2g} = 6H$$

So one can jump on moon six times as high as on earth.

Problem 69. A projectile of mass m is fired with velocity v at an angle θ with the horizontal. What is the change in momentum as it rises to the highest point of the trajectory ?

Solution. Horizontal component of velocity remains constant. The vertical component is $v \sin \theta$ at the point of projection and zero at the highest point.

Magnitude of change in momentum

$= m(v \sin \theta - 0) = mv \sin \theta$ (in the vertical direction)

Problem 70. A projectile of mass m is thrown with velocity v from the ground at an angle of 45° with the horizontal. What is the magnitude of change in momentum between leaving and arriving back at the ground ?

Solution. The vertical component of velocity at the point of projection is $v \sin \theta$ and at the point of return is $-v \sin \theta$.

$\therefore$ Magnitude of change in momentum

$$= m(v \sin \theta + v \sin \theta) = 2\, mv \sin \theta$$
$$= 2\, mv \sin 45° = \sqrt{2}\, mv$$

(in the vertical direction).

Problem 71. A block slides down a smooth inclined plane when released from the top while another falls freely from the same point. Which one of them will strike the ground earlier ?

Solution. Acceleration of the freely falling block is g while that of the block sliding down the smooth inclined plane is $g \sin \theta$. As $g > g \sin \theta$, so the block falling freely will reach the ground earlier.

Problem 72. Why does a tennis ball bounce higher on hills than in plains ?

Solution. Maximum height attained by a projectile

$$\propto \frac{1}{g}$$

As the value of g is less on hills than on plains, so a tennis ball bounces higher on hills than on plains.

Problem 73. Two projectiles A and B are projected with velocities $\sqrt{2}v$ and v respectively. They have the same range. If A is thrown at angle of 15° with the horizontal, then what is the angle of projection of B ?

Solution. Given $\quad R_A = R_B$

or $$\frac{(\sqrt{2}v)^2 \sin(2\times 15°)}{g} = \frac{v^2 \sin 2\theta}{g}$$

or $$\sin 2\theta = 1 = \sin 90°$$

or $$\theta = 45°.$$

Problem 74. A stone is thrown horizontally with a speed $\sqrt{2gh}$ from the top of a wall of height h. It strikes the level ground through the foot of the wall at a distance x from the wall. What is the value of x ?

Solution. As $\quad h = \frac{1}{2}gt^2$

$$\therefore \quad t = \sqrt{\frac{2h}{g}}$$

$$\text{Speed} = \frac{\text{Distance}}{\text{Time}}$$

$$\sqrt{2gh} = \frac{x}{\sqrt{2h/g}}$$

or $$x = \sqrt{2gh} \times \sqrt{\frac{2h}{g}} = 2h.$$

Problem 75. The velocity of a particle is constant in magnitude but not in direction. What is the nature of trajectory ?

Solution. The particle moves on a curved path. The instantaneous direction will be tangential to the path at every point.

Problem 76. What is the angle between velocity vector and acceleration vector in uniform circular motion ?

Solution. 90°.

Problem 77. For uniform circular motion, does the direction of centripetal acceleration depend upon the sense of rotation ?

Solution. The direction of centripetal acceleration does not depend on the clockwise or anticlockwise sense of rotation of the body. It always acts along the radius towards the centre of the circle.

Problem 78. If both the speed and radius of the circular path of a body are doubled, how will the centripetal acceleration change ?

Solution. Original centripetal acceleration, $a = \frac{v^2}{r}$

New centripetal acceleration, $a' = \frac{(2v)^2}{2r} = \frac{2v^2}{r} = 2a.$

Problem 79. A stone tied to the end of a string is whirled in a circle. If the string breaks, the stone flies away tangentially. Why ?

Solution. The instantaneous velocity of the stone going around the circular path is always along tangent to the circle. When the string breaks, the centripetal force ceases. Due to inertia, the stone continues its motion along tangent to the circular path.

Problem 80. When a knife is sharpened with the help of a rotating grinding stone, the spark always travels tangentially to it. Why ?

Solution. When a knife is sharpened, the red hot particles of the grinding stone get separated as sparks. As these particles are in circular motion, hence their velocities are directed tangentially to the grinding stone.

Short Answer Conceptual Problems

Problem 1. Which of the following quantities are independent of the choice of orientation of the coordinate axes :

$$\vec{a} + \vec{b},\ 3a_x + 2b_y,\ |\vec{a} + \vec{b} - \vec{c}|,$$

angle between $\vec{b}$ and $\vec{c}$, $\lambda\vec{a}$, where λ is a scalar ?

Solution. A vector, its magnitude and the angle between two vectors do not depend on the choice of the orientation of the coordinate axes, so $\vec{a} + \vec{b}$, $|\vec{a} + \vec{b} - \vec{c}|$, angle between $\vec{b}$ and $\vec{c}$ and $\lambda\vec{a}$ are independent of the orientation of the coordinate axes.

But the quantity $3a_x + 2b_y$ depends upon the magnitude of the components along x- and y-axes, so it will change with the change in coordinate axes.

Problem 2. Do $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ lie in the same plane. Give reason.

Solution. Yes, because $\vec{a} + \vec{b}$ is represented by the diagonal of the parallelogram drawn with $\vec{a}$ and $\vec{b}$ as adjacent sides. The diagonal passes through the common tail of $\vec{a}$ and $\vec{b}$. However, $\vec{a} - \vec{b}$ is represented by the other diagonal of the same parallelogram not passing through the common tail of $\vec{a}$ and $\vec{b}$. Thus both $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ lie in plane of the same parallelogram.

Problem 3. Can we apply the commutative and associative laws to vector subtraction also ?

Solution. (*i*) No, we cannot apply commutative law to vector subtraction because $\vec{a} - \vec{b} \neq \vec{b} - \vec{a}$

(*ii*) Yes, association law can be applied to vector subtraction because $(\vec{a} + \vec{b}) - \vec{c} = \vec{a} + (\vec{b} - \vec{c})$.

Problem 4. Can three vectors not in one plane give a zero resultant ? Can four vectors do so ?

Solution. No, three vectors not in one plane cannot give a zero resultant because the resultant of two vectors will not lie in the plane of the third vector and hence cannot cancel its effect.

The resultant of four vectors not in one plane may be a zero vector.

Problem 5. What is the difference between the following two data :

(*i*) 8 (5 km/hr, east) (*ii*) (8 hr) (5 km/hr, east) ?

Solution. (*i*) It is the product of a pure number and a vector (velocity), hence the unit of product is the same as that of vector *i.e.*, the product is a velocity of 40 km/hr, towards east.

(*ii*) It is the product of a scalar (time) and a vector (velocity). Hence the unit of the product will be hr × (km/hr). Thus the product is a displacement of magnitude 40 km, towards east.

Problem 6. Is $|\vec{a} + \vec{b}|$ greater than or less than $|\vec{a}| + |\vec{b}|$? Give reason.

Solution. $|\vec{a} + \vec{b}|^2 - (|\vec{a}| + |\vec{b}|)^2$

$= |\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}||\vec{b}|\cos\theta - |\vec{a}|^2 - |\vec{b}|^2 - 2|\vec{a}||\vec{b}|$

$= -2|\vec{a}||\vec{b}|(1 - \cos\theta)$

$= -2|\vec{a}||\vec{b}|\sin^2\dfrac{\theta}{2}$ = a negative quantity

Hence $|\vec{a} + \vec{b}| \leq |\vec{a}| + |\vec{b}|$.

Problem 7. Is $|\vec{a} - \vec{b}|$ greater than or less than $|\vec{a}| + |\vec{b}|$?

Solution. $|\vec{a} - \vec{b}|^2 - (|\vec{a}| + |\vec{b}|)^2$

$= |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|\cos\theta - |\vec{a}|^2 - |\vec{b}|^2 - 2|\vec{a}||\vec{b}|$

$= -4|\vec{a}||\vec{b}|\cos^2\dfrac{\theta}{2}$ = a negative quantity

Hence $|\vec{a} - \vec{b}| \leq |\vec{a}| + |\vec{b}|$.

Problem 8. Under what condition does the equality : $|\vec{A} + \vec{B}| = |\vec{A} - \vec{B}|$ hold good ?

Solution. $(\vec{A} + \vec{B}) = (\vec{A} - \vec{B})$

$\therefore \quad (\vec{A} + \vec{B})^2 = (\vec{A} - \vec{B})^2$

or $\quad A^2 + B^2 + 2\vec{A}.\vec{B} = A^2 + B^2 - 2\vec{A}.\vec{B}$

or $\quad 4\vec{A}.\vec{B} = 0 \quad$ or $\quad \vec{A}.\vec{B} = 0$

This implies that $\vec{A}$ and $\vec{B}$ are perpendicular to each other.

Problem 9. The sum and difference of two vectors are perpendicular to each other. Prove that the vectors are equal in magnitude.

Solution. As the vectors $\vec{A} + \vec{B}$ and $\vec{A} - \vec{B}$ are perpendicular to each other, therefore

$$(\vec{A} + \vec{B}).(\vec{A} - \vec{B}) = 0$$

$$\vec{A}.\vec{A} - \vec{A}.\vec{B} + \vec{B}.\vec{A} - \vec{B}.\vec{B} = 0$$

or $\quad A^2 - B^2 = 0 \Rightarrow A = B \qquad [\because \vec{A}.\vec{B} = \vec{B}.\vec{A}]$

Problem 10. Find the conditions for two vectors to be (*i*) parallel and (*ii*) perpendicular to each other. *[Punjab 91]*

Solution. (*i*) If $\vec{A} \parallel \vec{B}$, then $\theta = 0°$

and $\quad \vec{A} \times \vec{B} = |\vec{A}||\vec{B}|\sin 0°\, \hat{n} = \vec{0}$

Hence *if two vectors are parallel, then their cross product must be zero.*

(*ii*) If $\vec{A} \perp \vec{B}$, then $\theta = 90°$ and

$$\vec{A}.\vec{B} = |\vec{A}||\vec{B}|\cos 90° = 0$$

Hence *if two vectors are perpendicular, then their dot product must be zero.*

Problem 11. The resultant of two vectors $\vec{P}$ and $\vec{Q}$ is perpendicular to $\vec{P}$ and its magnitude is half that of $\vec{Q}$. What is the angle between $\vec{P}$ and $\vec{Q}$?

Solution. Clearly, $P^2 + \left(\dfrac{Q}{2}\right)^2 = Q^2$

or $\quad P^2 = \dfrac{3}{4}Q^2 \quad$ or $\quad P = \dfrac{\sqrt{3}}{2}Q$

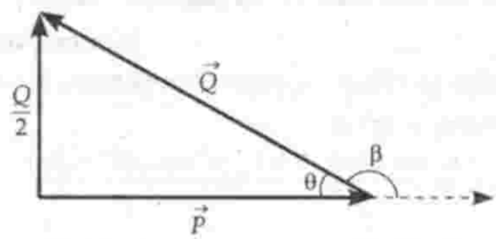


Fig. 4.91

$$\therefore \quad \tan\theta = \frac{Q/2}{P} = \frac{Q/2}{\sqrt{3}\,Q/2} = \frac{1}{\sqrt{3}}$$

or $\quad \theta = 30°$

Angle between $\vec{P}$ and $\vec{Q}$,

$$\beta = 180° - \theta = 180° - 30° = \mathbf{150°}.$$

Problem 12. The magnitude of vector $\vec{A}$ is 2.5 metre and is directed towards east. What will be the magnitudes and directions of the following :

(*a*) $-\vec{A}$ (*b*) $\vec{A}/2$ (*c*) $2.5\vec{A}$ (*d*) $4\vec{A}$?

Solution. (*a*) 2.5 m due west. (*b*) 1.25 due east.

(*c*) 6.25 m due west. (*d*) 10 m due east.

Problem 13. A person sitting in a moving train throws a ball vertically upwards. How will the ball appear to move to an observer (*i*) sitting inside the train (*ii*) standing outside the train ? Give reason.

Solution. (*i*) To the observer inside the train, the ball will appear to move *straight vertically upwards and then downwards* because with respect to an observer sitting inside the train, the ball has only one velocity acting vertically downward.

(*ii*) To the observer outside the train, the ball will appear to move along the *parabolic path* because with respect to the observer outside the train, the ball has both horizontal and vertical components of velocity.

Problem 14. A bob hung from the celling of a room by a string is performing simple harmonic oscillations. What will be the trajectory of the bob, if the string is cut, when bob is (*i*) at one of its extreme positions, (*ii*) at its mean position ?

Solution. (*i*) When the string is cut in one of the extreme positions of the bob, the bob will fall *vertically downwards*. The reason is that when the bob reaches its extreme positions, it comes to rest for a moment. If the string is cut, it will be simply the case of a body falling freely under gravity.

(*ii*) When the string is cut in mean position of the bob, then the bob will move on parabolic path. The reason is that when the bob is at its mean position, it has a velocity parallel to the horizontal. If the string is cut, it will be simply the case of a projectile fired parallel to the horizontal.

Problem 15. Prove that the horizontal range is same when angle of projection is (*i*) greater than 45° by certain value and (*ii*) less than 45° by the same value.

[Central Schools 08]

Solution. (*i*) When the angle of projection, $\theta = 45° + \alpha$, the horizontal range is

$$R = \frac{u^2 \sin 2(45° + \alpha)}{g} = \frac{u^2 \sin(90° + 2\alpha)}{g}$$

$$= \frac{u^2 \cos 2\alpha}{g}$$

(*ii*) When the angle of projection, $\theta = 45° - \alpha$, the horizontal range is

$$R' = \frac{u^2 \sin 2(45° - \alpha)}{g} = \frac{u^2 \sin(90° - 2\alpha)}{g}$$

$$= \frac{u^2 \cos 2\alpha}{g}$$

Clearly, $\quad R' = R$.

Problem 16. Two bombs of 5 kg and 10 kg are thrown from a cannon with the same velocity in the same direction. (*i*) Which bomb will reach the ground first ? (*ii*) If the bombs are thrown in the same direction with different velocity, what would be the effect ?

Solution. (*i*) Both the bombs will reach the ground simultaneously because the time of flight does not depend upon mass of the projectile.

(*ii*) On throwing with different velocities, the bomb thrown with lesser velocity will reach the ground earlier.

$$T = \frac{2u\sin\theta}{g} \quad i.e., \quad T \propto u$$

Problem 17. Two bodies are thrown with the same initial velocity at angles α and $(90° - \alpha)$ with the horizontal. What will be the ratio of (*i*) maximum heights attained by them and (*ii*) their horizontal ranges ?

Solution. (*i*) When the angle of projection α, the maximum height is

$$H_1 = \frac{u^2 \sin^2\alpha}{2g}$$

When the angle of projection is $(90° - \alpha)$, the maximum height is

$$H_2 = \frac{u^2 \sin^2(90° - \alpha)}{2g} = \frac{u^2 \cos^2\alpha}{2g}$$

$$\therefore \quad \frac{H_1}{H_2} = \frac{\sin^2\alpha}{\cos^2\alpha} = \tan^2\alpha = \mathbf{\tan^2\alpha : 1}.$$

(*ii*) When the angle of projection is α, the horizontal range is

$$R_1 = \frac{u^2 \sin 2\alpha}{g}$$

When the angle of projection is $(90° - \alpha)$, the horizontal range is

$$R_2 = \frac{u^2 \sin 2(90° - \alpha)}{g} = \frac{u^2 \sin(180° - 2\alpha)}{g}$$

$$= \frac{u^2 \sin 2\alpha}{g}$$

$$\therefore \quad R_1 : R_2 = \mathbf{1 : 1}.$$

Problem 18. At which points on the projectile trajectory is the (*i*) potential energy maximum (*ii*) kinetic energy minimum and (*iii*) total energy maximum ?

Solution. (*i*) P.E. of a projectile is maximum at its highest point because of its maximum height. It is given by

$$(\text{P.E.})_H = mg\,H = mg \cdot \frac{u^2 \sin^2 \theta}{2g} = \frac{1}{2}\, m\, u^2 \sin^2 \theta$$

(*ii*) K.E. of a projectile is minimum (not zero) at its highest point because of its minimum velocity.

$$(\text{K.E.})_H = \tfrac{1}{2}\, m\, u_H^2 = \tfrac{1}{2}\, m\,(u\cos\theta)^2$$
$$= \tfrac{1}{2}\, m\, u^2 \cos^2 \theta$$

(*iii*) Total energy at highest point

$$= (\text{P.E.})_H + (\text{K.E.})_H$$
$$= \tfrac{1}{2}\, mu^2 (\sin^2\theta + \cos^2\theta) = \tfrac{1}{2}\, m\, u^2$$

= Total energy at the point of projection

Hence total energy of a projectile is conserved at all points of its motion. The K.E. is maximum at the point of projection.

Problem 19. A body *A* of mass *m* is thrown with velocity *v* at an angle 30° to the horizontal and another body *B* of the same mass is thrown with velocity *v* at an angle of 60° to the horizontal, find the ratio of the horizontal range and maximum height of *A* and *B*.

Solution. *For body* $A : \theta = 30°$

Horizontal range,

$$R_A = \frac{u^2 \sin^2 (2 \times 30°)}{g} = \frac{u^2}{g} \times \frac{\sqrt{3}}{2}.$$

Maximum height,

$$H_A = \frac{u^2 \sin^2 30°}{2g} = \frac{u^2}{2g} \times \frac{1}{4}$$

For body $B :\ \theta = 60°$

$$\therefore \quad R_B = \frac{u^2 \sin (2 \times 60°)}{g} = \frac{u^2}{g} \times \frac{\sqrt{3}}{2}$$

and $$H_B = \frac{u^2 \sin^2 60°}{2g} = \frac{u^2}{2g} \times \frac{3}{4}$$

Hence $R_A : R_B = \mathbf{1:1}$ and $H_A : H_B = \mathbf{1:3}$.

Problem 20. A ball of mass *m* is thrown vertically up. Another ball of mass 2*m* is thrown at an angle θ with the vertical. Both of them remain in air for the same period of time. What is the ratio of the heights attained by the two balls ?

Solution. Time of flight of the ball thrown vertically upwards,

$$T_1 = \frac{2u_1}{g}$$

Time of flight of the ball thrown at an angle θ with the vertical,

$$T_2 = \frac{2u_2 \cos\theta}{g}$$

As $T_1 = T_2$

$$\therefore \quad \frac{2u_1}{g} = \frac{2u_2 \cos\theta}{g}$$

or $u_1 = u_2 \cos\theta$

Now $H_1 = \dfrac{u_1^2}{2g}$ and $H_2 = \dfrac{u_2^2 \cos^2\theta}{2g}$

$$\therefore \quad \frac{H_1}{H_2} = \frac{u_1^2}{u_2^2 \cos^2\theta} = \frac{u_2^2 \cos^2\theta}{u_2^2 \cos^2\theta} = \mathbf{1:1}.$$

Problem 21. A skilled gunman always keeps his gun slightly tilted above the line of sight while shooting. Why ?

Solution. When a bullet is fired from a gun with its barrel directed towards the target, it starts falling downwards on account of acceleration due to gravity. Due to which the bullet hits below the target. Just to avoid it, the barrel of the gun is lined up little above the target, so that the bullet, after travelling in parabolic path hits the distant target, as shown in Fig. 4.92.

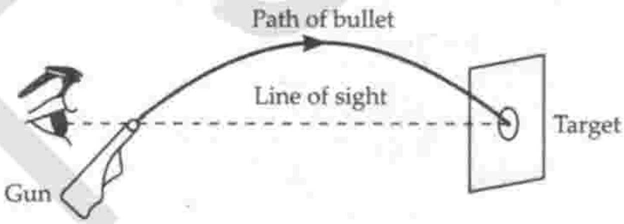


Fig. 4.92

Problem 22. Is it important in the long jump that how much height you take for jumping ? What factors determine the span of a jump ?

Solution. Yes, for the longest jump the player should throw himself at an angle of 45° with respect to horizontal. The vertical height for this angle is

$$H = u^2 \sin^2 45°/2g = u^2 / 4g,$$

where u is the velocity of projection. If the vertical height is different from $u^2/4g$, then the angle will be different from 45° and the horizontal distance covered will also be less.

Clearly, the span of jump depends upon

(*i*) the initial velocity of the jump,

(*ii*) the angle of projection.

Problem 23. A body is thrown horizontally with a velocity *v* from a tower of height *H*. After how much time and at what distance from the base of the tower will the body strike the ground ?

Solution. Clearly, $H = \dfrac{1}{2} g t^2$

$$\therefore \quad t = \sqrt{\frac{2H}{g}}.$$

Distance from the base of the tower,

$$x = vt = v\sqrt{2H/g}.$$

Problem 24. A ball is thrown horizontally and at the same time another ball is dropped down from the top of a tower (*i*) Will both the balls reach the ground at the same time ? (*ii*) Will both strike the ground with the same velocity ?

Solution. (*i*) Yes, because the initial vertical velocity of both the balls is zero, and both will cover the same vertical distance under the same vertical acceleration g.

(*ii*) No, because on striking the ground although their vertical velocities will be same but horizontal velocities will be different. Hence their resultant velocities will be different.

Problem 25. A horizontal stream of water leaves an opening in the side of a tank. If the opening is h metre above the ground and the stream hits the ground D metre away, then what is the speed of water as it leaves the tank in terms of g, h and D ?

Solution. The stream of water leaving the tank behaves like a horizontal projectile and follows parabolic path.

$$h = \frac{1}{2} g t^2$$

$$\therefore \quad t = \sqrt{\frac{2h}{g}}$$

$$v = \frac{D}{t} = D\sqrt{\frac{g}{2h}}.$$

Problem 26. A railway carriage moves over a straight track with acceleration a. A passenger in the carriage drops a stone. What is the acceleration of the stone w.r.t. the carriage and the earth ?

Solution. When the stone is dropped, it falls freely under the acceleration due to gravity g. With respect to the earth, the acceleration of the stone is g.

Inside the carriage, the stone possesses two accelerations :

(*i*) Horizontal acceleration 'a' due to the motion of the carriage.

(*ii*) Vertical downward acceleration 'g' due to gravity.

The acceleration of the stone w.r.t. the carriage

$$= \sqrt{a^2 + g^2}.$$

Problem 27. A glass marble slides from rest from the topmost point of a vertical circle of radius r along a smooth chord. Does the time of descent depend upon the chord chosen ?

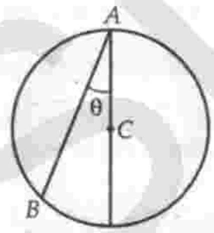


Fig. 4.93

Solution. As shown in Fig. 4.92, let C be the centre of the circle. Suppose chord AB makes angle θ with the vertical.

If the marble takes time t to slide along chord AB, then

$$AB = 0 \times t + \frac{1}{2} g \cos\theta \times t^2$$

[$\because$ Acceleration along $AB = g\cos\theta$]

$$\text{or} \quad t^2 = \frac{2\,AB}{g\cos\theta} = \frac{2 \times 2r\cos\theta}{g\cos\theta} = \frac{4r}{g}$$

$$\text{or} \quad t = 2\sqrt{\frac{r}{g}}$$

As t is independent of θ, so the time of descent does not depend on the chord chosen.

HOTS

Problems on Higher Order Thinking Skills

Problem 1. *Four persons K, L, M and N are initially at rest at the four corners of a square of side d. Each person now moves with a uniform speed v in such a way that K always moves directly towards L, L directly towards M, M directly towards N and N directly towards K. Show that the four persons meet at a time d/v.*

Solution. All the four persons will meet at the centre O of the square. Each person covers a displacement,

$$s = \frac{1}{2} \cdot \sqrt{d^2 + d^2} = \frac{d}{\sqrt{2}}$$

Component of velocity v towards O,

$$v' = v\cos 45° = v/\sqrt{2}$$

$\therefore$ Required time,

$$t = \frac{s}{v'} = \frac{d/\sqrt{2}}{v/\sqrt{2}} = \frac{d}{v}.$$

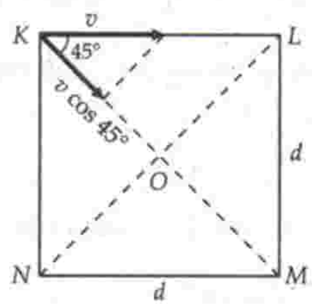


Fig. 4.94

Problem 2. *The x- and y- components of $\vec{A}$ are 4 m and 6 m respectively. The x- and y- components of vector $(\vec{A}+\vec{B})$ are 10 m and 9 m respectively. Calculate for the vector $\vec{B}$ (i) its x- and y- components (ii) its length and (iii) the angle it makes with the x-axis.* **[REC 89]**

Solution. Here $A_x = 4$ m, $A_y = 6$ m, $A_x + B_x = 10$ m, $A_y + B_y = 9$ m

(*i*) $B_x = 10 - 4 = \mathbf{6\ m}$, $\quad B_y = 9 - 6 = \mathbf{3\ m.}$

(*ii*) $B = \sqrt{B_x^2 + B_y^2} = \sqrt{36+9} = \mathbf{\sqrt{45}\ m.}$

(*iii*) $\theta = \tan^{-1}\dfrac{B_y}{B_x} = \tan^{-1}\dfrac{3}{6} = \mathbf{26.6°.}$

Problem 3. *A bird is at a point P whose coordinates are (4 m, −1 m, 5 m). The bird observes two points P_1 and P_2 having coordinates (−1 m, 2 m, 0 m) and (1 m, 1 m, 4 m) respectively. At time t = 0, it starts flying in a plane of three positions, with a constant speed of 5 ms^{-1} in a direction perpendicular to the straight line P_1P_2 till it sees P_1 and P_2 collinear at time t. Calculate t.*

Solution. The situation is shown in Fig. 4.95. The bird flies in a direction perpendicular to line P_1P_2. Suppose it reaches the point Q in time *t* (after starting from point P) where it sees P_1 and P_2 as collinear.

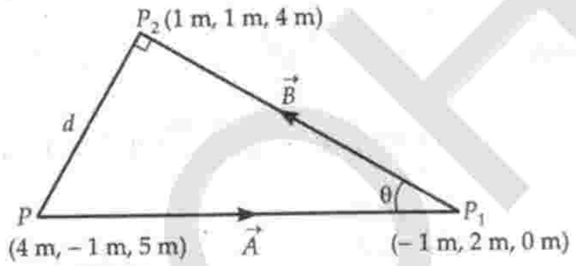


Fig. 4.95

Let $\vec{PP_1} = \vec{A}$, $\vec{P_1P_2} = \vec{B}$, $\angle PP_1P_2 = \theta$ and $PQ = d$.

As $|\vec{A}\times\vec{B}| = |\vec{A}||\vec{B}|\sin\theta$

$$\therefore \quad \sin\theta = \frac{|\vec{A}\times\vec{B}|}{|\vec{A}||\vec{B}|}$$

Now $\quad d = |\vec{A}|\sin\theta = |\vec{A}|\dfrac{|\vec{A}\times\vec{B}|}{|\vec{A}||\vec{B}|} = \dfrac{|\vec{A}\times\vec{B}|}{\vec{B}}$

But $\quad \vec{A} = (-1-4)\hat{i} + (2+1)\hat{j} + (0-5)\hat{k}$

$= -5\hat{i} + 3\hat{j} - 5\hat{k}$

and $\quad \vec{B} = (1+1)\hat{i} + (1-2)\hat{j} + (4-0)\hat{k}$

$= 2\hat{i} - \hat{j} + 4\hat{k}$

$$\therefore \quad \vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -5 & 3 & -5 \\ 2 & -1 & 4 \end{vmatrix}$$

$= (12-5)\hat{i} + (-10+20)\hat{j} + \hat{k}(5-6)$

$= 7\hat{i} + 10\hat{j} - \hat{k}$

$\therefore \quad |\vec{A}\times\vec{B}| = \sqrt{7^2+10^2+1^2} = 12.25\ m^2$

and $\quad |\vec{B}| = \sqrt{2^2+1^2+4^2} = 4.583$ m

$\therefore \quad d = \dfrac{12.25}{4.583} = \mathbf{2.67\ m}$

Time taken by bird to reach the point Q will be

$$t = \frac{d}{v} = \frac{2.67}{5} = \mathbf{0.5346\ s.}$$

Problem 4. *A person travelling eastward with a speed of 3 kmh^{-1} finds that wind seems to blow from north. On doubling his speed, the wind appears to flow from north-east. Find the magnitude of the actual velocity of the wind.*

Solution. The situation is shown in Fig. 4.96.

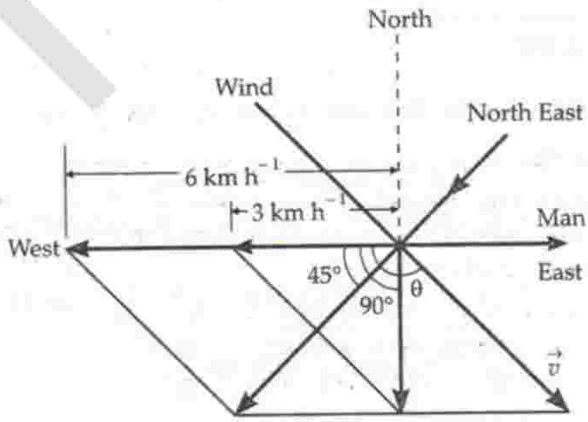


Fig. 4.96

Let the wind velocity be *v*.

In the first case, relative velocity of wind makes an angle of 90° with opposite velocity of man.

From parallelogram law,

$$\tan 90° = \frac{v\sin\theta}{3+v\cos\theta} = \infty$$

or $\quad 3 + v\cos\theta = 0 \quad$ ...(1)

In the second case, relative velocity of wind makes an angle of 45° with double opposite velocity of man

$$\therefore \quad \tan 45° = \frac{v\sin\theta}{6+v\cos\theta} = 1$$

or $\quad 6 + v\cos\theta = v\sin\theta \quad$ ...(2)

From (1), $\quad v\cos\theta = -3$

From (2), $\quad v\sin\theta = 6 + v\cos\theta = 6 - 3 = 3$

$$\therefore \quad v^2 = (v\cos\theta)^2 + (v\sin\theta)^2$$
$$= (-3)^2 + (3)^2 = 9 + 9 = 18$$

or $\quad v = \mathbf{3\sqrt{2}\ km\ h^{-1}}$.

Problem 5. *Consider a collection of a large number of particles each with speed v. The direction of velocity is randomly distributed in the collection. Show that the magnitude of the relative velocity between a pair of particles averaged over all the pairs in the collection is greater than v.* **[NCERT]**

Solution. As shown in Fig. 4.97, let $\vec{v}_1$ and $\vec{v}_2$ be velocities of any two particles and θ be the angle between them. As each particle has speed v, so

$$|\vec{v}_1| = |\vec{v}_2| = v$$

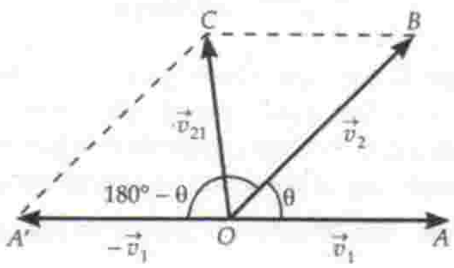


Fig. 4.97

The relative velocity $\vec{v}_{21}$ of particle 2 w.r.t. 1 is given by

$$v_{21} = \sqrt{|-\vec{v}_1|^2 + |\vec{v}_2|^2 + 2|-\vec{v}_1||\vec{v}_2|\cos(180° - \theta)}$$
$$= \sqrt{v^2 + v^2 - 2v\,v\cos\theta} = \sqrt{2v^2(1-\cos\theta)}$$
$$= \sqrt{2v^2 \times 2\sin^2\frac{\theta}{2}} = 2v\sin\frac{\theta}{2}$$
$$\left[1 - \cos 2\theta = 2\sin^2\theta \quad \therefore \quad 1 - \cos\theta = 2\sin^2\frac{\theta}{2}\right]$$

As the velocities of the particles are randomly distributed, so θ can vary from 0 to 2π. If $\bar{v}_{21}$ is the magnitude of the average velocity when averaged over all pairs, then

$$\bar{v}_{21} = \frac{\int_0^{2\pi} 2v\sin\frac{\theta}{2}\,d\theta}{\int_0^{2\pi} d\theta} = \frac{2v\left[\frac{-\cos\theta/2}{1/2}\right]_0^{2\pi}}{[\theta]_0^{2\pi}}$$
$$= \frac{-4v\left[\cos\frac{\theta}{2}\right]_0^{2\pi}}{2\pi - 0} = -\frac{2v}{\pi}[\cos\pi - \cos 0]$$
$$= -\frac{2v}{\pi}[-1-1] = \frac{4v}{\pi} = 1.273\,v$$

Clearly, $\bar{v}_{21} > v$

Problem 6. *A ball rolls off the top of a stairway with a constant horizontal velocity u. If the steps are h metre high and w metre wide, show that the ball will just hit the edge of nth step if* $n = \dfrac{2hu^2}{gw^2}$.

Solution. Refer to Fig. 4.98. *For nth step,*

net vertical displacement = nh

net horizontal displacement = $n\omega$

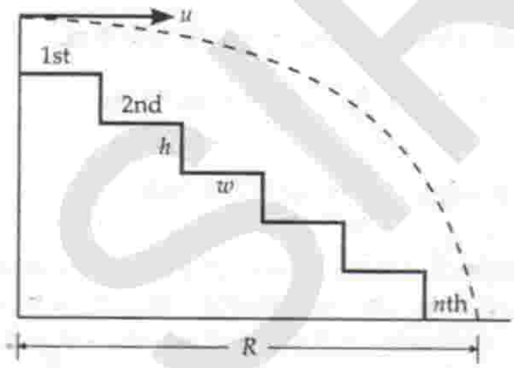


Fig. 4.98

Let t be the time taken by the ball to reach the nth step. Then

$$R = ut$$

or $\quad n\omega = ut$

or $\quad t = \dfrac{n\omega}{u}$

Also, $\quad y = u_y\,t + \dfrac{1}{2}gt^2$

or $\quad nh = 0 + \dfrac{1}{2}gt^2 = \dfrac{1}{2}g\left(\dfrac{n\omega}{u}\right)^2$

or $\quad n = \dfrac{2hu^2}{g\omega^2}$.

Problem 7. *Show that the motion of one projectile as seen from another projectile will always be a straight line motion.*

Solution. As shown in Fig. 4.99, suppose two projectiles are thrown from the origin O of the XY-plane with velocities u_1 and u_2, making angles θ_1 and θ_2

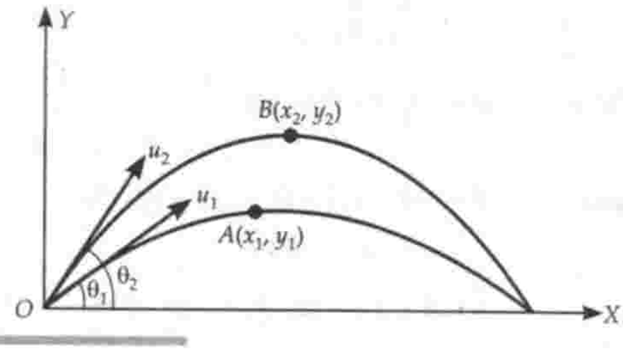


Fig. 4.99

respectively with X-axis. After time t, let the two projectiles occupy positions $A(x_1, x_1)$ and $B(x_2, y_2)$. Then

$$x_1 = u_1 \cos\theta_1 \,.\, t$$

and $$y_1 = u_1 \sin\theta_1 \,.\, t - \tfrac{1}{2} g t^2$$

Also, $$x_2 = u_2 \cos\theta_2 \,.\, t$$

and $$y_2 = u_2 \sin\theta_2 \,.\, t - \tfrac{1}{2} g t^2$$

$$\therefore \quad x_2 - x_1 = (u_2 \cos\theta_2 - u_1 \cos\theta_1)\, t$$

$$y_2 - y_1 = (u_2 \sin\theta_2 - u_1 \sin\theta_1)\, t$$

or $$\frac{y_2 - y_1}{x_2 - x_1} = \frac{u_2 \sin\theta_2 - u_1 \sin\theta_1}{u_2 \cos\theta_2 - u_1 \cos\theta_1} = m(\text{a constant})$$

If (x, y) be the coordinates of point B relative to the point A, then

$$x_2 - x_1 = x \quad \text{and} \quad y_2 - y_1 = y$$

$$\therefore \quad \frac{y}{x} = m \quad \text{or} \quad y = mx$$

This is the equation of a straight line. Hence the motion of a projectile as seen from another projectile is a straight line motion.

Problem 8. *A boy playing on the roof a 10 m high building throws a ball with a speed of 10 ms^{-1} at an angle of 30° with the horizontal. How far from the throwing point will the ball be at the height of 10 m from the ground ? Given g = 10 ms^{-2}.* **[AIEEE 03]**

Solution. When the ball thrown from the height of 10 m reaches again at the height of 10 m, it covers a distance equal to its horizontal range R given by

$$R = \frac{u^2 \sin 2\theta}{g}$$

But $u = 10 \text{ ms}^{-1}$, $\theta = 30°$, $g = 10 \text{ ms}^{-2}$

$$\therefore \quad R = \frac{(10)^2 \sin 60°}{10} = 10 \times 0.866 = \mathbf{8.66 \ m.}$$

Problem 9. *A ball is thrown from a point with a speed of v_0 at an angle of projection θ. From the same point and at the same instant, a person starts running with a constant speed of $v_0/2$ to catch the ball. Will the person be able to catch the ball ? If yes, what should be the angle of projection ?* **[AIEEE 04]**

Solution. Yes, the person will be able to catch up the ball if the horizontal component of the velocity of the ball is equal to the constant speed of the person, *i.e.*,

$$v_0 \cos\theta = \frac{v_0}{2} \quad \text{or} \quad \cos\theta = \frac{1}{2}$$

or $$\theta = \mathbf{60°}.$$

Problem 10. *A projectile can have the same range R for two angles of projection. If t_1 and t_2 be the time of flight in the two cases, then prove that*

$$t_1 t_2 = \frac{2R}{g}.$$

[AIEEE 05]

Solution. For equal range, the particle should either be projected at an angle θ or (90° – θ).

$$\therefore \quad t_1 = \frac{2u \sin\theta}{g}$$

and $$t_2 = \frac{2u \sin(90° - \theta)}{g} = \frac{2u \cos\theta}{g}$$

Hence $$t_1\, t_2 = \frac{2u \sin\theta}{g} \cdot \frac{2u \cos\theta}{g}$$

$$= \frac{2}{g} \frac{u^2 \sin 2\theta}{g} = \frac{2R}{g}.$$

Guidlines to NCERT Exercises

4.1. *State for each of the following physical quantities, if it is a scalar or a vector : volume, mass, speed, acceleration, density, number of moles, velocity, angular frequency, displacement, angular velocity.*

Ans. Scalars : volume, mass, speed, density, number of moles and angular frequency.

Vectors : Acceleration, velocity, displacement and angular velocity.

4.2. *Pick out the two scalar quantities in the following list : force, angular momentum, work, current, linear momentum, electric field, average velocity, magnetic moment, reaction as per Newton's third law, relative velocity.*

Solution. Work and current.

4.3. *Pick out the only vector quantity in the following list : Temperature, pressure, impulse, time, power, total path length, energy, gravitational potential, coefficient of friction, charge.*

Ans. Impulse.

4.4. *State with reasons, whether the following algebraic operations with scalar and vector physical quantities are meaningful :*

(*a*) *Adding any two scalars.*

(*b*) *Adding a scalar to a vector of the same dimensions.*

(*c*) *Multiplying any vector by any scalar.*

(*d*) *Multiplying any two scalars.*

(*e*) *Adding any two vectors.*

(*f*) *Adding a component of a vector to the same vector.*

Ans. (*a*) **No.** Only two such scalars can be added which represent the same physical quantity.

(*b*) **No.** A scalar cannot be added to a vector even of same dimensions because a vector has a direction while a scalar has no direction *e.g.*, speed cannot be added to velocity.

(*c*) **Yes.** We can multiply any vector by a scalar. For example, when mass (scalar) is multiplied with acceleration (vector), we get force (vector) *i.e.*, $\vec{F} = m\vec{a}$.

(*d*) **Yes.** We can multiply any two scalars. When we multiply power (scalar) with time (scalar), we get work done (scalar) *i.e.*, $W = Pt$.

(*e*) **No.** Only two vectors of same nature can be added by using the law of vector addition.

(*f*) **No.** A component of a vector can be added to the same vector only by using the law of vector addition. So the addition of a component of a vector to the same vector is not a meaningful algebraic operation.

4.5. *Read each statement below carefully and state with reasons, if it is true or false :*

(*a*) *The magnitude of a vector is always a scalar.*

(*b*) *Each component of a vector is always a scalar.*

(*c*) *The total path length is always equal to the magnitude of the displacement vector of a particle.*

(*d*) *The average speed of a particle (defined as total path length divided by the time taken to cover the path) is either greater or equal to the magnitude of average velocity of the particle over the same interval of time.*

(*e*) *Three vectors not lying in a plane can never add up to give a null vector.*

Ans. (*a*) **True.** The magnitude of a vector is a pure number and has no direction.

(*b*) **False.** Each component of a vector is also a vector.

(*c*) **False.** The displacement depends only on the end points while the path length depends on the actual path. The two quantities are equal only if the direction of motion of the object does not change. In all other cases, path length is greater than the magnitude of displacement.

(*d*) **True.** This is because the total path length is either greater than or equal to the magnitude of displacement over the same interval of time.

(*e*) **True.** This is because the resultant of two vectors will not lie in the plane of third vector and hence cannot cancel its effect to give null vector.

4.6. *Establish the following vector inequalities geometrically or otherwise :*

(*a*) $|\vec{a} + \vec{b}| \le |\vec{a}| + |\vec{b}|$ (*b*) $|\vec{a} + \vec{b}| \ge |\vec{a}| - |\vec{b}|$

(*c*) $|\vec{a} - \vec{b}| \le |\vec{a}| + |\vec{b}|$ (*d*) $|\vec{a} - \vec{b}| \ge |\vec{a}| - |\vec{b}|$

When does the equality sign above apply ?

Ans. (*a*) If θ be the angle between $\vec{a}$ and $\vec{b}$, then

$$|\vec{a} + \vec{b}| = \sqrt{|\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}||\vec{b}|\cos\theta}$$

Now $|\vec{a} + \vec{b}|$ will be maximum when

$$\cos\theta = 1 \text{ or } \theta = 0$$

$$\therefore \quad |\vec{a} + \vec{b}|_{max} = \sqrt{|\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}||\vec{b}|\cos 0^\circ}$$

$$= \sqrt{|\vec{a}|^2 + |\vec{b}|^2} = |\vec{a}| + |\vec{b}|$$

Hence $|\vec{a} + \vec{b}| \le |\vec{a}| + |\vec{b}|$

The equality sign is applicable when $\theta = 0°$ *i.e.*, when $\vec{a}$ and $\vec{b}$ are in the same direction.

(*b*) Again

$$|\vec{a} + \vec{b}| = \sqrt{|\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}||\vec{b}|\cos\theta}$$

The value of $|\vec{a} + \vec{b}|$ will be minimum when

$$\cos\theta = -1$$

or $$\theta = 180°$$

$$\therefore \quad |\vec{a} + \vec{b}|_{min} = \sqrt{|\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}||\vec{b}|\cos 180^\circ}$$

$$= \sqrt{(|\vec{a}| - |\vec{b}|)^2} = |\vec{a}| - |\vec{b}|$$

Hence $|\vec{a} + \vec{b}| \ge |\vec{a}| - |\vec{b}|$

The equality sign is applicable when $\theta = 180°$ *i.e.*, when $\vec{a}$ and $\vec{b}$ are in opposite directions.

(*c*) If θ is the angle between $\vec{a}$ and $\vec{b}$, then the angle between $\vec{a}$ and $-\vec{b}$ will be $(180° - \theta)$, as shown in Fig. 4.100.

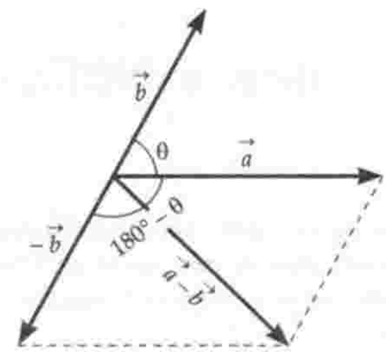


Fig. 4.100

$$\therefore \quad |\vec{a} - \vec{b}| = |\vec{a} + (-\vec{b})|$$

$$= \sqrt{|\vec{a}|^2 + |-\vec{b}|^2 + 2|\vec{a}||-\vec{b}|\cos(180° - \theta)}$$

$$= \sqrt{|\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|\cos\theta}$$

$$[\because \; |-\vec{b}| = |\vec{b}|, \cos(180° - \theta) = -\cos\theta]$$

$|\vec{a} - \vec{b}|$ will be maximum when $\cos\theta = -1$ or $\theta = 180°$

$$\therefore \quad |\vec{a} - \vec{b}|_{max} = \sqrt{|\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|\cos(180° - \theta)}$$

$$= \sqrt{(|\vec{a}| + |\vec{b}|)^2} = |\vec{a}| + |\vec{b}|$$

Hence $|\vec{a} - \vec{b}| \le |\vec{a}| + |\vec{b}|$

The equality sign is applicable when $\theta = 180°$.

(*d*) $|\vec{a} - \vec{b}|$ will be minimum when

$\cos\theta = 1$ or $\theta = 0°$

$$\therefore \quad |\vec{a} - \vec{b}|_{min} = \sqrt{|\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|\cos 0°}$$

$$= \sqrt{(|\vec{a}| - |\vec{b}|)^2} = |\vec{a}| - |\vec{b}|$$

Hence $|\vec{a} - \vec{b}| \ge |\vec{a}| - |\vec{b}|$

The equality sign is applicable when $\theta = 0°$.

4.7. *Given $\vec{a} + \vec{b} + \vec{c} + \vec{d} = 0$, which of the following statements are correct :*

(*a*) *$\vec{a}$, $\vec{b}$, $\vec{c}$, and $\vec{d}$ must each be a null vector,*

(*b*) *The magnitude of $(\vec{a} + \vec{c})$ equals the magnitude of $(\vec{b} + \vec{d})$,*

(*c*) The magnitude of $\vec{a}$ can never be greater than the sum of the magnitudes of $\vec{b}$, $\vec{c}$, and $\vec{d}$,

(*d*) $\vec{b} + \vec{c}$ must lie in the plane of $\vec{a}$ and $\vec{d}$ if $\vec{a}$ and $\vec{d}$ are not collinear, and in the line of $\vec{a}$ and $\vec{d}$, if they are collinear ?

Ans. (*a*) $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$ need not each be a null vector. The resultant of four non-zero vectors can be a null vector in many ways *e.g.*, the resultant of any three vectors may be equal to the magnitude of fourth vector but has the opposite direction. Hence the statement $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$ must each be a null vector, is **not correct**.

(*b*) Because $\vec{a} + \vec{b} + \vec{c} + \vec{d} = 0$, hence $\vec{a} + \vec{c} = -(\vec{b} + \vec{d})$ *i.e.*, the magnitude of $(\vec{a} + \vec{c})$ is equal to the magnitude of $(\vec{b} + \vec{d})$ but their directions are opposite. Hence the given statement is **correct**.

(*c*) Because, $\vec{a} + \vec{b} + \vec{c} + \vec{d} = 0$ or $\vec{a} = -(\vec{b} + \vec{c} + \vec{d})$. Hence, magnitude of vector $\vec{a}$ is equal to magnitude of vector $(\vec{b} + \vec{c} + \vec{d})$. The sum of the magnitudes of vectors $\vec{b}$, $\vec{c}$ and $\vec{d}$ may be greater than or equal to that of vector $\vec{a}$ (or vector $\vec{b} + \vec{c} + \vec{d}$). Hence the statement that the magnitude of $\vec{a}$ can never be greater than the sum of the magnitudes of $\vec{b}$, $\vec{c}$ and $\vec{d}$ is **correct**.

(*d*) Because $\vec{a} + \vec{b} + \vec{c} + \vec{d} = 0$, hence $(\vec{b} + \vec{c}) + \vec{a} + \vec{d} = 0$. The resultant sum of three vectors $\vec{b} + \vec{c}$, $\vec{a}$ and $\vec{d}$ can be zero only if $\vec{b} + \vec{c}$ is in plane of $\vec{a}$ and $\vec{d}$. In case $\vec{a}$ and $\vec{d}$ are collinear, $\vec{b} + \vec{c}$ must be in line of $\vec{a}$ and $\vec{d}$. Hence the given statement is **correct**.

4.8. *Three girls skating on a circular ice ground of radius 200 m start from a point P on the edge of the ground and reach a point Q diametrically opposite to P following different paths as shown in Fig. 4.101. What is the magnitude of the displacement vector for each ? For which girl is this equal to the actual length of path skated ?*

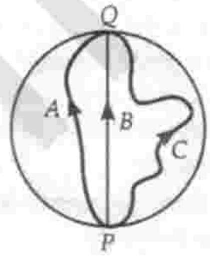


Fig. 4.101

Solution. Displacement of each girl = $\vec{PQ}$

Magnitude of displacement vector for each girl

$= |\vec{PQ}| = 2 \times \text{radius} = 2 \times 200 = \mathbf{400\ m}$.

For girl *B*, the magnitude of displacement vector = actual length of path.

4.9. *A cyclist travels from centre O of a circular park of radius 1 km and reaches point P. After cycling along 1/4 th of the circumference along PQ, he returns to the centre of the park along QO. If the total time taken is 10 minutes, calculate (i) net displacement (ii) average velocity and (ii) average speed of the cyclist.*

Ans. (*i*) Net displacement is zero as both initial and final positions are same.

(*ii*) Average velocity $= \dfrac{\text{displacement}}{\text{time taken}}$

As displacement is zero, average velocity of the cyclist is also **zero**.

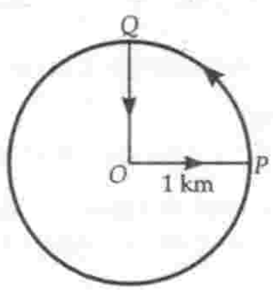


Fig. 4.102

(*iii*) Total distance covered

$$= OP + \text{Arc } PQ + OQ = r + \frac{2\pi r}{4} + r$$

$$= 1 + \frac{2 \times 22 \times 1}{7 \times 4} + 1 = \frac{25}{7} \text{ km}$$

Time taken $= 10 \text{ min} = \frac{1}{6}$ h

$$\text{Average speed} = \frac{\text{Total distance covered}}{\text{Time taken}}$$

$$= \frac{25/7 \text{ km}}{1/6 \text{ h}} = \mathbf{21.43\ kmh^{-1}}.$$

4.10. *On an open ground, a motorist follows a track that turns to his left by an angle of 60° after every 500 m. Starting from a given turn, specify the displacement of the motorist at the third, sixth and eighth turn. Compare the magnitude of the displacement with the total path length covered by the motorist in each case.*

Ans. As shown in Fig. 4.103, suppose motorist starts from point *A*.

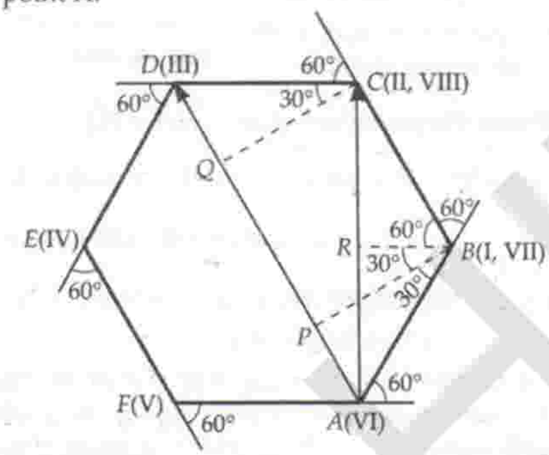


Fig. 4.103

Clearly, he will follow the hexagonal path *ABCDEFA*. The orders of the turns taken by him are indicated at the vertices of the hexagon.

(*i*) *At the third turn,* the motorist will be at *D*. The magnitude of displacement $\vec{AD}$ will be

$$= AP + PQ + QD$$

$$= AB \sin 30° + BC + CD \sin 30°$$

$$= 500 \times \tfrac{1}{2} + 500 + 500 \times \tfrac{1}{2} = 1000 \text{ m} = \mathbf{1\ km}.$$

The direction of $\vec{AD}$ is 60° left of the initial direction $\vec{AB}$.

Total path length

$$= AB + BC + CD = 500 \times 3 = 1500 \text{ m} = \mathbf{1.5\ km}.$$

(*ii*) *At the sixth turn,* the motorist comes back to the starting point *A*, so magnitude of displacement is *zero*.

Total path length

$$= AB + BC + CD + DE + EF + FA$$

$$= 500 \times 6 = 3000 \text{ m} = \mathbf{3\ km}.$$

(*iii*) *At the eighth turn,* the motorist will be at *C*. The magnitude of his displacement $\vec{AC}$ is

$$|\vec{AC}| = AR + RC = AB \sin 60° + BC \sin 60°$$

$$= 500 \times \frac{\sqrt{3}}{2} + 500 \times \frac{\sqrt{3}}{2} = 500\sqrt{3} = \mathbf{866\ m.}$$

The direction of $\vec{AC}$ is 30° left of the initial direction $\vec{AB}$.

Total path length $= 500 \times 8 = 4000$ m $=$ **4 km.**

4.11. *A passenger arriving in a new town wishes to go from the station to a hotel located 10 km away on a straight road from the station. A dishonest cabman takes him along a circuitous path 23 km long and reaches the hotel in 28 minutes. What is (i) the average speed of the taxi and (ii) the magnitude of average velocity ? Are the two equal ?*

Solution. Magnitude of displacement = 10 km

Total path length = 23 km

Time taken $= 28 \text{ min} = \frac{28}{60} \text{ h} = \frac{7}{15}$ h

$$(i) \text{ Average speed} = \frac{\text{Total path length}}{\text{Time taken}}$$

$$= \frac{23 \text{ km}}{\frac{7}{15} \text{ h}} = \mathbf{49.3\ kmh^{-1}}.$$

(*ii*) Magnitude of average velocity

$$= \frac{\text{Displacement}}{\text{Time taken}} = \frac{10 \text{ km}}{\frac{7}{15} \text{ h}} = \mathbf{21.43\ kmh^{-1}}.$$

Clearly, the average speed and magnitude of the average velocity are not equal. They will be equal only for straight path.

4.12. *Rain is falling vertically with a speed of 30 ms^{-1}. A woman rides a bicycle with a speed of 10 ms^{-1} in the north to south direction. What is the relative velocity of rain with respect to the woman ? What is the direction in which she should hold her umbrella to protect herself from the rain ?*

Solution. The situation is shown in Fig. 4.104.

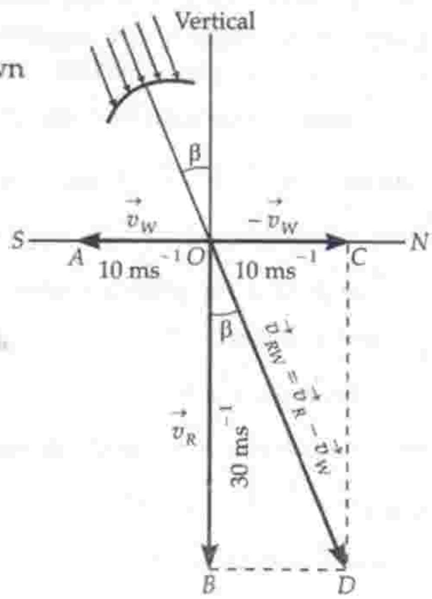


Fig. 4.104

Here

$\vec{OA} = \vec{v}_W$ = velocity of woman cyclist

$= 10\ ms^{-1}$, due south

$\vec{OB} = \vec{v}_R$ = velocity of rain

$= 30\ ms^{-1}$, vertically downward

$\vec{OC} = -\vec{v}_W$

= Opposite velocity of the woman cyclist.

$\vec{OD} = \vec{v}_R + (-\vec{v}_W) = \vec{v}_R - \vec{v}_W = \vec{v}_{RW}$

= Velocity of rain relative to woman cyclist

$$v_{RW} = OD = \sqrt{OC^2 + OB^2} = \sqrt{10^2 + 30^2}$$

$$= 10\sqrt{10} = \mathbf{31.6\ ms^{-1}}$$

If OD makes angle β with the vertical, then

$$\tan\beta = \frac{BD}{OB} = \frac{OC}{OB} = \frac{10}{30} = 0.3333 \quad \text{or} \quad \beta = \mathbf{18°26'}.$$

The woman should hold her umbrella at 18°26′ with the vertical in the direction of her motion *i.e.*, towards south.

4.13. *A man can swim with a speed of 4 kmh^{-1} in still water. How long does he take to cross the river 1 km wide, if the river flows steadily at 3 kmh^{-1} and he makes his strokes normal to the river current ? How far from the river does he go, when he reaches the other bank ?*

Solution. In Fig. 4.105, $\vec{v}_M$ and $\vec{v}_R$ represent the velocities of man and river. Clearly $\vec{v}$ is the resultant of these velocities. If the man begins to swim along AB, he will be deflected to the path AC by the flowing river.

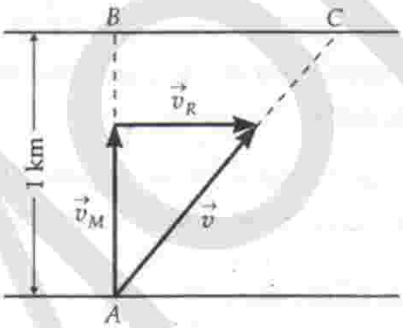


Fig. 4.105

Time taken to cover distance AC with velocity $\vec{v}$ will be same as the time taken to cover distance AB with velocity $\vec{v}_M$.

$\therefore$ Time taken by the man to cross the river is

$$t = \frac{AB}{v_M} = \frac{1\text{ km}}{4\text{ km h}^{-1}} = \frac{1}{4}\text{ h} = \mathbf{15\ min.}$$

Distance through which the man goes down the river is

$$BC = v_R \times t = 3\text{ kmh}^{-1} \times \frac{1}{4}\text{ h} = \mathbf{0.75\ km.}$$

4.14. *In a harbour, wind is blowing at the speed of 72 kmh^{-1} and the flag on the mast of a boat anchored in the harbour flutters along the N-E direction. If the boat starts moving at a speed of 51 kmh^{-1} to the north, what is the direction of the flag on the mast of the boat ?*

Solution. When the boat is stationary, the flag flutters along N-E direction. This shows that velocity of the wind is along N-E direction. When the boat moves, the flag flutters along the direction of relative velocity of wind w.r.t the boat. Thus $\vec{v}_W = \vec{OA}$ = wind velocity = 72 kmh^{-1}, due N-E direction

Boat velocity $= \vec{v}_B = \vec{OB} = 51$ kmh^{-1}, due north

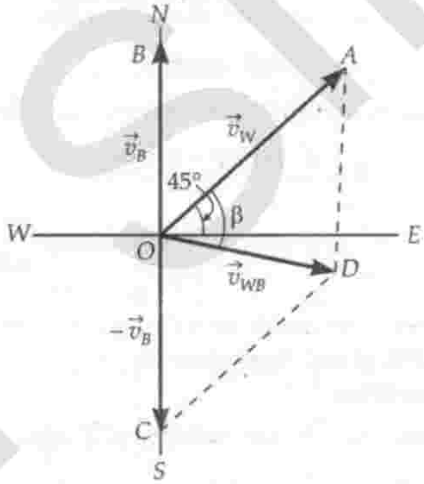


Fig. 4.106

Relative velocity of wind w.r.t. boat is given by

$$\vec{v}_{WB} = \vec{v}_W - \vec{v}_B = \vec{v}_W + (-\vec{v}_B)$$

$$= \vec{OA} + \vec{OC} = \vec{OD}$$

Clearly, the flag will flutter in the direction of $\vec{OD}$ on the mast of the moving boat.

Angle between $\vec{v}_W$ and $-\vec{v}_B$, $\theta = 45° + 90° = 135°$

If $\vec{v}_{WB}$ makes angle β with $\vec{v}_W$, then

$$\tan\beta = \frac{v_B \sin\theta}{v_W + v_B\cos\theta} = \frac{51\sin 135°}{72 + 51\cos 135°}$$

$$= \frac{51\sin 45°}{72 + 51(-\cos 45°)} = \frac{51\times\frac{1}{\sqrt{2}}}{72 - 51\times\frac{1}{\sqrt{2}}}$$

$$= \frac{51}{72\sqrt{2} - 51} = 1.0037$$

or $\beta = 45.01°$

Angle w.r.t east direction = 45.01° − 45° = 0.01°

Hence the flag will flutter almost in the east direction.

4.15. *The ceiling of a long hall is 25 m high. What is the maximum horizontal distance that a ball thrown with a speed of 40 ms^{-1} can go without hitting the ceiling of the hall ?*

Solution. Here $H = 25$ m, $u = 40\ \text{ms}^{-1}$

If the ball is thrown at an angle θ with the horizontal, then maximum height of flight,

$$H = \frac{u^2 \sin^2 \theta}{2g}$$

$$\therefore \quad 25 = \frac{(40)^2 \sin^2 \theta}{2 \times 9.8}$$

or $$\sin^2 \theta = \frac{25 \times 2 \times 9.8}{(40)^2} = 0.306$$

or $$\sin \theta = \sqrt{0.306} = 0.554$$

and $$\cos \theta = \sqrt{1 - \sin^2 \theta} = \sqrt{1 - 0.306}$$
$$= \sqrt{0.694} = 0.833$$

The maximum horizontal distance is given by

$$R = \frac{u^2 \sin 2\theta}{g} = \frac{2u^2 \sin \theta \cos \theta}{g}$$
$$= \frac{2 \times (40)^2 \times 0.554 \times 0.833}{9.8} = \mathbf{150.7\ m.}$$

4.16. *A cricketer can throw a ball to a maximum horizontal distance of 100 m. How high above the ground can the cricketer throw the same ball ?* **[Delhi 98]**

Solution. Let u be the velocity of projection. Then

$$R_{max} = \frac{u^2}{g} = 100 \text{ m}$$

or $$u^2 = 100\, g \quad \text{or} \quad u = \sqrt{100\, g}$$

For upward throw of the ball, we have

$$u = \sqrt{100\, g}, \ v = 0, \ a = -g, \ s = ?$$

As $v^2 - u^2 = 2as$

$\therefore \quad 0 - 100\, g = 2(-g)\, s$

or $$s = \frac{-100\, g}{-2g} = \mathbf{50\ m.}$$

Thus the cricketer can throw the same ball to a height of 50 m.

4.17. *A stone tied to the end of a string 80 cm long is whirled in a horizontal circle with a constant speed. If the stone makes 14 revolutions in 25 seconds, what is the magnitude and direction of acceleration of the stone ?*

Solution. Here $r = 80$ cm, $\nu = \frac{14}{25}$ rps

$$\therefore \quad \omega = 2\pi\nu = 2 \times \frac{22}{7} \times \frac{14}{25} = \frac{88}{25} \text{ rad s}^{-1}$$

The acceleration of the stone is

$$a = r\omega^2 = 80 \times \left(\frac{88}{25}\right)^2 = \mathbf{991.2\ cm\ s^{-2}}.$$

This acceleration is directed along the radius of the circular path towards the centre of the circle.

4.18. *An aircraft executes a horizontal loop of radius 1 km with a steady speed of 900 km h^{-1}. Compare its centripetal acceleration with the acceleration due to gravity.*

Solution. Here $r = 1$ km $= 1000$ m

$$v = 900 \text{ kmh}^{-1} = \frac{900 \times 5}{18} = 250 \text{ ms}^{-1}$$

Centripetal acceleration,

$$a = \frac{v^2}{r} = \frac{(250)^2}{1000} = 62.5 \text{ ms}^{-2}$$

$$\therefore \quad \frac{\text{Centripetal acceleration}}{\text{Acceleration due to gravity}} = \frac{62.5}{9.8} = \mathbf{6.38}.$$

4.19. *Read each statement below carefully and state, with reasons, if it is true or false :*

(*a*) *The net acceleration of a particle in circular motion is always along the radius of the circle towards the centre.*

(*b*) *The velocity vector of a particle at a point is always along the tangent to the path of the particle at that point.*

(*c*) *The acceleration vector of a particle in uniform circular motion averaged over one cycle is a null vector.*

Solution. (*a*) **False.** The net acceleration of a particle in circular motion is towards the centre only if its speed is constant.

(*b*) **True.** A particle released at any point of its path will always move along the tangent to the path at that point.

(*c*) **True.** For any two diametrically opposite points on the circumference, the acceleration vectors are equal and opposite. Hence the acceleration vector averaged over one complete cycle is a null vector.

4.20. *The position of a particle is given by*

$$\vec{r} = 3.0\, t\, \hat{i} - 2.0\, t^2\, \hat{j} + 4.0\, \hat{k} \text{ m}$$

where t is in seconds and the coefficients have the proper units for $\vec{r}$ to be in metres.

(a) Find the $\vec{v}$ and $\vec{a}$ of the particle. (b) What is the magnitude and direction of velocity of the particle at t = 2 s ? **[Delhi 10]**

Solution. (*a*) Given :

$$\vec{r}(t) = 3.0\, t\, \hat{i} - 2.0\, t^2\, \hat{j} + 4.0\, \hat{k} \text{ m}$$

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = \frac{d}{dt}(3.0\, t\, \hat{i} - 2.0\, t^2\, \hat{j} + 4.0\, \hat{k})$$
$$= \mathbf{3.0\, \hat{i} - 4.0\, t\, \hat{j}}$$

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = \frac{d}{dt}(3.0\, \hat{i} - 4.0\, t\, \hat{j}) = \mathbf{-4.0\, \hat{j}}.$$

(*b*) At $t = 2$ s, $\quad \vec{v} = 3.0\, \hat{i} - 8.0\, \hat{j}$

The magnitude of the velocity is

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{3^2 + (-8)^2} = \sqrt{73} = \mathbf{8.54\ ms^{-1}}.$$

The direction of velocity is given by

$$\theta = \tan^{-1}\left(\frac{v_y}{v_x}\right) = \tan^{-1}\left(-\frac{8}{3}\right)$$

$$= -\tan^{-1}(-2.6667)$$

$$\simeq -70^\circ \text{ with } x\text{-axis.}$$

4.21. *A particle starts from the origin at $t = 0$ s with a velocity of $10.0\,\hat{j}$ m/s and moves in the x-y plane with a constant acceleration of $(8.0\,\hat{i} + 2.0\,\hat{j})$ ms^{-2}. (a) At what time is the x-coordinate of the particle 16 m ? What is the y-coordinate of the particle at the time ? (b) What is the speed of the particle at the time ?*

Solution. (*a*) Here initial velocity,

$$\vec{v}_0 = 10.0\,\hat{j}\ \text{ms}^{-1}$$

Acceleration,

$$\vec{a} = (8.0\,\hat{i} + 2.0\,\hat{j})\ \text{ms}^{-2}$$

The position of the particle at any instant t will be

$$\vec{r}(t) = \vec{v}_0 t + \frac{1}{2}\vec{a}\, t^2$$

$$= 10.0\,\hat{j}\, t + \frac{1}{2}(8.0\,\hat{i} + 2.0\,\hat{j})\, t^2$$

or $x(t)\,\hat{i} + y(t)\,\hat{j} = 4.0\,t^2\,\hat{i} + (10.0\,t + 1.0\,t^2)\,\hat{j}$

$\therefore \quad x(t) = 4.0\,t^2$

$\quad\quad y(t) = 10.0\,t + 1.0\,t^2$

Given $x(t) = 16$ m, $t = ?$

$4.0\,t^2 = 16$

$\Rightarrow \quad$ **$t = 2$ s.**

At $t = 2$ s, $y = 10.0 \times 2 + 1.0 \times 2^2 = 24$ m.

(*b*) Velocity,

$$\vec{v} = \frac{d\vec{r}}{dt}$$

$$= \frac{d}{dt}[4.0\,t^2\,\hat{i} + (10.0\,t + 1.0\,t^2)\,\hat{j}]$$

$$= 8.0\,t\,\hat{i} + (10.0 + 2.0\,t)\,\hat{j}$$

At $t = 2$ s, $\vec{v} = 16.0\,\hat{i} + 14.0\,\hat{j}$

Speed, $\quad v = \sqrt{v_x^2 + v_y^2} = \sqrt{16^2 + 14^2}$

$$= \sqrt{256 + 196} = \sqrt{452} = \mathbf{21.26\ ms^{-1}}$$

4.22. (*a*) *If $\hat{i}$ and $\hat{j}$ are unit vectors along X- and Y-axis respectively, then what is the magnitude and direction of $\hat{i} + \hat{j}$ and $\hat{i} - \hat{j}$?*

(*b*) *Find the components of $\vec{a} = 2\,\hat{i} + 3\,\hat{j}$ along the directions of vectors $\hat{i} + \hat{j}$ and $\hat{i} - \hat{j}$.*

Solution. (*a*) In Fig. 4.107, $\vec{OA} = \hat{i}$, $\vec{AB} = \hat{j}$, $\vec{AC} = -\hat{j}$.

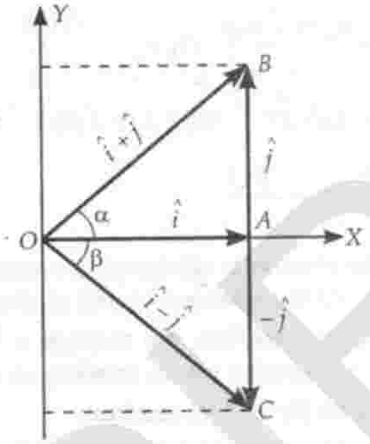


Fig. 4.107

Applying Δ law of vector addition in ΔOAB,

$$\vec{OB} = \vec{OA} + \vec{AB} = \hat{i} + \hat{j}$$

$\therefore \quad |\hat{i} + \hat{j}| = OB = \sqrt{OA^2 + AB^2} = \sqrt{1^2 + 1^2} = \mathbf{\sqrt{2}}$.

Direction of $\hat{i} + \hat{j}$ is given by

$$\tan\alpha = \frac{BA}{OA} = \frac{1}{1} = 1$$

$\therefore \quad$ **$\alpha = 45°$.**

Again, $\quad \vec{OC} = \vec{OA} + \vec{AC} = \hat{i} + (-\hat{j}) = \hat{i} - \hat{j}$

$\therefore \quad |\hat{i} - \hat{j}| = OC = \sqrt{OA^2 + AC^2} = \sqrt{1^2 + 1^2} = \sqrt{2}$.

Direction of $\hat{i} - \hat{j}$ is given by

$$\tan\beta = \frac{AC}{OA} = \frac{-1}{1} = -1$$

$\therefore \quad$ **$\beta = -45°$.**

(*b*) Given $\vec{a} = 2\hat{i} + 3\hat{j}$ and let $\vec{b} = \hat{i} + \hat{j}$ and $\hat{c} = \hat{i} - \hat{j}$. Then component of $\vec{a}$ in the direction $\vec{b}$

$$= (a\cos\theta)\,\hat{b}$$

$$= \frac{ab\cos\theta}{b}\frac{\vec{b}}{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\vec{b}$$

$$= \frac{(2\hat{i} + 3\hat{j})\cdot(\hat{i} + \hat{j})}{[\sqrt{1^2 + 1^2}]^2}(\hat{i} + \hat{j})$$

$$= \frac{2\times 1 + 3\times 1}{2}(\hat{i} + \hat{j}) = \frac{5}{2}(\hat{\mathbf{i}} + \hat{\mathbf{j}}).$$

Component of $\vec{a}$ in the direction of $\vec{c}$

$$= \frac{\vec{a}\cdot\vec{c}}{|\vec{c}|^2}\vec{c} = \frac{(2\hat{i} + 3\hat{j})\cdot(\hat{i} - \hat{j})}{[\sqrt{1^2 + 1^2}]^2}(\hat{i} - \hat{j})$$

$$= \frac{2\times 1 + 3\times(-1)}{2}(\hat{i} - \hat{j}) = -\frac{1}{2}(\hat{\mathbf{i}} - \hat{\mathbf{j}}).$$

4.23. *For any arbitrary motion in space, which of the following relations are true :*

(*a*) $\vec{v}_{av} = \frac{\vec{v}(t_1) + \vec{v}(t_2)}{2}$ (*b*) $\vec{v}_{av} = \frac{\vec{r}(t_2) - \vec{r}(t_1)}{t_2 - t_1}$

(*c*) $\vec{v}(t) = \vec{v}(0) + \vec{a}\,t$ (*d*) $\vec{r}(t) = \vec{r}(0) + \vec{v}(0)\,t + \frac{1}{2}\,\vec{a}\,t^2$

(*e*) $\vec{a}_{av} = \frac{\vec{v}(t_2) - \vec{v}(t_1)}{t_2 - t_1}$.

Ans. As the motion is arbitrary, the acceleration may not be uniform. So the relations (*c*) and (*d*) cannot be true.

For an arbitrary motion, the average velocity cannot be defined as in equation (*a*), so relation (*a*) is not true.

Only relations (*b*) and (*e*) are true.

4.24. *Read each statement below carefully and state, with reasons and examples, if it is true or false :*

A scalar quantity is one that (a) is conserved in a process, (b) can never take negative values, (c) must be dimensionless, (d) does not vary from one point to another in space, (e) has the same value for observers with different orientations of axes.

Solution. (*a*) **False.** Kinetic energy (scalar) is not conserved in an inelastic collision. Moreover, vector quantities like linear momentum, angular momentum, etc., are also conserved.

(*b*) **False.** Scalar quantities such as electric potential, temperature, etc., can take negative values.

(*c*) **False.** Scalar quantities like mass, density, energy, etc., are not dimensionless.

(*d*) **False.** Density (scalar) varies from point to point in the atmosphere.

(*e*) **True.** The mass (scalar) of a body as measured by different observers with different orientations of axes has the same value.

4.25. *An aircraft is flying at a height of 3400 m above the ground. If the angle subtended at a ground observation point by the aircraft positions 10 s apart is 30°, what is the speed of the aircraft ?* **[Central Schools 12]**

Solution. Let A and B represent the two aircraft positions separated 10 s apart (Fig. 4.108).

Then $\tan 15° = \frac{x}{3400}$

or $x = 3400 \tan 15°$

$= 3400 \times 0.2679$

$= 910.86$ m.

Speed of aircraft

$= \frac{910.86 \text{ m}}{5 \text{ s}} = \mathbf{182.2 \ ms^{-1}}$.

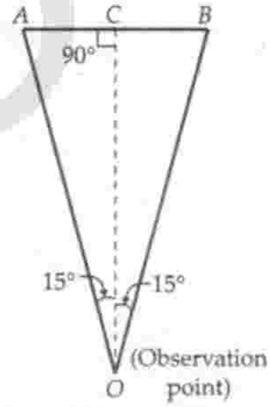


Fig. 4.108

4.26. *A vector has magnitude and direction. Does it have a location in space ? Can it vary with time ? Will two equal vectors $\vec{a}$ and $\vec{b}$ at different locations in space necessarily have identical physical effects ? Give examples in support of your answer.*

Solution. In addition to magnitude and direction, each vector also has a definite location in space. For example, a velocity vector has definite location at every point of uniform circular motion.

A vector can vary with time. For example, increase in velocity produces acceleration.

Two equal vectors $\vec{a}$ and $\vec{b}$ having different locations may not produce identical physical effects. For example, two equal forces (vectors) acting at two different points may not produce equal turning effects.

4.27. *A vector has both magnitude and direction. Does it mean that anything that has magnitude and direction is necessarily a vector ? The rotation of a body can be specified by the direction of the axis of rotation, and the angle of rotation about the axis. Does that make any rotation a vector ?*

Solution. No, anything that has both magnitude and direction is not necessarily a vector. It must obey the laws of vector addition.

Rotation is not generally considered a vector even though it has magnitude and direction because the addition of two finite rotations does not obey commutative law. However, infinitesimally small rotations obey commutative law and hence an infinitesimally small rotation is considered a vector.

4.28. *Can you associate vectors with (a) the length of a wire bent into a loop, (b) a plane area, (c) a sphere ? Explain.*

Solution. Out of these, only a plane area can be associated with a vector. The direction of this *area vector* is taken normal to the plane.

4.29. *A bullet fired at an angle of 30° with the horizontal hits the ground 3 km away. By adjusting the angle of projection, can one hope to hit a target 5 km away ? Assume the muzzle speed to be fixed and neglect air resistance.*

Solution. In the first case, $R = 3$ km $= 3000$ m, $\theta = 30°$

Horizontal range, $R = \frac{u^2 \sin 2\theta}{g}$

$\therefore \quad 3000 = \frac{u^2 \sin 60°}{g}$

or $\quad \frac{u^2}{g} = \frac{3000}{\sin 60°} = \frac{3000 \times 2}{\sqrt{3}} = 2000\sqrt{3}$

Maximum horizontal range,

$$R_{max} = \frac{u^2}{g} = 2000\sqrt{3} \text{ m} = 3464 \text{ m} = \mathbf{3.46 \ km.}$$

But distance of the target (5 km) is greater than the maximum horizontal range of 3.46 km, so the target cannot be hit by adjusting the angle of projection.

4.30. *A fighter plane flying horizontally at an altitude of 1.5 km with a speed 720 kmh^{-1} passes directly overhead an antiaircraft gun. At what angle from the vertical should the gun be fired for the shell muzzle speed 600 ms^{-1} to hit the plane ? At what maximum altitude should the pilot fly the plane to avoid being hit ? Take $g = 10$ ms^{-2}.*

Solution. Speed of plane = 720 kmh^{-1} = 200 ms^{-1}.

The shell moves along curve *OL*. The plane moves along *PL*. Let them hit after a time *t*.

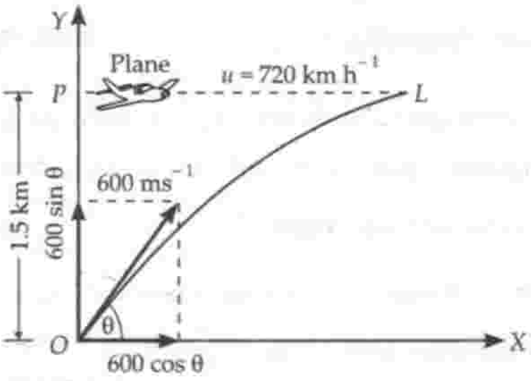


Fig. 4.109

For hitting, horizontal distance travelled by the plane

= Horizontal distance travelled by the shell.

or Horizontal velocity of plane × *t*

= Horizontal velocity of shell × *t*

$$200 \times t = 600 \cos\theta \times t$$

$$\cos\theta = \frac{200}{600} = \frac{1}{3} \quad \text{or} \quad \theta = 70°30'$$

The shell should be fired at an angle of 70°30′ with the horizontal or 19°30′ with the vertical.

The maximum height of flight of the shell is

$$h = \frac{u^2 \sin^2\theta}{2g} = \frac{u^2(1-\cos^2\theta)}{2g}$$

$$= \frac{(600)^2 \times (1-\frac{1}{9})}{2\times 10} = 16000 \text{ m} = \mathbf{16\ km.}$$

Thus the pilot should fly the plane at a minimum altitude of 16 km to avoid being hit by the shell.

4.31. *A cyclist is riding with a speed of 27 km h^{-1}. As he approaches a circular turn on the road of radius 80 m, he applies brakes and reduces his speed at the constant rate 0.5 ms^{-2}. What is the magnitude and direction of the net acceleration of the cyclist on the circular turn ?*

Solution. Here $r = 80$ m

$$v = 27 \text{ kmh}^{-1} = \frac{27\times 5}{18} \text{ ms}^{-1} = 7.5 \text{ ms}^{-1},$$

Centripetal acceleration, $a_c = \frac{v^2}{r} = \frac{(7.5)^2}{80} = 0.7 \text{ ms}^{-2}$

Suppose the cyclist applies brakes at the point *A* of the circular turn, then, tangential acceleration a_T (negative) will act opposite to velocity.

Given $a_T = 0.5 \text{ ms}^{-2}$

As the accelerations a_C and a_T are perpendicular to each other, so the net acceleration of the cyclist is

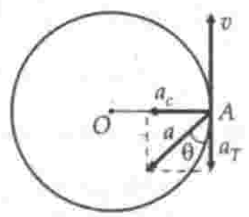


Fig. 4.110

$$a = \sqrt{a_C^2 + a_T^2} = \sqrt{(0.7)^2 + (0.5)^2}$$

$$= \sqrt{0.49 + 0.25} = \sqrt{0.74} = \mathbf{0.86\ ms^{-2}}.$$

If θ is the angle between the total acceleration and the velocity of the cyclist, then,

$$\tan\theta = \frac{a_C}{a_T} = \frac{0.7}{0.5} = 1.4 \quad \text{or} \quad \theta = \mathbf{54°28'}.$$

4.32. (*a*) *Show that for a projectile the angle between the velocity and the X-axis as a function of time is given by*

$$\theta(t) = \tan^{-1}\frac{(v_{oy} - gt)}{v_{ox}}$$

(*b*) *Show that the projection angle θ_0 for a projectile launched from the origin is given by :*

$$\theta_0 = \tan^{-1}\left(\frac{4h_m}{R}\right)$$

where the symbols have their usual meaning.

Ans. (*a*) As shown in Fig. 4.111, suppose the projectile is thrown with a velocity v_0 at an angle θ_0 with the *X*-axis.

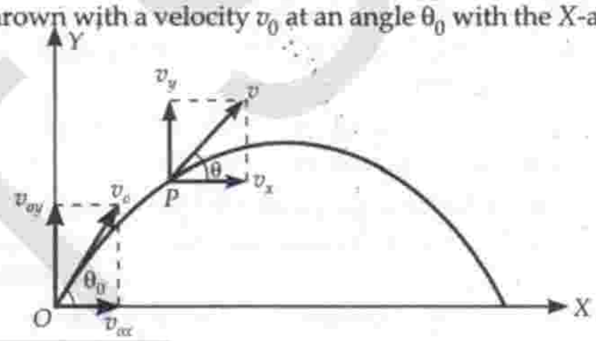


Fig. 4.111

Let v_{ox} and v_{oy} be the components of velocity v_0 along *X*- and *Y*-directions respectively.

At any time *t*, suppose the particle is at point *P*. Its velocity *v* has components v_x and v_y along X- and Y-directions. Then

$v_x = v_{0x}$ [Horizontal component remains unchanged]

and $v_y = v_{0y} - gt$

If the velocity *v* makes angle θ with X-axis, then

$$\tan\theta = \frac{v_y}{v_x} = \frac{v_{0y} - gt}{v_{0x}} \quad \therefore \quad \theta = \tan^{-1}\left(\frac{v_{0y} - gt}{v_{0x}}\right)$$

(*b*) Maximum height attained by the projectile ,

$$h_m = \frac{v_0^2 \sin^2\theta_0}{2g}$$

Horizontal range of the projectile,

$$R = \frac{v_0^2 \sin 2\theta_0}{g} = \frac{v_0^2 \times 2\sin\theta_0\cos\theta_0}{g}$$

$$\therefore \quad \frac{h_m}{R} = \frac{v_0^2 \sin^2\theta_0}{2g} \times \frac{g}{v_0^2 \times 2\sin\theta_0\cos\theta_0} = \frac{\tan\theta_0}{4}$$

or $$\tan\theta_0 = \frac{4h_m}{R}$$

Hence, $$\theta_0 = \tan^{-1}\frac{4h_m}{R}.$$

Text Based Exercises

Type A : Very Short Answer Questions **1 Mark Each**

1. Does a scalar quantity depend on the frame of reference chosen ?
2. A given quantity has both magnitude and direction. Is it necessarily a vector ?
3. When is the sum of two vectors (*i*) maximum and (*ii*) minimum ?
4. When can the addition of two vectors be zero ?
5. Is the magnitude of $(\vec{A}+\vec{B})$ same as that of $(\vec{B}+\vec{A})$?
6. Under what condition the sum and difference of two vectors will be equal in magnitude ?
7. What is a resultant vector ?
8. The magnitude of the resultant of two vectors of magnitudes 5 and 3 is 2. What is the angle between the two vectors ?
9. Can three vectors of different magnitudes be combined to give a zero resultant ?
10. What is the minimum number of forces acting on an object in a plane that can produce a zero resultant force ? **[Delhi 04]**
11. What is the minimum number of forces of equal magnitude whose vector sum is zero ?
12. Does it make a sense to call a physical quantity a vector, when its magnitude is zero ?
13. Give an example of a zero vector.
14. If $|\vec{P}|>|\vec{Q}|$, what is the angle between the maximum resultant and minimum resultant of the two vectors $\vec{P}$ and $\vec{Q}$?
15. Is the division of vectors defined ?
16. Can the sum of two vectors be a scalar ?
17. Can any of the components of a given vector have greater magnitude than that of the vector itself ?
18. Fifteen vectors, each of magnitude 5 units, are represented by the sides of a closed polygon, all taken in same order. What will be their resultant ?
19. Under what condition is the scalar product of two non-zero vectors zero ?
20. Give one example of dot product of vectors. Also give its unit. **[Delhi 98]**
21. What is the magnitude of the vector $2\hat{i}-3\hat{j}+\sqrt{3}\,\hat{k}$?
22. Show that $\vec{A}\cdot\vec{A}=A^2$.
23. Can the magnitude of the resultant vector of two given vectors be less than the magnitude of either vector ?
24. The magnitudes of vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ are 12, 5 and 13 units respectively and $\vec{A}+\vec{B}=\vec{C}$. What is the angle between $\vec{A}$ and $\vec{B}$? **[Delhi 96]**
25. If $\vec{A}$, $\vec{B}$ and $\vec{C}$ are mutually perpendicular vectors, then what is the value of $\vec{A}\cdot(\vec{B}+\vec{C})$?
26. What is the angle between $(\hat{i}+\hat{j})$ and $(\hat{i}-\hat{j})$?
27. A unit vector is represented by $a\hat{i}+b\hat{j}+c\hat{k}$. If the values of a and b are 0.6 and 0.8 respectively, find the value of c. **[Delhi 05]**
28. Given four forces :
$$\vec{F}_1=3\hat{i}-\hat{j}+9\hat{k},\qquad \vec{F}_2=2\hat{i}-2\hat{j}+16\hat{k},$$
$$\vec{F}_3=9\hat{i}+\hat{j}+18\hat{k}\quad\text{and}\quad \vec{F}_4=\hat{i}+2\hat{j}-18\hat{k}.$$
If all these forces act on a particle at rest at the origin of a coordinate system, then identify the plane in which the particle would begin to move ?
29. Two vectors of magnitudes 3 and 4 give a resultant of magnitude 2. What must be the dot product of these two vectors ?
30. Two vectors of magnitudes 5 and 3 give a resultant of magnitude 2. What must be the angle between the two vectors ?
31. When vectors are added or substrated, their resultant is a vector. Is it also true in case of multiplication of two vectors ?
32. Give an example of a physical quantity which can be represented as a vector product of two vectors.
33. What is the angle between the vectors $\vec{A}\times\vec{B}$ and $\vec{B}\times\vec{A}$?
34. What is the angle between two vectors if the ratio of their dot product and the magnitude of cross product is $\sqrt{3}$?
35. If $\vec{A}\times\vec{B}=\vec{0}$, what can be said about the non-zero vectors $\vec{A}$ and $\vec{B}$? **[Delhi 97]**
36. What will be the angle between $\vec{A}$ and $\vec{B}$, if the area of the parallelogram drawn with $\vec{A}$ and $\vec{B}$ as its adjacent sides is $\frac{1}{2}AB$?

37. Two vectors of magnitude A and $\sqrt{3}A$ are perpendicular to each other. What is the angle which their resultant makes with $\vec{A}$?
38. If $\vec{A}+\vec{B}+\vec{C}+\vec{D}=\vec{0}$, can the magnitude of $\vec{A}+\vec{B}+\vec{C}$ be equal to the magnitude of $\vec{D}$?
39. What are horizontal and vertical components of acceleration of a body thrown horizontally with uniform speed ?
40. Velocity of a projectile is $10\ ms^{-1}$. At what angle to the horizontal should it be projected so that it covers maximum horizontal distance ? **[Delhi 99]**
41. Can a body possess velocity at the same time in horizontal and vertical directions ?
42. Name the quantity which remains unchanged during the flight of an oblique projectile.
43. A football is thrown in a parabolic path. Is there any point at which the acceleration is perpendicular to the velocity ?
44. A projectile fired from the ground follows a parabolic path. The speed of the projectile is minimum at the top of its path. State whether this statement is true or false.
45. Is it true that the instantaneous velocity of a projectile is tangential to its parabolic path ?
46. A body is projected at an angle of 45° with a velocity of $9.8\ ms^{-1}$. What will be its horizontal range ?
47. At what angle to the horizontal should a body be projected so that the maximum height reached is equal to the horizontal range ?
48. Why does the direction of motion of a projectile become horizontal at the highest point of its trajectory ?
49. A projectile is fired with kinetic energy 1 kJ. If its range is maximum, what will be its kinetic energy at the highest point of its trajectory ?
50. A body is projected so that it has maximum range R. What is the maximum height reached during the flight ?
51. What is the maximum vertical height to which a cricketer can throw a ball if he can throw it to a maximum horizontal distance of 160 m ?
52. Two bodies are projected at angles θ and $(90° - \theta)$ to the horizontal with the same speed. Find the ratio of their time of flight.
53. A particle moves in a plane with uniform acceleration in a direction different from its initial velocity. What will be the trajectory followed by the particle ?
54. Name two quantities which have maximum values when the maximum height attained by the projectile is the largest.
55. The range of a projectile when launched at an angle of 15° to the horizontal is 1.5 km. What is the range of the projectile when launched at an angle of 45° to the horizontal ?
56. What are the angles of projection of a projectile projected with velocity $30\ ms^{-1}$, so that the horizontal range is 45 m ? Take $g = 10\ ms^{-2}$.
57. A bomb is dropped from an aeroplane flying horizontally with a speed of $720\ kmh^{-1}$, at an altitude of 980 m. After what time, the bomb will hit the ground ?
58. A ball is thrown at an angle of 45° to the horizontal with kinetic energy K. What is the kinetic energy at the highest point of trajectory ?
59. A ball is thrown at an angle θ with the horizontal. Its time of flight is 2 s. Its horizontal range is 100 m. What is the horizontal component of velocity of projection ? Take $g = 10\ ms^{-2}$.
60. The maximum range of a projectile is $2/\sqrt{3}$ times its actual range. What is the angle of projection for actual range ? **[Central Schools 08]**
61. What is the angular velocity of the hour hand of a clock ?
62. What furnishes the centripetal acceleration for the earth to go round the sun ?
63. Express the unit vector $\hat{A}$ in a mathematical form. **[Himachal 03]**
64. Find a unit vector parallel to the vector $3\hat{i}+7\hat{j}+4\hat{k}$. **[Chandigarh 02]**
65. Evaluate $\hat{k}\cdot\hat{k}$. **[Himachal 01]**
66. Can two non-zero vectors give zero resultant when they multiply with each other ? If yes, give condition for the same. **[Delhi 06]**
67. A vector $\vec{A}$ is expressed as $\vec{A} = A_x\hat{i} + A_y\hat{j}$ where A_x and A_y are its components along x-axis and y-axis respectively. If vector $\vec{A}$ makes an angle θ with x-axis, then θ is given by which expression ? **[Delhi 08]**
68. What is the value of $\vec{A}\times\vec{A}$? **[Central Schools 12]**

Answers

1. No. A scalar quantity does not depend on the frame of reference.
2. No. The given quantity will be a vector only if it obeys the laws of vector addition.

3. (*i*) When two vectors have same direction, their sum is maximum.
   (*ii*) When two vectors have opposite directions, their sum is minimum.
4. The addition of two vectors can be zero only when these vectors have equal magnitude and opposite directions.
5. Yes. $|\vec{A}+\vec{B}|=|\vec{B}+\vec{A}|$.
6. When the two vectors are perpendicular to each other.
7. The resultant of two or more vectors is that single vector which produces the same effect as the individual vectors together produce.
8. 180°, *i.e.*, the two vectors have opposite directions.
9. Yes, when three vectors are represented by the three sides of a triangle taken in order, they will have resultant null-vector.
10. Three forces, provided they can be represented by the three sides of a triangle taken in the same order.
11. Two forces, provided the two forces have opposite directions.
12. Yes. A zero vector has a definite physical significance.
13. The displacement of a ball thrown up and received back by the thrower is a zero vector.
14. 0°.
15. No.
16. No, the sum of two vectors is always a vector.
17. Yes, provided the components are not rectangular.
18. Zero vector. The vector sum of all the vectors represented by the sides of a closed polygon taken in the same order is zero.
19. When they are mutually perpendicular to each other.
20. Work done is equal to the dot product of force and displacement vectors.
    $W=\vec{F}\cdot\vec{S}$
    The SI unit of work is joule.
21. $|2\hat{i}-3\hat{j}+\sqrt{3}\hat{k}|=\sqrt{2^2+(-3)^2+(\sqrt{3})^2}$
    $=\sqrt{4+9+3}=\mathbf{4}$
22. $\vec{A}\cdot\vec{A}=|\vec{A}||\vec{A}|\cos 0°=|\vec{A}|^2=A^2$.
23. Yes. This is possible when the angle θ between the two given vectors lies between 90° and 270°, because in that case $\cos\theta$ is negative.
24. Clearly $12^2+5^2=13^2$ *i.e.*, $A^2+B^2=C^2$. Hence the angle between $\vec{A}$ and $\vec{B}$ is 90°.
25. $\vec{A}\cdot(\vec{B}+\vec{C})=\vec{A}\cdot\vec{B}+\vec{A}\cdot\vec{C}$
    $=AB\cos 90°+AC\cos 90°=\mathbf{0}$.
26. **90°**, because $(\hat{i}+\hat{j})\cdot(\hat{i}-\hat{j})=1.1-1.1=0$.
27. $\sqrt{0.6^2+0.8^2+c^2}=1 \quad \therefore \quad c=0$.
28. Resultant force,
    $\vec{F}=\vec{F}_1+\vec{F}_2+\vec{F}_3+\vec{F}_4=15\hat{i}+0\hat{j}+25\hat{k}$.
    Clearly, the particle will move in the X-Z plane.
29. As $3^2+4^2=5^2$, so the two vectors are perpendicular to each other. Their dot product will be **zero**.
30. As $5-3=2$, this indicates that the two vectors have opposite directions and angle between them = **180°**.
31. May or may not be. The dot product of two vectors gives a scalar quantity while the cross product gives a vector quantity.
32. Torque $\vec{\tau}$ is the vector product of position vector $\vec{r}$ and force vector $\vec{F}$, *i.e.*, $\vec{\tau}=\vec{r}\times\vec{F}$.
33. 180°, because $\vec{A}\times\vec{B}$ and $\vec{B}\times\vec{A}$ are antiparallel.
34. $\dfrac{\vec{A}\cdot\vec{B}}{|\vec{A}\times\vec{B}|}=\dfrac{AB\cos\theta}{AB\sin\theta}=\cot\theta=\sqrt{3}$.
    or $\tan\theta=\dfrac{1}{\sqrt{3}} \Rightarrow \theta=\mathbf{30°}$.
35. The vectors $\vec{A}$ and $\vec{B}$ are parallel vectors.
36. Area of parallelogram
    $=|\vec{A}\times\vec{B}|=AB\sin\theta=\dfrac{1}{2}AB$
    $\therefore \quad \sin\theta=\dfrac{1}{2} \Rightarrow \theta=\mathbf{30°}$.
37. Here $\tan\theta=\dfrac{\sqrt{3}A}{A}=\sqrt{3} \Rightarrow \theta=\mathbf{60°}$.
38. Clearly, $\vec{A}+\vec{B}+\vec{C}=\vec{0}-\vec{D}=-\vec{D}$
    Hence the magnitude of $(\vec{A}+\vec{B}+\vec{C})$ will be equal to that of $\vec{D}$.
39. For a body thrown horizontally with a uniform speed,
    acceleration along horizontal = 0
    acceleration along vertical = g.
40. At an angle of 45° to the horizontal.
41. Yes, a body in projectile motion possesses both horizontal and vertical velocities.
42. Horizontal component of velocity.
43. Yes, at the highest point of the parabolic path.

44. The statement is true. The horizontal component of the velocity will remain constant throughout the motion whereas the vertical component of the velocity will be zero at the top. Hence the resultant velocity of the projectile will be minimum at the top.

45. Yes.

46. Horizontal range will be maximum at 45°. It is given by

$$R_{max} = \frac{u^2}{g} = \frac{9.8 \times 9.8}{9.8} = \mathbf{9.8\ m}.$$

47. Maximum height = Horizontal range

or $$\frac{u^2 \sin^2 \theta}{2g} = \frac{u^2 \sin 2\theta}{g}$$

On solving, $\tan \theta = 4$ or $\theta = \mathbf{76°}$.

48. At the highest point, the vertical component of projectile velocity becomes zero. The projectile has only the horizontal component of velocity.

49. K.E. at the highest point

$= \text{Initial K.E.} \times \cos^2 \theta = 1\text{ kJ} \times \cos^2 45° = \mathbf{500\ J}.$

50. $H_{max} = R/4.$

51. Max. horizontal range $= u^2/g = 160$ m

Max. vertical height $= u^2/2g = 80$ m.

52. $T_1 = \frac{2u \sin \theta}{g}$ and $T_2 = \frac{2u \sin (90° - \theta)}{g} = \frac{2u \cos \theta}{g}$

$\therefore \quad \frac{T_1}{T_2} = \frac{\sin \theta}{\cos \theta} = \mathbf{\tan \theta : 1}.$

53. Parabolic path.

54. (*i*) Vertical component of initial velocity.

(*ii*) Time of flight.

55. When $\theta = 15°$,

$$R = \frac{u^2 \sin 30°}{g} = \frac{u^2}{g} \times \frac{1}{2} = 1.5 \quad \text{or} \quad \frac{u^2}{g} = 3\text{ km}$$

When $\theta = 45°$, $R = \frac{u^2 \sin 90°}{g} = \frac{u^2}{g} = 3$ km.

56. $R = \frac{u^2 \sin 2\theta}{g} \quad \therefore \quad 45 = \frac{30^2 \sin 2\theta}{10}$

or $\sin 2\theta = 1/2 = \sin 30°$ or $\sin 150°$

$\therefore \quad \theta = \mathbf{15°}$ or $\mathbf{75°}$.

57. $t = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2 \times 980}{9.8}} = 10\sqrt{2} = \mathbf{14.14\ s}$

58. K.E. at the highest point

$= K \cos^2 \theta = K \cos^2 45° = \mathbf{K/2}$

59. $T = \frac{2u \sin \theta}{g}$ and $R = \frac{2u^2 \sin \theta \cos \theta}{g}$

Horizontal component

$= u \cos \theta = \frac{R}{T} = \frac{100\text{ m}}{2\text{ s}} = \mathbf{50\ ms^{-1}}.$

60. $R_{max} = \frac{2}{\sqrt{3}} R$ or $\frac{u^2}{g} = \frac{2}{\sqrt{3}} \times \frac{u^2 \sin 2\theta}{g}$

or $\sin 2\theta = \frac{\sqrt{3}}{2} = \sin 60° \quad \therefore \quad \theta = \mathbf{30°}$

61. $\omega = \frac{2\pi \text{ radian}}{12 \text{ hour}} = \frac{\pi}{6} \text{ rad h}^{-1}.$

62. The gravitational pull of the sun on the earth.

63. $\hat{A} = \frac{\vec{A}}{|\vec{A}|}.$

64. The unit vector parallel to the vector $3\hat{i} + 7\hat{j} + 4\hat{k}$

$$= \frac{3\hat{i} + 7\hat{j} + 4\hat{k}}{\sqrt{3^2 + 7^2 + 4^2}} = \frac{3\hat{i} + 7\hat{j} + 4\hat{k}}{\sqrt{74}}.$$

65. $\hat{k} \cdot \hat{k} = (1)(1) \cos 0° = 1$

66. Yes. For example, the cross product of two non-zero vectors will be zero when angle $\theta = 0°$ or $180°$.

67. $\tan \theta = \frac{A_y}{A_x}.$

68. $\vec{A} \times \vec{A} = \vec{0}.$

Type B : Short Answer Questions

2 or 3 Marks Each

1. Distinguish between position and displacement vectors.
2. Define (*i*) unit vector (*ii*) null vector (*iii*) cross product of two vectors $\vec{A}$ and $\vec{B}$. **[Manipur 98]**
3. State and prove the commutative property of vector addition. **[Himachal 07 ; Central Schools 03]**
4. State and prove the polygon law of vector addition. **[Himachal 01C, 07, 07C, 09C ; Chandigarh 03]**
5. Give one example of dot product of vectors. Also give its unit. **[Delhi 98]**
6. Suppose you have two forces $\vec{F}$ and $\vec{F}$. How would you combine them in order to have resultant force of magnitudes

   (*a*) zero, (*b*) $2\vec{F}$, and (*c*) $\vec{F}$? **[Delhi 03]**
7. Given that

   $\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$

   and $\vec{B} = B_x \hat{i} + B_y \hat{j} + B_z \hat{k}$.

   Find $\vec{A} \times \vec{B}$. **[NCERT]**

8. Find the scalar and vector product of two vectors,
$\vec{a} = (3\hat{i} - 4\hat{j} + 5\hat{k})$ and
$\vec{b} = (-2\hat{i} + \hat{j} + 3\hat{k})$. [NCERT]

9. Find the work done in moving a particle along a vector $\vec{S} = (4\hat{i} - \hat{j} + 7\hat{k})$ if the applied force is $\vec{F} = (\hat{i} + 2\hat{j} - \hat{k})$ newton. $\vec{S}$ is in metre. [Delhi 99]

10. The angle between vectors $\vec{A}$ and $\vec{B}$ is 60°. What is the ratio of $\vec{A} \cdot \vec{B}$ and $|\vec{A} \times \vec{B}|$? [Central Schools 05]

11. State parallelogram law of vector addition. Show that resultant of two vectors $\vec{A}$ and $\vec{B}$ inclined at an angle θ is $R = \sqrt{A^2 + B^2 + 2AB\cos\theta}$. [Delhi 03 ; Himachal 03, 04, 07, 09C ; Chandigarh 08]

12. A man moving in rain holds his umbrella inclined to the vertical even though the rain drops are falling vertically downwards. Why ? [Delhi 1999]

13. Obtain an expression for the area of a triangle in terms of the cross product of two vectors representing the two sides of the triangle. [NCERT]

14. Show that two dimensional uniform velocity motion is equivalent to two one dimensional uniform velocity motion along two coordinate axes. [Himachal 07C, 08C]

15. Define uniform acceleration. Show that in two-dimensional motion with uniform acceleration, each rectangular component of velocity is similar to that of uniformly accelerated motion along one dimension.

16. Write an expression for the instantaneous position vector of an object having two dimensional motion under uniform acceleration. Hence, show that a two dimensional uniformly accelerated motion is a combination of two rectangular uniformly accelerated motions.

17. Show that for a particle moving in two dimensions,
$$\vec{r'} = \vec{r} + \vec{v}(t'-t) + \frac{1}{2}\vec{a}(t'-t)^2$$
where $\vec{r}$ and $\vec{r'}$ are the position vectors of the particle at time t and t' respectively. [Himachal 06C]

18. A projectile is fired with a velocity 'u' making an angle θ with the horizontal. Show that its trajectory is a parabola. [Central Schools 12 ; Delhi 06, 08]

19. A projectile is projected with velocity u making angle θ with horizontal direction, find :
(*a*) time of flight
(*b*) horizontal range [Central Schools 04]

20. Show that there are two angles of projection for which the horizontal range is same for a projectile. [Delhi 95 ; Himachal 06]

21. Show that there are two angles of projection for a projectile to have the same horizontal range. What will be the maximum heights attained in the two cases ? Compare the two heights for $\theta = 30°$ and 60°. [Himachal 07C]

22. Find the angle of projection at which the horizontal range and maximum height of a projectile are equal.

23. What is a projectile ? Find its horizontal range and show it is maximum for an angle of 45°. [Himachal 01C]

24. A projectile is fired at an angle θ with the horizontal with velocity u. Deduce the expression for the maximum height reached by it. [Delhi 12]

25. A projectile is fired with velocity 'u', making an angle θ with the horizontal from the surface of earth. Prove that the projectile will hit the surface of earth with same velocity and at the same angle. [Chandigarh 07]

26. Prove the following statement "For Elevations which exceed or fall short of 45° by equal amounts, the ranges are equal". [Central Schools 08]

27. Prove that the maximum horizontal range is four tims the maximum height attained by the projectile when fired at an inclination so as to have maximum horizontal range. [Himachal 06, 07 ; Chandigarh 08]

28. Justify the statement that a uniform circular motion is an accelerated motion. [Delhi 99]

29. Derive the relation between linear displacement and angular displacement. [Himachal 07]

30. Establish a relation between linear velocity and angular velocity in a uniform circular motion and explain the direction of linear velocity. [Himachal 06, 07, 09 ; Chandigarh 03]

31. Define angular acceleration. Establish its relation with linear acceleration. [Himachal 06]

32. Derive an expression for the acceleration of a body moving in a circular path of radius 'r' with uniform speed 'v'. [Delhi 97]

33. Define Angular velocity and Angular acceleration. Derive an expression for centripetal acceleration of a uniform circular motion of an object. [Delhi 98, BIT Ranchi 99]

34. Derive an expression for centripetal acceleration of an object in uniform circular motion in a plane. What will be the direction of the velocity and acceleration at any instant ? [Himachal 04, 06C ; Chandigarh 08 ; Central Schools 08]

35. For a uniform circular motion show that :
(*i*) $v = rw$ (*ii*) $a = rw^2$ [Central Schools 12]

Answers

1. Refer answer to Q. 5 on page 4.2.
2. Refer answer to Q. 7 on page 4.2 and Q. 33 on page 4.25.
3. Refer answer to Q. 15 on page 4.5.
4. Refer answer to Q. 14 on page 4.5.
5. Refer answer to Very Short Answer Q. 20 on page 4.82.
6. (*i*) When the two forces act in opposite directions, the magnitude of their resultant is zero.

   (*ii*) When the two forces act in the same direction, the magnitude of their resultant = Magnitude of $2\vec{F}$.

   (*iii*) When angle between $\vec{F}$ and $\vec{F}$ is 120°, the magnitude of their resultant = Magnitude of $\vec{F}$. For details, refer to the solution of Example 5 on page 4.9.
7. Refer answer to Q. 38 on page 4.27.
8. $\vec{a}\cdot\vec{b} = -25$ and $\vec{a}\times\vec{b} = 7\hat{i}-\hat{j}-5\hat{k}$.
9. $W = \vec{F}\cdot\vec{S}$

   $= (\hat{i}+2\hat{j}-\hat{k})\cdot(4\hat{i}-\hat{j}+7\hat{k})$

   $= 4-2-7 = \mathbf{-5\ J}$.
10. $\dfrac{\vec{A}\cdot\vec{B}}{|\vec{A}\times\vec{B}|} = \dfrac{AB\cos\theta}{AB\sin\theta}$

    $= \cot\theta = \cot 60° = \dfrac{1}{\sqrt{3}}$.
11. Refer answer to Q. 19 on page 4.7.
12. Refer answer to Q. 48 on page 4.37.
13. Refer answer to Q. 34 on page 4.25.
14. Refer answer to Q. 45 on page 4.34.
15. Refer answer to Q. 46(*a*) on page 4.34.
16. Refer answer to Q. 46(*b*) on page 4.35.
17. Refer answer to Q. 46(*b*) on page 4.35.
18. Refer answer to Q. 53 on page 4.44.
19. Refer answer to Q. 53 on page 4.44.
20. Refer answer to Q. 54 on page 4.45.
21. Refer answer to Q.54 on page 4.45.

    Max. height attained in first case,
    $$H_1 = \frac{u^2\sin^2\theta}{2g}$$
    Max. height attained in second case,
    $$H_2 = \frac{u^2\sin^2(90°-\theta)}{2g} = \frac{u^2\cos^2\theta}{2g}$$
    Again,
    $$\frac{H(30°)}{H(60°)} = \frac{u^2\sin^2 30°}{2g}\cdot\frac{2g}{u^2\sin^2 60°}$$
    $$= \frac{(1/2)^2}{(\sqrt{3}/2)^2} = \frac{1}{3} = \mathbf{1:3}.$$
22. Refer to the solution of Example 76 on page 4.48.
23. Refer answer to Q. 54 on page 4.45.
24. Refer answer to Q. 53 on page 4.44
25. Refer answer to Q. 54 on page 4.45.
26. Refer to the solution of Problem 15 on page 4.68.
27. Refer to the solution of Example 77 on page 4.48.
28. Refer answer to Q. 56 on page 4.55.
29. Refer answer to Q.57(*i*) on page 4.56.
30. Refer answer to Q. 59 on page 4.56.
31. Refer answer to Q.60 on page 4.57.
32. Refer answer to Q. 61 on page 4.57.
33. Refer answer to Q. 57 and Q. 58 on page 4.56 and Q. 61 on page 4.57.
34. Refer answer to Q. 61 on page 4.57.
35. (*i*) Refer answer to Q. 59 on page 4.56.

    (*ii*) Refer answer to Q. 61 on page 4.57.

Type C : Long Answer Questions — 5 Marks Each

1. What is meant by resolution of a vector ? Prove that a vector can be resolved along two given directions in one and only one way.
2. State the parallelogram law of vector addition and find the magnitude and direction of the resultant of two vectors $\vec{P}$ and $\vec{Q}$ inclined at an angle θ with each other. What happens, when θ = 0° and θ = 90° ? **[Meghalaya 96 ; Himachal 02, 04, 06, 09]**
3. (*a*) Analytically, find the resultant $\vec{R}$ of two vectors $\vec{A}$ and $\vec{B}$ inclined at an angle θ.

   (*b*) Find the angle between two vectors $\vec{P}$ and $\vec{Q}$ if resultant of the vectors is given by $R^2 = P^2 + Q^2$. **[Central Schools 07]**
4. State triangle law of vector addition. Give analytical treatment to find the magnitude and direction of a resultant vector by using this law. **[Himachal 03, 08]**

5. (*a*) Pick out only the vector quantities from the following : Temperature, pressure, impulse, time, power, charge.
   (*b*) Show by drawing a neat diagram that the flight of a bird is an example of composition of vectors.
   (*c*) A man is travelling at 10.8 km h^{-1} in a topless car on a rainy day. He holds his umbrella at an angle 37° to the vertical to protect himself from the rain which is falling vertically downwards. What is the velocity of the rain ?
   [Given $\cos 37° = \frac{4}{5}$] **[Central Schools 08]**
6. Establish the following vector inequalities :
   (*i*) $|\vec{a} + \vec{b}| \leq |\vec{a}| + |\vec{b}|$
   (*ii*) $|\vec{a} - \vec{b}| \leq |\vec{a}| + |\vec{b}|$
   When does the equality sign apply ?
7. What is a projectile ? Derive the expression for the trajectory, time of flight, maximum height and horizontal range for a projectile thrown upwards, making an angle θ with the horizontal direction. **[Himachal 07C ; Central Schools 07]**
8. What is a projectile ? Show that its path is parabolic. Also find the expressions for (*i*) maximum height attained and (*ii*) time of flight. **[Himachal 03, 04]**
9. Define projectile. Show that the path of projectile is parabola. Find the angle of projection at which the horizontal range and maximum height of the projectile are equal. **[Central Schools 05]**
10. (*a*) Show that for two complementary angles of projection of a projectile thrown with the same velocity, the horizontal ranges are equal.
    (*b*) For what angle of projection of a projectile, is the range maximum ?
    (*c*) For what angle of projection of a projectile, are the horizontal range and maximum height attained by the projectile equal ? **[Manipur 99]**
11. (*a*) What is projectile motion ?
    (*b*) The maximum range of projectile is $2/\sqrt{3}$ times actual range. What is the angle of projection for the actual range ?
    (*c*) Two balls are thrown with the same initial velcoity at angles α and (90° − α) with the horizontal. What will be the ratio of the maximum heights attained by them ? **[Central Schools 08]**
12. Define centripetal acceleration. Derive an expression for the centripetal acceleration of a body moving with uniform speed v along a circular path of radius r. Explain how it acts along the radius towards the centre of the circular path. **[Himachal 04, 08, 09C]**

Answers

1. Refer answer to Q. 20 on page 4.14.
2. Refer answer to Q. 19 on page 4.7.
3. (*a*) Refer answer to Q. 19 on page 4.7.
   (*b*) Refer to the hint of Problem 14 on page 4.14.
4. Refer answer to Q. 18 on page 4.7.
5. (*a*) Impulse.
   (*b*) Refer answer to Q.17(*a*) on page 4.6.
   (*c*) $v_R = 10.8 \text{ km h}^{-1} = 3 \text{ ms}^{-1}$

   Given : $\cos 37° = \frac{4}{5}$ $\quad \therefore \quad \tan 37° = \frac{3}{4}$

   But $\tan 37° = \frac{v_R}{v_M}$ or $\frac{3}{4} = \frac{v_R}{3\text{ms}^{-1}}$

   or $v_R = \frac{9}{4} = \mathbf{2.25 \text{ ms}^{-1}}$.
6. Refer to solution of Example 10 on page 4.10.
7. Refer answer to Q. 53 on page 4.43.
8. Refer answer to Q. 53 on page 4.43.
9. Refer answer to Q. 53 on page 4.43 and see solution of Example 76 on page 4.48.
10. (*a*) Refer answer to Q. 54 on page 4.44.
    (*b*) θ = 45°
    (*c*) θ = 75°58′.
11. (*a*) See point 46 of Glimpses.
    (*b*) Refer to the answer of Q.60 on page 4.85.
    (*c*) Refer to the solution of Problem 17(*i*) on page 4.78.
12. Refer answer to Q. 61 on page 4.57.

Competition Section

Motion in a Plane

GLIMPSES

1. **Scalars.** The physical quantities which have only magnitude and no direction are called scalars *e.g.*, mass, length, time, speed, work, power, etc.
2. **Vectors.** The physical quantities which have both magnitude and direction are called vectors *e.g.*, displacement, velocity, acceleration, force, momentum, etc.
3. **Representation of a vector.** A vector is represented by a straight line with an arrowhead over it. The length of the line gives the magnitude and the arrowhead gives the direction of the vector.
4. **Position vector.** A vector which gives position of an object with reference to the origin of a coordinate system is called position vector.
5. **Displacement vector.** It is that vector which tells how much and in which direction an object has changed its position in a given time interval.
6. **Polar vectors.** These are the vectors which have a starting point or a point of application *e.g.*, displacement, force, velocity, etc.
7. **Axial vectors.** The vectors which represent rotational effect and act along the axis of rotation in accordance with right hand screw rule are called axial vectors *e.g.*, torque, angular momentum, etc.
8. **Equal vectors.** Two vectors are said to be equal if they have the same magnitude and direction.
9. **Negative vector.** The negative of a vector is defined as another vector having the same magnitude but having an opposite direction.
10. **Zero vector.** A vector having zero magnitude and an arbitrary direction is called a zero or null vector.
11. **Collinear vectors.** The vectors which either act along the same line or along parallel lines are called collinear vectors.
12. **Coplanar vectors.** The vectors which act in the same plane are called coplanar vectors.
13. **Modulus of a vector.** The magnitude or length of a vector is called its modulus.

    Modulus of vector $\vec{A} = |\vec{A}| = A$
14. **Fixed vector.** The vector whose initial vector is fixed is called a *fixed vector* or *localised vector*.
15. **Unit vector.** A unit vector is a vector of unit magnitude drawn in the direction of a given vector. A unit vector in the direction of $\vec{A}$ is given by

    $$\hat{A} = \frac{\vec{A}}{|\vec{A}|}$$
16. **Free vector.** A vector whose initial point is not fixed is called a *free vector* or *non-localised vector*.
17. **Co-initial vectors.** The vectors which have the same initial point are called co-initial vectors.
18. **Co-terminus vectors.** The vectors which have the common terminal point are called co-terminus vectors.
19. **Properties of zero vector.** A zero vector has the following properties :

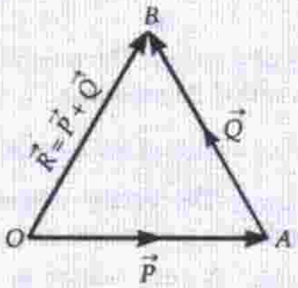
    $$\vec{A} + \vec{O} = \vec{A}\,; \quad \lambda\vec{O} = \vec{O}\,; \quad 0\vec{A} = \vec{O}$$
20. **Multiplication of vector by a real number.** When a vector $\vec{A}$ is multiplied by a real number λ, we get another vector $\lambda\vec{A}$. The magnitude of $\lambda\vec{A}$ is λ times the magnitude of $\vec{A}$. If λ is positive, then the direction of $\lambda\vec{A}$ is same as that of $\vec{A}$. If λ is negative, then the direction of $\lambda\vec{A}$ is opposite to that of $\vec{A}$.
21. **Multiplication of a vector by a scalar.** When a vector $\vec{A}$ is multiplied by a scalar λ, which has

certain units, the units of $\lambda\vec{A}$ are obtained by multiplying the units of $\vec{A}$ by the units of λ.

22. **Composition of vectors.** The resultant of two or more vectors is that single vector which produces the same effect as the individual vectors together would produce. The process of adding two or more vectors is called the composition of vectors.

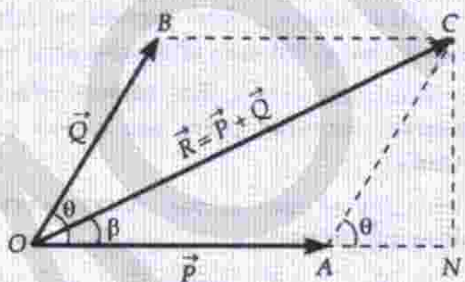
23. **Triangle law of vector addition.** If two vectors can be represented both in magnitude and direction by the two sides of a triangle taken in the same order, then their resultant is represented completely both in magnitude and direction by the third side of the triangle taken in the opposite order. In the figure below,

$$\vec{OA} + \vec{AB} = \vec{OB} \quad \text{or} \quad \vec{P} + \vec{Q} = \vec{R}$$



24. **Parallelogram law of vector addition.** If two vectors acting simultaneously at a point can be represented both in magnitude and direction by the two adjacent sides of a parallelogram, then their resultant is represented completely both in magnitude and direction by the diagonal of the parallelogram passing through that point. In the figure,

$$\vec{OA} + \vec{OB} = \vec{OC} \quad \text{or} \quad \vec{P} + \vec{Q} = \vec{R}$$



The magnitude of the resultant $\vec{R}$ is given by

$$R = \sqrt{P^2 + Q^2 + 2PQ\cos\theta}$$

where θ is the angle between $\vec{P}$ and $\vec{Q}$. If $\vec{R}$ makes angle β with $\vec{P}$, then

$$\tan\beta = \frac{Q\sin\theta}{P + Q\cos\theta}$$

25. **Polygon law of vector addition.** If a number of vectors are represented both in magnitude and direction by the sides of an open polygon taken in the same order, then their resultant is represented both in magnitude and direction by the closing side of the polygon taken in opposite order.

26. **Properties of vector addition** :

(i) Vectors representing physical quantities of same nature can only be added.

(ii) Vector addition is *commutative*.

$$\vec{A} + \vec{B} = \vec{B} + \vec{A}$$

(iii) Vector addition is *associative*.

$$(\vec{A} + \vec{B}) + \vec{C} = \vec{A} + (\vec{B} + \vec{C})$$

27. **Subtraction of vectors.** The subtraction of a vector $\vec{B}$ from vector $\vec{A}$ is defined as the addition of vector $-\vec{B}$ to $\vec{A}$. Thus $\vec{A} - \vec{B} = \vec{A} + (-\vec{B})$.

28. **Resolution of a vector.** The process of splitting a vector into two or more vectors is known as resolution of the vector. The vectors into which the given vector is splitted are called component vectors. A vector $\vec{A}$ can be resolved into components along two given vectors $\vec{a}$ and $\vec{b}$ lying in the same plane in one and only one way :

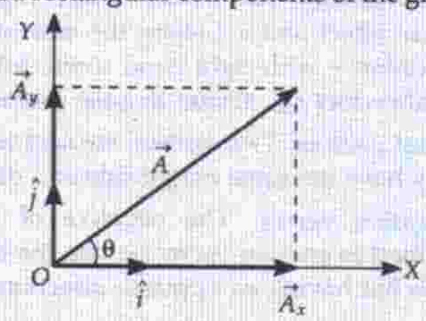
$$\vec{A} = \lambda\vec{a} + \mu\vec{b}$$

where λ and μ are real numbers.

29. **Orthogonal triad of unit vectors : Base vectors.** The unit vectors $\hat{i}, \hat{j}, \hat{k}$ are vectors of unit magnitude and point in the direction of the *x*-, *y*-, and *z*-axes, respectively in a right-handed coordinate system. These are collectively known as the orthogonal triad of unit vectors or base vectors.

$$|\hat{i}| = |\hat{j}| = |\hat{k}| = 1$$

30. **Rectangular components of a vector.** When a vector is resolved along two mutually perpendicular directions, components so obtained are called rectangular components of the given vector.



As shown in figure, if $\vec{A}$ makes angle θ with *X*-axis and $\vec{A}_x$ and $\vec{A}_y$ are the rectangular components of $\vec{A}$ along X- and Y-axis respectively, then

$$\vec{A} = \vec{A}_x + \vec{A}_y = A_x\hat{i} + A_y\hat{j}$$

Also, $A_x = A\cos\theta,\ \ A_y = A\sin\theta$

$$A = \sqrt{A_x^2 + A_y^2} \text{ and } \tan\theta = \frac{A_y}{A_x}.$$

Any vector in three dimensions can be expressed in terms of its rectangular components as

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$

Its magnitude, $A = \sqrt{A_x^2 + A_y^2 + A_z^2}$.

31. Scalar or dot product. The scalar or dot product of two vectors $\vec{A}$ and $\vec{B}$ is defined as the product of the magnitudes of $\vec{A}$ and $\vec{B}$ and cosine of the angle θ between them. Thus

$$\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos\theta = AB\cos\theta$$

It can be positive, negative or zero depending upon the value of θ.

32. Geometrical interpretation of scalar product. The scalar product of two vectors can be interpreted as the product of magnitude of one vector and component of the other vector along the first vector.

33. Properties of dot product of two vectors :

(i) For parallel vectors,

$$\theta = 0^\circ,\quad \cos\theta = 1,\quad \vec{A}\cdot\vec{B} = AB$$

(ii) For antiparallel vectors,

$$\theta = 180^\circ,\quad \cos\theta = -1,\quad \vec{A}\cdot\vec{B} = -AB$$

(iii) For perpendicular vectors,

$$\theta = 90^\circ,\quad \cos\theta = 0,\quad \vec{A}\cdot\vec{B} = 0$$

(iv) $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$ *(Commutative law)*

(v) $\vec{A}\cdot(\vec{B}+\vec{C}) = \vec{A}\cdot\vec{B} + \vec{A}\cdot\vec{C}$ *(Distributive law)*

(vi) $\vec{A}\cdot\vec{A} = A^2$

(vii) $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$

(viii) $\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$.

34. Dot product in Cartesian co-ordinates. For

$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$ and $\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$,

$\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z$

Angle θ between $\vec{A}$ and $\vec{B}$ is given by

$$\cos\theta = \frac{\vec{A}\cdot\vec{B}}{|\vec{A}||\vec{B}|} = \frac{A_xB_x + A_yB_y + A_zB_z}{\sqrt{A_x^2 + A_y^2 + A_z^2}\sqrt{B_x^2 + B_y^2 + B_z^2}}$$

35. Examples of dot product.

(i) Work done, $W = \vec{F}\cdot\vec{S}$

(ii) Instantaneous power, $P = \vec{F}\cdot\vec{v}$

36. Vector or cross product. For two vectors $\vec{A}$ and $\vec{B}$ inclined at an angle θ, the vector or cross product is defined as

$$\vec{A}\times\vec{B} = AB\sin\theta\,\hat{n}$$

where $\hat{n}$ is a unit vector perpendicular to the plane of $\vec{A}$ and $\vec{B}$ and its direction is that in which a right handed screw advances when rotated from $\vec{A}$ to $\vec{B}$.

37. Geometrical interpretation of vector product. The magnitude of the vector product of two vectors is equal to (i) the area of the parallelogram formed by the two vectors as its adjacent sides and (ii) twice the area of the triangle formed by the two vectors as its adjacent sides.

38. Properties of cross product of two vectors :

(i) For parallel or antiparallel vectors,

$$\theta = 0^\circ \text{ or } 180^\circ$$

$$\vec{A}\times\vec{B} = \vec{0}$$

(ii) $\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$ *(Anti-commutative law)*

(iii) $\vec{A}\times(\vec{B}+\vec{C}) = \vec{A}\times\vec{B} + \vec{A}\times\vec{C}$ *(Distributive law)*

(iv) $\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = \vec{0}$

(v) $\hat{i}\times\hat{j} = \hat{k},\ \ \hat{j}\times\hat{k} = \hat{i},\ \ \hat{k}\times\hat{i} = \hat{j}$

(vi) Unit vector perpendicular to the plane of $\vec{A}$ and $\vec{B}$ is given by

$$\hat{n} = \frac{\vec{A}\times\vec{B}}{|\vec{A}\times\vec{B}|}$$

(vii) Angle θ between $\vec{A}$ and $\vec{B}$ is given by

$$\sin\theta = \frac{|\vec{A}\times\vec{B}|}{|\vec{A}||\vec{B}|}\text{d}$$

39. Cross product in Cartesian co-ordinates.

$$\vec{A}\times\vec{B} = (A_x\hat{i} + A_y\hat{j} + A_z\hat{k})\times(B_x\hat{i} + B_y\hat{j} + B_z\hat{k})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$= (A_yB_z - A_zB_y)\hat{i} - (A_zB_x - A_xB_z)\hat{j} + (A_xB_y - A_yB_x)\hat{k}.$$

40. Examples of cross product. (*i*) Moment of force or torque,

$$\vec{\tau} = \vec{r} \times \vec{F}$$

(*ii*) Angular momentum,

$$\vec{L} = \vec{r} \times \vec{p}.$$

41. Position and displacement vectors. The *position vector* of an object in *x-y* plane is given by

$$\vec{r} = x\,\hat{i} + y\,\hat{j}$$

and the *displacement* from position $\vec{r}$ to position $\vec{r}\,'$ is given by

$$\Delta\vec{r} = \vec{r}\,' - \vec{r} = (x' - x)\,\hat{i} + (y' - y)\,\hat{j} = \Delta x\,\hat{i} + \Delta y\,\hat{j}$$

42. Velocity vector. If an object undergoes a displacement $\Delta\vec{r}$ in time Δt, its *average velocity* is given by

$$\bar{v} = \frac{\Delta\vec{r}}{\Delta t}$$

The *velocity* of an object at time *t* is the limiting value of the average velocity as Δt tends to zero. Thus

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta\vec{r}}{\Delta t} = \frac{d\vec{r}}{dt}$$

In component form, we have

$$\vec{v} = v_x\,\hat{i} + v_y\,\hat{j} + v_z\,\hat{k}$$

where $v_x = \frac{dx}{dt}$, $v_y = \frac{dy}{dt}$, $v_z = \frac{dz}{dt}$ and

$$|\vec{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2}.$$

When position of a particle is plotted on a coordinate system, $\vec{v}$ is always tangent to the curve representing the path of the particle.

43. Acceleration vector. If the velocity of an object changes from $\vec{v}$ to $\vec{v}\,'$ in time Δt, then its average acceleration is given by

$$\bar{a} = \frac{\vec{v}\,' - \vec{v}}{\Delta t} = \frac{\Delta\vec{v}}{\Delta t}$$

The *acceleration* $\vec{a}$ at any time *t* is the limiting value of $\bar{a}$ as Δt tends to zero. So

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta\vec{v}}{\Delta t} = \frac{d\vec{v}}{dt}$$

In component form, we have

$$\vec{a} = a_x\,\hat{i} + a_y\,\hat{j} + a_z\,\hat{k}$$

where, $a_x = \frac{dv_x}{dt}$, $a_y = \frac{dv_y}{dt}$, $a_z = \frac{dv_z}{dt}$

and $$|\vec{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}$$

44. Equations of motion in vector form. For motion with constant acceleration,

(*i*) $\vec{v} = \vec{v}_0 + \vec{a}\,t$

(*ii*) $\vec{r} = \vec{r}_0 + \vec{v}_0\,t + \frac{1}{2}\vec{a}\,t^2$

(*iii*) $v^2 - v_0^2 = 2\vec{a}\cdot(\vec{r} - \vec{r}_0)$

The motion in a plane with uniform acceleration can be treated as the superposition of two separate simultaneous one-dimensional motions along two perpendicular directions.

45. Relative velocity. The relative velocity of an object *A* with respect to object *B*, when both are in motion, is the rate of change of position of object *A* with respect to object *B*.

(*i*) If the two objects are moving with velocities $\vec{v}_A$ and $\vec{v}_B$ with respect to the ground, then

Relative velocity of *A* w.r.t. *B*,

$$\vec{v}_{AB} = \vec{v}_A - \vec{v}_B$$

Relative velocity of *B* w.r.t. *A*,

$$\vec{v}_{BA} = \vec{v}_B - \vec{v}_A$$

(*ii*) If the two objects *A* and *B* are moving with velocities $\vec{v}_A$ and $\vec{v}_B$ inclined at an angle θ, then magnitude of the relative velocity of *A* w.r.t. *B* is given by

$$v_{AB} = \sqrt{|\vec{v}_A|^2 + |-\vec{v}_B|^2 + 2\,|\vec{v}_A|\,|-\vec{v}_B|\cos(180° - \theta)}$$

$$= \sqrt{v_A^2 + v_B^2 - 2\,v_A\,v_B\cos\theta}$$

If the velocity $\vec{v}_{AB}$ makes angle β with the velocity $\vec{v}_A$, then

$$\tan\beta = \frac{|-\vec{v}_B|\sin(180° - \theta)}{|\vec{v}_A| + |-\vec{v}_B|\cos(180° - \theta)}$$

$$= \frac{v_B\sin\theta}{v_A - v_B\cos\theta}$$

46. Projectile motion. Any body projected into space, such that it moves under the effect of gravity alone is called a projectile. The path followed by a projectile is called its trajectory which is always a parabola.

A projectile executes two independent motions simultaneously :

(*i*) uniform horizontal motion and

(*ii*) uniform accelerated downward motion.

47. Projectile fired horizontally. Suppose a body is projected horizontally with velocity *u* from a height *h* above the ground. Let it reach the point (*x*, *y*) after time *t*.

Then

(*i*) Position of the projectile after time t :

$$x = ut, \quad y = \frac{1}{2} g t^2$$

(*ii*) Equation of trajectory : $y = \frac{g}{2u^2} \cdot x^2$

(*iii*) Velocity after time t :

$$v = \sqrt{u^2 + g^2 t^2} \; ; \; \beta = \tan^{-1} \frac{gt}{u}$$

(*iv*) Time of flight, $T = \sqrt{\frac{2h}{g}}$

(*v*) Horizontal range, $R = u \times T = u\sqrt{\frac{2h}{g}}$.

48. Projectile fired at an angle with the horizontal. Suppose a projectile is fired with velocity u at an angle θ with the horizontal. Let it reach the point (x, y) after time t. Then

(*i*) Components of initial velocity :

$$u_x = u\cos\theta, \quad u_y = u\sin\theta$$

(*ii*) Components of acceleration at any instant :

$$a_x = 0, \; a_y = -g$$

(*iii*) Position after time t :

$$x = (u\cos\theta)\,t, \quad y = (u\sin\theta)\,t - \frac{1}{2} g t^2$$

(*iv*) Equation of trajectory :

$$y = x\tan\theta - \frac{g}{2u^2\cos^2\theta} \cdot x^2$$

(*v*) Maximum height, $H = \frac{u^2\sin^2\theta}{2g}$

(*vi*) Time of flight, $T = \frac{2u\sin\theta}{g}$

(*vii*) Horizontal range, $R = \frac{u^2\sin 2\theta}{g}$

(*viii*) Maximum horizontal range is attained at $\theta = 45°$ and its value is

$$R_{max} = \frac{u^2}{g}$$

(*ix*) Velocity after time t :

$$v_x = u\cos\theta, \quad v_y = u\sin\theta - gt$$

$$\therefore \quad v = \sqrt{v_x^2 + v_y^2} \quad \text{and} \quad \tan\beta = \frac{v_y}{v_x}$$

(*x*) The velocity with which the projectile reaches the horizontal plane through the point of projection is same as the velocity of projection.

49. Uniform circular motion. When a body moves along a circular path with uniform speed, its motion is said to uniform circular motion.

50. Angular displacement. It is the angle swept out by a radius vector in a given time interval.

$$\theta\,(\text{rad}) = \frac{\text{Arc}}{\text{Radius}} = \frac{s}{r}$$

51. Angular velocity. The angle swept out by the radius vector per second is called angular velocity.

$$\omega = \frac{\theta}{t} \quad \text{or} \quad \omega = \frac{\theta_2 - \theta_1}{t_2 - t_1}$$

52. Time period and frequency. Time taken for one complete revolution is called time period (T). The number of revolutions completed per second is called frequency

$$\omega = \frac{2\pi}{T} = 2\pi\nu$$

53. Relationship between v and ω. It is given by

$$v = r\omega$$

i.e. Linear velocity = Radius × angular velocity.

54. Angular acceleration and its relation with linear acceleration. The rate of change of angular velocity is called angular acceleration. It is given by

$$\alpha = \frac{\omega_2 - \omega_1}{t_2 - t_1}$$

Also, $a = r\alpha$

i.e. Linear acceleration

= Radius × angular acceleration

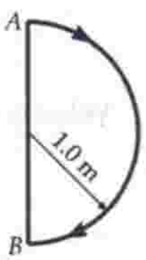
55. Centripetal acceleration. A body moving along a circular path is acted upon by an acceleration directed towards the centre along the radius. This acceleration is called centripetal acceleration. It is given by

$$a = \frac{v^2}{r} = r\omega^2.$$

IIT Entrance Exam

MULTIPLE CHOICE QUESTIONS WITH ONE CORRECT ANSWER

1. In 1.0 s, a particle goes from point A to point B, moving in a semicircle of radius 1.0 m as shown in the figure.



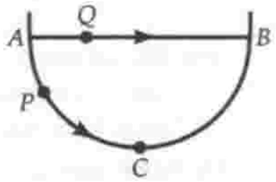
The magnitude of the average velocity is

(*a*) 3.14 m/s (*b*) 2.0 m/s

(*c*) 1.0 m/s (*d*) zero [IIT 99]

2. A particle P is sliding down a frictionless hemispherical bowl. It passes the point A at $t = 0$. At this instant of time, the horizontal component of its velocity is v. A bead Q of the same mass as P is ejected

from A at $t=0$ along the horizontal string AB with speed v. Friction between the bead and the string may be neglected. Let t_P and t_Q be the respective times taken by P and Q to reach the point B. Then



(*a*) $t_P < t_Q$ (*b*) $t_P = t_Q$

(*c*) $t_P > t_Q$ (*d*) $\frac{t_P}{t_Q} = \frac{\text{Lenght of arc } ACB}{\text{Lenght of chord } AB}$

[IIT 93]

3. A river is flowing from west to east at a speed of 5 metre per minute. A man on the south bank of the river, capable of swimming at 10 metre per minute in still water, wants to swim across the river in the shortest time. He should swim in a direction

(*a*) due north (*b*) 30° east of north

(*c*) 30° west of north (*d*) 60° east of north [IIT 83]

4. A boat which has a speed of 5 km/hr in still water crosses a river of width 1 km along the shortest possible path in 15 minutes. The velocity of the river water in km/hr is

(*a*) 1 (*b*) 3

(*c*) 4 (*d*) $\sqrt{41}$ [IIT 88]

MULTIPLE CHOICE QUESTIONS WITH ONE OR MORE THAN ONE CORRECT ANSWER

5. The coordinates of a particle moving in a plane are given by $x(t) = a\cos pt$ and $y(t) = b\sin pt$, where a, b $(< a)$ and p are positive constants of appropriate dimensions. Then,

(*a*) the path of the particle is an ellipse.

(*b*) the velocity and acceleration of the particle are normal to each other at $t = \pi/2p$

(*c*) the acceleration of the particle is always directed towards a focus

(*d*) the distance travelled by the particle in time interval $t=0$ to $t = \pi/2p$ is a. [IIT 99]

Answers and Explanations

1. (*b*) $|\text{Average velocity}| = \frac{|\text{Displacement}|}{\text{Time}}$

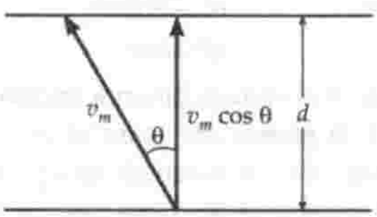
$$= \frac{AB}{t} = \frac{2r}{t} = \frac{2 \times 1.0\,\text{m}}{1\,\text{s}} = \mathbf{2\ m/s}.$$

2. (*a*) As the particle P moves from A to C, its horizontal velocity increases from v. Again the horizontal velocity decreases to v as P moves from C to B. But the horizontal velocity P remains greater than or equal to v. But the horizontal velocity of bead Q remains constant equal to v. For same horizontal displacement AB, P takes takes smaller time than Q *i.e.*,

$$t_P < t_Q.$$

3. (*a*) Time taken to cross the river

$$t = \frac{d}{v_m \cos\theta}$$

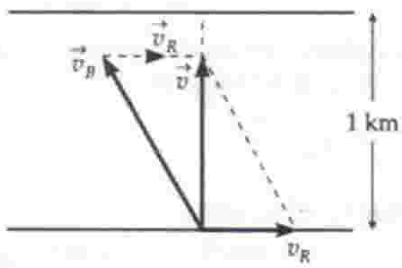


For t to be minimum, $\cos\theta$ should be maximum

i.e., $\cos\theta = 1$ or $\theta = 0°$

Hence the man should swim due north.

4. (*b*)



Resultant velocity of boat,

$$v = \frac{1\text{ km}}{15\text{ min}} = \frac{1\text{ km}}{(1/4)\text{h}} = 4\text{ km h}^{-1}.$$

Velocity of river,

$$v_R = \sqrt{v_B^2 - v^2} = \sqrt{5^2 - 4^2} = \mathbf{3\ km/h}.$$

5. (*a*), (*b*) and (*c*)

$$x = a\cos pt \quad \Rightarrow \quad \frac{x}{a} = \cos pt$$

$$y = b\sin pt \quad \Rightarrow \quad \frac{y}{b} = \sin pt$$

$$\therefore \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Hence the path of the particle is an ellipse. Option (*a*) holds good.

Now, $\vec{r} = x\hat{i} + y\hat{j}$

or $\vec{r} = a\cos pt\,\hat{i} + b\sin pt\,\hat{j}$

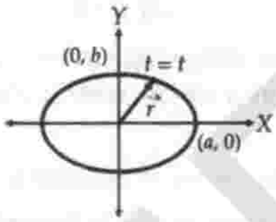
$$\therefore \quad \vec{v} = \frac{d\vec{r}}{dt} = -ap\sin pt\,\hat{i} + bp\cos pt\,\hat{j}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = -ap^2\cos pt\,\hat{i} - bp^2\sin pt\,\hat{j}$$

At $t = \dfrac{\pi}{2p}$, we have

$$\vec{v}\cdot\vec{a} = \left(-ap\sin\frac{\pi}{2}\hat{i}\right)\cdot\left(-bp^2\sin\frac{\pi}{2}\hat{j}\right) = 0 \quad [\hat{i}\cdot\hat{j} = 0]$$

Thus the velocity and acceleration are perpendicular to each other at $t = \dfrac{\pi}{2p}$. Option (*b*) holds good.



Clearly, $\vec{a} = -p(\vec{r})$

Thus the acceleration is always directed towards the origin. Option (*c*) holds good.

At $t = 0$, the particle is at $(a, 0)$.

At $t = \dfrac{\pi}{2p}$, the particle is at $(0, b)$.

Thus the distance covered is one-fourth of the elliptical path. It is greater than *a* and not equal to *a*. Option (*d*) does not hold good.

AIEEE

1. Two forces are such that the sum of their magnitudes is 18 N and their resultant has magnitude 12 N and is perpendicular to the smaller force. Then the magnitudes of the forces are

(*a*) 12 N, 6 N (*b*) 13 N, 5 N

(*c*) 10 N, 8 N (*d*) 16 N, 2N [AIEEE 02]

2. A particle is moving eastward with a velocity of $5\,\text{ms}^{-1}$. In 10 s, the velocity of the particle changes to $5\,\text{ms}^{-1}$ northward. The average acceleration in this time is

(*a*) $\frac{1}{\sqrt{2}}\,\text{ms}^{-2}$ towards north-west

(*b*) $\frac{1}{\sqrt{2}}\,\text{ms}^{-2}$ towards north-east

(*c*) $\frac{1}{2}\,\text{ms}^{-2}$ towards north-west

(*d*) zero [AIEEE 05]

3. The coordinates of a moving particle at any time *t* are given by $x = \alpha t^3$ and $y = \beta t^3$. The speed of the particle at time *t* is given by

(*a*) $3t\sqrt{\alpha^2+\beta^2}$ (*b*) $3t^2\sqrt{\alpha^2+\beta^2}$

(*c*) $t^2\sqrt{\alpha^2+\beta^2}$ (*d*) $\sqrt{\alpha^2+\beta^2}$

where the letters have their usual meanings. [AIEEE 03]

4. A boy playing on the roof of a 10 m high building throws a ball with a speed of $10\,\text{ms}^{-1}$ at an angle of 30° with the horizontal. How far from the throwing point will the ball be at the height of 10 m from the ground ? Given that

$$g = 10\,\text{ms}^{-2},\ \sin 30° = 1/2,\ \cos 30° = \sqrt{3}/2.$$

(*a*) 5.20 m (*b*) 4.33 m

(*c*) 2.60 m (*d*) 8.66 m [AIEEE 03]

5. A projectile can have the same range *R* for two angles of projection. If t_1 and t_2 be the times of flight in the two cases, then the product of the two times of flight is directly proportional to

(*a*) $1/R^2$ (*b*) $1/R$

(*c*) R (*d*) R^2 [AIEEE 04]

6. A ball is thrown from a point with a speed v_0 at an angle of projection θ. From the same point and at the same instant, a person starts running with a constant speed $v_0/2$ to catch the ball. Will the person be able to catch the ball ? If yes, what should be the angle of projection ?

(*a*) Yes, 60° (*b*) Yes, 30°

(*c*) No (*d*) Yes, 45° [AIEEE 04]

7. A particle is projected at 60° to the horizontal with a kinetic energy K. The kinetic energy at the highest point is

(a) $K/2$ (b) K
(c) zero (d) $K/4$ [AIEEE 07]

8. Which of the following statements is false for a particle moving in a circle with a constant angular speed ?

(a) The velocity vector is tangent to the circle
(b) The acceleration vector is tangent to the circle
(c) The acceleration vector points to the centre of the circle
(d) The velocity and acceleration vectors are perpendicular to each other [AIEEE 04]

9. A particle is acted upon by a force of constant magnitude, which is always perpendicular to the velocity of the particle. The motion of the particle takes place in a plane. It follows that

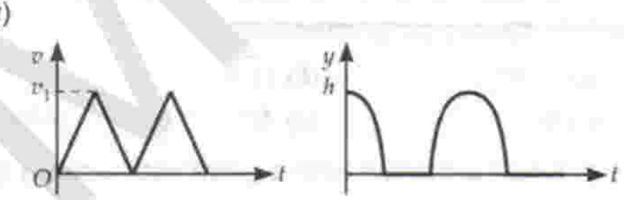
(a) its velocity is constant
(b) its acceleration is constant
(c) its kinetic energy is constant
(d) it moves in a circular path [AIEEE 04]

10. A particle has an initial velocity of $3\hat{i}+4\hat{j}$ and an acceleration of $0.4\hat{i}+0.3\hat{j}$. Its speed after 10 s is

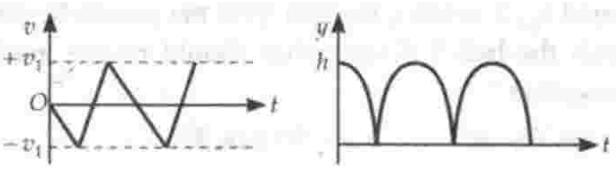
(a) $7\sqrt{2}$ units (b) 7 units
(c) 8.5 units (d) 10 units [AIEEE 09]

11. Consider a rubber ball freely falling from a height $h = 4.9$ m onto a horizontal elastic plate. Assume that the duration of collision is negligible and the collision with the plate is totally elastic. Then the velocity as a function of time the height as function of time will be

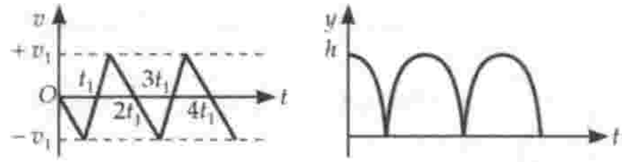
(a)



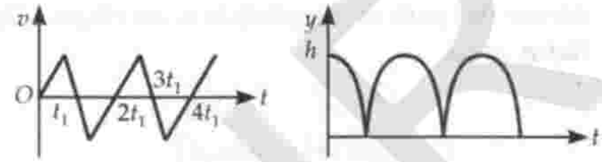
(b)



(c)



(d)



[AIEEE 09]

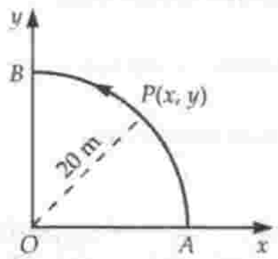
12. A particle is moving with velocity $\vec{v} = K(y\hat{i} + x\hat{j})$, where K is a constant. The general equation for its path is

(a) $y^2 = x^2 +$ constant (b) $y = x^2 +$ constant
(c) $y^2 = x +$ constant (d) $xy =$ constant [AIEEE 2010]

13. For a particle in uniform circular motion, the acceleration $\vec{a}$ at a point $P(R, \theta)$ on the circle of radius R is (Here θ is measured from the x-axis)

(a) $\frac{v^2}{R}\hat{i} + \frac{v^2}{R}\hat{j}$ (b) $-\frac{v^2}{R}\cos\theta\,\hat{i} + \frac{v^2}{R}\sin\theta\,\hat{j}$
(c) $-\frac{v^2}{R}\sin\theta\,\hat{i} + \frac{v^2}{R}\cos\theta\,\hat{j}$
(d) $-\frac{v^2}{R}\cos\theta\,\hat{i} - \frac{v^2}{R}\sin\theta\,\hat{j}$ [AIEEE 2010]

14. A point P moves in counter-clockwise direction on a circular path as shown in the figure. The movement of P is such that it sweeps out a length



$s = t^3 + 5$, where s in metres and t is in seconds. The radius of the path is 20 m. The acceleration of P when $t = 2$ s is nearly

(a) $14\ m/s^2$ (b) $13\ m/s^2$
(c) $12\ m/s^2$ (d) $7.2\ m/s^2$ [AIEEE 2010]

Answers and Explanations

1. (*b*) Refer to the solution of Example 9 on Page 4.10.

2. (*a*) Refer to the solution of Example 56 on page 4.36.

3. (*b*) As $x = \alpha t^3$

$$\therefore \quad v_x = \frac{dx}{dt} = 3\alpha t^2$$

Also, $y = \beta t^3$

$$\therefore \quad v_y = \frac{dy}{dt} = 3\beta t^2$$

Hence
$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(3\alpha t^2)^2 + (3\beta t^2)^2}$$
$$= 3t^2\sqrt{\alpha^2 + \beta^2}$$

4. (*d*) Refer to the solution of Problem 9 on Page 4.73.

5. (*c*) Refer to the solution of Problem 10 on Page 4.73.

6. (*a*) Refer to the solution of Problem 9 on Page 4.73.

7. (*d*) Initial K.E., $K = \frac{1}{2}mv^2$

Velocity at the highest point

= Horizontal component of v

$$= v\cos 60^\circ = \frac{v}{2}$$

$\therefore$ K.E. at the highest point

$$= \frac{1}{2}m\left(\frac{v}{2}\right)^2 = \frac{K}{4}.$$

8. (*b*) The acceleration vector is along the radius of the circle. It is not tangent to the circle.

9. (*c*), (*d*) Such a force makes the particle move along a circular path with a constant speed. So the K.E. also is constant. But velocity and acceleration continuously change due to change in direction.

10. (*a*) $\vec{v} = \vec{u} + \vec{a}\,t$

$$= (3\hat{i} + 4\hat{j}) + 10(0.4\hat{i} + 0.3\hat{j})$$
$$= 7\hat{i} + 7\hat{j}$$

Speed $= |\vec{v}| = \sqrt{7^2 + 7^2} = \mathbf{7\sqrt{2}}$ **units.**

11. (*c*) We use $v = u + at$

For downward motion,

$v = 0 - gt = -gt$ (negative velocity)

For upward motion,

$v = +gt$ (positive velocity)

Just after collision velocity is upward and then becomes zero after some time and then negative. The same process repeats, so v-t given in option (*c*) is correct.

Again, $s = ut + \frac{1}{2}gt^2$

$$h = 4.9 - \frac{1}{2}gt^2$$

So, h-t graph is a downward parabola as given in option (*c*). Moreover, the collision is perfectly elastic. So, the ball reaches the same height again and again with the same velocity.

12. (*a*) $\vec{v} = v_x\hat{i} + v_y\hat{j} = K(y\hat{i} + x\hat{j})$

$$\therefore \quad v_x = \frac{dx}{dt} = Ky$$
$$v_y = \frac{dy}{dt} = Kx$$

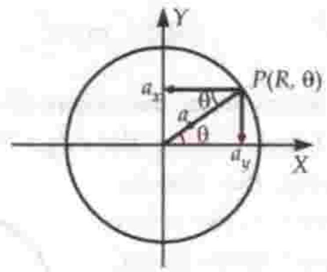
Here $\frac{dx}{dy} = \frac{y}{x}$ or $y\,dy = x\,dx$

or $\int y\,dy = \int x\,dx \Rightarrow y^2 = x^2 + \text{constant}$

13. (*d*) For a particle in uniform circular motion,

$a = \frac{v^2}{R}$, towards the centre of the circle

$$\therefore \quad \vec{a} = \frac{v^2}{R}(-\cos\theta\,\hat{i} - \sin\theta\,\hat{j})$$
$$= -\frac{v^2}{R}\cos\theta\,\hat{i} - \frac{v^2}{R}\sin\theta\,\hat{j}$$



14. (*a*) $s = t^3 + 5$

$$v = \frac{ds}{dt} = 3t^2$$
$$a_t = \frac{dv}{dt} = 6t$$

At $t = 2$ s ; $v = 12$ m/s

Now $a_c = \frac{v^2}{R} = \frac{(12)^2}{20} = \frac{144}{20} = 7.2$ m/s

$$a_t = 6 \times 2 = 12 \text{ m/s}$$

$$\therefore \quad a = \sqrt{a_c^2 + a_t^2} = \sqrt{12^2 + (7.2)^2} \approx \mathbf{14\,m/s^2}$$

DCE and G.G.S. Indraprastha University Engineering Entrance Exam

(Similar Questions)

1. A body moves 6 m north, 8 m east and 10 m vertically upwards, the resultant displacement from its initial position is

(a) $10\sqrt{2}$ m (b) 10 m

(c) $\frac{10}{\sqrt{2}}$ m (d) 20 m [DCE 2K]

2. When $\vec{A}.\vec{B} = -|\vec{A}||\vec{B}|$, then

(a) $\vec{A}$ and $\vec{B}$ are perpendicular to each other

(b) $\vec{A}$ and $\vec{B}$ act in the same direction

(c) $\vec{A}$ and $\vec{B}$ act in the opposite direction

(d) $\vec{A}$ and $\vec{B}$ can act in any direction [IPUEE 05]

3. If the angle between the vectors $\vec{A}$ and $\vec{B}$ is θ, the value of the product $(\vec{B} \times \vec{A}) . \vec{A}$ is equal to

(a) $BA^2 \cos\theta$ (b) $BA^2 \sin\theta$

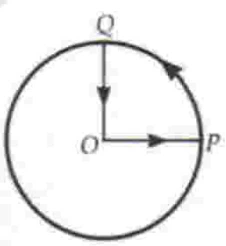
(c) $BA^2 \sin\theta \cos\theta$ (d) zero [DCE 06]

4. A particle moves along a parabolic path $y = 9x^2$ in such a way that the x-component of velocity remains constant and has a value $\frac{1}{3}$ m/s. The acceleration of the projectile is

(a) $\frac{1}{3}\hat{j}$ m/s^2 (b) $3\hat{j}$ m/s^2

(c) $\frac{2}{3}\hat{j}$ m/s^2 (d) $2\hat{j}$ m/s^2 [DCE 05]

5. A cyclist starts from the centre O of a circular park of radius 1 km, reaches the edge P of the park, then cycles along the circumference and returns to the centre along QO as shown in the figure. If the round trip takes 10 min, the net displace- ment and average speed of the cyclist (in metre and kilometre per hour) are



(a) 0, 1 (b) $\frac{\pi+4}{2}$, 0

(c) 21.4, $\frac{\pi+4}{2}$ (d) 0, 21.4 [IPUEE 07]

6. A cyclist moves in such a way that he takes 60° turn after every 100 metres. What is the displacement when he takes seventh turn ?

(a) 100 m (b) 200 m

(c) $100\sqrt{3}$ m (d) $100/\sqrt{3}$ m [DCE 99]

7. An object is moving in a circle of radius 100 m with a constant speed of 31.4 m/s. What is its average speed for one complete revolution ?

(a) zero (b) 31.4 m/s

(c) 3.14 m/s (d) $\sqrt{2} \times 31.4$ m/s [DCE 04]

8. A car runs at a constant speed on a circular track of radius 100 m, taking 62.8 s for every circular lap. The average velocity and average speed for each circular lap respectively are

(a) 0, 0 (b) 0, 10 m/s

(c) 10 m/s, 10 m/s (d) 10 m/s, 0 [DCE 06]

9. Two boys are standing at the ends A and B of a ground, where $AB = a$. The boy at B starts running simultaneously with velocity v and catches the other boy in a time t, where t is

(a) $a/\sqrt{v^2 + v_1^2}$ (b) $\sqrt{a^2/(v^2 - v_1^2)}$

(c) $a/(v - v_1)$ (d) $a/(v + v_1)$ [DCE 01]

10. The driver of a car moving towards a rocket launching pad with a speed of 6 m/s observes that the rocket is moving with speed of 10 m/s. The upward speed of the rocket as seen by the stationary observer is nearly

(a) 4 m/s (b) 6 m/s

(c) 8 m/s (d) 11 m/s [IPUEE 04]

11. Consider a collection of large number of particles each with speed v. The direction of velocity is randomly distributed in the collection. The magnitude of the relative velocity between a pair of particles averaged over all the pairs in the collection is

(a) $\frac{4v}{\pi}$ (b) greater than $\frac{4v}{\pi}$

(c) less than $\frac{4v}{\pi}$ (d) zero [DCE 05]

12. Two bullets are fired simultaneously, horizontally and with different speeds from the same place. Which bullet will hit the ground first ?

(a) The faster one (b) Depends on their mass

(c) The slower one

(d) Both will reach simultaneously [IPUEE 05]

13. Three particles A, B and C are projected from the same point with the same initial speeds making angles 30°, 45° and 60° respectively with the horizontal. Which of the following statements is correct ?

(a) A, B and C have unequal ranges

(b) Ranges of A and C are equal and less than that of B

(c) Ranges of A and C are equal and greater than that of B

(d) A, B and C have equal ranges. [DCE 05]

14. A particle is projected at an angle 45°. The relation between range and maximum height attained by the particle is

(a) $R = 4H$ (b) $4R = H$

(c) $2H = R$ (d) none of these [DCE 07]

15. For a projectile, (range)^2 is 48 times of (maximum height)^2 obtained. The angle of projection is

(a) 60° (b) 30°

(c) 45° (d) 75° [DCE 06]

16. If two balls are projected at an angle of 60° and 45° and the total heights reached are same, then their initial velocities are in the ratio of

(a) $\sqrt{3} : \sqrt{2}$ (b) $\sqrt{2} : \sqrt{3}$

(c) 3 : 2 (d) 2 : 3 [DCE 98]

17. One projectile moving with velocity v in space gets burst into 2 parts of masses in the ratio of 1 : 2. The smaller part becomes stationary, the velocity of other part is

(a) uv (b) $\frac{3v}{2}$

(c) $\frac{4v}{3}$ (d) $\frac{2v}{3}$ [DCE 99]

18. A body executing uniform circular motion has at any instant its velocity vector and acceleration vector

(a) along the same direction

(b) in opposite direction

(c) normal to each other

(d) not related to each other [DCE 2K, 03]

19. The circular motion of a particle with constant speed is

(a) simple harmonic but not periodic

(b) periodic and simple harmonic

(c) neither periodic nor simple harmonic

(d) periodic but not simple harmonic [DCE 01]

20. A particle moves with constant speed v along a circular path of radius r and completes the circle in time T. The acceleration of the particle is

(a) $\frac{2\pi v}{T}$ (b) $\frac{2\pi r}{T}$

(c) $\frac{2\pi r^2}{T}$ (d) $\frac{2\pi v^2}{T}$ [IPUEE 04]

21. A stone tied to the end of a string 1 m long is whirled in a horizontal circle with a constant speed. If the stone makes 22 revolutions in 44 s, what is the magnitude and direction of acceleration of the stone ?

(a) $\frac{\pi^2}{4}$ ms^{-2} and direction along the radius towards the centre

(b) π^2 ms^{-2} and direction along the radius away from the centre

(c) π^2 ms^{-2} and direction along the radius towards the centre

(d) π^2 ms^{-2} and direction along the tangent to the circle [DCE 01]

22. For vectors $\vec{A}$ and $\vec{B}$ making an angle 'θ' which one of the following relations is correct ?

(a) $\vec{A} \times \vec{B} = \vec{B} \times \vec{A}$ (b) $\vec{A} \times \vec{B} = AB \sin\theta$

(c) $\vec{A} \times \vec{B} = AB\cos\theta$ (d) $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$

[DCE 09]

23. For an object thrown at 45° to the horizontal, the maximum height (H) and horizontal range (R) are related as

(a) $R = 16H$ (b) $R = 8H$

(c) $R = 4H$ (d) $R = 2H$ [DCE 09]

24. A particle is projected with certain velocity at two different angles of projections with respect to horizontal plane so as to have the same range 'R' on a horizontal plane. If 't_1' and 't_2' are the time taken for the two paths, then which one of the following relations is correct ?

(a) $t_1 t_2 = 2R/g$ (b) $t_1 t_2 = R/g$

(c) $t_1 t_2 = R/2g$ (d) $t_1 t_2 = 4R/g$. [DCE 09]

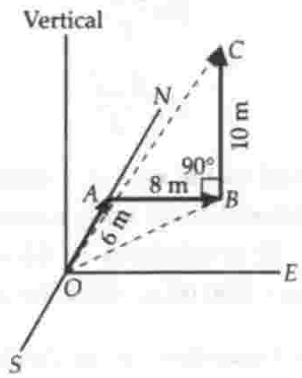
25. A body is tied with a string and is given a circular motion with a velocity v in radius r. The magnitude of the acceleration is

(a) v/r (b) v^2/r

(c) v/r^2 (d) v^2/r^2 [DCE 09]

Answers and Explanations

1. (*a*) The situation is shown below



$$OC = \sqrt{OA^2 + AB^2 + BC^2}$$
$$= \sqrt{6^2 + 8^2 + 10^2}$$
$$= \mathbf{10\sqrt{2}\ m.}$$

2. (*d*) $\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos\theta = -|\vec{A}||\vec{B}|$

$\therefore \quad \cos\theta = -1$

or $\quad \theta = \mathbf{180°}.$

3. (*d*) As scalar triple product is cyclic,

$$(\vec{B}\times\vec{A})\cdot\vec{A} = (\vec{A}\times\vec{B})\cdot\vec{A} = (\vec{A}\times\vec{A})\cdot\vec{B}$$
$$= \vec{0}\cdot\vec{B} = 0.$$

4. (*d*) Given :

$$v_x = \frac{dx}{dt} = \frac{1}{3}\,\text{ms}^{-2}$$
$$y = 9x^2$$
$$\therefore \quad v_y = \frac{dy}{dt} = 18x\frac{dx}{dt}$$
$$= 18x \times \frac{1}{3} = 6x$$
$$a_y = \frac{dv_y}{dt} = 6\frac{dx}{dt} = 6 \times \frac{1}{3} = 2\,\text{ms}^{-2}$$
$$\therefore \quad \vec{a}_y = 2\,\hat{j}\ \text{m/s}^2.$$

5. (*d*) As the cyclist returns to initial position O after one round trip, his displacement is zero.

Total distance travelled

$$= OP + OQ + \text{Arc}\,PQ$$
$$= r + r + \tfrac{1}{2}\pi r$$
$$= \left(2 + \frac{\pi}{2}\right)r = \frac{4+\pi}{2} \times 1\ \text{km}$$

Time taken $= 10\ \text{min} = \frac{1}{6}\,\text{h}$

$$\text{Average speed} = \frac{4+\pi}{2} \times \frac{6}{1} = (12 + 3\pi)\ \text{kmh}^{-1}$$
$$= \mathbf{21.4\ kmh^{-1}}.$$

6. (*a*) When the cyclist takes seventh turn, he is at 100 m distance from the initial position.

7. (*b*) Average speed = constant speed

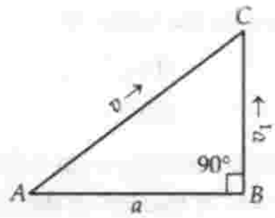
$$= \mathbf{31.4\ m/s.}$$

8. (*b*) $\text{Average velocity} = \dfrac{\text{Displacement}}{\text{Time}} = \dfrac{0}{t} = \mathbf{0}.$

$$\text{Average speed} = \frac{\text{Distance}}{\text{time}} = \frac{2\pi r}{t}$$
$$= \frac{2 \times 3.14 \times 100}{62.8} = \mathbf{10\ ms^{-1}}.$$

9. (*b*) Distance covered by A in time t, $AC = vt$

Distance covered by B in time t, $BC = v_1 t$



By Pythagoras theorem,

$$AC^2 = AB^2 + BC^2$$

or $\quad (vt)^2 = a^2 + (v_1 t)^2$

or $\quad t^2(v^2 - v_1^2) = a^2$

or $\quad t = \sqrt{\dfrac{a^2}{v^2 - v_1^2}}.$

10. (*d*) Observed speed,

$$v = \sqrt{(10)^2 + (6)^2}$$
$$= \sqrt{136} = \mathbf{11\ ms^{-1}}.$$

11. (*a*) Refer to the solution of Problem 5 on Page 4.72.

12. (*d*) Both bullets will reach the ground simultaneously.

$$t = \sqrt{\frac{2h}{g}}$$

Time t does not depend on horizontal speed.

13. (*b*) $R_A = \frac{u^2 \sin 60°}{g} = \frac{\sqrt{3}}{2}\frac{u^2}{g}$

$$R_B = \frac{u^2 \sin 90°}{g} = \frac{u^2}{g}$$

$$R_C = \frac{u^2 \sin^2 120°}{g} = \frac{\sqrt{3}}{2}\frac{u^2}{g}$$

Ranges of A and C are equal but less than that of B.

14. (*a*) $R = \frac{u^2 \sin(2 \times 45°)}{g} = \frac{u^2}{g}$

$$H = \frac{u^2 \sin^2 45°}{2g} = \frac{u^2}{4g}$$

$\therefore$ $R = 4H$.

15. (*b*) Given : $(\text{Range})^2 = 48\,(\text{Max. Height})^2$

or $$\left(\frac{u^2 \sin 2\theta}{g}\right)^2 = 48\left(\frac{u^2 \sin^2\theta}{2g}\right)^2$$

or $$u^2 \frac{\sin 2\theta}{g} = 4\sqrt{3}\,\frac{u^2 \sin^2\theta}{2g}$$

or $$2\sin\theta\cos\theta = 2\sqrt{3}\sin^2\theta$$

or $$\tan\theta = \frac{1}{\sqrt{3}}$$

$\therefore$ $$\theta = 30°.$$

16. (*b*) Given : $H_1 = H_2$

or $$\frac{u_1^2 \sin^2 60°}{2g} = \frac{u_2^2 \sin^2 45°}{2g}$$

or $$\frac{u_1}{u_2} = \frac{\sin 45°}{\sin 60°} = \frac{1/\sqrt{2}}{\sqrt{3}/2} = \frac{\sqrt{2}}{\sqrt{3}}.$$

17. (*b*) By conservation of momentum,

$$mv = m_1 v_1 + m_2 v_2$$

$$mv = \frac{m}{3} \times 0 + \frac{2m}{3} \times v_2$$

or $$v_2 = \frac{3v}{2}.$$

18. (*c*) Velocity is along the tangent to the circular path and acceleration vector is along the radius.

19. (*d*) Circular motion with constant speed is periodic but not simple harmonic.

20. (*a*) For circular motion,

$$v = \frac{2\pi r}{T}$$

or $$r = \frac{vT}{2\pi}$$

Acceleration in circular motion,

$$a = \frac{v^2}{r} = \frac{v^2}{vT/2\pi} = \frac{2\pi v}{T}.$$

21. (*c*) $a = r\omega^2 = r\left(\frac{2\pi n}{t}\right)^2$

$$= r \times \frac{4\pi^2 n^2}{t^2} = \frac{1 \times 4 \times \pi^2 \times (22)^2}{(44)^2}$$

$$= \pi^2\ \mathbf{ms^{-2}}.$$

22. (*d*) The cross product of two vectors is anticommutative *i.e.*,

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}.$$

23. (*c*) As $R = \frac{u^2}{g}$ and $H = \frac{u^2}{4g}$ $\therefore$ $\mathbf{R = 4H}$.

24. (*a*) As $t_1 = \frac{2u\sin\theta}{g}$

and $$t_2 = \frac{2u\cos\theta}{g}$$

$\therefore$ $$t_1 t_2 = \frac{4u^2 \sin\theta\cos\theta}{g} = \frac{2u^2 \sin 2\theta}{g} = \frac{2R}{g}.$$

25. (*b*) Centripetal acceleration,

$$a_c = \frac{v^2}{r}.$$

AIIMS Entrance Exam

1. Two equal vectors have a resultant equal to either. The angle between them is

(*a*) 60° (*b*) 90°
(*c*) 100° (*d*) 120° [AIIMS 01]

2. Angle between two vectors of magnitudes 12 and 18 units, when their resultant is 24 units, is

(*a*) 63°51′ (*b*) 75°52′
(*c*) 82°31′ (*d*) 89°16′ [AIIMS 96]

3. A body A is dropped vertically from the top of a tower. If another identical body B is thrown horizontally from the same point at the same instant, then

(*a*) A will reach the ground earlier than B

(*b*) B will reach the ground earlier than A

(*c*) both A and B will reach the ground simultaneously

(*d*) either A or B. [AIIMS 94]

4. Which of the following is constant in a projectile motion ?

(*a*) Horizontal component of the velocity

(*b*) Vertical component of the velocity

(*c*) Velocity of projection

(*d*) All of these. [AIIMS 96]

5. At the uppermost point of a projectile, its velocity and acceleration are at an angle of

(*a*) $0°$ (*b*) $45°$

(*c*) $90°$ (*d*) $180°$ [AIIMS 02]

Two projectiles are projected with the same velocity. If one is projected at an angle of $30°$ and the other at $60°$ to the horizontal, then the ratio of maximum height reached, is

(*a*) 3 : 1 (*b*) 1 : 3

(*c*) 1 : 2 (*d*) 2 : 1 [AIIMS 01]

7. A body is projected at such an angle that the horizontal range is three times the greatest height. The angle of projection is

(*a*) $25°8'$ (*b*) $33°7'$

(*c*) $42°8'$ (*d*) $53°8'$ [AIIMS 98]

8. The direction of the angular velocity vector is along

(*a*) the tangent to the circular path

(*b*) the inward radius

(*c*) the outward radius

(*d*) the axis of rotation [AIIMS 04, 07]

9. A stone tied to the end of string 80 cm long is whirled in a horizontal circle with a constant speed. If the stone makes 14 revolutions in 25 s, what is the magnitude of acceleration of the stone ?

(*a*) 680 cm s^{-2} (*b*) 720 cm s^{-2}

(*c*) 860 cm s^{-2} (*d*) 990 cm s^{-2} [AIIMS 01]

10. If the equation for the displacement of a particle moving on a circular path is given by $\theta = 2t^3 + 0.5$, where θ is in radians and t in seconds, then, the angular velocity of the particle at $t = 2$ s is

(*a*) 8 rad s^{-1} (*b*) 12 rad s^{-1}

(*c*) 24 rad s^{-1} (*d*) 36 rad s^{-1} [AIIMS 98]

11. A body is projected horizontally with a velocity of $4\sqrt{2}$ m/sec. The velocity of the body after 0.7 seconds will be nearly (Take $g = 10 \text{ m/s}^2$)

(*a*) 10 m/sec (*b*) 9 m/sec

(*c*) 19 m/sec (*d*) 11 m/sec [AIIMS 2009]

12. Two projectiles of same mass have their maximum kinetic energies in ratio 4 : 1 and ratio of their maximum heights is also 4 : 1, then what is the ratio of their ranges ?

(*a*) 2 : 1 (*b*) 4 : 1

(*c*) 8 : 1 (*d*) 16 : 1 [AIIMS 2010]

13. The position of a particle moving in the x-y plane at any time t is given by ; $x = (3t^2 - 6t)$ metres ; $y = (t^2 - 2t)$ metres. Select the correct statement.

(*a*) acceleration is zero at $t = 0$

(*b*) velocity is zero at $t = 0$

(*c*) velocity is zero at $t = 1$s

(*d*) velocity and acceleration of the particle are never zero. [AIIMS 2010]

Assertions and Reasons

Directions. In the following questions, a statement of assertion (A) is followed by a statement of reason (R). Mark the correct choice as :

(*a*) If both assertion and reason are true and reason is the correct explanation of assertion

(*b*) If both assertion and reason are true but reason is not the correct explanation of assertion

(*c*) If assertion is true but reason is false

(*d*) If both assertion and reason are false

14. Assertion. Two balls of different masses are thrown vertically upward with same speed. They will pass through their point of projection in the downward direction with the same speed.

Reason. The maximum height and downward velocity attained at the point of projection are independent of the mass of the ball. [AIIMS 2010]

15. Assertion. In javelin throw, the athlete throws the projectile at an angle slightly more than 45°.

Reason. The maximum range does not depend upon angle of projection. [AIIMS 2010]

16. **Assertion.** Generally the path of a projectile from the earth is parabolic but it is elliptical for projectiles going to a very great height.

Reason. Up to ordinary height the projectile moves under a uniform gravitational force, but for great heights, projectile moves under a variable force.

[AIIMS 1998]

Answers and Explanations

1. (*d*) Refer to the solution of Example 5 on page 4.9.

2. (*b*) As $R^2 = P^2 + Q^2 + 2PQ\cos\theta$

$\therefore \quad 24^2 = 12^2 + 18^2 + 2 \times 12 \times 18 \cos\theta$

or $\quad 576 - (144 + 324) = 432 \cos\theta$

or $\quad \cos\theta = \dfrac{108}{432} = 0.25.$

$\therefore \quad \theta = \cos^{-1}(0.25) = \mathbf{75°52'}.$

3. (*c*) Both *A* and *B* will reach the ground simultaneously. Both have zero initial vertical velocity, cover the same height under the acceleration due to gravity '*g*'.

4. (*a*) Horizontal component of velocity is not affected by the force of gravity.

5. (*c*) At the highest point, the projectile has only a horizontal velocity while the acceleration due to gravity acts vertically downwards.

6. (*b*) Maximum height, $H = \dfrac{u^2 \sin^2\theta}{2g}$

$$\therefore \quad \frac{H_1}{H_2} = \frac{\sin^2 30°}{\sin^2 60°} = \frac{(1/2)^2}{(\sqrt{3}/2)^2} = \frac{1}{3} = \mathbf{1:3}.$$

7. (*d*) Here $\quad R = 3H$

or $\quad \dfrac{u^2 \sin 2\theta}{g} = 3\dfrac{u^2 \sin^2\theta}{2g}$

or $\quad \dfrac{u^2 \times 2\sin\theta\cos\theta}{g} = 3\dfrac{u^2\sin^2\theta}{2g}$

or $\quad \tan\theta = \dfrac{4}{3} = 1.3333$

$$\theta = \tan^{-1}(1.3333) = \mathbf{53°8'}.$$

8. (*d*) $\quad \vec{v} = \vec{\omega} \times \vec{r}$

As linear velocity vector $\vec{v}$ is along the tangent to the circular path and angular velocity vector $\vec{\omega}$ is perpendicular to $\vec{v}$, so $\vec{\omega}$ is along the axis of rotation.

9. (*d*) Refer answer to NCERT Exercise 4.17.

10. (*c*) $\quad \theta = 2t^3 + 0.5$

$$\omega = \frac{d\theta}{dt} = 6t^2$$

At $t = 2$ s,

$$\omega = 6 \times 4 = \mathbf{24\ rad\ s^{-1}}.$$

11. (*b*) After 0.7 s,

$$v_x = 4\sqrt{2}\ \text{m/s}$$

$$v_y = 0 + gt = 10 \times 0.7 = 7\ \text{m/s}$$

$$\therefore \quad v = \sqrt{v_x^2 + v_y^2} = \sqrt{32 + 49} = \mathbf{9\ m/s}$$

12. (*b*) $\dfrac{K_1^{\max}}{K_2^{\max}} = \dfrac{\frac{1}{2}mu_1^2}{\frac{1}{2}mu_2^2} = \dfrac{u_1^2}{u_2^2} = \dfrac{4}{1}$

$$\frac{H_1}{H_2} = \frac{u_1^2\sin^2\theta_1}{2g} \cdot \frac{2g}{u_2^2\sin^2\theta_2} = \frac{4}{1}$$

or $\quad \dfrac{u_1^2}{u_2^2} \cdot \dfrac{\sin^2\theta_1}{\sin^2\theta_2} = \dfrac{4}{1} \quad$ or $\quad \dfrac{4}{1} \cdot \dfrac{\sin^2\theta_1}{\sin^2\theta_2} = \dfrac{4}{1}$

$\therefore \quad \sin\theta_1 = \sin\theta_2 \Rightarrow \theta_1 = \theta_2$

$\Rightarrow \quad \dfrac{R_1}{R_2} = \dfrac{u_1^2\sin^2\theta_1}{u_2^2\sin^2\theta_2} = \dfrac{u_1^2}{u_2^2} = \dfrac{4}{1} = \mathbf{4:1} \quad [\theta_1 = \theta_2]$

13. (*c*) $\quad x = 3t^2 - 6t$

$\therefore \quad v_x = \dfrac{dx}{dt} = 6t - 6$

At $t = 1$, $v_x = 6 \times 1 - 6 = 0$

Also, $\quad y = t^2 - 2t$

$\therefore \quad v_y = \dfrac{dy}{at} = 2t - 2$

At $t = 1$s, $v_y = 2 \times 1 - 2 = 0.$

14. (*a*) Maximum height attained,

$$h = ut - \frac{1}{2}gt^2$$

Downward velocity attained,

$$v = \sqrt{u^2 - 2gh}$$

Both h and v are independent of mass m.

15. (*d*) When a body is projected from a place above the earth's surface, the angle of projection must be slightly less than 45° for the maximum horizontal angle.

16. (*a*) Assertion is true and the reason is its correct explanation. At greater heights, the gravitational force varies in inverse proportion to the square of the distance of the projectile from the centre of the earth. Under such a variable force, the path of the projectile is elliptical

CBSE PMT Prelims and Final Exams

(Similar Questions)

1. If $|\vec{A}+\vec{B}|=|\vec{A}|+|\vec{B}|$, then angle between A and B will be

(*a*) 90° (*b*) 120°
(*c*) 0° (*d*) 60° [CBSE PMT 01]

2. A bus is moving on a straight road towards north with a uniform speed of 50 km/hour, then it turns left through 90°. If the speed remains unchanged after turning, the increase in the velocity of bus in the turning process is

(*a*) 70.7 km/hr along south-west direction
(*b*) zero
(*c*) 50 km/hr along west
(*d*) 70.7 km/hr along north-west direction

[CBSE PMT 90]

3. If a unit vector is represented by $0.5\hat{i}-0.8\hat{j}+c\hat{k}$ then the value of c is

(*a*) $\sqrt{0.01}$ (*b*) $\sqrt{0.11}$
(*c*) 1 (*d*) $\sqrt{0.39}$ [CBSE PMT 99]

4. The angle between the two vectors $\vec{A}=3\hat{i}+4\hat{j}+5\hat{k}$ and $\vec{B}=3\hat{i}+4\hat{j}-5\hat{k}$ will be

(*a*) 90° (*b*) 180°
(*c*) zero (*d*) 45° [CBSE PMT 94]

5. If a vector $2\hat{i}+3\hat{j}+8\hat{k}$ is perpendicular to the vector $4\hat{j}-4\hat{i}+\alpha\hat{k}$, then the value of α is

(*a*) 1/2 (*b*) −1/2
(*c*) 1 (*d*) −1 [CBSE PMT 05]

6. The magnitudes of vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ are 3, 4 and 5 units respectively. If $\vec{A}+\vec{B}=\vec{C}$, the angle between $\vec{A}$ and $\vec{B}$ is

(*a*) $\pi/2$ (*b*) $\cos^{-1}0.6$
(*c*) $\tan^{-1}7/5$ (*d*) $\pi/4$ [CBSE PMT 88]

7. The vectors $\vec{A}$ and $\vec{B}$ are such that $|\vec{A}+\vec{B}|=|\vec{A}-\vec{B}|$. The angle between the two vectors is

(*a*) 45° (*b*) 90°
(*c*) 60° (*d*) 75°

[CBSE PMT 91, 96, 2K]

8. The vector sum of two forces is perpendicular to their vector differences. In that case, the forces

(*a*) are equal to each other
(*b*) are equal to each other in magnitude
(*c*) are not equal to each other in magnitude
(*d*) cannot be predicted

[CBSE PMT 03]

9. $\vec{A}$ and $\vec{B}$ are two vectors and θ is the angle between them, if $|\vec{A}\times\vec{B}|=\sqrt{3}(\vec{A}.\vec{B})$, the value of θ is

(*a*) 45° (*b*) 30°
(*c*) 90° (*d*) 60° [CBSE PMT 07]

10. If $|\vec{A}\times\vec{B}|=\sqrt{3}\,\vec{A}.\vec{B}$, then the value of $|\vec{A}+\vec{B}|$ is

(*a*) $(A^2+B^2+AB)^{1/2}$ (*b*) $\left(A^2+B^2+\frac{AB}{\sqrt{3}}\right)^{1/2}$
(*c*) $A+B$ (*d*) $(A^2+B^2+\sqrt{3}AB)^{1/2}$

[CBSE PMT 04]

11. If the angle between the vectors $\vec{A}$ and $\vec{B}$ is θ, the value of the product $(\vec{B}\times\vec{A}).\vec{A}$ is equal to

(*a*) $BA^2\sin\theta$ (*b*) $BA^2\cos\theta$
(*c*) $BA^2\sin\theta\cos\theta$ (*d*) zero

[CBSE PMT 89, 05]

12. The resultant of $\vec{A}\times 0$ will be equal to

(*a*) zero (*b*) A
(*c*) zero vector (*d*) unit vector

[CBSE PMT 92]

13. A boat is sent across a river with a velocity of $8\,\text{km h}^{-1}$. If the resultant velocity of boat is $10\,\text{km h}^{-1}$, then velocity of river is

(a) $12.8\,\text{km h}^{-1}$ (b) $6\,\text{km h}^{-1}$
(c) $8\,\text{km h}^{-1}$ (d) $10\,\text{km h}^{-1}$

[CBSE PMT 93, 94]

14. The width of river is 1 km. The velocity of boat is 5 km / h The boat covered the width of river in shortest time 15 min. Then the velocity of river stream is

(a) 3 km/hr (b) 4 km/hr
(c) $\sqrt{29}$ km/hr (d) $\sqrt{41}$ km/hr

[CBSE PMT 98, 2K]

15. A person aiming to reach exactly opposite point on the bank of a stream is swimming with a speed of 0.5 m/s at an angle of $120°$ with the direction of flow of water. The speed of water in the stream are

(a) 0.25 m/s (b) 0.5 m/s
(c) 1.0 m/s (d) 0.433 [CBSE PMT 99]

16. A car runs at a constant speed of a circular track of radius 100 m, taking 62.8 seconds for every circular lap. The average velocity and average speed for each circular lap respectively is

(a) 10 m/s, 0 (b) 0, 0
(c) 0, 10 m/s (d) 10 m/s, 10 m/s

[CBSE PMT 01]

17. A particle starting from the origin (0, 0) moves in a straight line in the (x, y) plane. Its coordinates at a later time are $(\sqrt{3}, 3)$. The path of the particle makes with the x-axis an angle of

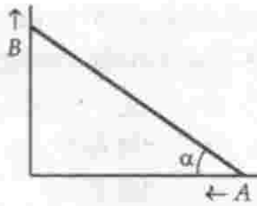
(a) $45°$ (b) $60°$
(c) $0°$ (d) $30°$ [CBSE PMT 07]

18. Two boys are standing at the ends A and B of a ground where $AB = a$. The boy at B starts running in a direction perpendicular to AB with velocity v_1. The boy at A starts running simultaneously with velocity v and catches the other in a time t, where t is

(a) $\dfrac{a}{\sqrt{v^2 + v_1^2}}$ (b) $\dfrac{a}{v + v_1}$

(c) $\dfrac{a}{v - v_1}$ (d) $\sqrt{\dfrac{a^2}{v^2 - v_1^2}}$ [CBSE PMT 05]

19. Two particles A and B are connected by a rigid rod AB. The rod slides along perpendicular rails as shown here.



The velocity of A to the left is 10 m/s. What is the velocity of B when angle $\alpha = 60°$?

(a) 10 m/s (b) 9.8 m/s
(c) 5.8 m/s (d) 17.3 m/s [CBSE PMT 98]

20. The position vector of a particle is

$$\vec{r} = (a\cos\omega t)\hat{i} + (a\sin\omega t)\hat{j}$$

The velocity of the particle is

(a) directed towards the origin
(b) directed away from the origin
(c) parallel to the position vector
(d) perpendicular to the position vector

[CBSE PMT 95]

21. Two projectiles of same mass and with same velocity are thrown at an angle $60°$ and $30°$ with the horizontal, then which will remain same

(a) time of flight
(b) range of projectile
(c) maximum height acquired
(d) all of them [CBSE PMT 2K]

22. If a body A of mass M is thrown with velocity v at an angle $30°$ to the horizontal and another body B of the same mass is thrown with the same speed at an angle of $60°$ to the horizontal, the ratio of horizontal range of A to B will be

(a) 1 : 3 (b) 1 : 1
(c) $1 : \sqrt{3}$ (d) $\sqrt{3} : 1$ [CBSE PMT 90, 92]

23. For angles of projection of a projectile at angle $(45° - \theta)$ and $(45° + \theta)$, the horizontal range described by the projectile are in the ratio of

(a) 2 : 1 (b) 1 : 1
(c) 2 : 3 (d) 1 : 2 [CBSE PMT 06]

24. The maximum range of a gun of horizontal tarrain is 16 km. If $g = 10\ \text{ms}^{-2}$, then muzzle velocity of a shell must be

(a) $160\,\text{ms}^{-1}$ (b) $200\sqrt{2}\ \text{ms}^{-2}$
(c) $400\,\text{ms}^{-1}$ (d) $800\,\text{ms}^{-1}$

[CBSE PMT 90]

25. A cricket ball is hit at 45° to the horizontal with a kinetic energy E. The kinetic energy at the highest point is

(a) 0 (b) $E/2$
(c) $E/\sqrt{2}$ (d) E [CBSE PMT 97]

26. A particle A is dropped from a height and another particle B is projected in horizontal direction with speed of 5 m/sec from the same height, then correct statement is

(a) particle A will reach at ground first with respect to particle B
(b) particle B will reach at ground first with respect to particle A
(c) both particles will reach at ground simultaneously
(d) both particles will reach at ground with same speed [CBSE PMT 02]

27. A man throws balls with the same speed vertically upwards one after the other at an interval of 2 seconds. What should be the speed of the throw so that more than two balls are in the sky at any time? (Given $g = 9.8\,\text{m/s}^2$)

(a) more than 19.6 m/s
(b) atleast 9.8 m/s
(c) any speed less than 19.6 m/s
(d) only with speed 19.6 m/s [CBSE PMT 03]

28. A man is slipping on a frictionless inclined plane and a bag falls down from the same height. Then the velocity of both is related as

(a) $v_B > v_m$
(b) $v_B < v_m$
(c) $v_B = v_m$
(d) v_B and v_m can't be related [CBSE PMT 2K]

29. The angular speed of a flywheel making 120 revolutions/minute is

(a) 4π rad/s (b) $4\pi^2$ rad/s
(c) π rad/s (d) 2π rad/s [CBSE PMT 95]

30. Two racing cars of masses m_1 and m_2 are moving in circles of radii r_1 and r_2 respectively. Their speeds are such that each makes a complete circle in the same time t. The ratio of the angular speeds of the first to the second car is

(a) $r_1 : r_2$ (b) $m_1 : m_2$
(c) $1 : 1$ (d) $m_1 m_2 : r_1 r_2$
[CBSE PMT 99]

31. Two particles having mass M and m are moving in a circular path having radius R and r. If their time periods are same, then the ratio of angular velocity will be

(a) $\frac{r}{R}$ (b) $\frac{R}{r}$
(c) 1 (d) $\sqrt{\frac{R}{r}}$ [CBSE PMT 01]

32. A body is whirled in a horizontal circle of radius 20 cm. It has an angular velocity of 10 rad/s

What is its linear velocity at any point on circular path ?

(a) 20 m/s (b) $\sqrt{2}$ m/s
(c) 10 m/s (d) 2 m/s [CBSE PMT 96]

33. What is the value of linear velocity, if

$$\vec{r} = 3\hat{i} - 4\hat{j} + \hat{k} \text{ and } \vec{\omega} = 5\hat{i} - 6\hat{j} + 6\hat{k} \text{ ?}$$

(a) $4\hat{i} - 13\hat{j} + 6\hat{k}$ (b) $18\hat{i} + 13\hat{j} - 2\hat{k}$
(c) $6\hat{i} + 2\hat{j} - 3\hat{k}$ (d) $6\hat{i} - 2\hat{j} + 8\hat{k}$
[CBSE PMT 99]

34. An electric fan has blades of length 30 cm measured from the axis of rotation. If the fan is rotating at 120 rpm, the acceleration of a point on the tip of the blade is

(a) $1600\ \text{ms}^{-2}$ (b) $47.4\ \text{ms}^{-2}$
(c) $23.7\ \text{ms}^{-2}$ (d) $50.55\ \text{ms}^{-2}$ [CBSE PMT 90]

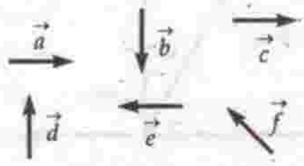
35. A particle moves along a circle of radius $\left(\frac{20}{\pi}\right)$m with constant tangential acceleration. If the velocity of the particle is 80 m/s at the end of the second revolution after motion has begun, the tangential acceleration is

(a) $40\ \text{m/s}^2$ (b) $640\pi\ \text{m/s}^2$
(c) $160\pi\ \text{m/s}^2$ (d) $40\pi\ \text{m/s}^2$ [CBSE PMT 03]

36. A stone tied to the end of a string of 1 m length is whirled in a horizontal circle with a constant speed. If the stone makes 22 revolutions in 44 seconds, what is the magnitude and direction of acceleration of the stone ?

(a) $\pi^2\ \text{ms}^{-2}$ and direction along the radius towards the centre
(b) $\pi^2\ \text{ms}^{-2}$ and direction along the radius away from the centre
(c) $\pi^2\ \text{ms}^{-2}$ and direction along the tangent to the circle
(d) $\pi^2/4\ \text{ms}^{-2}$ and direction along the radius towards the centre. [CBSE PMT 05]

37. Six vectors, $\vec{a}$ through $\vec{f}$ have the magnitudes and directions indicated in the figure. Which of the following statements is true ?



(*a*) $\vec{b}+\vec{c}=\vec{f}$ (*b*) $\vec{d}+\vec{c}=\vec{f}$

(*c*) $\vec{d}+\vec{e}=\vec{f}$ (*d*) $\vec{b}+\vec{e}=\vec{f}$ [CBSE Pre 2010]

38. A particle has initial velocity $(3\hat{i}+4\hat{j})$ and has acceleration $(0.4\hat{i}+0.3\hat{j})$. Its speed after 10 s is

(*a*) 7 units (*b*) $7\sqrt{2}$ units

(*c*) 8.5 units (*d*) 10 units [CBSE Pre 2010]

39. A body is moving with velocity 30 m/s towards east. After 10 seconds its velocity becomes 40 m/s towards north. The average acceleration of the body is

(*a*) 1 m/s^2 (*b*) 7 m/s^2

(*c*) $\sqrt{7}\text{ m/s}^2$ (*d*) 5 m/s^2 [CBSE Pre 2011]

40. A missile is fired for maximum range with an initial velocity of 20 m/s. If $g=10\text{ m/s}^2$, the range of the missile is

(*a*) 40 m (*b*) 50 m

(*c*) 60 m (*d*) 20 m [CBSE Pre 2011]

41. A projectile is fired at an angle of 45° with the horizontal. Elevation angle of the projectile at its highest point as seen from the point of projection, is

(*a*) $\tan^{-1}\left(\frac{\sqrt{3}}{2}\right)$ (*b*) 45°

(*c*) 60° (*d*) $\tan^{-1}\frac{1}{2}$ [CBSE Final 2011]

42. A particle moves in a circle of radius 5 cm with constant speed and time period 0.2π s. The acceleration of the particle is

(*a*) 15 m/s^2 (*b*) 25 m/s^2

(*c*) 36 m/s^2 (*d*) 5 m/s^2 [CBSE Pre 2011]

Answers and Explanations

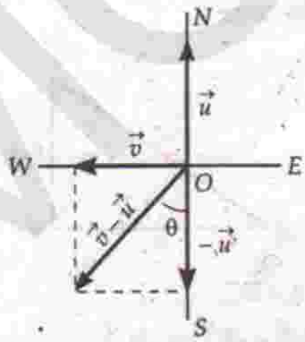
1. (*c*) When $\theta=0°$,

$$|\vec{A}+\vec{B}|=\sqrt{|\vec{A}|^2+|\vec{B}|^2+2|\vec{A}||\vec{B}|\cos 0°}$$

$$=|\vec{A}|+|\vec{B}|$$

2. (*a*) Initial velocity, $u=50\text{ kmh}^{-1}$, due north

Final velocity, $v=50\text{ kmh}^{-1}$, due west



Change in velocity $=\vec{v}-\vec{u}=\vec{v}+(-\vec{u})$, along south-west.

$$|\vec{v}-\vec{u}|=\sqrt{50^2+50^2}=\sqrt{5000}$$

$$=\mathbf{70.71\ km\ h^{-1}}.$$

3. (*b*)
$$|0.5\hat{i}-0.8\hat{j}+c\hat{k}|=1$$
$$\sqrt{(0.5)^2+(0.8)^2+c^2}=1$$
or
$$0.25+0.64+c^2=1$$
$$c=\sqrt{\mathbf{0.11}}.$$

4. (*a*)
$$\vec{A}=3\hat{i}+4\hat{j}+5\hat{k}$$
$$\vec{B}=3\hat{i}+4\hat{j}-5\hat{k}$$
$$\vec{A}\cdot\vec{B}=3\times3+4\times4+5\times(-5)=0$$
$$\Rightarrow\quad \vec{A}\perp\vec{B}.$$

5. (*b*)
$$\vec{A}\perp\vec{B}\Rightarrow\vec{A}\cdot\vec{B}=0$$
$$(2\hat{i}+3\hat{j}+8\hat{k})\cdot(4\hat{j}-4\hat{i}+\alpha\hat{k})=0$$
$$2\times(-4)+3\times4+8\alpha=0$$
as
$$\alpha=-\frac{\mathbf{1}}{\mathbf{2}}.$$

6. (*a*) $\vec{A}+\vec{B}=\vec{C}$

$$\therefore\quad(\vec{A}+\vec{B})\cdot(\vec{A}+\vec{B})=\vec{C}\cdot\vec{C}$$

$$|\vec{A}|^2+|\vec{B}|^2+2|\vec{A}||\vec{B}|\cos\theta=|\vec{C}|^2$$

$$9+16+2\times3\times4\cos\theta = 25$$
$$\cos\theta = 0$$
or $$\theta = \frac{\pi}{2}.$$

7. (*b*) Refer to the solution of Example 37 on Page 4.23.

8. (*a*) Given : $(\vec{F_1}+\vec{F_2})\perp(\vec{F_1}-\vec{F_2})$

$\therefore$ $$(\vec{F_1}+\vec{F_2}).(\vec{F_1}-\vec{F_2}) = 0$$
or $$F_1^2 - \vec{F_1}.\vec{F_2} + \vec{F_2}.\vec{F_1} - F_2^2 = 0$$
or $$F_1^2 = F_2^2$$
$\Rightarrow$ $$F_1 = F_2.$$

9. (*d*) Given : $|\vec{A}\times\vec{B}| = \sqrt{3}(\vec{A}.\vec{B})$

$\therefore$ $$AB\sin\theta = \sqrt{3}\,AB\cos\theta$$
or $$\tan\theta = \sqrt{3}$$
$\Rightarrow$ $$\theta = \mathbf{60^\circ}.$$

10. (*a*) From above problem, $\theta = 60^\circ$

$\therefore$ $$|\vec{A}\times\vec{B}| = \sqrt{|\vec{A}| + |\vec{B}|^2 + 2|\vec{A}||\vec{B}|\cos 60^\circ}$$
$$= (A^2 + B^2 + AB)^{1/2}$$

11. (*d*) As triple scalar product is cyclic.

$$(\vec{B}\times\vec{A})\,\vec{A} = (\vec{A}\times\vec{B}).\vec{A}$$
$$= (\vec{A}\times\vec{A}).\vec{B} = \vec{0}.\vec{B} = \vec{0}$$

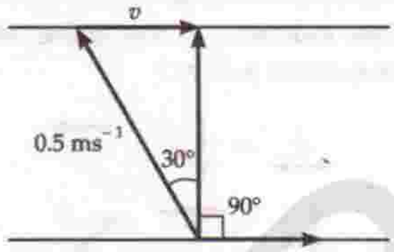
12. (*c*) The resultant of $\vec{A}\times 0$ is a vector of zero magnitude. The product of a vector with a scalar gives a vector.

13. (*b*) Resultant velocity

$$= \sqrt{v_B^2 + v_R^2 + 2v_Bv_R\cos\theta}$$
$$(10)^2 = 8^2 + V_R^2 + 2\times8\times V_R\cos 90^\circ$$
$$V_R = \sqrt{100-64}$$
$$= \mathbf{6\ km\ h^{-1}}$$

14. (*a*) $$v_{\text{Resultant}} = \frac{1\text{ km}}{15\text{ min}}$$
$$= \frac{1\text{km}}{1/4\text{ h}} = 4\text{ km h}^{-1}$$
$$v_{\text{River}} = \sqrt{v_{\text{Resultant}}^2 - v_{\text{Boat}}^2}$$
$$= \sqrt{5^2-4^2}$$
$$= \mathbf{3\ kmh^{-1}}.$$

15. (*a*) Let v be the speed of river water.



Clearly, $$\frac{v}{0.5\,\text{ms}^{-1}} = \sin 30^\circ$$
$\therefore$ $$v = 0.5\times0.5 = \mathbf{0.25\ ms^{-1}}.$$

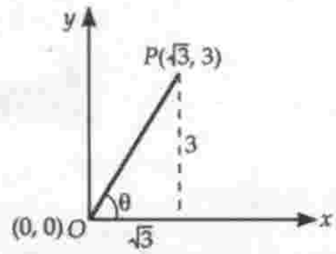
16. (*c*) Distance covered in one lap $= 2\pi r$

$$\text{Average speed} = \frac{\text{Distance}}{\text{Time}} = \frac{2\pi r}{t}$$
$$= \frac{2\times3.14\times100}{62.8} = 10\text{ m.}$$

Net displacement for one lap = 0.

$$\text{Average velocity} = \frac{\text{Displacement}}{\text{Time}} = \frac{0}{t} = 0.$$

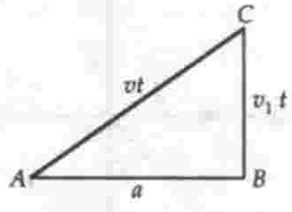
17. (*b*) The situation is shown in the figure.



If the path OP makes angle θ with the x-axis, then

$$\tan\theta = \frac{3}{\sqrt{3}} = \sqrt{3}$$
$\therefore$ $$\theta = \mathbf{60^\circ}.$$

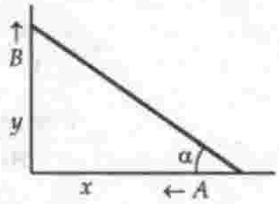
18. (*d*) Two boys stand at the points A and B separated by distance a.



From right ΔABC,

$$AB^2 + BC^2 = AC^2$$
$$a^2 + (v_1t)^2 = (vt)^2$$
$$a^2 = (v^2 - v_1^2)t^2$$
$$t = \frac{a}{\sqrt{v^2 - v_1^2}}$$

19. (*d*) From the figure,



$$\frac{y}{x} = \tan\alpha$$

$$y = x\tan\alpha$$

Differentiating w.r.t. time t,

$$\frac{dy}{dx} = \frac{dx}{dt}\tan\alpha$$

$$v_y = v_x \tan\alpha = 10\tan 60^\circ$$
$$= 10\sqrt{3} = 10 \times 1.732 = \mathbf{17.32\ ms^{-1}}.$$

20. (*d*) Position vector,

$$\vec{r} = (a\cos\omega t)\hat{i} + (a\sin\omega t)\hat{j}$$

Velocity vector,

$$\vec{v} = \frac{d\vec{r}}{dt} = (-a\omega\sin\omega t)\hat{i} + (a\omega\cos\omega t)\hat{j}$$

$$\vec{v}.\vec{r} = (-a\omega\sin\omega t)(a\cos\omega t) + (a\omega\cos\omega t)(a\sin\omega t)$$
$$= 0$$

$$\Rightarrow \vec{v} \perp \vec{r}.$$

21. (*b*) Here $\theta_1 + \theta_2 = 60^\circ + 30^\circ = 90^\circ$.

Hence horizontal range R is same.

22. (*b*) Same reason as in the above problem.

Hence $R_1 : R_2 = 1 : 1$.

23. (*b*) Here $\theta_1 + \theta_2 = 45^\circ - \theta + 45^\circ - \theta = 90^\circ$.

Hence horizontal range R is same and $R_1 : R_2 = \mathbf{1 : 1}$.

24. (*c*) $R_{max} = \dfrac{u^2}{g}$

or $$1600\ \text{m} = \frac{u^2}{10\ \text{ms}^{-2}}$$

$$u = \sqrt{160000} = \mathbf{400\ ms^{-1}}.$$

25. (*b*) Initial kinetic energy,

$$E = \frac{1}{2}mv^2$$

At the highest point, the particle has only the horizontal component of velocity $u\cos 45^\circ$. Therefore, its kinetic energy becomes

$$E' = \frac{1}{2}m(u\cos 45^\circ)^2$$
$$= \frac{1}{2}mu^2 \times \frac{1}{2} = \frac{E}{2}.$$

26. (*c*) Initially, both particles have zero vertical velocity. So, both particles take same time to fall through same height.

27. (*a*) Time of flight for vertical projection,

$$T = \frac{2u\sin 90^\circ}{g} = \frac{2u}{g}$$

Time interval between two balls thrown = 2 s

For a minimum of three balls to remain in air, time of flight of first ball must be greater than 4 s.

$$\therefore \quad T > 4\ \text{s} \quad \text{or} \quad \frac{2u}{g} > 4\ \text{s}$$

or $$u > 2g \quad \text{or} \quad u > \mathbf{19.6\ m/s}.$$

28. (*c*) Vertical acceleration of both is equal to g. Also horizontal velocity is constant.

29. (*a*) $\omega = \dfrac{\theta}{t} = \dfrac{2\pi \times 120\ \text{rad}}{60\ \text{s}} = \mathbf{4\pi\ rad\ s^{-1}}.$

30. (*c*) $t = \dfrac{2\pi}{\omega_1} = \dfrac{2\pi}{\omega_2} \quad \therefore \quad \dfrac{\omega_1}{\omega_2} = \dfrac{\mathbf{1}}{\mathbf{1}}.$

31. (*c*) Same reason as in the above problem.

32. (*d*) Here $r = 20\ \text{cm} = 0.20\ \text{m}$,

$$\omega = 10\ \text{rad s}^{-1}.$$

$$\therefore \quad v = r\omega = 0.20 \times 10 = \mathbf{2\ ms^{-1}}.$$

33. (*b*) $\vec{v} = \vec{\omega} \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5 & -6 & 6 \\ 3 & -4 & 1 \end{vmatrix}$

$$= \hat{i}(-6+24) - \hat{j}(5-18) + \hat{k}(-20+18)$$
$$= 18\hat{i} + 13\hat{j} - 2\hat{k}$$

34. (*b*) $\omega = \dfrac{2\pi \times 120\ \text{rad}}{60\ \text{s}} = 4\pi\ \text{rad s}^{-1}$

$$a = \omega^2 r = (4\pi)^2 \times 0.30 = \mathbf{47.4\ ms^{-2}}.$$

35. (*a*) Here $r = \dfrac{20}{\pi}\ \text{m}$, $v = 80\ \text{ms}^{-1}$, $\omega_0 = 0$

$$\theta = 2\ \text{rev} = 4\pi\ \text{rad}$$

As $$\omega^2 = \omega_0^2 + 2\alpha\theta$$

$$\therefore \quad \left(\frac{v}{r}\right)^2 = 0 + 2\left(\frac{a}{r}\right)\theta$$

or $$a = \frac{v^2}{2r\theta} = \frac{(80)^2}{2 \times 20/\pi \times 4\pi} = \mathbf{40\ ms^{-2}}.$$

36. (*a*) Period of revolution,

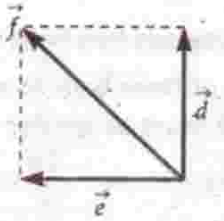
$$T = \frac{44}{22} = 2\ \text{s}$$

$$a = \omega^2 r = \left(\frac{2\pi}{T}\right)^2 r = \left(\frac{2\pi}{2}\right)^2 \times 1 = \pi^2\ \text{ms}^{-2}$$

The centripetal acceleration acts along the radius towards the centre.

37. (c) Clearly,

$$\vec{d} + \vec{e} = \vec{f}$$

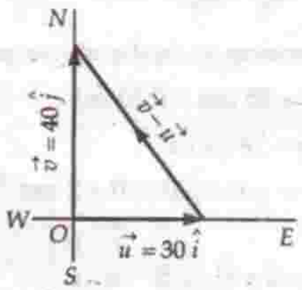


38. (b) $\vec{v} = \vec{u} + \vec{a}\,t$

$$= (3\hat{i} + 4\hat{j}) + (0.4\,\hat{i} + 0.3\,\hat{j}) \times 10 = 7\hat{i} + 7\hat{j}$$

$\therefore$ $v = \sqrt{v_x^2 + v_y^2} = \sqrt{7^2 + 7^2} = 7\sqrt{2}$ **units**

39. (d) $a_{av} = \dfrac{\text{Change in velocity}}{\text{Time taken}}$



$$= \frac{|\vec{v} - \vec{u}|}{t} = \frac{|40\hat{j} - 30\hat{i}|}{10}$$

$$= \frac{\sqrt{40^2 + 30^2}}{10} = \frac{50}{10} = \mathbf{5\,m/s^2}$$

40. (a) $R_{max} = \dfrac{u^2}{g} = \dfrac{20 \times 20}{10} = 40$ m

41. (d) At $\theta = 45°$,

$$y = \frac{u^2 \sin^2 45°}{2g} = \frac{u^2}{4g}$$

$$x = \frac{1}{2} \cdot \frac{u^2 \sin^2 90°}{g} = \frac{u^2}{2g}$$

$\therefore$ $\tan\beta = \dfrac{y}{x} = \dfrac{1}{2}$

$\Rightarrow$ $\beta = \tan^{-1}\left(\dfrac{1}{2}\right)$

42. (d) Acceleration $= r\omega^2 = r\left(\dfrac{2\pi}{T}\right)^2$

$$= 5 \times 10^{-2} \times \left(\frac{2\pi}{0.2\,\pi}\right)^2 = 5 \times 10^{-2} \times 100$$

$$= \mathbf{5\,ms^{-2}}$$

Delhi PMT and VMMC Entrance Exam

(Similar Questions)

1. If a unit vector is represented by $0.5\hat{i} + 0.8\hat{j} + c\hat{k}$, then the value of c is

(a) 1 (b) $\sqrt{0.8}$
(c) $\sqrt{0.11}$ (d) $\sqrt{0.01}$ [DPMT 94]

2. Two vectors are perpendicular if

(a) $\vec{A}.\vec{B} = 1$ (b) $\vec{A} \times \vec{B} = 0$
(c) $\vec{A}.\vec{B} = 0$ (d) $\vec{A}.\vec{B} = AB$ [DPMT 07]

3. The angle between the two vectors $\vec{A} = 5\hat{i} + 5\hat{j}$ and $\vec{B} = 5\hat{i} - 5\hat{j}$ will be

(a) zero (b) 90°
(c) 45° (d) 0° [DPMT 99]

4. Two equal vectors have a resultant equal to either of the two. The angle between them is

(a) 90° (b) 60°
(c) 120° (d) 0° [VMMC 03]

5. If $|\vec{A} \times \vec{B}| = |\vec{A}\,.\,\vec{B}|$, then angle between $\vec{A}$ and $\vec{B}$ will be

(a) 30° (b) 60°
(c) 45° (d) 90° [DPMT 05]

6. If $\vec{P} + \vec{Q} = \vec{R}$ and $|\vec{P}| = |\vec{Q}| = |\vec{R}|$, then angle between $\vec{P}$ and $\vec{Q}$ is

(a) 30° (b) 60°
(c) 90° (d) 120° [DPMT 05]

7. If $|\vec{A} \times \vec{B}| = \sqrt{3}\,\vec{A} \cdot \vec{B}$, then the value of $|\vec{A} + \vec{B}|$ is

(a) $(A^2 + B^2 + AB)^{1/2}$

(b) $\left(A^2 + B^2 + \dfrac{AB}{\sqrt{3}}\right)^{1/2}$

(c) $A + B$

(d) $(A^2 + B^2 + \sqrt{3}AB)^{1/2}$ [VMMC 07]

8. Given the speed of a boat in still water is u and the velocity of a river is v, the time taken by the boat to reach a place downstream at a distance d and come back to the original place is

(a) $\frac{2vd}{v^2-u^2}$ (b) $\frac{2ud}{u^2-v^2}$

(c) $\frac{vd}{v^2-u^2}$ (d) $\frac{ud}{u^2-v^2}$ [DPMT 06]

9. Which of the following is true regarding projectile motion ?

(a) horizontal velocity of projectile is constant

(b) vertical velocity of projectile is constant

(c) acceleration is not constant

(d) momentum is constant [DPMT 07]

10. An aeroplane flying horizontally with a speed of 360 kmh^{-1} releases a bomb at a height of 490 m from the ground. If $g = 9.8 \, ms^{-2}$, it will strike the ground at

(a) 10 km (b) 100 km

(c) 1 km (d) 16 km [VMMC 03]

11. A bomber plane moves horizontally with a speed of 500 m/s and a bomb released from it, strikes the ground in 10 sec. Angle at which it strikes the ground ($g = 10 \, m/s^2$) will be

(a) $\tan^{-1}\left(\frac{1}{5}\right)$ (b) $\tan\left(\frac{1}{5}\right)$

(c) $\tan^{-1}(1)$ (d) $\tan^{-1}(5)$

[DPMT 97 ; VMMC 04]

12. If range and height of a projectile are equal, then angle of projection with the horizontal is

(a) 60° (b) $\tan^{-1}(4)$

(c) 30° (d) 45° [VMMC 02]

13. The horizontal range of projectile is $4\sqrt{3}$ times of its maximum height.

The angle of projection will be

(a) 40° (b) 90°

(c) 30° (d) 45° [DPMT 03]

14. Two bodies are projected with the same velocity. If one is projected at an angle of 30° and the other at 60° to the horizontal, then ratio of maximum heights reached is

(a) 3 : 1 (b) 1 : 2

(c) 1 : 3 (d) 2 : 1 [DPMT 96]

15. Two bodies are thrown up at angles of 45° and 60°, respectively, with the horizontal. If both bodies attain same vertical height, then the ratio of velocities with which these are thrown is

(a) $\sqrt{2/3}$ (b) $2/\sqrt{3}$

(c) $\sqrt{3/2}$ (d) $\sqrt{3}/2$ [DPMT 02, 05]

16. A bullet is fired from a gun with a speed of 1000 m/s in order to hit a target 100 m away. At what height above the target should be gun be aimed ? (The resistance of air is negligible and $g = 10 \, m/s^2$)

(a) 5 cm (b) 15 cm

(c) 9 cm (d) 23 cm [DPMT 92]

17. A cricketer hits a ball with a velocity 25 m/s at 60° above the horizontal. How far above the ground it passes over a fielder 50 m from the bat (assume the ball is struck very close to the ground) ?

(a) 8.2 m (b) 9.0 m

(c) 11.6 m (d) 12.7 m [VMMC 05]

18. A ball is projected upwards from the top of tower with a velocity $50 \, ms^{-1}$ making an angle 30° with the horizontal. The height of tower is 70 m. After how many seconds from the instant of throwing will the ball reach the ground ?

(a) 2 s (b) 5 s

(c) 7 s (d) 9 s [DPMT 04]

19. The angular speed of a fly-wheel making 120 revolutions/minute is

(a) π rad/sec (b) 4π rad/sec

(c) 2π rad/sec (d) $4\pi^2$ rad/sec [DPMT 98]

20. The angular speed of the earth around the sun is

(a) $\frac{2\pi}{365 \times 24 \times 60 \times 60}$ rad s^{-1}

(b) $\frac{365 \times 24 \times 60 \times 60}{2\pi}$ rad s^{-1}

(c) $\frac{2\pi}{24 \times 60}$ rad/sec (d) $\frac{2\pi}{60}$ rad s^{-1} [DPMT 06]

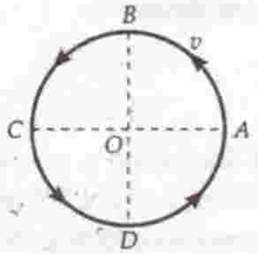
21. What is the value of linear velocity, if

$\vec{\omega} = 3\hat{i} - 4\hat{j} + \hat{k}$ and $\vec{r} = 5\hat{i} - 6\hat{j} + 6\hat{k}$?

(a) $-6\hat{i} - 2\hat{j} + 3\hat{k}$ (b) $4\hat{i} - 13\hat{j} + 6\hat{k}$

(c) $-18\hat{i} - 13\hat{j} + 2\hat{k}$ (d) $6\hat{i} - 2\hat{j} + 8\hat{k}$ [DPMT 96]

22. Figure along side shows a body of mass m moving with a uniform speed v along a circle of radius r.



The change in velocity in going from A to B is

(a) $v\sqrt{2}$ (b) $v/\sqrt{2}$

(c) v (d) zero [DPMT 04]

23. If a cycle wheel of radius 4 m completes one revolution in two seconds, then acceleration of the cycle is

(a) $\pi\,\mathrm{m/s^2}$ (b) $2\pi^2\,\mathrm{m/s^2}$

(c) $\pi^2\,\mathrm{m/s^2}$ (d) $4\pi^2\mathrm{m/s^2}$

24. A stone tied to the end of string 80 cm long is whirled in horizontal circle with a constant speed. If the stone makes 14 revolutions in 25 sec, what is the magnitude of acceleration of the stone ?

(a) 680 cm/s^2 (b) 860 cm/s^2

(c) 720 cm/s^2 (d) 990 cm/s^2 [DPMT 97]

25. The three initial and final positions of a man on the x-axis are given as

(i) $(-8\,\mathrm{m}, 7\,\mathrm{m})$ (ii) $(7\,\mathrm{m}, -3\,\mathrm{m})$

(iii) $(-7\,\mathrm{m}, 3\,\mathrm{m})$

Which pair gives the negative displacement ?

(a) (i) (b) (ii)

(c) (iii) (d) (i) and (iii) [DPMT 09]

26. A bird flies from $(-3m, 4m, -3m)$ to $(7m, -2m, -3m)$ in the xyz coordinates. The bird's displacement in unit vectors is given by

(a) $(4\hat{i}+2\hat{j}-6\hat{k})$ (b) $(10\hat{i}-6\hat{j})$

(c) $(4\hat{i}-2\hat{j})$ (d) $(10\hat{i}+6\hat{j}-6\hat{k})$

[DPMT 09]

27. A boat crosses a river from port A to port B, which are just on the opposite side. The speed of the water is v_W and that of boat is v_B relative to water. Assume $v_B = 2v_W$. What is the time taken by the boat, if it has to cross the river directly on the AB line ?

(a) $\dfrac{2D}{v_B\sqrt{3}}$ (b) $\dfrac{\sqrt{3}D}{2v_B}$

(c) $\dfrac{D}{v_B\sqrt{2}}$ (d) $\dfrac{D\sqrt{2}}{v_B}$ [DPMT 09]

28. A coastguard ship locates a pirate ship at a distance 560 m. It fires a cannon ball with an initial speed 82 m/s. At what angle from horizontal the ball must be fired so that it hits the pirate ship ?

(a) 54° (b) 125°

(c) 27° (d) 18° [DPMT 09]

29. An object moves at a constant speed along a circular path in a horizontal XY-plane, with the centre at the origin. When the object is at $x = -2$ m, its velocity is $-(4\,\mathrm{m/s})\hat{j}$. What is the object's acceleration when it is at $y = 2\,\mathrm{m}$?

(a) $-(8\,\mathrm{m/s^2})\hat{j}$ (b) $-(8\,\mathrm{m/s^2})\hat{i}$

(c) $(-4\,\mathrm{m/s^2})\hat{j}$ (d) $(4\,\mathrm{m/s^2})\hat{i}$ [DPMT 09]

30. Which of the following is correct relation between an arbitrary vector $\vec{A}$ and null vector $\vec{O}$?

(a) $\vec{A}+\vec{O}+\vec{A}\times\vec{O}=\vec{A}$

(b) $\vec{A}+\vec{O}+\vec{A}\times\vec{O}\neq\vec{A}$

(c) $\vec{A}+\vec{O}+\vec{A}\times\vec{O}=\vec{O}$

(d) none of these [DPMT 2011]

31. An object is being thrown at a speed of 20 m/s in a direction 45° above the horizontal. The time taken by the object to return to the same level is

(a) $20/g$ (b) $20g$

(c) $20\sqrt{2}/g$ (d) $20\sqrt{2}\,g$ [DPMT 2011]

Answers and Explanations

1. (c) $\sqrt{0.5^2+0.8^2+c^2}=1$

or $c=\sqrt{1-0.89}=\sqrt{\mathbf{0.11}}$.

2. (c) For two perpendicular vectors,

$\vec{A}\cdot\vec{B}=AB\cos 90°=\mathbf{0}$.

3. *(b)* $\vec{A}\cdot\vec{B}=(5\hat{i}+5\hat{j})\cdot(5\hat{i}-5\hat{j})=0$

$\therefore \quad \theta = 90°$.

4. *(c)* Refer to the solution of Example 5 on Page 4.9.

5. *(c)* Given : $|\vec{A}\times\vec{B}|=|\vec{A}\cdot\vec{B}|$

$\therefore \quad AB\sin\theta = AB\cos\theta$

or $\quad \tan\theta = 1$

$\therefore \quad \theta = 45°$

6. *(d)* As $\vec{R}=\vec{P}+\vec{Q}$

and $\quad |\vec{P}|=|\vec{Q}|=|\vec{R}|$,

$$R^2 = P^2 + Q^2 + 2PQ\cos\theta$$

or $\quad P^2 = P^2 + P^2 + 2P^2\cos\theta$

or $\quad \cos\theta = -\frac{1}{2}$

$\therefore \quad \theta = 120°$.

7. *(a)* Given : $|\vec{A}\times\vec{B}|=\sqrt{3}\,\vec{A}\cdot\vec{B}$

$\therefore \quad AB\sin\theta = \sqrt{3}\times AB\cos\theta$

or $\quad \tan\theta = \sqrt{3}$

$\therefore \quad \theta = 60°$

Now $\quad |\vec{A}+\vec{B}| = \sqrt{A^2+B^2+2AB\cos 60°}$

$$= \sqrt{A^2+B^2+2AB\times\frac{1}{2}}$$

$$= (A^2+B^2+AB)^{1/2}$$

8. *(b)* Velocity downstream $= u + v$

Velocity upstream $= u - v$.

Time taken in one round trip

$$= \frac{d}{u+v}+\frac{d}{u-v}=\frac{2ud}{u^2-v^2}.$$

9. *(a)* Horizontal velocity of a projectile is not affected by gravity.

10. *(c)* Time taken by the bomb to fall through a height of 490 m,

$$t=\sqrt{\frac{2h}{g}}=\sqrt{\frac{2\times 490}{9.8}}=10 \text{ s}.$$

Distance at which the bomb strikes the ground

$= \text{Horizontal velocity} \times \text{time}$

$= 360 \text{ kmh}^{-1} \times 10 \text{ s}$

$= 360 \text{ km h}^{-1}\times\frac{10}{3600}\text{ h} = \mathbf{1\ km}.$

11. *(a)* Height through which the bomb falls,

$$h=\frac{1}{2}gt^2=\frac{1}{2}\times 10\times(10)^2=500 \text{ m}$$

Horizontal distance moved by the bomb in 10 s

$$= 500 \text{ ms}^{-1}\times 10 \text{ s} = 5000 \text{ m}$$

$\therefore \quad \tan\theta = \frac{500}{5000}=\frac{1}{5}$

or $\quad \theta = \tan^{-1}\left(\frac{1}{5}\right)$.

12. *(d)* Horizontal range = Max. height

$$\frac{u^2\sin 2\theta}{g}=\frac{u^2\sin^2\theta}{2g}$$

or $\quad 2\sin\theta\cos\theta = \frac{\sin^2\theta}{2} \quad$ or $\quad \tan\theta = 4$

$\therefore \quad \theta = \tan^{-1}(4)$

13. *(c)* Refer to the solution of Problem 15 on page 4.100 .

14. *(c)* $\quad H=\frac{u^2\sin^2\theta}{2g}$

i.e., $\quad H \propto \sin^2\theta$

$\therefore \quad \frac{H_1}{H_2}=\frac{\sin^2 30°}{\sin^2 60°}=\frac{(1/2)^2}{(\sqrt{3}/2)^2}=\frac{1}{3}=\mathbf{1:3}.$

15. *(c)* $\quad H_1 = H_2$

$$\frac{u_1^2\sin^2 45°}{2g}=\frac{u_2^2\sin^2 60°}{2g}$$

or $\quad \frac{u_1}{u_2}=\frac{\sin 60°}{\sin 45°}=\frac{\sqrt{3}/2}{1/\sqrt{2}}=\sqrt{\frac{3}{2}}$.

16. *(a)* Tike taken by the bullet to cover a horizontal distance of 100 m,

$$t=\frac{100 \text{ m}}{1000 \text{ ms}^{-1}}=\frac{1}{10}\text{ s}$$

Height at which the gun should be aimed,

$$h=\frac{1}{2}gt^2=\frac{1}{2}\times 10\times\left(\frac{1}{10}\right)^2 \text{ m}$$

$$= 5\times 10^{-2}\text{ m} = \mathbf{5\ cm}.$$

17. *(a)* $\quad u_x = 25\cos 60° = 12.5 \text{ ms}^{-1}$

$$u_y = 25\sin 60° = 12.5\sqrt{3}\text{ ms}^{-1}$$

Time taken by the ball to cover 50 m distance

$$t=\frac{50 \text{ m}}{u_x}=\frac{50 \text{ m}}{12.5 \text{ ms}^{-1}}=4 \text{ s}$$

Vertical distance covered by the ball in 4 s,

$$h = u_y t - \frac{1}{2}gt^2$$

$$= 12.5\sqrt{3}\times 4 - \frac{1}{2}\times 9.8\times(4)^2$$

$$= 50\sqrt{3} - 78.4 = 86.6 - 78.4 = \mathbf{8.2\ m}.$$

18. (*c*) $u_y = 50\sin 30° = 25\ \text{ms}^{-1}$

For vertical motion of the ball,

$$\therefore \quad h = u_y t - \frac{1}{2}gt^2$$

$$-70 = 25 \times t - \frac{1}{2} \times 10 \times t^2$$

or $\quad -14 = 5t - t^2$

or $\quad t^2 - 5t - 14 = 0$

or $\quad (t+2)(t-7) = 0$

$\therefore \quad t = 7$ **s.** $\quad (t \neq -2)$

19. (*b*) $\omega = \frac{2\pi n}{t} = \frac{2\pi \times 120}{60\,\text{s}}$

$= \mathbf{4\pi\ rad\ s^{-1}}$.

20. (*a*) Earth completes one revolution around the sun in one year.

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{365 \times 24 \times 60 \times 60}\ \text{rad s}^{-1}.$$

21. (*c*) $\vec{v} = \vec{\omega} \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -4 & 1 \\ 5 & -6 & 6 \end{vmatrix}$

$= \mathbf{-18\hat{i} - 13\hat{j} + 2\hat{k}}$.

22. (*a*) Velocity at A, $\vec{v} = v\hat{j}$

Velocity at B, $\vec{v}_2 = -v\hat{i}$

Change in velocity,

$$\Delta\vec{v} = \vec{v}_2 - \vec{v}_1 = -v\hat{i} - v\hat{j}$$

$$|\Delta\vec{v}| = \sqrt{(-v)^2 + (-v)^2} = \mathbf{v\sqrt{2}}.$$

23. (*d*) $\omega = 2\pi\nu = 2\pi\left(\frac{1}{2}\right) = \pi\,\text{rad s}^{-1}$

$a = r\omega^2$

$= 4\,\text{m} \times (\pi\,\text{rad s}^{-1})^2 = \mathbf{4\pi^2 ms^{-2}}$.

24. (*d*) $a = r\omega^2 = r\left(\frac{2\pi n}{t}\right)^2$

$$= \frac{80 \times 4 \times (22/7)^2 \times (14)^2}{(25)^2}$$

$\simeq$ **990 cm/s²**.

25. (*b*) Displacement is negative only for pair (*ii*) *i.e.*, $(7m, -3m)$.

$$\Delta x = x_f - x_i = -3m - 7m = -10\,m$$

26. (*b*) Bird's displacement is

$$\Delta\vec{r} = \vec{r}_2 - \vec{r}_1$$

$$= (7\hat{i} - 2\hat{j} - 3\hat{k}) - (-3\hat{i} + 4\hat{j} - 3\hat{k})$$

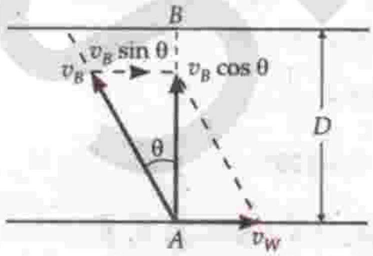
$$= 10\hat{i} - 6\hat{j}.$$

27. (*a*) To move straight along AB,

$$v_B \sin\theta = v_W$$

$$\sin\theta = \frac{v_W}{v_B} = \frac{v_W}{2v_W} = \frac{1}{2}$$

$\therefore \quad \theta = 30°$



Time taken to cross the river,

$$t = \frac{D}{v_B\cos\theta} = \frac{D}{v_B\cos 30°} = \frac{2D}{v_B\sqrt{3}}.$$

28. (*c*) Horizontal range,

$$R = \frac{u^2 \sin 2\theta}{g}$$

$$\therefore \quad 560 = \frac{82 \times 82 \sin 2\theta}{10}$$

$$\sin 2\theta = \frac{5600}{6724} = 0.82$$

or $\quad 2\theta = 53.8° \Rightarrow \theta \simeq 27°.$

29. (*a*) Clearly, radius of the circular path is 2 m and velocity has magnitude equal to 4 m/s. Hence

$$a_c = \frac{v^2}{r} = \frac{4 \times 4}{2} = 8\ \text{m/s}^2$$

When the object is at $y = 2$ m, the acceleration a_c acts towards the centre *i.e.*, towards the origin along –ve y-axis.

$$\therefore \quad \vec{a}_c = \mathbf{-(8\ m/s^2)\hat{j}}.$$

30. (*a*) $\vec{A} + \vec{O} + \vec{A} \times \vec{O} = \vec{A} + \vec{O} + \vec{O} = \vec{A}$

31. (*c*) $T = \frac{2u\sin\theta}{g}$

$$= \frac{2 \times 20 \times \sin 45°}{g} = \frac{20\sqrt{2}}{g}.$$