CHAPTER

7

SYSTEMS OF PARTICLES & ROTATIONAL MOTION

7.1 INTRODUCTORY CONCEPTS

1. What is meant by a particle, a system and internal and external forces ?

Particle. *A particle is defined as an object whose mass is finite but whose size and internal structure can be neglected.*

System. *A system is a collection of a very large number of particles which mutually interact with one another.* A body of finite size can be regarded as a system because it is composed of a large number of particles interacting with one another.

Internal forces. *The mutual forces exerted by the particles of a system on one another are called internal forces.* These forces are responsible for holding together the particles as a single object.

External forces. *The outside force exerted on an object by any external agency is called an external force. Such a force changes the velocity of an object.*

7.2 CENTRE OF MASS

2. What do you mean by centre of mass of a system ? How does it differ from centre of gravity ?

Centre of mass. Newton's laws of motion are applicable to point objects. The introduction of the concept of centre of mass enables us to apply them equally well to the motion of finite or extended objects. *The centre of mass of a body is a point where the whole mass of the body is supposed to be concentrated for describing its translatory motion.*

*The **centre of mass** of a system of particles is that single point which moves in the same way in which a single particle having the total mass of the system and acted upon by the same external force would move.*

If a single force acts on a body and the line of action of the force passes through the centre of mass, the body will have only linear acceleration and no angular acceleration. For example, consider a hammer resting

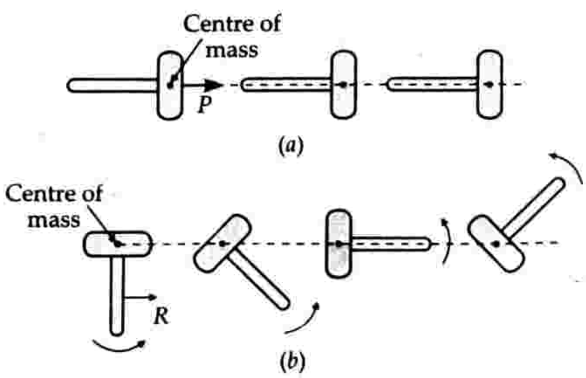


Fig. 7.1 Concept of centre of mass.

on a plane surface. If a force P is applied on the hammer in such a way that its line of action passes through the centre of mass of the hammer, then the hammer moves along a straight line path, as shown in Fig. 7.1(a). But when a force R is applied along a line not passing through its centre of mass, then the hammer rotates about its centre of mass, as shown in Fig. 7.1(b).

Centre of mass vs. centre of gravity. The centre of mass of body is point where whole mass of the body may be assumed to be concentrated for describing its translatory motion. On the other hand, the centre of gravity is a point at which the resultant of the gravitational forces on all the particles of the body acts *i.e.*, a point where whole weight may be assumed to act. In a uniform gravitational field such as that of the earth on a small body, the centre of gravity coincides with the centre of mass. But in the case of Mount Everest, the centre of gravity lies a little below its centre of mass because the gravitational force decreases with altitude.

7.3 CENTRE OF MASS OF A TWO-PARTICLE SYSTEM

3. *Write an expression for the location of centre of mass of a two particle system. Discuss the result.*

Centre of mass of a two particle-system. Consider a system of two particles P_1 and P_2 of masses m_1 and m_2. Let $\vec{r}_1$ and $\vec{r}_2$ be their position vectors with respect to the origin O, as shown in Fig. 7.2.

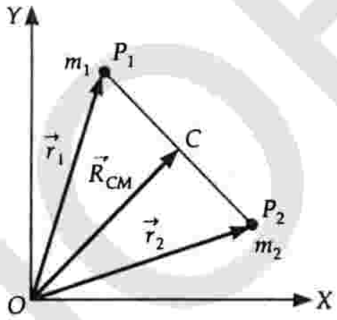


Fig. 7.2 Centre of mass of a two-particle system.

The position vector $\vec{R}_{CM}$ of the centre of mass C of the two-particle system is given by

$$\vec{R}_{CM} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}$$

Discussion. (i) The above equation shows that *the position vector of a system of particles is the weighted average of the position vectors of the particles making the system,* each particle making a contribution proportional to its mass.

(ii) We can write the above equation as

$$(m_1 + m_2)\, \vec{R}_{CM} = m_1 \vec{r}_1 + m_2 \vec{r}_2$$

Thus *the product of the total mass of the system and the position vector of its centre of mass is equal to the sum of the products of individual masses and their respective position vectors.*

(iii) If $m_1 = m_2 = m$ (say), then

$$\vec{R}_{CM} = \frac{\vec{r}_1 + \vec{r}_2}{2}$$

Thus *the centre of mass of two equal masses lies exactly at the centre of the line joining the two masses.*

(iv) If (x_1, y_1) and (x_2, y_2) are the coordinates of the locations of the two particles, the coordinates of their centre of mass are given by

$$x_{CM} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

and $$y_{CM} = \frac{m_1 y_1 + m_2 y_2}{m_1 + m_2}.$$

7.4 CENTRE OF MASS OF A TWO PARTICLE SYSTEM FROM *ab-initio*

4. *Derive an expression for the centre of mass of a two particle system from ab-initio.*

Derivation of expression for the centre of mass of a two-particle system. Consider a system of two particles P_1 and P_2 of masses m_1 and m_2. Let $\vec{r}_1$ and $\vec{r}_2$ be their position vectors at any instant t with respect to the origin O, as shown in Fig. 7.3.

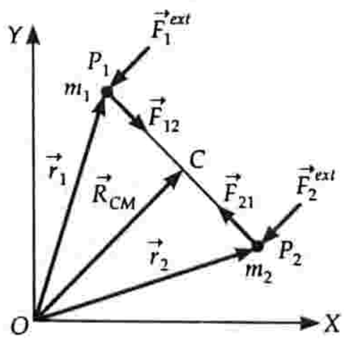


Fig. 7.3 Equations of motion of a two-particle system.

The velocity and acceleration vectors of the two particles are

$$\vec{v}_1 = \frac{d\vec{r}_1}{dt} \quad \text{and} \quad \vec{a}_1 = \frac{d\vec{v}_1}{dt} = \frac{d^2\vec{r}_1}{dt^2}$$

$$\vec{v}_2 = \frac{d\vec{r}_2}{dt} \quad \text{and} \quad \vec{a}_2 = \frac{d\vec{v}_2}{dt} = \frac{d^2\vec{r}_2}{dt^2} \qquad \ldots(1)$$

Total force $\vec{F}_1$ acting on particle P_1 is the sum of the internal force $\vec{F}_{12}$ due to P_2 and external force $\vec{F}_1^{ext}$ on it. Thus

$$\vec{F}_1 = \vec{F}_{12} + \vec{F}_1^{ext}$$

Similarly, total force acting on particle P_2 is given by

$$\vec{F}_2 = \vec{F}_{21} + \vec{F}_2^{ext}$$

According to Newton's second law of motion, the equations of motion for the two particles can be written as

$$m_1 \vec{a}_1 = \vec{F}_1 = \vec{F}_{12} + \vec{F}_1^{ext} \qquad ...(2)$$

$$m_2 \vec{a}_2 = \vec{F}_2 = \vec{F}_{21} + \vec{F}_2^{ext} \qquad ...(3)$$

On adding equations (2) and (3), we get

$$m_1 \vec{a}_1 + m_2 \vec{a}_2 = \vec{F}_{12} + \vec{F}_{21} + \vec{F}_1^{ext} + \vec{F}_2^{ext}$$

According to Newton's third law, the internal forces mutually exerted by the two particles are equal and opposite, *i.e.*,

$$\vec{F}_{12} = -\vec{F}_{21} \quad \text{or} \quad \vec{F}_{12} + \vec{F}_{21} = 0$$

$$\therefore \quad m_1 \vec{a}_1 + m_2 \vec{a}_2 = \vec{F} \qquad ...(4)$$

where $\vec{F} = \vec{F}_1^{ext} + \vec{F}_2^{ext}$ is the total external force acting on the system.

Suppose the total mass of the two-particle system is M. Then

$$M = m_1 + m_2$$

Let us assume that the total external force $\vec{F}$ acting on the system of mass M produces acceleration $\vec{a}_{CM}$. Then according to Newton's second law,

$$M \vec{a}_{CM} = \vec{F} \qquad ...(5)$$

From equations (4) and (5), we get

$$M\vec{a}_{CM} = m_1 \vec{a}_1 + m_2 \vec{a}_2$$

$$= m_1 \frac{d^2 \vec{r}_1}{dt^2} + m_2 \frac{d^2 \vec{r}_2}{dt^2} \qquad \text{[Using (1)]}$$

$$= \frac{d^2}{dt^2}(m_1 \vec{r}_1 + m_2 \vec{r}_2)$$

or

$$\vec{a}_{CM} = \frac{1}{M}\frac{d^2}{dt^2}(m_1 \vec{r}_1 + m_2 \vec{r}_2)$$

or

$$\frac{d^2 \vec{R}_{CM}}{dt^2} = \frac{d^2}{dt^2}\left(\frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}\right) \qquad [\because M = m_1 + m_2]$$

The acceleration $\vec{a}_{CM}$ is called the acceleration vector of the centre of mass of the system and $\vec{R}_{CM}$ is called the position vector of the centre of mass.

$$\text{Clearly,} \qquad \vec{R}_{CM} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}. \qquad ...(6)$$

This equation defines the position of the centre of mass of a system of two particles of masses m_1 and m_2 and having position vectors $\vec{r}_1$ and $\vec{r}_2$. Here the entire mass of the two-particle system may be supposed to be concentrated. Clearly, Newton's second law, as applied to the individual particles of the system, also holds for the entire system provided the external force acts at the centre of mass as defined by the above equation. Newton's third law helps us to get rid of the mutual internal forces between the particles. Hence *while applying Newton's second law to the motion of the centre of mass, we need to consider only the external forces acting on the system.*

7.5 CENTRE OF MASS OF *n*-PARTICLE SYSTEM

5. *Write an expression for the position vector of the centre of mass of n-particle system. Also write the equations of motion which govern the motion of the centre of mass.*

Centre of mass of *n*-particle system. Consider a system of n particles having masses $m_1, m_2, m_3, ..., m_n$ and position vectors $\vec{r}_1, \vec{r}_2, \vec{r}_3, ..., \vec{r}_n$ relative to the origin O, as shown in Fig. 7.4.

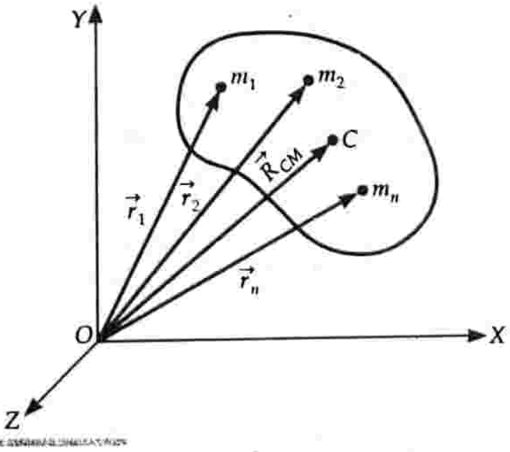


Fig. 7.4 Centre of mass of *n*-particle system.

The total mass of the system is

$$m_1 + m_2 + m_3 + ... + m_n = M \qquad \text{(say)}$$

The position vector $\vec{R}_{CM}$ of the centre of mass C can be obtained by adding the products $m_1 \vec{r}_1, m_2 \vec{r}_2, ..., m_n \vec{r}_n$ and dividing it by the total mass of the system

Thus

$$\vec{R}_{CM} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2 + ... + m_n \vec{r}_n}{m_1 + m_2 + ... + m_n}$$

or

$$\vec{R}_{CM} = \frac{\sum_{i=1}^{n} m_i \vec{r}_i}{\sum_{i=1}^{n} m_i} = \frac{\sum_{i=1}^{n} m_i \vec{r}_i}{M} \qquad ...(1)$$

Clearly, $\vec{R}_{CM}$ *is the weighted average of the position vectors of all the particles of the system, the contribution of each particle being proportional to its mass.*

Cartesian coordinates of the centre of mass. If x_{CM}, y_{CM} and z_{CM} are the cartesian coordinates of the centre of mass of the *n*-particle system, then

$$x_{CM} = \frac{m_1 x_1 + m_2 x_2 + ... + m_n x_n}{m_1 + m_2 + ... + m_n} = \frac{\Sigma mx}{M} \qquad ...2(i)$$

$$y_{CM} = \frac{m_1 y_1 + m_2 y_2 + ... + m_n y_n}{m_1 + m_2 + ... + m_n} = \frac{\Sigma my}{M} \qquad ...2(ii)$$

$$z_{CM} = \frac{m_1 z_1 + m_2 z_2 + ... + m_n z_n}{m_1 + m_2 + ... + m_n} = \frac{\Sigma mz}{M} \qquad ...2(iii)$$

Equations of motion for the centre of mass. Let $\vec{F}_1$, $\vec{F}_2$, $\vec{F}_3$, ..., $\vec{F}_n$ be the external forces acting on the particles of masses $m_1, m_2, m_3, ..., m_n$ respectively. Let $\vec{F}_{tot}$ be the vector sum of all these external forces acting on the system. If $\vec{a}_{CM}$ is the acceleration of the centre of mass of the system, then the motion of the centre of mass is governed by the equation

$$M\vec{a}_{CM} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + ... + \vec{F}_n$$

or

$$M\vec{a}_{CM} = \vec{F}_{tot} \qquad ...(3)$$

where

$$\vec{a}_{CM} = \frac{d^2 \vec{R}_{CM}}{dt^2}$$

The equation (3) shows that *the centre of mass of the system moves as if the entire mass of the system is concentrated at this point and the total external force acts on this point.* The internal forces between various particles cancel out in pairs in accordance with Newton's third law of motion. The definition of centre of mass given by equation (1) holds even though there may not be any actual matter present at the centre of mass.

Note **In case of a body with a continuous mass distribution, we can replace the summations in equations (2) by the following integrals :**

$$\Sigma mx \rightarrow \int x\, dm$$

$$\Sigma my \rightarrow \int y\, dm$$

$$\Sigma mz \rightarrow \int z\, dm$$

Then the coordinates of the centre of mass of a body of mass M will be

$$x_{CM} = \frac{1}{M} \int x\, dm,$$

$$y_{CM} = \frac{1}{M} \int y\, dm,$$

$$z_{CM} = \frac{1}{M} \int z\, dm$$

The equivalent vector representation for the centre of mass will be

$$\vec{R}_{CM} = \frac{1}{M} \int \vec{r}\, dm$$

If we choose, the centre of mass as the origin of our coordinate system, then

$$\vec{R}_{CM}(x, y, z) = 0$$

or

$$\int \vec{r}\, dm = 0$$

or

$$\int x\, dm = \int y\, dm = \int z\, dm = 0$$

7.6 MOMENTUM CONSERVATION AND CENTRE OF MASS MOTION

6. *Show that in the absence of any external force, the velocity of the centre of mass remains constant.*

Velocity of *CM* is constant in the absence of external force. Suppose an external force $\vec{F}_{tot}$ acts on a system of mass M and produces an acceleration $\vec{a}_{CM}$ in its centre of mass. Then

$$\vec{F}_{tot} = M\vec{a}_{CM}$$

In the absence of any external force, $\vec{F}_{tot} = 0$, so

so

$$M\vec{a}_{CM} = 0$$

or

$$\vec{a}_{CM} = 0$$

or

$$\frac{d\vec{v}_{CM}}{dt} = 0$$

As the derivative of a constant is zero, so

$$\vec{v}_{CM} = \text{constant}$$

where $\vec{v}_{CM}$ is the velocity of the centre of mass. Hence *in the absence of any external force, the centre of mass of system moves with a uniform velocity.* This is Newton's first law of motion.

The position vector of the centre of mass at any instant t is given by

$$\vec{R}_{CM}(t) = \vec{R}_{CM}(0) + \vec{v}_{CM}\, t$$

7. *Show that the total linear momentum of a system of particles is conserved in the absence of any external force. Also show that total linear momentum of the system is equal to the product of the total mass of the system and the velocity of its centre of mass.*

Momentum conservation and centre-of-mass motion. Consider a system of n particles of masses $m_1, m_2, m_3, \ldots, m_n$. Suppose the forces $\vec{F}_1, \vec{F}_2, \vec{F}_3, \ldots, \vec{F}_n$ exerted on them produce accelerations $\vec{a}_1, \vec{a}_2, \vec{a}_3, \ldots, \vec{a}_n$ respectively. In the absence of any external force,

$$\vec{F}_{tot} = 0$$

or $$\vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \ldots + \vec{F}_n = 0$$

or $$m_1 \vec{a}_1 + m_2 \vec{a}_2 + m_3 \vec{a}_3 + \ldots + m_n \vec{a}_n = 0$$

or $$m_1 \frac{d\vec{v}_1}{dt} + m_2 \frac{d\vec{v}_2}{dt} + m_3 \frac{d\vec{v}_3}{dt} + \ldots + m_n \frac{d\vec{v}_n}{dt} = 0$$

or $$\frac{d}{dt}(m_1 \vec{v}_1 + m_2 \vec{v}_2 + m_3 \vec{v}_3 + \ldots + m_n \vec{v}_n) = 0$$

or $$m_1 \vec{v}_1 + m_2 \vec{v}_2 + m_3 \vec{v}_3 + \ldots + m_n \vec{v}_n = \text{constant}$$

or $$\vec{P} = \vec{p}_1 + \vec{p}_2 + \vec{p}_3 + \ldots + \vec{p}_n = \text{constant}$$

where $\vec{P}$ is the total linear momentum of the system. Hence *if no net external force acts on a system, the total linear momentum of the system is conserved. This is the law of conservation of linear momentum.*

Now the position vector of the centre of mass of n-particle system is given by

$$\vec{R}_{CM} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2 + m_3 \vec{r}_3 + \ldots + m_n \vec{r}_n}{m_1 + m_2 + m_3 + \ldots + m_n}$$

or $$\vec{R}_{CM} = \frac{1}{M}(m_1 \vec{r}_1 + m_2 \vec{r}_2 + m_3 \vec{r}_3 + \ldots + m_n \vec{r}_n)$$

Differentiating both sides w.r.t. time t, we get

$$\frac{d\vec{R}_{CM}}{dt} = \frac{1}{M}\left(m_1 \frac{d\vec{r}_1}{dt} + m_2 \frac{d\vec{r}_2}{dt} + m_3 \frac{d\vec{r}_3}{dt} + \ldots + m_n \frac{d\vec{r}_n}{dt} \right)$$

$$= \frac{1}{M}(m_1 \vec{v}_1 + m_2 \vec{v}_2 + m_3 \vec{v}_3 + \ldots + m_n \vec{v}_n)$$

$$= \frac{1}{M}(\vec{p}_1 + \vec{p}_2 + \vec{p}_3 + \ldots + \vec{p}_n)$$

or $$\vec{v}_{CM} = \frac{1}{M}\vec{P} \quad \text{or} \quad \vec{P} = M\vec{v}_{CM}$$

This equation shows that the *total linear momentum of a system of particles is equal to the product of the total mass of the system and the velocity of its centre of mass.*

Examples based on Centre of Mass

FORMULAE USED

1. For a system of N particles, the position vector of centre mass is

$$\vec{R}_{CM} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2 + \ldots + m_N \vec{r}_N}{m_1 + m_2 + \ldots + m_N} = \frac{\sum_{i=1}^{N} m_i \cdot \vec{r}_i}{M}$$

2. The position vector of the centre mass of a two particle system is

$$\vec{R}_{CM} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}$$

3. The Cartesian co-ordinates of the centre of mass are given by

$$x = \frac{m_1 x_1 + m_2 x_2 + \ldots + m_N x_N}{m_1 + m_2 + \ldots + m_N} = \frac{\sum_{i=1}^{N} m_i x_i}{M}$$

$$y = \frac{m_1 y_1 + m_2 y_2 + \ldots + m_N y_N}{m_1 + m_2 + \ldots + m_N} = \frac{\sum_{i=1}^{N} m_i y_i}{M}$$

$$z = \frac{m_1 z_1 + m_2 z_2 + \ldots + m_N z_N}{m_1 + m_2 + \ldots + m_N} = \frac{\sum_{i=1}^{N} m_i z_i}{M}$$

4. For a continuous mass distribution,

$$\vec{R}_{CM} = \frac{1}{M}\int \vec{r}\, dm$$

where dm is the mass of small element located at position $\vec{r}$.

Also $x_{CM} = \frac{1}{M}\int x\,dm, \quad y_{CM} = \frac{1}{M}\int y\, dm,$

$$z_{CM} = \frac{1}{M}\int z\,dm$$

5. The algebraic sum of the moments of masses of various particles of a system about its centre of mass is zero.

6. Velocity of CM of a two particle system is

$$\vec{v}_{CM} = \frac{m_1 \vec{v}_1 + m_2 \vec{v}_2}{m_1 + m_2}$$

UNITS USED

Masses of various particles of a system are in kg and their distances from the axis of rotation are in metre.

EXAMPLE 1. *Find the centre of mass of a triangular lamina.* **[NCERT]**

Solution. As shown in Fig. 7.5, divide the lamina (ΔLMN) into narrow strips, each parallel to the base MN.

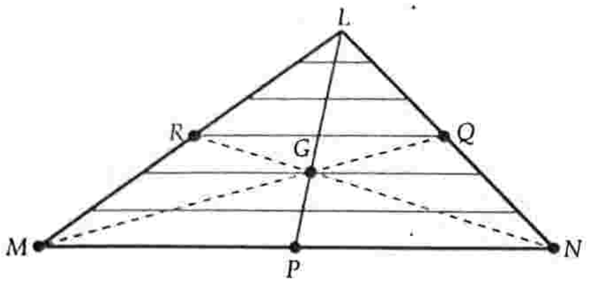


Fig. 7.5

By symmetry, the CM of each strip lies at its mid-point. By joining these midpoints, we get the median *LP*. The CM of triangular lamina must lie on the median *LP*.

By similar arguments as above, we can say that the CM of lamina (ΔLMN) lies on the medians *MQ* and *NR*. This means that CM lies on the point of concurrence of the medians *i.e.*, on the centroid *G* of the triangle.

Example 2. *Three masses 3, 4 and 5 kg are located at the corners of an equilateral triangle of side 1 m. Locate the centre of mass of the system.*

Solution. Suppose the equilateral triangle lies in the *XY*-plane with mass 3 kg at the origin. Let (x, y) be the co-ordinates of CM.

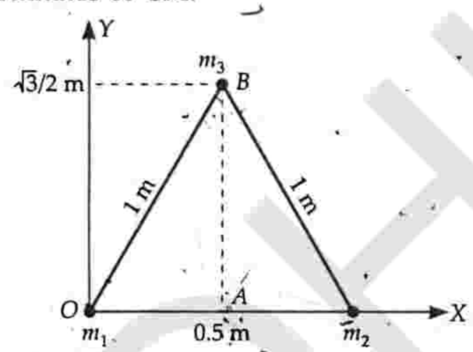


Fig. 7.6

Clearly, $AB = \sqrt{OB^2 - OA^2} = \sqrt{1^2 - \left(\frac{1}{2}\right)^2} = \frac{\sqrt{3}}{2}$ m

Now $x_1 = 0,\ x_2 = 1\text{ m},\ x_3 = OA = 0.5\text{ m}$

$m_1 = 3\text{ kg},\ m_2 = 4\text{ kg},\ m_3 = 5\text{ kg}$

$$\therefore\quad x = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{m_1 + m_2 + m_3}$$

$$= \frac{3 \times 0 + 4 \times 1 + 5 \times 0.5}{3 + 4 + 5} = \frac{6.5}{12} = \mathbf{0.54\ m.}$$

Again, $y_1 = 0,\ y_2 = 0,\ y_3 = AB = \frac{\sqrt{3}}{2}$

$$\therefore\quad y = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3}{m_1 + m_2 + m_3}$$

$$= \frac{3 \times 0 + 4 \times 0 + 5 \times (\sqrt{3}/2)}{3 + 4 + 5} = \frac{5 \times \sqrt{3}}{2 \times 12} = \mathbf{0.36\ m.}$$

Thus the co-ordinates of CM are (0.54 m, 0.36 m).

Example 3. *Two particles of masses 100 g and 300 g at a given time have positions $2\hat{i} + 5\hat{j} + 13\hat{k}$ and $-6\hat{i} + 4\hat{j} - 2\hat{k}$ m respectively and velocities $10\hat{i} - 7\hat{j} - 3\hat{k}$ and $7\hat{i} - 9\hat{j} + 6\hat{k}$ ms^{-1} respectively. Determine the instantaneous position and velocity of CM.*

Solution. Here $m_1 = 100\text{ g} = 0.1\text{ kg}$,

$m_2 = 300\text{ g} = 0.3\text{ kg}$,

$\vec{r}_1 = 2\hat{i} + 5\hat{j} + 13\hat{k}$ m,

$\vec{r}_2 = -6\hat{i} + 4\hat{j} - 2\hat{k}$ m

The position of CM will be

$$\vec{R}_{CM} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}$$

$$= \frac{0.1(2\hat{i} + 5\hat{j} + 13\hat{k}) + 0.3(-6\hat{i} + 4\hat{j} - 2\hat{k})}{0.1 + 0.3}$$

$$= \frac{-16\hat{i} + 17\hat{j} + 7\hat{k}}{4}\text{ m}$$

Again, $\vec{v}_1 = 10\hat{i} - 7\hat{j} - 3\hat{k}$ ms^{-1},

$\vec{v}_2 = 7\hat{i} - 9\hat{j} + 6\hat{k}$ ms^{-1}

The velocity of CM will be

$$\vec{v}_{CM} = \frac{m_1 \vec{v}_1 + m_2 \vec{v}_2}{m_1 + m_2}$$

$$= \frac{0.1(10\hat{i} - 7\hat{j} - 3\hat{k}) + 0.3(7\hat{i} - 9\hat{j} + 6\hat{k})}{0.1 + 0.3}$$

$$= \frac{31\hat{i} - 34\hat{j} + 15\hat{k}}{4}\text{ ms}^{-1}.$$

Example 4. *If three point masses m_1, m_2 and m_3 are situated at the vertices of an equilateral triangle of side a, then what will be the co-ordinates of the centre of mass of this system ?*

Solution. Let the point mass m_1 lie at the origin of the co-ordinate system, as shown in Fig. 7.7.

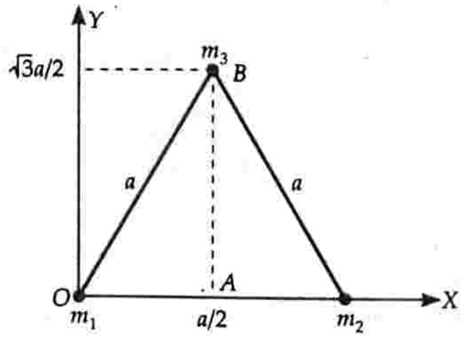


Fig. 7.7

Then

Co-ordinates of m_1 : $x_1 = 0, y_1 = 0$

Co-ordinates of m_2 : $x_2 = a, y_2 = 0$

Co-ordinates of m_3 : $x_3 = \frac{a}{2}, y_3 = \frac{\sqrt{3}}{2} a$

$$\therefore \quad x_{CM} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{m_1 + m_2 + m_3} = \frac{m_1 \times 0 + m_2 a + m_3 (a/2)}{m_1 + m_2 + m_3} = \frac{m_2 a + m_3 (a/2)}{m_1 + m_2 + m_3}$$

and

$$y_{CM} = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3}{m_1 + m_2 + m_3} = \frac{m_1 \times 0 + m_2 \times 0 + m_3 (\sqrt{3}\, a/2)}{m_1 + m_2 + m_3} = \frac{m_3 \sqrt{3}\, a}{2(m_1 + m_2 + m_3)}.$$

EXAMPLE 5. *Find the position of the centre of mass of the T-shaped plate from O in Fig. 7.8.*

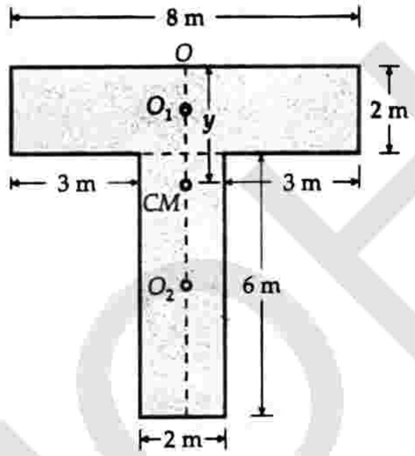


Fig. 7.8

Solution. Let mass per unit area of the plate be σ.

Mass of horizontal portion $= 8 \times 2\, \sigma = 16\, \sigma$

Mass of vertical portion $= 6 \times 2\, \sigma = 12\, \sigma$

The centres O_1 and O_2 of these portions lie at distances 1 m and $2 + 3 = 5$ m from the point O. The CM of the *T*-shaped plate will lie at distance y from the point O which is given by

$$y = \frac{16\,\sigma \times 1 + 12\,\sigma \times 5}{16\,\sigma + 12\,\sigma} = \frac{76\,\sigma}{28\,\sigma} = \mathbf{2.71\ m.}$$

EXAMPLE 6. *Find the centre of mass of a uniform* L-*shaped lamina (a thin flat plate) with dimensions as shown in Fig. 7.9. The mass of lamina is 3 kg.* **[NCERT]**

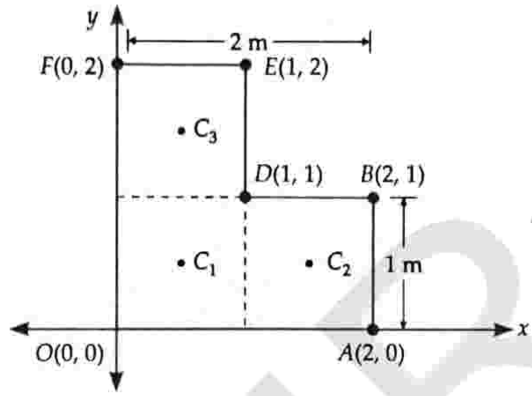


Fig. 7.9

Solution. We divide the L-shaped lamina into three squares each of length 1 m, as shown in Fig. 7.9. As the lamina is uniform, the mass of each square is 1 kg. By symmetry, the centres of mass of the three squares lie at their geometric centres C_1, C_2 and C_3. Their coordinates are as follows :

	C_1	C_2	C_3
x	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{1}{2}$
y	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$

The coordinates of the centre of mass of the *L*-shaped lamina can be obtained as follows :

$$x_{CM} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{m_1 + m_2 + m_3} = \frac{1(1/2) + 1(3/2) + 1(1/2)}{1 + 1 + 1} = \frac{5}{6}\ \text{m}$$

$$y_{CM} = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3}{m_1 + m_2 + m_3} = \frac{1 \times (1/2) + 1(1/2) + 1(3/2)}{1 + 1 + 1} = \frac{5}{6}\ \text{m}$$

Clearly, the centre of mass lies on the line of symmetry *OD*.

EXAMPLE 7. *A circular plate of uniform thickness has a diameter of* 56 *cm. A circular portion of diameter* 42 *cm is removed from one edge of the plate. Find CM of the remaining portion.*

Solution. The situation is shown in Fig. 7.10. Let O be CM of the original circular plate, O_1 that of the circular portion removed and O_2 that of the remaining shaded portion. Let m be the mass per unit area of the plate.

Mass of original plate,

$$M = \pi (28)^2\, m$$

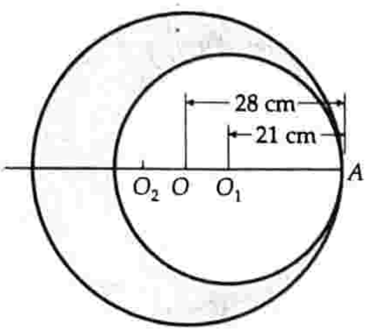


Fig. 7.10

Mass of the circular portion removed,

$$m_1 = \pi (21)^2\, m$$

Mass of the shaded part,

$$m_2 = M - m_1 = \pi [(28)^2 - (21)^2]\, m = 343\, \pi\, m$$

Masses m_1 and m_2 may be assumed to be concentrated at O_1 and O_2 respectively and O is their CM.

$\therefore$ Moment of m_1 about O = Moment of m_2 about O

or $\quad m_1 \times O_1O = m_2 \times O_2O$

or $\quad O_2O = \dfrac{m_1}{m_2} \times O_1O$

$$= \frac{441\, \pi\, m}{343\, \pi\, m} \times (28 - 21) = \mathbf{9\ cm.}$$

Example 8. *A square of side 4 cm and uniform thickness is divided into four equal squares, as shown in Fig. 7.11. If one of the squares is cut off, find the position of the centre of mass of the remaining portion from O.*

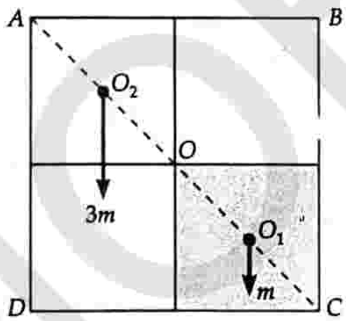


Fig. 7.11

Solution. Let mass of each small square be m. Total mass of the square will be $4m$ which acts at its centre of mass O. Let O_1 be CM of cut off square (shaded square of mass m) and O_2 be CM of the remaining unshaded portion of mass $3\,m$.

Now $\quad AC = \sqrt{AB^2 + BC^2} = \sqrt{4^2 + 4^2} = 4\sqrt{2}$ cm

$$OC = \frac{1}{2}\, AC = \frac{4\sqrt{2}}{2} = 2\sqrt{2}\ \text{cm}$$

$$OO_1 = \frac{1}{2}\, OC = \frac{2\sqrt{2}}{2} = \sqrt{2}\ \text{cm}$$

Moment of unshaded portion about O

= Moment of shaded portion about O

$$3\,m \times OO_2 = m \times OO_1$$

$$OO_2 = \frac{1}{3} \times OO_1 = \frac{1}{3} \times \sqrt{2} = \frac{\sqrt{2}}{3}\ \textbf{cm.}$$

Example 9. *Show that the centre of mass of a uniform rod of mass M and length L lies at the middle point of the rod.*

Solution. As shown in Fig. 7.12, suppose the rod is placed along X-axis with its left end at the origin O. Consider a small element of thickness dx at distance x from O.

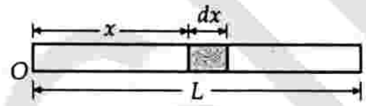


Fig. 7.12

Mass of the small element $= \dfrac{M}{L} \cdot dx$

Position of the centre of mass is given by

$$x_{CM} = \frac{1}{M}\int_0^L x\, dm$$

$$= \frac{1}{M}\int_0^L x \cdot \frac{M}{L} \cdot dx = \frac{1}{L}\int_0^L x\, dx$$

$$= \frac{1}{L}\left[\frac{x^2}{2}\right]_0^L = \frac{L}{2}.$$

Example 10. *Determine the position of the centre of mass of a hemisphere of radius R.*

Solution. Let ρ be the density of the material of the hemisphere. Take its centre O as the origin. The hemisphere can be assumed to be made of up a large number of co-axial discs. Consider one such elementary disc of radius y and thickness dx at a distance x from the origin.

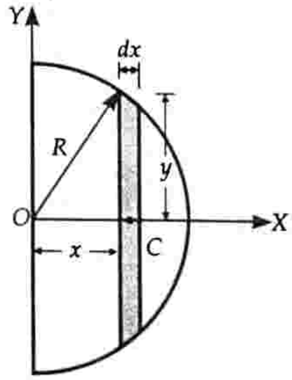


Fig. 7.13

Mass of the elementary disc = Volume × density

$$dm = \pi y^2\, dx \times \rho = \pi (R^2 - x^2)\, dx \cdot \rho$$

The coordinates of the centre of mass of the hemisphere can be determined as follows :

$$x_{CM} = \frac{1}{M}\int x\, dm = \frac{1}{M}\int_0^R x\, \pi (R^2 - x^2)\, \rho\, dx$$

$$= \frac{\pi\rho}{M}\int_0^R (R^2x - x^3)\,dx = \frac{\pi\rho}{M}\left[R^2\frac{x^2}{2} - \frac{x^4}{4}\right]_0^R$$

$$= \frac{\pi\rho}{M}\left[\frac{R^4}{2} - \frac{R^4}{4}\right] = \frac{\pi\rho}{M}\left[\frac{R^4}{4}\right]$$

$$= \frac{\pi\rho}{\frac{2}{3}\pi R^3\rho}\left(\frac{R^4}{4}\right) = \frac{3}{8}R \qquad \left[\because M = \frac{2}{3}\pi R^3 \times \rho\right]$$

Similarly,

$$y_{CM} = \int y\,dm = 0 \text{ and } z_{CM} = \int z\,dm = 0$$

Hence the coordinates of the centre of mass of the hemisphere are $\left(\frac{3}{8}\mathbf{R}, \mathbf{0}, \mathbf{0}\right)$

EXAMPLE 11. *Determine the coordinates of the centre of mass of a right circular solid cone of base radius R and height h.*

Solution. Let ρ be the density of the material of the cone. Take centre of base O as the origin. The cone can be assumed to be made up of a large number of circular discs of different radii and mass. Consider one such elementary disc of radius x and mass dm at a height y from the base.

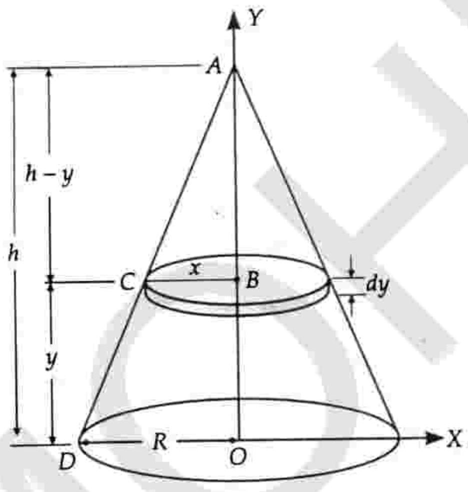


Fig. 7.14

In similar triangles AOD and ABC, we have

$$\frac{h}{h-y} = \frac{R}{x} \text{ or } x = \frac{R(h-y)}{h}$$

Mass of elementary disc is

$$dm = \text{Volume} \times \text{density}$$

$$= \pi x^2\,dy \times \rho = \pi\rho\,\frac{R^2(h-y)^2}{h^2}\,dy$$

The coordinates of the centre of mass can be determined as follows :

$$x_{CM} = \frac{1}{M}\int x\,dm = 0 \qquad [\because x = 0]$$

$$y_{CM} = \frac{1}{M}\int y\,dm = \frac{1}{M}\int_0^h y \cdot \pi\rho\,\frac{R^2(h-y)^2}{h^2}\,dy$$

$$= \frac{\pi\rho R^2}{Mh^2}\int_0^h y(h-y)^2\,dy$$

$$= \frac{\pi\rho R^2}{Mh^2}\int_0^h y(h^2 + y^2 - 2hy)\,dy$$

$$= \frac{\pi\rho R^2}{Mh^2}\int_0^h (h^2y + y^3 - 2hy^2)\,dy$$

$$= \frac{\pi\rho R^2}{Mh^2}\left[h^2\left|\frac{y^2}{2}\right|_0^h + \left|\frac{y^4}{4}\right|_0^h - 2h\left|\frac{y^3}{3}\right|_0^h\right]$$

$$= \frac{\pi\rho R^2}{Mh^2}\left[\frac{h^4}{2} + \frac{h^4}{4} - \frac{2h^4}{3}\right] = \frac{\pi\rho R^2}{Mh^2}\left[\frac{h^4}{12}\right]$$

$$= \frac{\pi\rho R^2}{\frac{1}{3}\pi R^2 h\rho \times h^2} \times \frac{h^4}{12} = \frac{h}{4} \qquad \left[\because M = \frac{1}{3}\pi r^2 h\rho\right]$$

Similarly,

$$z_{CM} = \frac{1}{m}\int z\,dm = 0 \qquad [\because z = 0]$$

Hence the coordinates of the centre of mass of the right circular solid cone are $\left(\mathbf{0}, \frac{\mathbf{h}}{\mathbf{4}}, \mathbf{0}\right)$.

✖ PROBLEMS FOR PRACTICE

1. Two bodies of masses 1 kg and 2 kg are located at (1, 2) and (− 1, 3) respectively. Calculate the co-ordinates of the centre of mass. **[Central Schools 04, 08]**
$\left[\textbf{Ans.}\left(-\frac{1}{3}, \frac{8}{3}\right)\right]$

2. The distance between the centres of carbon and oxygen atoms in the carbon monoxide gas molecule is 1.13 Å. Locate the centre of mass of the gas molecule relative to the carbon atom. **[Central Schools 11]**
(**Ans.** 0.6457 Å from C-atom)

3. Three blocks of uniform thickness and masses m, m and 2 m are placed at three corners of a triangle having co-ordinates (2.5, 1.5), (3.5, 1.5) and (3, 3) respectively. Find the centre of mass of the system. **[Delhi 06]** [**Ans.** (3, 2.25)]

4. Find the centre of mass of three particles at the vertices of an equilateral triangle. The masses of the particles are 100 g, 150 g and 200 g respectively. Each side of the equilateral triangle is 0.5 m long. **[NCERT]**
(**Ans.** $\frac{5}{18}$ m, $\frac{1}{3\sqrt{3}}$ m)

5. Three particles each of mass m are placed at three corners of an equilateral triangle of length l. Find the position of centre of mass in terms of coordinates. **[Chandigarh 02]**

$$\left[\textbf{Ans. } \left(\frac{l}{2}, \frac{l}{2\sqrt{3}}\right)\right]$$

6. Three point masses of 1 kg, 2 kg and 3 kg lie at (1, 2), (0, –1) and (2, –3) respectively. Calculate the coordinates of the centre of mass of the system. **[Central Schools 12]**

$$\left[\textbf{Ans. } \left(\frac{7}{6}, \frac{-3}{2}\right)\right]$$

7. Two bodies of masses 10 kg and 2 kg are moving with velocities $2\hat{i} - 7\hat{j} + 3\hat{k}$ and $-10\hat{i} + 35\hat{j} - 3\hat{k}$ ms^{-1} respectively. Find the velocity of the centre of mass of the system. (**Ans.** $2\hat{k}$ ms^{-1})

8. Three particles of masses 0.50 kg, 1.0 kg and 1.5 kg are placed at the corners of a right angle triangle, as shown in Fig. 7.15. Locate the centre of mass of the system. (**Ans.** 1.33 cm, 1.5 cm)

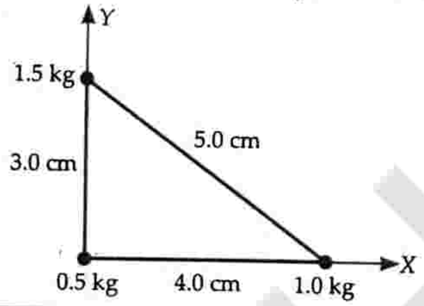


Fig. 7.15

9. Four particles of masses m, m, 2 m and 2 m are placed at the four corners of a square of side a. Find the centre of mass of the system.

$$\left[\textbf{Ans. } \left(\frac{a}{2}, \frac{2}{3}a\right) \text{ with first mass } m \text{ at the origin}\right]$$

10. Four particles of masses m, $2m$, $3m$ and $4m$ respectively are placed at the corners of a square of side a, as shown in Fig. 7.16. Locate the centre of mass.

$$\left[\textbf{Ans. } \left(\frac{a}{2}, \frac{7a}{10}\right)\right]$$

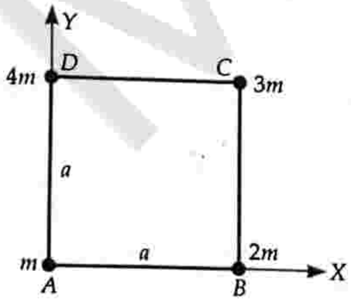


Fig. 7.16

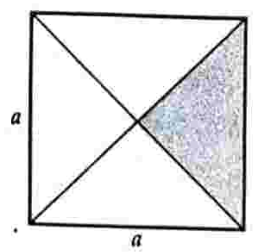


Fig. 7.17

11. From a square sheet of uniform density, a portion is removed shown shaded in Fig. 7.17. Find the centre of mass of the remaining portion if the side of the square is a.

$$\left[\textbf{Ans. } \left(\frac{7}{18}a, \frac{a}{2}\right)\right]$$

12. The centre of mass of three particles of masses 1 kg, 2 kg and 3 kg lies at the point (3 m, 3 m, 3 m). Where should a fourth particle of mass 4 kg be positioned so that the centre of mass of the four particle system lies at the point (1 m, 1 m, 1 m) ?

[**Ans.** (– 2 m, – 2 m, – 2 m]

✖ HINTS

2. $$x_{CM} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} = \frac{12 \times 0 + 16 \times 1.13}{12 + 16} = \mathbf{0.6457\ Å}$$

3. $$x_{CM} = \frac{m(2.5) + m(3.5) + 2m(3)}{m + m + 2m} = 3$$

$$y_{CM} = \frac{m(1.5) + m(1.5) + 2m(3)}{m + m + 2m} = 2.25.$$

4. The position coordinates for the three particles are shown in Fig. 7.18.

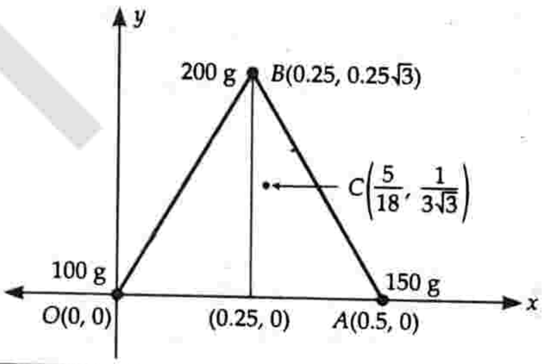


Fig. 7.18

$$x_{CM} = \frac{100(0) + 150(0.5) + 200(0.25)}{100 + 150 + 200} = \frac{5}{18}\ \mathbf{m}$$

$$y_{CM} = \frac{100(0) + 150(0) + 200(0.25\sqrt{3})}{100 + 150 + 200} = \frac{1}{3\sqrt{3}}\ \mathbf{m.}$$

5. $$x_{CM} = \frac{m(0) + m(l) + m(l/2)}{m + m + m} = \frac{l}{2}.$$

$$y_{CM} = \frac{m(0) + m(0) + m(\sqrt{3}\,l/2)}{m + m + m} = \frac{l}{2\sqrt{3}}.$$

9. $$x_{CM} = \frac{m \times 0 + m \times a + 2m \times a + 2m \times 0}{m + m + 2m + 2m} = \frac{3ma}{6m} = \frac{a}{2}$$

$$y_{CM} = \frac{m \times 0 + m \times 0 + 2m \times a + 2m \times a}{m + m + 2m + 2m} = \frac{4ma}{6m} = \frac{2}{3}a.$$

10. $$x_{CM} = \frac{m \times 0 + 2m \times a + 3m \times a + 4m \times 0}{m + 2m + 3m + 4m} = \frac{a}{2}$$

$$y_{CM} = \frac{m \times 0 + 2m \times 0 + 3m \times a + 4m \times a}{m + 2m + 3m + 4m} = \frac{7a}{10}.$$

12. Suppose the 4 kg mass be placed at point (x, y, z). Then the centre of mass of 6 kg (1 kg + 2 kg + 3 kg) mass and 4 kg mass must lie at the point (1 m, 1 m, 1 m).

$$\therefore \quad x_{CM} = \frac{6 \times 3 + 4 \times x}{6 + 4} = 1 \text{ m} \quad \text{or} \quad x = -2 \text{ m}$$

$$y_{CM} = \frac{6 \times 3 + 4 \times y}{6 + 4} = 1 \text{ m} \quad \text{or} \quad y = -2 \text{ m}$$

$$z_{CM} = \frac{6 \times 3 + 4 \times z}{6 + 4} = 1 \text{ m} \quad \text{or} \quad z = -2 \text{ m}.$$

7.7 EXAMPLES OF BINARY SYSTEMS IN NATURE

8. *Discuss the motion of the centre of mass of the following binary systems in nature : (i) Binary stars, (ii) Diatomic molecules, and (iii) Earth-moon system.*

(*i*) **Binary stars.** *Two stars bound to each other by the gravitational force and orbiting around their common centre of mass are called binary stars.* Fig. 7.19(*a*) shows binary stars S_1 and S_2 of equal mass moving in circular orbits around their common centre of mass, which is at rest.

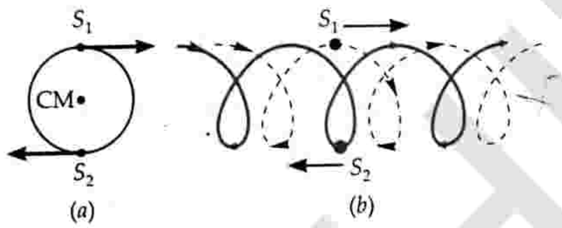


Fig. 7.19 Orbits of binary stars of equal mass, when their (*a*) CM is at rest, (*b*) CM is in uniform motion.

When no external force acts on the system, the centre of mass of the double star moves like a free particle. The orbits of the two stars are slightly complicated, as shown in Fig. 7.19(*b*). But these are just the combination of *two* motions : (*i*) the uniform motion of the centre of mass CM in a straight line and (*ii*) the circular orbits of the two stars around their CM. However, the two stars always remain on the opposite sides of the CM.

(*ii*) **Diatomic molecule.** A symmetric diatomic molecule like O_2 is also an example of binary system. The internal binding force between the two oxygen atoms is due to the chemical bond which can be regarded as a spring. When there is no external force (*i.e.*, no collisions between the molecules themselves or with the walls of the vessel), the centre of mass of the molecule moves with uniform velocity in a straight line, as shown in Fig. 7.19(*c*). The molecule can also have vibrational and rotational motions again due to the internal forces. Even then the centre of mass moves like a free particle.

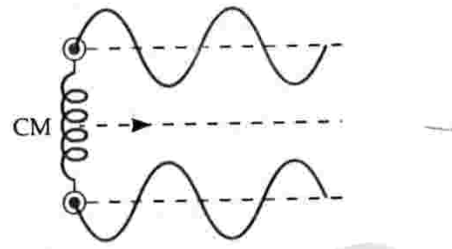


Fig. 7.19. (*c*) Uniform motion of the CM of moving, vibrating symmetric diatomic molecule O_2.

(*iii*) **Earth-moon system.** The moon moves around the earth in a circular orbit and the earth moves around the sun in an elliptical orbit. It will be more correct to say that the centre of mass of the earth-moon system moves around the sun in an elliptical orbit, not the earth and moon themselves. As the mass of the earth is nearly 80 times the mass of the moon, so the centre of mass divides the earth-moon (*E-M*) line in the ratio 1 : 80. In fact, this point lies inside the earth, as shown in Fig. 7.19(*d*).

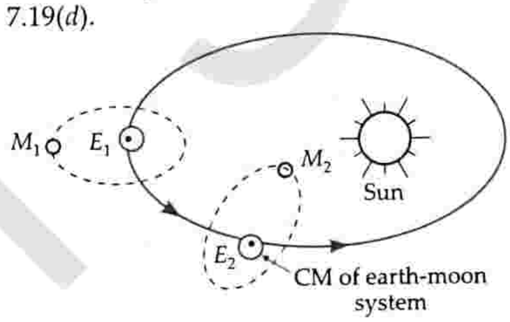


Fig. 7.19. (*d*) Motion of CM of the earth-moon system in an elliptical orbit.

Here the mutual forces of gravitation between the earth and moon are internal forces while the Sun's attraction of both earth and moon are the external forces acting on the centre of mass of the earth-moon system.

7.8 SOME OTHER EXAMPLES OF THE CM MOTION

9. *Discuss the trajectory of the motion of the centre of mass of a fire cracker that explodes in air.*

Motion of the CM of fire crackers exploding in air. Initially, a fire cracker moves along a parabolic path. It explodes in flight. Each fragment will follow its

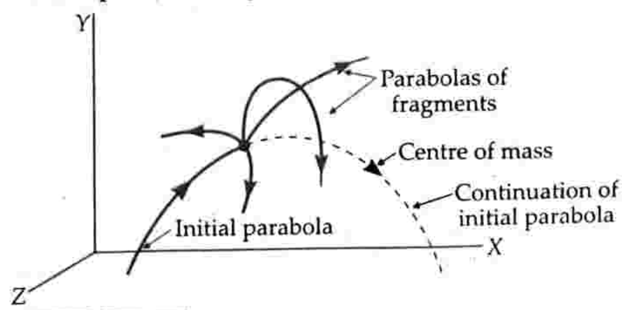


Fig. 7.20 Trajectory of the centre of mass of exploding cracker.

own parabolic path. Since the explosion is caused by internal forces only, the centre of mass of all the fragments will continue to move along the same parabolic path of the cracker as before explosion.

7.9 RIGID BODIES

10. *What is a rigid body ? Give examples.*

Rigid body. *A body is said to be rigid if it does not undergo any change in its size and shape, however large the external force may be acting on it.* More appropriately, *a rigid body is one whose constituent particles retain their relative positions even when they move under the action of an external force.*

A rigid body cannot be deformed. If the body undergoes some displacement, every particle in it suffers the same displacement. If the body rotates through a certain angle, every particle of it rotates through the same angle about the axis of rotation. No body can be perfectly rigid. In practice, solid bodies of steel, glass etc. ; can be regarded as rigid for moderate forces.

7.10 CENTRE OF MASS OF A RIGID BODY

11. *State the factors on which the position of the centre of mass of a rigid body depends.*

Centre of mass of a rigid body. *The centre of mass of a rigid body is a point at a fixed position with respect to the body as a whole.*

The position of centre of mass of a rigid body depends on *two* factors : (*i*) The geometrical shape of the body. (*ii*) The distribution of mass in the body.

12. *How can we locate the centre of mass of rigid bodies of regular geometrical shape and having uniform mass distribution ? Give examples.*

Centre of mass of regular bodies. *For bodies having regular geometrical shape and uniform mass (or density) distribution, the centre of mass lies at their geometrical centre.* Fig. 7.21 shows the positions of the centre of mass of some regular bodies.

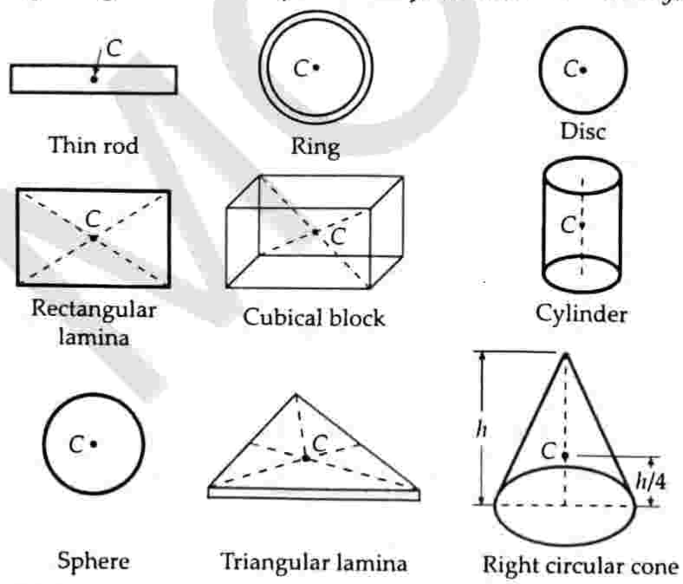


Fig. 7.21 Centres (C) of mass of some regular bodies.

Table 7.1 Centres of mass of some regular bodies

S. No.	Shape of body	Position of centre of mass
1.	Long thin rod	Middle point of the rod
2.	Thin circular ring	Geometrical centre of the ring
3.	Circular disc	Geometrical centre of the disc
4.	Rectangular lamina	Point of intersection of diagonals
5.	Rectangular cubical block	Point of intersection of diagonals
6.	Cylinder	Middle point of the axis
7.	Solid or hollow sphere	Geometrical centre of the sphere
8.	Triangular lamina	Point of intersection of the medians
9.	Right circular cone	A point on its axis at a distance of $\frac{h}{4}$ from its base, h = height of the cone

7.11 ROTATIONAL MOTION OF A RIGID BODY

13. *What do you mean by rotational motion of a rigid body ?*

Rotational motion of a rigid body. *A body is said to possess rotational motion if all its particles move along circles in parallel planes. The centres of these circles lie on a fixed line perpendicular to the parallel planes and this line is called the* ***axis of rotation***.

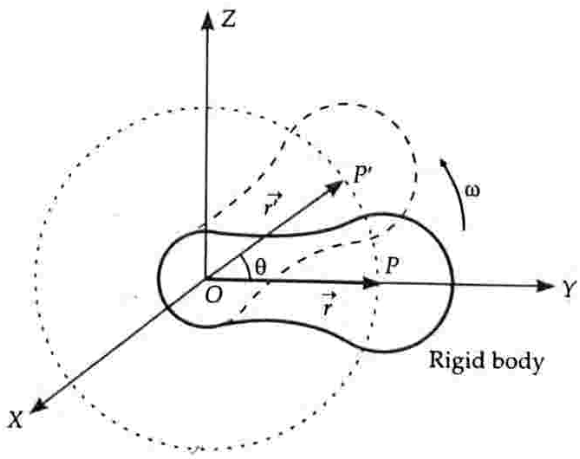


Fig. 7.22 Rotational motion about Z-axis.

Fig. 7.22 shows a rigid body being rotated anti-clockwise about Z-axis of an inertial frame of reference. Let P be any particle of the body and $\vec{r}$ be its position vector. As the body rotates, the particle P moves along a circle of radius r whose centre lies on the axis of rotation. The radius vector $\vec{r}$ sweeps out an angle θ in certain time t. Similarly, all other particles of body move along circles with their centres on Z-axis and their radius vectors sweep the same angle θ in time t. This implies that *all the particles have the same angular velocity* $\omega (= \theta / t)$ which is also the angular velocity of the body.

7.12 EQUATIONS OF ROTATIONAL MOTION

14. *Derive the three equations of rotational motion under constant angular acceleration from first principles.*

Derivation of first equation of motion. Consider a rigid body rotating about a fixed axis with constant angular acceleration α. By definition,

$$\alpha = \frac{d\omega}{dt}$$

or $$d\omega = \alpha\, dt \qquad ...(1)$$

At $t = 0$, let $\omega = \omega_0$

At $t = t$, let $\omega = \omega$

Integrating equation (1) within the above limits of time and angular velocity, we get

$$\int_{\omega_0}^{\omega} d\omega = \int_0^t \alpha\, dt = \alpha \int_0^t dt$$

or $$[\omega]_{\omega_0}^{\omega} = \alpha\,[t]_0^t$$

or $$\omega - \omega_0 = \alpha\,(t - 0)$$

or $$\omega = \omega_0 + \alpha t \qquad ...(2)$$

Derivation of second equation of motion. Let ω be the angular velocity of a rigid body at any instant t. By definition,

$$\omega = \frac{d\theta}{dt}$$

or $$d\theta = \omega\, dt \qquad ...(3)$$

At $t = 0$, let $\theta = 0$

At $t = t$, let $\theta = \theta$

Integrating equation (3) within the above limits of time and angular displacement, we get

$$\int_0^{\theta} d\theta = \int_0^t \omega\, dt = \int_0^t (\omega_0 + \alpha t)\, dt \qquad \text{[Using (2)]}$$

$$= \omega_0 \int_0^t dt + \alpha \int_0^t t\, dt$$

or $$[\theta]_0^{\theta} = \omega_0\,[t]_0^t + \alpha\left[\frac{t^2}{2}\right]_0^t$$

or $$\theta - 0 = \omega_0\,(t - 0) + \frac{\alpha}{2}(t^2 - 0)$$

or $$\theta = \omega_0 t + \frac{1}{2}\alpha t^2 \qquad ...(4)$$

Derivation of third equation of motion. The angular acceleration α may be expressed as

$$\alpha = \frac{d\omega}{dt} = \frac{d\omega}{d\theta}\cdot\frac{d\theta}{dt} = \frac{d\omega}{d\theta}\cdot\omega$$

or $$\omega\, d\omega = \alpha\, d\theta \qquad ...(5)$$

At $t = 0$, $\theta = 0$ and $\omega = \omega_0$ (initial angular velocity)

At $t = t$, $\theta = \theta$ and $\omega = \omega$ (final angular velocity)

Integrating equation (5) within the above limits of θ and ω, we get

$$\int_{\omega_0}^{\omega} \omega\, d\omega = \int_0^{\theta} \alpha\, d\theta = \alpha \int_0^{\theta} d\theta$$

or $$\left[\frac{\omega^2}{2}\right]_{\omega_0}^{\omega} = \alpha\,[\theta]_0^{\theta}$$

or $$\frac{\omega^2}{2} - \frac{\omega_0^2}{2} = \alpha\,(\theta - 0)$$

or $$\omega^2 - \omega_0^2 = 2\alpha\theta$$

Examples based on Equations of Rotational Motion

FORMULAE USED

For a body in rotational motion under constant angular acceleration, the equations of motion can be written as

1. $\omega = \omega_0 + \alpha\, t$
2. $\theta = \omega_0\, t + \frac{1}{2}\alpha\, t^2$
3. $\omega^2 - \omega_0^2 = 2\,\alpha\, t$

UNITS USED

Angular displacement θ is in rad, initial angular velocity ω_0 and final angular velocity ω are in rad s^{-1}, angular acceleration α in rad s^{-2}.

EXAMPLE 12. *On the application of a constant torque, a wheel is turned from rest through 400 radians in 10 s. (i) Find angular acceleration. (ii) If same torque continues to act, what will be angular velocity of the wheel after 20 s from start ?*

Solution. (*i*) Here $\theta = 400$ rad, $\omega_0 = 0$, $t = 10$ s, $\alpha = ?$

As $\quad \theta = \omega_0 t + \frac{1}{2} \alpha t^2$

$\therefore \quad 400 = 0 + \frac{1}{2} \alpha (10)^2$

or $\quad \alpha = \frac{2 \times 400}{100} = \mathbf{8\ rad\ s^{-2}}.$

(*ii*) Here $\omega_0 = 0,\ \alpha = 8\ \text{rad s}^{-2},\ t = 20\ \text{s},\ \omega = ?$

$$\omega = \omega_0 + \alpha t = 0 + 8 \times 20 = \mathbf{160\ rad\ s^{-1}}.$$

EXAMPLE 13. *The angular speed of a motor wheel is increased from 1200 rpm to 3120 rpm in 16 seconds. (i) What is its angular acceleration, assuming the acceleration to be uniform ? (ii) How many revolutions does the wheel make during this time ?* **[NCERT]**

Solution. Here

$$\nu_0 = 1200\ \text{rpm} = \frac{1200}{60} = 20\ \text{rps}$$

$$\nu = 3120\ \text{rpm} = \frac{3120}{60} = 52\ \text{rps}$$

$$\therefore \quad \omega_0 = 2\pi\nu_0 = 2\pi \times 20 = 40\pi\ \text{rads}^{-1}$$

$$\omega = 2\pi\nu = 2\pi \times 52 = 104\pi\ \text{rad s}^{-1}$$

(*i*) Angular acceleration,

$$\alpha = \frac{\omega - \omega_0}{t} = \frac{104\pi - 40\pi}{16} = \mathbf{4\pi\ rad\ s^{-2}}.$$

(*ii*) The angular displacement in time *t*,

$$\theta = \omega_0 t + \frac{1}{2} \alpha t^2$$

$$= 40\pi \times 16 + \frac{1}{2} \times 4\pi \times (16)^2$$

$$= (640\pi + 512\pi)\ \text{rad} = 1152\pi\ \text{rad}.$$

Number of revolutions completed in 16 s

$$= \frac{\theta}{2\pi} = \frac{1152\pi}{2\pi} = \mathbf{576}.$$

EXAMPLE 14. *A constant torque is acting on a wheel. If starting from rest, the wheel makes n rotations in t second, show that the angular acceleration is given by*

$$\alpha = \frac{4\pi n}{t^2}\ \textit{rad s}^{-2}.$$

Solution. Initial angular velocity, $\omega_0 = 0$

Number of rotations completed in t second $= n$

$\therefore$ Angular displacement in t second, $\theta = 2\pi n$

As $\quad \theta = \omega_0 t + \frac{1}{2} \alpha t^2$

$\therefore \quad 2\pi n = 0 + \frac{1}{2} \alpha t^2$

or $\quad \alpha = \mathbf{\frac{4\pi n}{t^2}\ rad\ s^{-2}}.$

EXAMPLE 15. *The radius of a wheel of a car is 0.4 m. The car is accelerated from rest by an angular acceleration of 1.5 rad s^{-2} for 20 s. How much distance the wheel covers in this time interval and what will be its linear velocity ?*

Solution. Here $r = 0.4\ \text{m},\ \omega_0 = 0,$

$$\alpha = 1.5\ \text{rad s}^{-2},\ t = 20\ \text{s}$$

Angular displacement,

$$\theta = \omega_0 t + \frac{1}{2} \alpha t^2$$

$$= 0 + \frac{1}{2} \times 1.5 \times (20)^2 = 300\ \text{rad}$$

Distance covered by the wheel,

$$s = r\theta = 0.4 \times 300 = \mathbf{120\ m}.$$

After 20 s, angular velocity of the wheel is

$$\omega = \omega_0 + \alpha t = 0 + 1.5 \times 20 = \mathbf{30\ rad\ s^{-1}}$$

$\therefore$ Linear velocity of the wheel is

$$v = r\omega = 0.4 \times 30 = \mathbf{12\ ms^{-1}}.$$

✖ PROBLEMS FOR PRACTICE

1. A gridstone has a constant acceleration of 4 rad s^{-1}. Starting from rest, calculate the angular speed of the grindstone 2.5 s later. (**Ans.** 10 rad s^{-1})
2. The speed of a motor increases from 600 rpm to 1200 rpm in 20 s. What is its angular acceleration and how many revolutions does it make during this time ? (**Ans.** π rad s^{-2}, 300)
3. On the application of a constant torque, a wheel is turned from rest through an angle of 200 rad in 8 s. (*i*) What is its angular acceleration ? (*ii*) If the same torque continues to act, what will be the angular velocity of the wheel after 16 s from the start ? [**Ans.** (*i*) 6.25 rad s^{-2} (*ii*) 100 rad s^{-1}]
4. The motor of an engine is rotating about its axis with an angular velocity of 100 rpm. It comes to rest in 15 s after being switched off. Assuming constant angular deceleration, calculate the number of revolutions made by it before coming to rest. (**Ans.** 12.5)
5. A car is moving at a speed of 72 kmh^{-1}. The diameter of its wheels is 0.50 m. If the wheels are stopped in 20 rotations by applying brakes, calculate the angular retardation produced by the brakes. (**Ans.** – 25.5 rad s^{-2})
6. A flywheel rotating at 420 rpm slows down at a constant rate of 2 rad s^{-2}. What time is required to stop the flywheel ? **[Central Schols 12]**

✖ HINTS

4. Here $\nu_0 = 100\ \text{rpm} = \frac{100}{60}\ \text{rps}$

$$\therefore \quad \omega_0 = 2\pi\nu_0 = 2\pi \times \frac{100}{60} = \frac{10\pi}{3}\ \text{rad s}^{-1}$$

$$\omega = 0, \quad t = 15\ \text{s}$$

$$\therefore \quad \alpha = \frac{\omega - \omega_0}{t} = \frac{0 - 10\pi/3}{15} = -\frac{2\pi}{9}\ \text{rad s}^{-2}$$

Now $\theta = \omega_0 t + \frac{1}{2}\alpha t^2 = \frac{10\pi}{3} \times 15 - \frac{1}{2} \times \frac{2\pi}{9} \times (15)^2$

$= 25\pi$ rad

Number of revolutions completed before coming to rest

$= \frac{\theta}{2\pi} = \frac{25\pi}{2\pi} = \mathbf{12.5.}$

5. Here $v_0 = 72 \text{ kmh}^{-1} = \frac{72 \times 5}{18} = 20 \text{ ms}^{-1}$, $r = 0.25$ m

Angular speed, $\omega_0 = \frac{v_0}{r} = \frac{20}{0.25} = 80 \text{ rad s}^{-1}$

Angular displacement in 20 rotations,

$\theta = 2\pi n = 2\pi \times 20 = 40\pi$ rad

Also, final angular velocity, $\omega = 0$

As $\omega^2 - \omega_0^2 = 2\alpha\theta$

$\therefore \; 0 - (80)^2 = 2\alpha \times 40\pi$

or $\quad \alpha = -\frac{(80)^2}{80\pi} = \mathbf{-25.5 \text{ rad s}^{-2}}$.

6. $v_0 = 420$ rpm $= 7$ rps

$\omega_0 = 2\pi v_0 = 2 \times \frac{22}{7} \times 7 = 44 \text{ rad s}^{-1}$,

$\omega = 0, \quad \alpha = -2 \text{ rad s}^{-2}$

$t = \frac{\omega - \omega_0}{\alpha} = \frac{0 - 44}{-2} = \mathbf{22 \text{ s}}.$

7.13 MOMENT OF FORCE OR TORQUE

15. *On what factors does the turning effect of a force depend ? What is the turning effect of force called ?*

Turning effect of force. To open a door, we apply a force on its handle. The door turns on its hinges. *The larger the force, the more is its turning effect.* Also, we can easily notice that it is easier to open a door by applying a force near the end than near the hinges. This is because the turning effect of the same force is larger when its distance from the axis of rotation is more.

This turning effect of force is called **moment of force or torque**. It depends on *two* factors :

(*i*) The magnitude of the force.

(*ii*) The perpendicular distance of the line of action of the force from the axis of rotation. It is called *lever arm* or *moment arm*.

Thus, greater the magnitude of the force, or greater the perpendicular distance between the line of action of the force and the axis of rotation, the greater is the moment of force, or greater is the turning effect.

16. *Define the term torque or moment of force. Give its units and dimensions.*

Torque or moment of force. *The torque or moment of force is the turning effect of the force about the axis of rotation. It is measured as the product of the magnitude of the force and the perpendicular distance between the line of action of the force and the axis of rotation.*

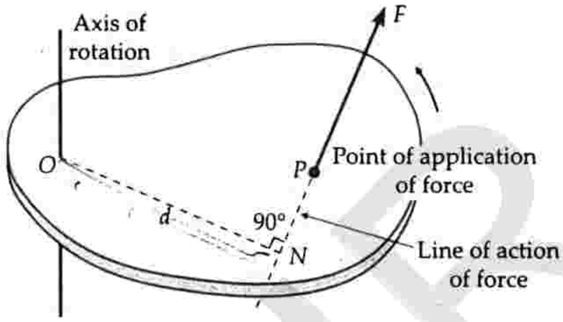


Fig. 7.23 Torque or moment of force.

Fig. 7.23 shows a body free to rotate about a vertical axis through *O*. A horizontal force *F* applied on it at point *P* rotates it about this axis. If *d* is the perpendicular distance of the line of action of the force from the axis of rotation, then the torque or moment of force *F* about the axis of rotation is

$$\tau = F \times ON$$

or $$\tau = F \times d$$

or Torque = Force × Lever arm

Dimensions of torque. As

Torque = Force × distance,

so $$[\tau] = [MLT^{-2}][L] = [ML^2T^{-2}].$$

Units of torque. The SI unit of torque is newton metre (Nm) and its CGS unit is dyne cm.

Sign convention. The moment of force is taken *positive* if the turning tendency of the force is *anticlockwise* and *negative* if it is *clockwise*.

7.14 ROTATIONAL EQUILIBRIUM AND THE PRINCIPLE OF MOMENTS

17. *State and explain the principle of moments of rotational equilibrium.*

Principle of moments. *When a body is in rotational equilibrium, the sum of the clockwise moments about any point is equal to the sum of the anticlockwise moments about that point or the algebraic sum of moments about any point is zero.*

As shown in Fig. 7.24, consider a uniform rod free to rotate on a pivot *O*. Two weights W_1 and W_2 are hung from it at distances d_1 and d_2 from the pivot *O*.

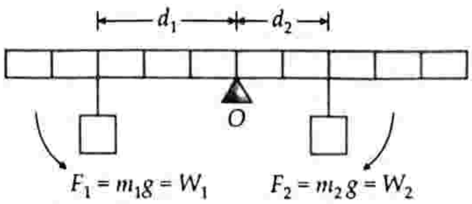


Fig. 7.24 Principle of moments.

Anticlockwise moment about $O = F_1 \times d_1 = W_1 \times d_1$

Clockwise moment about $O \quad = F_2 \times d_2 = W_2 \times d_2$

According to the principle of moments, the rod will be horizontal or in rotational equilibrium if

Anticlockwise moment = Clockwise moment

or $\quad F_1 \times d_1 = F_2 \times d_2$

or $\quad W_1 \times d_1 = W_2 \times d_2$

i.e., Load × load arm = Effort × effort arm

This is sometimes called the ***lever principle***.

18. *What is a couple ? What effect does it have on a body ? Show that the moment of couple is same irrespective of the point of rotation of a body.*

Couple. *A pair of equal and opposite forces acting on a body along two different lines of action constitute a couple.* A couple has a turning effect, but no resultant force acts on a body. So it cannot produce translational motion. When we steer a bicycle round a bend with our both hands on the handle-bars, we apply a couple.

Moment of a couple. The moment of couple can be found by taking the moments of the two forces about any point and then adding them.

In Fig. 7.25, two opposite forces, each of magnitude F act at two points A and B of a rigid body, which can rotate about point O. The turning tendency of the two forces is anticlockwise.

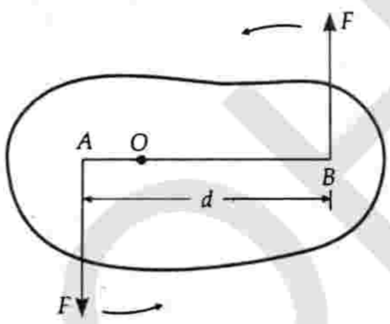


Fig. 7.25 Moment of couple.

Moment or torque of the couple about O is

$$\tau = F \times AO + F \times OB$$
$$= F(AO + OB) = F \times AB$$

or $\quad \tau = Fd \qquad [\because AB = d, \text{ say}]$

Moment of a couple = Force × perpendicular distance between two forces

Hence *the moment of a couple is equal to the product of either of the forces and the perpendicular distance, called the arm of the couple, between their lines of action.* Note that the torque exerted by couple about O does not depend on the position of O. Hence *torque or moment of a couple is independent of the choice of the fulcrum or the point of rotation.*

Notably, a couple can only be balanced by an equal and opposite couple.

19. *Obtain an expression for the work done by a torque. Hence write the expression for power.*

Work done by a torque. As shown in Fig. 7.26, suppose a body undergoes an angular displacement $\Delta\theta$ under the action of a tangential force F.

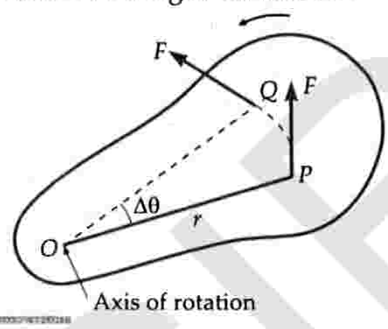


Fig. 7.26 Work done by a torque.

The work done in the rotational motion of the body or the work done by the torque is

$$\Delta W = F \times \text{distance along the arc } PQ$$

But $\quad \Delta\theta = \dfrac{\text{Arc}}{\text{Radius}} = \dfrac{\text{Arc } PQ}{r}$

$\therefore \quad \text{Arc } PQ = r\,\Delta\theta$

Hence $\quad \Delta W = Fr\,\Delta\theta$

or $\quad \Delta W = \tau\,\Delta\theta$

i.e., Work done by a torque = Torque × angular displacement

In case the torque applied is not constant, but variable, the total work done by the torque is given by

$$W = \int_{\theta_1}^{\theta_2} \tau\, d\theta$$

Power delivered by a torque. We know that

$$\Delta W = \tau\,\Delta\theta$$

Dividing both sides by Δt, we get

$$\frac{\Delta W}{\Delta t} = \tau \frac{\Delta\theta}{\Delta t}$$

or $\quad P = \tau\omega$

i.e., Power = Torque × Angular velocity.

20. *Explain how torque can be expressed as a vector product of two vectors. How is the direction of torque determined ?*

Torque acting on a particle. Consider a particle P in the X-Y plane. Suppose its position vector is $\vec{OP} = \vec{r}$ with respect to the origin O of an inertial frame, as shown in Fig. 7.27. Let $\vec{F}$ be the force acting on the particle. *The torque acting on the particle is defined as the* *vector product of position and force vectors.* Thus the torque of $\vec{F}$ about O is defined as

$$\vec{\tau} = \vec{r} \times \vec{F}$$

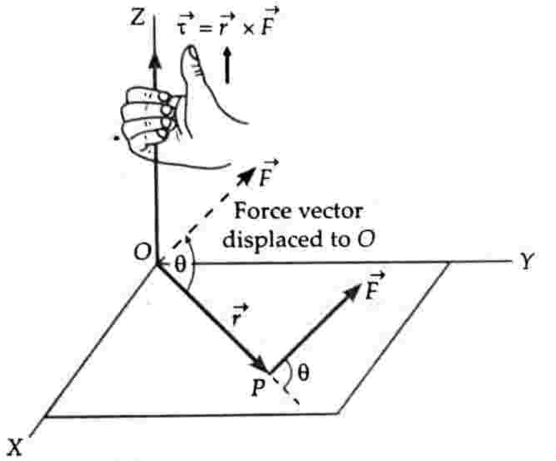


Fig. 7.27 Torque as the vector product of $\vec{r}$ and $\vec{F}$.

Torque $\vec{\tau}$ is a vector quantity. Its magnitude is given by

$$\tau = rF \sin \theta$$

The direction of torque $\vec{\tau}$ is perpendicular to the plane containing vectors $\vec{r}$ and $\vec{F}$ and its sense is determined by *right hand rule.* If we curl the fingers of our right hand in the direction in which vector $\vec{r}$ must be rotated to move into the position of vector $\vec{F}$ through the smaller angle between them, then the extended thumb gives the sense of torque $\vec{\tau}$. In the present case, $\vec{\tau}$ acts along Z-axis.

Special cases. (*i*) **When $\theta = 0°$ or $180°$**, the line of action of the force passes through the origin. In this case $\sin \theta = 0$, so that $\tau = 0$.

(*ii*) **When $\theta = 90°$**, $\tau = Fr \sin 90° = Fr$ and is maximum. This explains why it is easier to open or close a door when we apply force perpendicular to the door at its outer or free edge.

(*iii*) **When r is maximum**, torque due to the force is maximum. This explains why we can open or close a door easily by applying force near the outer edge of the door (at maximum distance from the hinges). For this reason, a handle or knob is provided near the free edge of the door.

21. *Show that the magnitude of torque = magnitude of force × moment arm. Also show that only the angular component of the force is responsible for producing torque.*

Dependence of torque on moment arm. Consider a particle capable of rotation in the X-Y plane about the origin O. Suppose the force vector makes an angle θ with the position vector $\vec{r}$ of the particle, as shown in Fig. 7.28.

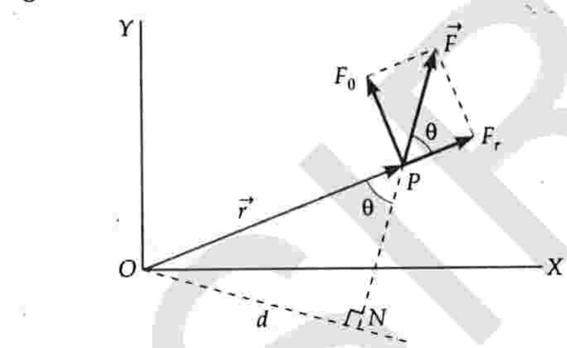


Fig. 7.28 Dependence of torque on moment arm.

Draw ON perpendicular to the line of action of the force. Then from the right angled ΔONP, we have

$$\frac{ON}{OP} = \sin \theta \quad \text{or} \quad \frac{d}{r} = \sin \theta \quad \text{or} \quad d = r \sin \theta$$

This is the perpendicular distance of the line of action of force from the axis of rotation through the point O and is called *moment arm* or *lever arm* of the force.

The magnitude of the torque of force F is given by

$$\tau = rF \sin \theta = F\,(r \sin \theta) = Fd$$

i.e., Torque = Magnitude of force × moment arm

Now the force $\vec{F}$ can be resolved into two rectangular components :

1. *Radial component* F_r along the direction of position vector $\vec{r}$.
2. *Angular or tangential component* F_θ perpendicular to $\vec{r}$.

Clearly, $F_r = F \cos \theta$ and $F_\theta = F \sin \theta$

$\therefore$ $\tau = F \sin \theta \, . \, r = F_\theta \, r$

i.e. Torque = Angular component of the force × its distance from the axis of rotation

Hence *torque due to a force is only due to its angular component.*

The radial component of the force does not contribute to the torque.

22. *Write an expression for torque in three-dimensional motion. Hence write the expressions for the rectangular components of torque.*

Rectangular components of torque. For three dimensional motion, the position vector $\vec{r}$, and torque vector $\vec{\tau}$ can be written in terms of their rectangular components as follows :

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\vec{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$$

and $$\vec{\tau} = \tau_x\hat{i} + \tau_y\hat{j} + \tau_z\hat{k}$$

Now $$\vec{\tau} = \vec{r} \times \vec{F}$$

$$= (x\hat{i} + y\hat{j} + z\hat{k}) \times (F_x\hat{i} + F_y\hat{j} + F_z\hat{k})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix}$$

or $$\tau_x\hat{i} + \tau_y\hat{j} + \tau_z\hat{k} = \hat{i}(yF_z - zF_y) + \hat{j}(zF_x - xF_z) + \hat{k}(xF_y - yF_x)$$

Comparing the coefficients of $\hat{i}$, $\hat{j}$ and $\hat{k}$ on the two sides of the above equation, we get the rectangular components of torque $\vec{\tau}$ as follows :

$$\tau_x = xF_z - zF_y\ ; \quad \tau_y = zF_x - xF_z\ ; \quad \tau_z = xF_y - yF_x.$$

7.15 ANGULAR MOMENTUM

23. *Define the term angular momentum. Give its units and dimensions.*

Angular momentum. In linear motion, the linear momentum of a body gives a measure of its translatory motion. Analogous to it in rotational motion, the angular momentum gives a measure of the turning motion of the body.

The angular momentum of a particle rotating about an axis is defined as the moment of the linear momentum of the particle about that axis. It is measured as the product of linear momentum and the perpendicular distance of its line of action from the axis of rotation.

Angular momentum
= Linear momentum × its perpendicular distance from the axis of rotation.

$$L = pd$$

Dimensions of angular momentum

$$= L \times MLT^{-1} = [ML^2T^{-1}]$$

SI unit of angular momentum is kg m^2s^{-1}.

CGS unit of angular momentum is g cm^2s^{-1}.

24. *Explain how angular momentum can be expressed as the vector product of two vectors. How is its direction determined ?*

Angular momentum of a particle. Consider a particle P of mass m rotating about an axis through O in the X-Y plane. Suppose the particle has linear momentum $\vec{p}$ which makes angle θ with its position vector $\vec{OP} = \vec{r}$. The angular momentum $\vec{L}$ of the particle about the origin O is defined as the vector product of the vectors $\vec{r}$ and $\vec{p}$. Thus

$$\vec{L} = \vec{r} \times \vec{p}$$

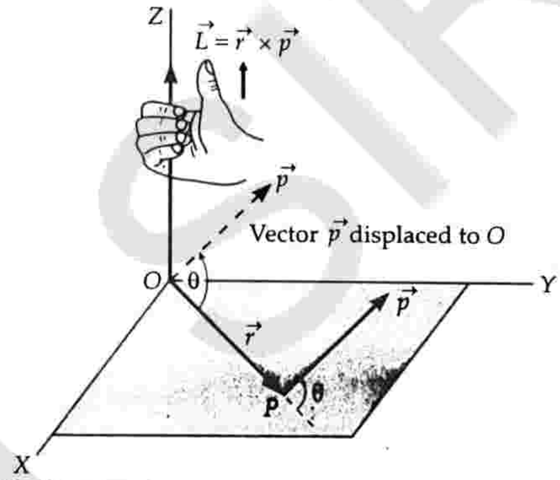


Fig. 7.29 Angular momentum as the vector product of $\vec{r}$ and $\vec{p}$.

Angular momentum is a vector quantity. Its magnitude is given by

$$L = rp \sin\theta$$

The direction of angular momentum $\vec{L}$ is perpendicular to the plane of vectors $\vec{r}$ and $\vec{p}$ in the sense given by *right hand rule*. Thus in the present case, $\vec{L}$ points in Z-direction.

Special cases. (*i*) If $\theta = 0°$ or $180°$, $\sin\theta = 0$

$\therefore \quad L = rp \times 0 = 0$

Hence *the angular momentum is zero if the line of action of linear momentum passes through the point of rotation.*

(*ii*) If $\theta = 90°$, $\sin 90° = 1$

$\therefore \quad L = rp \times 1 = rp =$ maximum

Hence *the angular momentum is maximum and is equal to rp or mvr, if the line of action of the linear momentum is perpendicular to the position vector.*

25. *Show that the angular momentum of a particle is the product of its linear momentum and the moment arm. Also show that the angular momentum is produced only by the angular component of linear momentum.*

Physical meaning of angular momentum. Consider a particle P of mass m whose position vector relative to the origin O is $\vec{r}$. Suppose the momentum vector $\vec{p}$ of the particle makes angle θ with the position vector $\vec{r}$, as shown in Fig. 7.30.

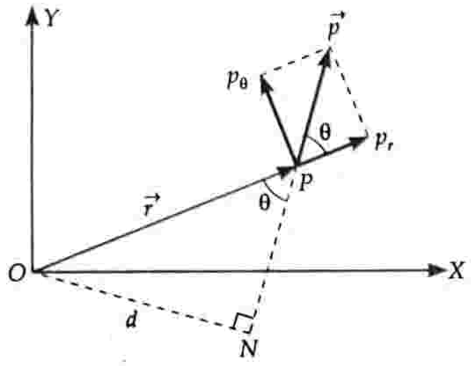


Fig. 7.30 Physical meaning of angular momentum.

Draw ON perpendicular to the line of action of linear momentum $\vec{p}$. From right angled ΔONP, we get

$$\frac{ON}{OP} = \frac{d}{r} = \sin\theta \quad \text{or} \quad d = r\sin\theta.$$

This is the perpendicular distance of the line of action of linear momentum from the point of rotation O and is called *moment arm* of the momentum.

The magnitude of the angular momentum about the point O is

$$L = rp\sin\theta = p(r\sin\theta) = pd$$

i.e., Angular momentum = Linear momentum × moment arm

This is the physical meaning of angular momentum. According to it, *angular momentum is the moment of linear momentum and is a measure of the turning motion of the object*. In contrast to it, we know that

Torque = Force × moment arm

Thus *torque is the moment of force and is a measure of the turning effect of force.*

Moreover, as shown in Fig. 7.30, the momentum vector $\vec{p}$ can be resolved into *two* rectangular components :

1. *Radial component*, p_r along the direction of position vector $\vec{r}$.
2. *Angular or tangential component*, p_θ perpendicular to $\vec{r}$.

Clearly, $p_r = p\cos\theta$ and $p_\theta = p\sin\theta$

$\therefore \quad L = (p\sin\theta)\, r = p_\theta\, r$

or Angular momentum = Angular component of linear momentum × its distance from the axis of rotation.

Hence *only the angular component and not the radial component of the linear momentum contributes towards the angular momentum.*

26. *Express angular momentum in terms of the rectangular components of linear momentum and position vectors.*

Angular momentum in terms of rectangular components. For motion in three dimensions, the position vector $\vec{r}$ and linear momentum vector $\vec{p}$ can be written in terms of their rectangular components as follows :

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

and $$\vec{p} = p_x\hat{i} + p_y\hat{j} + p_z\hat{k}$$

But angular momentum is the cross product of $\vec{r}$ and $\vec{p}$, so

$$\vec{L} = \vec{r} \times \vec{p}$$

$$= (x\hat{i} + y\hat{j} + z\hat{k}) \times (p_x\hat{i} + p_y\hat{j} + p_z\hat{k})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix}$$

or $$L_x\hat{i} + L_y\hat{j} + L_z\hat{k} = \hat{i}(yp_z - zp_y) + \hat{j}(zp_x - xp_z) + \hat{k}(xp_y - yp_x)$$

Comparing the coefficients of $\hat{i}$, $\hat{j}$ and $\hat{k}$ on the two sides of the above equation, we obtain the rectangular components of vector $\vec{L}$ as follows :

$$L_x = yp_z - zp_y \; ; \; L_y = zp_x - xp_z \text{ and } L_z = xp_y - yp_x .$$

7.16 RELATION BETWEEN TORQUE AND ANGULAR MOMENTUM

27. *Deduce the relation between torque and angular momentum.*

Relation between torque and angular momentum. We know that

Torque, $\vec{\tau} = \vec{r} \times \vec{F}$

Angular momentum, $\vec{L} = \vec{r} \times \vec{p}$

Differentiating both sides w.r.t. time t, we get

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{p}) = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt}$$

$$= \vec{v} \times \vec{p} + \vec{r} \times \vec{F} \qquad \left[\because \frac{d\vec{p}}{dt} = \vec{F}\right]$$

$$= 0 + \vec{\tau} \qquad [\because \vec{v} \times \vec{p} = \vec{v} \times m\vec{v} = 0]$$

$$\therefore \qquad \vec{\tau} = \frac{d\vec{L}}{dt}$$

Thus *the torque acting on a particle is equal to its rate of change of angular momentum*. This equation is the rotational analogue of Newton's second law of linear motion which states that the rate of change of linear momentum of a particle is equal to the force acting on it *i.e.*,

$$\vec{F} = \frac{d\vec{p}}{dt}.$$

7.17 GEOMETRICAL MEANING OF ANGULAR MOMENTUM

28. *Prove that the angular momentum of a particle is equal to twice the product of its mass and areal velocity. How does it lead to Kepler's second law of planetary motion ?*

Geometrical meaning of angular momentum. Consider a particle of mass m rotating in the X-Y plane about the origin O. Let $\vec{r}$ and $(\vec{r} + \Delta\vec{r})$ be the position vectors of the particle at instants t and $(t + \Delta t)$ respectively, as shown in Fig. 7.31. The displacement of the particle in small time Δt is

$$\vec{PQ} = (\vec{r} + \Delta\vec{r}) - \vec{r} = \Delta\vec{r}$$

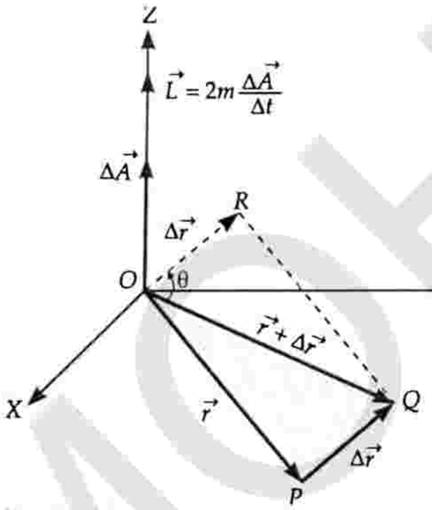


Fig. 7.31 Geometrical meaning of angular momentum.

If $\vec{v}$ is the velocity of the particle at point P, then the small displacement covered in time Δt may be expressed as

$$\Delta\vec{r} = \vec{v}\,\Delta t$$

Complete the parallelogram $OPQR$. Then

$$\vec{OR} = \vec{PQ} = \Delta\vec{r}$$

Area of the parallelogram $OPQR = \vec{r} \times \Delta\vec{r}$

$\therefore$ Area of $\Delta OPQ = \frac{1}{2}(\vec{r} \times \Delta\vec{r})$

The shaded area of ΔOPQ represents the area swept by the position vector in time Δt. By right hand rule, its direction is along Z-axis. If this area is represented by $\Delta\vec{A}$, then

$$\Delta\vec{A} = \frac{1}{2}(\vec{r} \times \Delta\vec{r}) = \frac{1}{2}(\vec{r} \times \vec{v}\,\Delta t)$$

If $\vec{p}$ is the linear momentum of the particle, then

$$\vec{p} = m\vec{v} \qquad \text{or} \qquad \vec{v} = \frac{\vec{p}}{m}$$

$$\therefore \qquad \Delta\vec{A} = \frac{1}{2}\vec{r} \times \frac{\vec{p}}{m}\Delta t$$

or $$\frac{\Delta\vec{A}}{\Delta t} = \frac{1}{2m}(\vec{r} \times \vec{p})$$

But $\vec{r} \times \vec{p} = \vec{L}$, the angular momentum of the particle about Z-axis, so we have

$$\frac{\Delta\vec{A}}{\Delta t} = \frac{\vec{L}}{2m} = \frac{1}{2} \times \text{Angular momentum per unit mass}$$

or $$\vec{L} = 2m\frac{\Delta\vec{A}}{\Delta t}$$

The quantity $\Delta\vec{A}/\Delta t$ is the *area swept out by the position vector per unit time and is called the areal velocity of the particle*. Thus

Angular momentum = 2 × Mass × areal velocity

This is the geometrical meaning of angular momentum. So geometrically, *the angular momentum of a particle is equal to twice the product of its mass and areal velocity*. Equivalently, we can say that the *areal velocity of a particle is just half its angular momentum per unit mass*.

Kepler's second law of planetary motion. A planet revolves around the sun under the influence of gravitational force which acts towards the sun *i.e.*, the force is purely radial and angular component F_θ of the force is zero.

As torque, $\tau = r\,F_\theta$,

therefore $\quad \tau = 0 \quad$ or $\quad \frac{dL}{dt} = 0$

or $\quad L = \text{constant}$

or $\quad 2m.\frac{\Delta A}{\Delta t} = \text{constant} \quad [\because L = 2m \times \text{areal velocity}]$

or $\quad \frac{\Delta A}{\Delta t} = \text{constant} \quad [\because m \text{ of planet is constant}]$

This means that the areal velocity of a planet is constant. This is *Kepler's second law of planetary motion which states that the line joining the planet to the sun sweeps out equal areas in equal intervals of time.*

7.18 TORQUE AND ANGULAR MOMENTUM FOR A SYSTEM OF PARTICLES

29. *Prove that the rate of change of total angular momentum of a system of particles about a reference point is equal to the total torque acting on the system.*

Torque and angular momentum for a system of particles. Consider a system of n particles. Let $\vec{L}_1, \vec{L}_2, \vec{L}_3, ...$ be the angular momenta of the various particles about the origin O of a reference frame. The total angular momentum of the system about the point O is given by the vector sum of angular momenta of all the individual particles. Thus

$$\vec{L} = \vec{L}_1 + \vec{L}_2 + ... + \vec{L}_n$$
$$= \vec{r}_1 \times \vec{p}_1 + \vec{r}_2 \times \vec{p}_2 + ... + \vec{r}_n \times \vec{p}_n$$

or $$\vec{L} = \sum_{i=1}^{n} \vec{L}_i = \sum_{i=1}^{n} \vec{r}_i \times \vec{p}_i$$

Similarly, the total torque acting on the system is equal to the vector sum of the torques of all the particles about the origin O. Thus

$$\vec{\tau}^{\,tot} = \vec{\tau}_1 + \vec{\tau}_2 + ... + \vec{\tau}_n$$
$$= \vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2 + ... + \vec{r}_n \times \vec{F}_n$$

or $$\vec{\tau}^{\,tot} = \sum_{i=1}^{n} \vec{\tau}_i = \sum_{i=1}^{n} \vec{r}_i \times \vec{F}_i$$

The total torque acting on the system is due sources : (*i*) Torques exerted on the particles system by mutual *internal forces* between the r (*ii*) Torques exerted on the individual partic system by the *external forces.*

According to Newton's third law, torque on the system due to internal f because the forces between any two partic and opposite and directed along the line two particles. Hence the total torque is due forces only. So we have

$$\vec{\tau}^{\,tot} = \vec{\tau}^{\,ext} = \sum_{i=1}^{n} \vec{\tau}_i^{\,ext}$$

This is in accordance with the common experience – bodies do not start spinning on their own without external forces acting on them. Hence if the angular momentum of a system changes with time, this change can be due to the torques produced by external forces only. So we can write

$$\vec{\tau}^{\,ext} = \frac{d\vec{L}}{dt}$$

Thus *the rate of change of total angular momentum of a system of particles about a fixed point is equal to the total external torque acting on the system about that point.*

Examples based on Torque, Power of a Torque, Work done by a Torque and Angular Momentum

FORMULAE USED

1. Torque = Force × its perpendicular distance from the axis of rotation
   or $\tau = Fd$
2. Torque, $\tau = rF \sin\theta$ or $\vec{\tau} = \vec{r} \times \vec{F}$
3. Power of a torque = Torque × Angular velocity
   or $P = \tau\omega$
4. Work done by a torque
   = Torque × Angular displacement
   or $W = \tau\theta$
5. Angular momentum = Linear momentum × its perpendicular distance from the axis of rotation
   or $L = pd$
6. Angular momentum, $L = rp \sin\theta$ or $\vec{L} = \vec{r} \times \vec{p}$
7. For a particle of mass m moving with uniform speed v along a circle of radius r, $L = mvr.$
8. Torque = Rate of change of angular momentum

centre of the rod and the weight W_1 is suspended at point P. Let R_1 and R_2 be the reactions of the support at the edges.

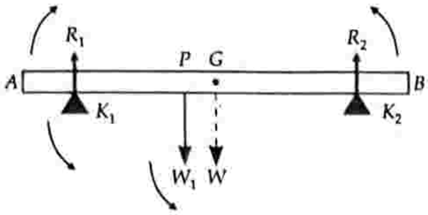


Fig. 7.32

Here $AB = 70$ cm, $AG = 35$ cm, $AP = 30$ cm, $PG = 5$ cm, $AK_1 = BK_2 = 10$ cm, $K_1G = K_2G = 25$ cm

W = Weight of rod = 4.00 kg wt = $4g$ N

W_1 = Weight of suspended mass

= 6.00 kg wt = $6\,g$ N

Clearly, the reactions R_1 and R_2 act vertically upward. For translational equilibrium of the rod,

$$R_1 + R_2 = W_1 + W = 4g + 6g = 10g$$
$$= 10 \times 9.8 = 98 \text{ N} \quad ...(1)$$

For rotational equilibrium of the rod,

Clockwise moment = Anticlockwise moment

$R_1 \times K_1G = W_1 \times$ P'

or $R_1 \times 0.2$'

or R_1 – P

or

On '

K

3. Reaction force F_2 of the floor. This force can be resolved into two components : the normal reaction N and the force of friction f. This friction prevents the ladder sliding away from the wall and hence acts towards the wall.

For translational equilibrium, balancing the forces in the vertical direction,

$$N = W$$

Balancing the forces in the horizontal direction,

$$f = F_1$$

For rotational equilibrium, we consider the moments about the point A.

Clockwise moment = Anticlockwise moment

$$F_1 \times BC = W \times AE$$

or $$F_1 \times 2\sqrt{2} = W \times \frac{1}{2}$$

But $$W = 20g = 20 \times 9.8 = 196.0 \text{ N}$$

$\therefore$ $$N = W = 196.0 \text{ N}$$

$$F_1 = \frac{W}{4\sqrt{2}} = \frac{196.0}{4\sqrt{2}} = 34.6 \text{ N}$$

$$f = F_1 = 34.6 \text{ N}$$

and $$F_2 = \sqrt{f^2 + N^2} = \sqrt{34.6^2 + 196^2} = 199.0 \text{ N}$$

If the force F_2 makes angle α with the horizontal,

$$\tan\alpha = \frac{N}{f} = 4\sqrt{2} = 5.6568$$

$\alpha \simeq 80°$

particle of mass m is released from point P X-axis from origin O and falls vertically as shown in Fig. 7.34.

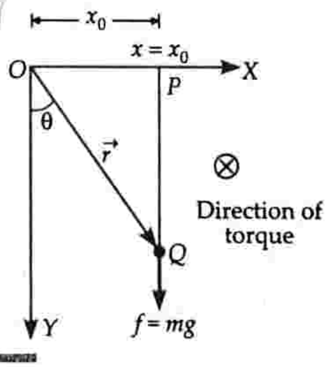


Fig. 7.34

(*i*) *Find the torque τ acting on the particle at a time t when it is at point Q with respect to O.*

(*ii*) *Find the angular momentum L of the particle about O at this time t.*

(*iii*) *Show that* $\tau = \frac{dL}{dt}$ *in this example.* **[NCERT]**

Solution. (*i*) The force of gravity, $F = mg$ produces the torque τ. Let $\vec{r}$ be the position vector of Q. Then the magnitude of the torque is given by

$$\tau = rF \sin\theta$$

$$= r \times mg \times \frac{x_0}{r} = mgx_0. \qquad \left[\because \sin\theta = \frac{x_0}{r}\right]$$

The direction of the torque is directed into the plane of paper and perpendicular to it, as shown by ⊗.

(*ii*) The magnitude of the angular momentum is

$$L = rp \sin\theta = rmv \sin\theta$$

But the velocity v at point Q is given by

$$v = u + at = 0 + gt = gt$$

$$\therefore \quad L = rmgt \cdot \frac{x_0}{r} = mgx_0\, t$$

The direction of angular momentum is the same as that of torque.

(*iii*) Now $L = mgx_0 t$

Differentiating both sides with respect to t, we get

$$\frac{dL}{dt} = \frac{d}{dt}(mgx_0 t) = mgx_0 = \tau$$

Hence the relation $\tau = \frac{dL}{dt}$ holds in this example.

EXAMPLE 20. *An electron of mass 9×10^{-31} kg revolves in a circle of radius 0.53 Å around the nucleus of hydrogen with a velocity of 2.2×10^6 ms^{-1}. Show that its angular momentum is equal to $h/2\pi$, where h is Planck's constant of value 6.6×10^{-34} Js.* **[NCERT]**

Solution. Here $m = 9 \times 10^{-31}$ kg,

$$r = 0.53 \text{ Å} = 0.53 \times 10^{-10} \text{ m},$$

$$v = 2.2 \times 10^6 \text{ ms}^{-1}, h = 6.6 \times 10^{-34} \text{ Js}$$

Angular momentum,

$$L = mvr$$

$$= 9 \times 10^{-31} \times 2.2 \times 10^6 \times 0.53 \times 10^{-10}$$

$$= 1.0494 \times 10^{-34} \text{ Js}$$

Also, $$\frac{h}{2\pi} = \frac{6.6 \times 10^{-34}}{2 \times 3.142} = 1.0503 \times 10^{-34} \text{ Js}$$

Hence $$L \simeq \frac{h}{2\pi}.$$

EXAMPLE 21. *Show that the angular momentum of a satellite of mass M_s revolving around the earth having mass M_e in an orbit of radius r is*

$$L = \sqrt{GM_e M_s^2 r}.$$

Solution. Let the satellite revolve around the earth with orbital speed v. Then

Centripetal force on the satellite = Gravitational force between the earth and the satellite

or $$\frac{M_s v^2}{r} = G\frac{M_e M_s}{r^2}$$

or $$v^2 = \frac{GM_e}{r} \quad \text{or} \quad v = \sqrt{\frac{GM_e}{r}}$$

As satellite is considered a point mass, its angular momentum is

$$L = M_s vr = M_s \sqrt{\frac{GM_e}{r}} \cdot r = \sqrt{GM_e M_s^2 r}.$$

✖ PROBLEMS FOR PRACTICE

1. Determine the angular momentum of a car of mass 1500 kg moving in a circular track of radius 50 m with a speed of 40 ms^{-1}. (**Ans.** 3×10^6 kgm^2 s^{-1})
2. Mass of an electron is 9.0×10^{-31} kg. It revolves around the nucleus of an atom in a circular orbit of radius 4.0 Å with a speed of 6.0×10^6 ms^{-1}. Calculate the angular momentum of the electron. (**Ans.** 2.16×10^{-33} kgm^2 s^{-1})

7.19 EQUILIBRIUM OF RIGID BODIES

30. *Define a rigid body. Name two kinds of motion which a rigid body can execute. What is meant by the term equilibrium ? For the equilibrium of bodies, two conditions need to be satisfied. State them.*

Equilibrium of rigid bodies. A rigid body is one for which the distances between different particles do not change, even though they move. Under the influence of an external force, a rigid body can execute two kinds of motion : (*i*) *translational motion* in which all particles move with the same velocity and (*ii*) *rotational motion* about an axis. The word 'equilibrium' is related to the Latin word 'Libra' which means equilibrium.

A rigid body is said to be in equilibrium if both the linear momentum and angular momentum of the rigid body remain constant with time. Hence for a body in equilibrium, the linear acceleration of its centre of mass would be zero and also the angular acceleration of the rigid body about any axis would be zero.

A body under the action of several forces will be in equilibrium, if it possesses the following two equilibria simultaneously :

(*i*) **Translational equilibrium.** *The resultant of all the external forces acting on the body must be zero,* otherwise they would produce linear acceleration. Hence for translation equilibrium,

$$\Sigma \vec{F}_{ext} = 0$$

or $$\Sigma F_x = 0, \quad \Sigma F_y = 0 \quad \text{and} \quad \Sigma F_z = 0$$

Applying Newton's second law,

$$\Sigma \vec{F}_{ext} = M\vec{a}_{CM} = M\frac{d\vec{v}_{CM}}{dt} = 0$$

or $$\frac{d\vec{v}_{CM}}{dt} = 0 \quad \text{or} \quad \vec{v}_{CM} = \text{constant.}$$

This implies that a body in translational equilibrium will be either at rest ($v = 0$) or in uniform motion (v = constant). If the body is at rest, it is said to be in ***static equilibrium***. It the body is in uniform motion along a straight path, it is in ***dynamic equilibrium***.

(*ii*) **Rotational equilibrium.** *The resultant of torques due to all the forces acting on the body about any point must be zero,* otherwise they would produce angular acceleration. Hence for rotational equilibrium,

$$\Sigma \vec{\tau}_{ext} = \Sigma \vec{r}_i \times \vec{F}_i^{ext} = 0$$

For rotational equilibrium, the choice of the point of rotation (or fixed point) is unimportant. If the total torque is zero about any point, then it will be zero about any other point when the body is in equilibrium.

31. *Obtain an expression for the work done on a rigid body executing both translational and rotational motions. Hence deduce the condition for the equilibrium of the rigid body.*

Expression for work done in combined rotational and translational motions. A rigid body can have only two kinds of motion : translational and rotational. In translational motion, all the particles of the rigid body move with the same velocity v_0, without rotating.

In small time interval Δt, each particle covers a displacement given by

$$\Delta\vec{s} = \vec{v}_0\,\Delta t$$

In rotational motion, every particle of the rigid body rotates about the axis of rotation with the same angular velocity $\vec{\omega}$. As shown in Fig. 7.35, we choose the origin O on the axis of rotation. Clearly, the particle velocity $\vec{v}$ has magnitude,

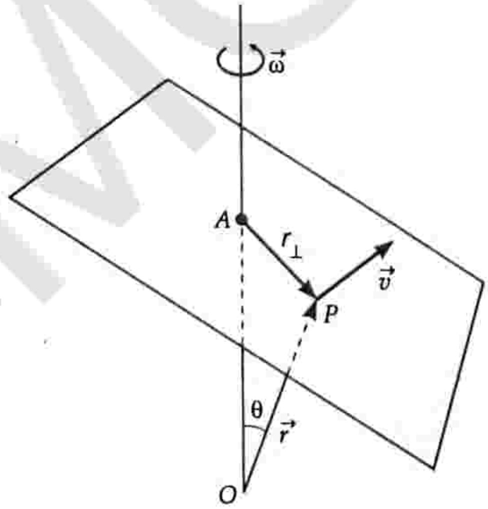


Fig. 7.35 Rotational motion of any particle of a rigid body.

$$v = r_\perp\,\omega = \omega r \sin\theta$$

where $r_\perp = r\sin\theta$ is the perpendicular distance of the particle from the axis of rotation and θ is the angle between $\vec{r}$ and the axis of rotation. In vector notation,

$$\vec{v} = \vec{\omega} \times \vec{r}$$

The direction of $\vec{v}$ is perpendicular to both $\vec{\omega}$ and $\vec{r}$. The direction of $\vec{\omega}$ is determined by right hand rule. If we curl the fingers of our right in the direction of rotation of the particle, then the extended thumb gives the direction of the angular velocity vector $\vec{\omega}$.

Displacement covered by the rotating particle in small time interval Δt

$$= \vec{v}\,\Delta t = \vec{\omega} \times \vec{r}\,\Delta t = (\vec{\omega}\,\Delta t) \times \vec{r} = \Delta\vec{\phi} \times \vec{r}$$

where $\Delta\vec{\phi} = \vec{\omega}\,\Delta t$ = the small angular displacement.

So when a rigid body rotates with angular velocity $\vec{\omega}$ and translates with velocity $\vec{v}_0$, the displacement of any point of $\vec{r}$ of the rigid body is given by

$$\Delta\vec{r} = \vec{v}_0\Delta t + \vec{\omega}\Delta t \times \vec{r} = \Delta\vec{s} + \Delta\vec{\phi} \times \vec{r}$$

The work done by an external force $\vec{F}$ acting on point $\vec{r}$ is given by

$$\Delta W = \vec{F}\,.\,\Delta\vec{r} = \vec{F}\,.\,(\Delta\vec{s} + \Delta\vec{\phi} \times \vec{r})$$

$$= \vec{F}\,.\,\Delta\vec{s} + \vec{F}\,.\,(\Delta\vec{\phi} \times \vec{r})$$

As the scalar triple product is cyclic, so

$$\vec{F}\,.\,(\Delta\vec{\phi} \times \vec{r}) = \vec{r}\,.\,(\vec{F} \times \Delta\vec{\phi}) = \Delta\vec{\phi}\,.\,(\vec{r} \times \vec{F})$$

$$= \Delta\vec{\phi}\,.\,\vec{\tau} = \vec{\tau}\,.\,\Delta\vec{\phi}$$

where $\vec{\tau} = \vec{r} \times \vec{F}$, is the torque acting on the particle.

$$\therefore \quad \Delta W = \vec{F}\,.\,\Delta\vec{s} + \vec{\tau}\,.\,\Delta\vec{\phi}$$

When a number of forces $\vec{F}_i$ act on different points $\vec{r}_i$ of the rigid body, the total work done on the rigid body will be

$$W = (\Sigma\vec{F}_i)\,.\,\Delta\vec{s} + (\Sigma\vec{\tau}_i)\,.\,\Delta\vec{\phi}$$

For the rigid body to be in equilibrium, the work done in this displacement plus rotation should be zero for all choices of $\Delta \vec{s}$ and $\Delta \vec{\phi}$. We therefore obtain the condition :

$$\Sigma \vec{F}_i = 0 \quad \text{and} \quad \Sigma \vec{\tau}_i = 0$$

Hence *for a rigid body in equilibrium, the sum of the forces acting on it must be zero and the sum of the torques acting on it must be zero.*

32. *Distinguish between stable, unstable and neutral equilibria of a body.*

(*i*) **Stable equilibrium.** *A body is said to be in stable equilibrium if it tends to regain its equilibrium position after being slightly displaced and released.* In stable equilibrium, a body has *minimum potential energy* and its centre of mass goes higher when it is slightly displaced.

(*ii*) **Unstable equilibrium.** *A body is said to be in unstable equilibrium if it gets further displaced from its equilibrium position after being slightly displaced and released.* In unstable equilibrium, a body possesses *maximum potential energy* and its centre of mass goes lower on being slightly displaced.

(*iii*) **Neutral equilibrium.** *If a body stays in equilibrium position even after being slightly displaced and released, it is said to be in neutral equilibrium.* When a body is slightly displaced from its position of neutral equilibrium, its centre of mass is neither raised nor lowered and its potential energy remains constant.

7.20 MOMENT OF INERTIA AND ITS PHYSICAL SIGNIFICANCE

33. *Define moment of inertia of a body. Give its units and dimensions. Explain the physical significance of moment of inertia.*

Moment of inertia. According to Newton's first law of motion, every body continues in its state of rest or of uniform linear motion, unless an external force acts on it to change that state. This inability of a body to change by itself its state of rest or of linear uniform motion is called *inertia*. Similarly, a body rotating about a given axis tends to maintain its state of uniform rotation, unless an external torque is applied on it to change that state. This property of a body by virtue of which it opposes the torque tending to change its state of rest or of uniform rotation about an axis is called *rotational inertia or moment of inertia*.

The moment of inertia of a rigid body about a fixed axis is defined as the sum of the products of the masses of the particles constituting the body and the squares of their respective distances from the axis of rotation.

Consider a rigid body rotating with uniform angular velocity ω about a vertical axis through *O*, as shown in Fig. 7.36. Suppose the body consists of *n* particles of masses $m_1, m_2, m_3, \ldots, m_n$ situated at

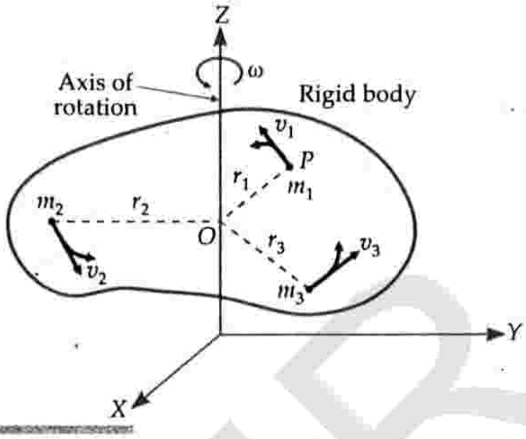


Fig. 7.36 M.I. and rotational K.E. of a rigid body.

distances $r_1, r_2, r_3, \ldots, r_n$ respectively from the axis of rotation. The moment of inertia of the body about the axis *OZ* is given by

$$I = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \ldots + m_n r_n^2$$

or

$$I = \sum_{i=1}^{n} m_i r_i^2$$

The dimensional formula of moment of inertia is $[ML^2T^0]$. The SI unit of moment of inertia is kg m^2 and its CGS unit is g cm^2.

Physical significance of moment of inertia. The mass of a body resists a change in its state of linear motion, it is a measure of its inertia in linear motion. Similarly, the moment of inertia of a body about an axis of rotation resists a change in its rotational motion. The greater the moment of inertia of a body, the greater is the torque required to change its state of rotation. Thus moment of inertia of a body can be regarded as the measure of rotational inertia of the body. The moment of inertia of a body plays the same role in the rotational motion as the mass plays in linear motion. That is why moment of inertia is called the rotational analogue of mass in linear motion.

34. *State the factors on which the moment of inertia of a body depends.*

Factors on which the moment of inertia depends. The moment of inertia of a body is the measure of the manner in which its different parts are distributed at different distances from the axis of rotation. Unlike mass, it is not a fixed quantity as it depends on the position and orientation of the axis of rotation with respect to the body as a whole.

The moment of inertia of a body depends on

(*i*) Mass of the body
(*ii*) Size and shape of the body.
(*iii*) Distribution of mass about the axis of rotation.
(*iv*) Position and orientation of the axis of rotation w.r.t. the body.

35. *Mention some practical applications which make use of the property of moment of inertia.*

Practical applications of moment of inertia. (*i*) A heavy wheel, called *flywheel*, is attached to the shaft of steam engine, automobile engine, etc. Because of its large moment of inertia, the flywheel opposes the sudden increase or decrease of the speed of the vehicle. It allows a gradual change in the speed and prevents jerky motions and hence ensures smooth ride for the passengers.

(*ii*) In a bicycle, bullock-cart, etc., the moment of inertia is increased by concentrating most of the mass at the rim of the wheel and connecting the rim to the axle through the spokes. Even after we stop paddling, the wheels of a bicycle continue to rotate for some time due to their large moment of inertia.

7.21 RELATION BETWEEN ROTATIONAL K.E. AND MOMENT OF INERTIA

36. *A body is rotating with uniform angular velocity ω about an axis. Establish the formula for its kinetic energy of rotation. Define moment of inertia of the body with respect to the axis of rotation on this basis.*

Relation between rotational kinetic energy and moment of inertia. As shown in Fig. 7.36, consider a rigid body rotating about an axis OZ with uniform angular velocity ω. The body may be assumed to consist of n particles of masses $m_1, m_2, m_3, \ldots, m_n$; situated at distances $r_1, r_2, r_3, \ldots, r_n$ from the axis of rotation. As the angular velocity ω of all the n particles is same, so their linear velocities are

$$v_1 = r_1\,\omega,\ v_2 = r_2\,\omega,\ v_3 = r_3\,\omega, \ldots,\ v_n = r_n\,\omega$$

Hence the total kinetic energy of rotation of the body about the axis OZ is

Rotational K.E.

$$= \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 + \frac{1}{2} m_3 v_3^2 + \ldots + \frac{1}{2} m_n v_n^2$$

$$= \frac{1}{2} m_1 r_1^2 \omega^2 + \frac{1}{2} m_2 r_2^2 \omega^2 + \frac{1}{2} m_3 r_3^2 \omega^2 + \ldots + \frac{1}{2} m_n r_n^2 \omega^2$$

$$= \frac{1}{2}(m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \ldots + m_n r_n^2)\,\omega^2$$

$$= \frac{1}{2}(\Sigma\, mr^2)\,\omega^2$$

But $\Sigma\, mr^2 = I$, the moment of inertia of the body about the axis of rotation.

$\therefore$ Rotational K.E. $= \frac{1}{2} I \omega^2$

When $\omega = 1$, rotational K.E. $= \frac{1}{2} I$

or $\quad I = 2 \times$ Rotational K.E.

Hence the moment of inertia of a rigid body about an axis of rotation is numerically equal to twice the rotational kinetic energy of the body when it is rotating with unit angular velocity about that axis.

7.22 RADIUS OF GYRATION

37. *Define radius of gyration of a body rotating about an axis. Derive an expression for it. On what factors does it depend ?*

Radius of gyration. For any body capable of rotation about a given axis, it is possible to find a radial distance from the axis where, if whole mass of the body is concentrated, its moment of inertia will remain unchanged. This radial distance is called ***radius of gyration*** and is denoted by k.

The radius of gyration of a body about its axis of rotation may be defined as the distance from the axis of rotation at which, if the whole mass of the body were concentrated, its moment of inertia about the given axis would be the same as with the actual distribution of mass.

The radius of gyration k is a geometrical property of the body and the axis of rotation. It gives a measure of the manner in which the mass of a rotating body is distributed with respect to the axis of rotation.

k has the dimensions of length L and is measured in metre or cm.

Expression for k. Suppose a rigid body consists of n particles of mass m each, situated at distances $r_1, r_2, r_3, \ldots, r_n$ from the axis of rotation AB.

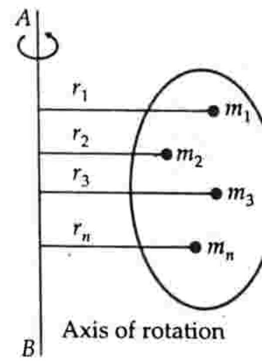


Fig. 7.37 Radius of gyration.

The moment of inertia of the body about the axis AB is

$$I = mr_1^2 + mr_2^2 + mr_3^2 + \ldots + mr_n^2$$

$$= m(r_1^2 + r_2^2 + r_3^2 + \ldots + r_n^2)$$

$$= m \times n\,\frac{(r_1^2 + r_2^2 + r_3^2 + \ldots + r_n^2)}{n}$$

or

$$I = M\,\frac{(r_1^2 + r_2^2 + r_3^2 + \ldots + r_n^2)}{n}$$

where $M = m \times n =$ total mass of the body.

If k is the radius of gyration about the axis AB, then

$$I = Mk^2$$

$$\therefore \quad Mk^2 = M\left(\frac{r_1^2 + r_2^2 + r_3^2 + \ldots + r_n^2}{n}\right)$$

or
$$k = \sqrt{\frac{r_1^2 + r_2^2 + r_3^2 + \ldots + r_n^2}{n}}$$

= Root mean square distance

Hence the radius of gyration of a body about an axis of rotation may also be defined as the root mean square distance of its particles from the axis of rotation.

Factors on which radius of gyration of a body depends :

(*i*) Position and direction of the axis of rotation.

(*ii*) Distribution of mass about the axis of rotation.

7.23 THEOREMS OF PARALLEL AND PERPENDICULAR AXES

38. *State and prove the theorem of perpendicular axes.*

Theorem of perpendicular axes. *The moment of inertia of a plane lamina about an axis perpendicular to its plane is equal to the sum of the moments of inertia of the lamina about any two mutually perpendicular axes in its own plane and intersecting each other at the point where the perpendicular axis passes through the lamina.*

Proof. Consider a plane lamina lying in the *XOY* plane. It can be assumed to be made up of large number of particles. Consider one such particle of mass *m* situated at point *P(x, y)*. Clearly, the distances of the particle from X-, Y- and Z-axes are *y*, *x* and *r* respectively such that

$$r^2 = y^2 + x^2$$

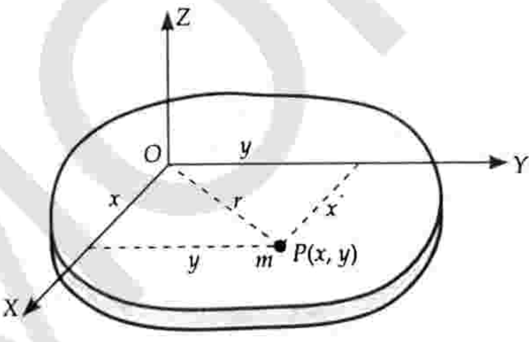


Fig. 7.38 Theorem of perpendicular axes.

Moment inertia of the particle about *X*-axis

$$= my^2$$

$\therefore$ Moment of inertia of whole lamina about *X*-axis is

$$I_x = \Sigma\, my^2$$

Moment of inertia of whole lamina about *Y*-axis is

$$I_y = \Sigma\, mx^2$$

Moment of inertia of whole lamina about *Z*-axis is

$$I_z = \Sigma\, mr^2 = \Sigma\, m(y^2 + x^2)$$
$$= \Sigma\, my^2 + \Sigma\, mx^2$$

or
$$I_z = I_x + I_y$$

This proves the theorem of perpendicular axes.

39. *State and prove the theorem of parallel axes.*

Theorem of parallel axes. *The moment of inertia of a body about any axis is equal to its moment of inertia about a parallel axis through its centre of mass plus the product of the mass of the body and the square of the perpendicular distance between the two parallel axes.*

Proof. Let *I* be the moment of inertia of a body of mass *M* about an axis *PQ*. Let *RS* be a parallel axis passing through the centre of mass *C* of the body and at distance *d* from *PQ*. Let I_{CM} be the moment of inertia of the body about the axis *RS*.

Consider a particle *P* of mass *m* at distance *x* from *RS* and so at distance $(x + d)$ from *PQ*.

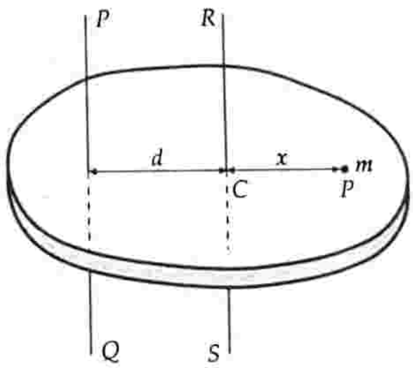


Fig. 7.39 Theorem of parallel axes.

Moment of inertia of the particle about axis *PQ*

$$= m(x + d)^2$$

$\therefore$ Moment of inertia of the whole body about the axis *PQ* is

$$I = \Sigma\, m(x + d)^2 = \Sigma\, m(x^2 + d^2 + 2xd)$$
$$= \Sigma\, mx^2 + \Sigma\, md^2 + \Sigma\, 2\, mxd$$

Now $\Sigma\, mx^2 = I_{CM}$

$$\Sigma\, md^2 = (\Sigma\, m)\, d^2 = Md^2$$
$$\Sigma\, 2\, mxd = 2d\, \Sigma\, mx = 2d \times 0 = 0$$

This is because a body can balance itself about its centre of mass, so the algebraic sum of moments ($\Sigma\, mx$) of masses of all its particles about the axis *RS* is zero.

Hence $\quad I = I_{CM} + Md^2$

This proves the theorem of parallel axes.

7.24 MOMENT OF INERTIA OF A THIN CIRCULAR RING

40. *Derive an expression for the moment of inertia of a thin uniform circular ring about (a) an axis through its centre and perpendicular to its plane (b) its diameter (c) a tangent in its plane (d) a tangent perpendicular to its plane.*

(*a*) **M.I. of a ring about an axis through its centre and perpendicular to its plane.** Consider a thin uniform circular ring of radius R and mass M. As shown in Fig. 7.40, we wish to determine its moment of inertia I about an axis YY' passing through its centre O and perpendicular to it. The ring can be imagined to be made of a large number of small elements. Consider one such element of length dx.

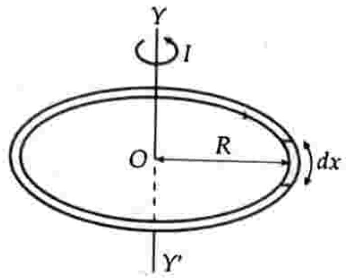


Fig. 7.40 M.I. of a ring about central axis.

Length of the ring = circumference $= 2\pi R$

Mass per unit length of ring $= \dfrac{M}{2\pi R}$

Mass of the small element $= \dfrac{M}{2\pi R}\, dx$

Moment of inertia of the small element about the axis YY',

$$dI = \left(\frac{M}{2\pi R}\, dx\right) R^2 = \frac{MR}{2\pi}\, dx$$

The small elements lie along the entire circumference of the ring *i.e.*, from $x = 0$ to $x = 2\pi R$. Hence the moment of inertia of the whole ring about the axis YY' will be

$$I = \int_0^{2\pi R} \frac{MR}{2\pi}\, dx = \frac{MR}{2\pi} \int_0^{2\pi R} dx$$

$$= \frac{MR}{2\pi} [\, x\,]_0^{2\pi R} = \frac{MR}{2\pi}(2\pi R - 0)$$

or $\quad I = MR^2.$

(*b*) **M.I. of a ring about any diameter**. According to the theorem of perpendicular axes, the moment of inertia about an axis YY' through O and perpendicular to the ring is equal to sum of its moments of inertia about two perpendicular diameters AB and CD, as shown in Fig. 7.41

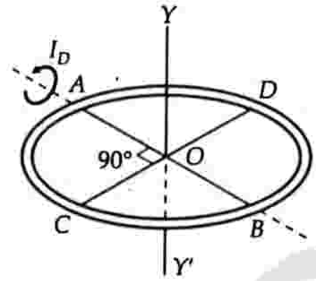


Fig. 7.41 M.I. of a ring about any diameter.

$$I_{AB} + I_{CD} = I_{YY'}$$

$$I_D + I_D = MR^2$$

or $$I_D = \tfrac{1}{2} MR^2.$$

Here I_D is the M.I. of the ring about any diameter.

(*c*) **M.I. of a ring about a tangent in its plane.** Refer to Fig. 7.42. Let I_T be the moment of inertia of the ring about the tangent EBF. Applying the theorem of parallel axes, we get

$$I_T = \text{M.I. about diameter } CD + MR^2$$

$$= \tfrac{1}{2} MR^2 + MR^2$$

or $$I_T = \tfrac{3}{2} MR^2.$$

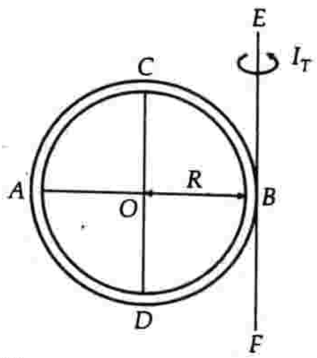


Fig. 7.42 M.I. of a ring about a tangent in its plane.

(*d*) **M.I. of a ring about a tangent perpendicular to its plane.** Let I'_T be the moment of inertia of the ring

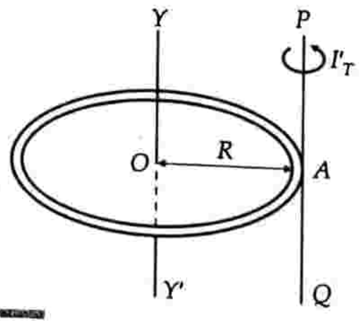


Fig. 7.43 M.I. of a ring about a tangent perpendicular to its plane.

about the axis PAQ tangent to the plane of the ring. Applying the theorem of parallel axes,

$$I_{PQ} = I_{YY'} + MR^2$$
$$= MR^2 + MR^2$$

or $$I'_T = 2\ MR^2.$$

Note We can determine the radius of gyration (k) of the ring about any axis by equating its M.I. about that axis to Mk^2. For example, the radius of gyration of a thin ring about any diameter is given by

$$I_D = \tfrac{1}{2}\ MR^2 = Mk^2$$

or $$k = R/\sqrt{2}.$$

7.25 MOMENT OF INERTIA OF A UNIFORM CIRCULAR DISC

41. *Derive an expression for the moment of inertia of a disc about (a) an axis through the centre and perpendicular to its plane, (b) its diameter, (c) a tangent in its own plane, (d) a tangent perpendicular to its plane.*

(*a*) **M.I. of a circular disc about an axis through its centre and perpendicular to its plane.** As shown in Fig. 7.44, consider a uniform disc of mass M and radius R. Suppose YY' is an axis passing through the centre O of the disc and perpendicular to its plane.

Area of the disc $= \pi R^2$

Mass per unit area of the disc $= \dfrac{M}{\pi R^2}$

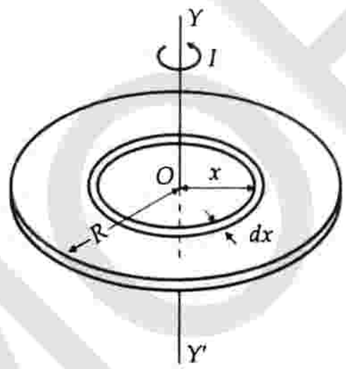


Fig. 7.44 M.I. of a disc about a central axis.

We can imagine the disc to be made up of a large number of concentric rings, whose radii vary from O to R. Let us consider one such concentric ring of radius x and width dx.

Area of the ring

$= \text{Circumference} \times \text{Width} = 2\pi x \times dx$

Mass of the concentric ring,

$$m = \left(\frac{M}{\pi R^2}\right) 2\pi x\ dx = \frac{2 M x\ dx}{R^2}$$

Moment of inertia of the concentric ring about the axis YY'

$$dI = mx^2 = \frac{2\ Mx\ dx}{R^2} \times x^2 = \frac{2\ Mx^3\ dx}{R^2}$$

The moment of inertia of the whole disc about the axis YY' can be obtained by integrating the above expression between the limits 0 to R.

$$I = \int_0^R \frac{2\ Mx^3\ dx}{R^2} = \frac{2\ M}{R^2} \int_0^R x^3\ dx$$

$$= \frac{2\ M}{R^2}\left[\frac{x^4}{4}\right]_0^R = \frac{2\ M}{4\ R^2}[R^4 - 0] = \frac{M}{2\ R^2} \times R^4$$

or $$I = \tfrac{1}{2}\ MR^2$$

(*b*) **M.I. of a disc about any diameter.** In Fig. 7.45, AB and CD are two mutually perpendicular diameters in the plane of the disc. Applying the theorem of perpendicular axes, we get

$$I_{AB} + I_{CD} = I_{YY'}$$

or $$I_D + I'_D = \tfrac{1}{2}\ MR^2$$

or $$I_D = \tfrac{1}{4}\ MR^2$$

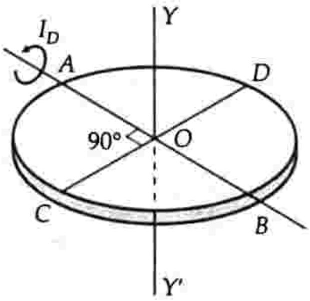


Fig. 7.45 M.I. of disc about any diameter.

(*c*) **M.I. of a disc about a tangent in its plane.** Let I_T the moment of inertia of the disc about a tangent EBF in the plane of the disc. This tangent is parallel to the diameter CD of the disc. Applying the *theorem of parallel axes*, we get

$$I_T = \text{Moment of inertia of disc about } CD + MR^2$$

or $$I_T = \tfrac{1}{4}\ MR^2 + MR^2 = \tfrac{5}{4}\ MR^2$$

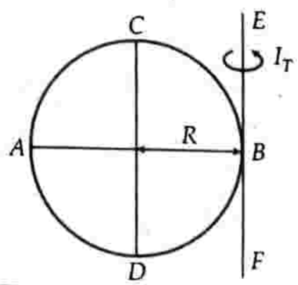


Fig. 7.46 M.I. of a disc about a tangent in its plane.

(*d*) **M.I. of a disc about a tangent perpendicular to its plane.** In Fig. 7.47, let I_T' be the moment of inertia of disc about the tangent PAQ perpendicular to the plane of the disc. Applying the theorem of parallel axes, we get

$$I_{PQ} = I_{YY'} + MR^2 = \frac{1}{2} MR^2 + MR^2$$

or $$I_T' = \frac{3}{2} MR^2$$

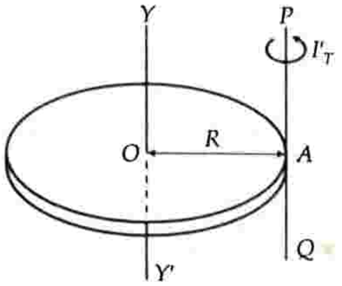


Fig. 7.47 M.I. of a disc about a tangent perpendicular to its plane.

Moreover, the radius of gyration (k) in this case is given by

$$Mk^2 = \frac{3}{2} MR^2 \quad \text{or} \quad k = \sqrt{\frac{3}{2}}\, R.$$

7.26 MOMENT OF INERTIA OF A THIN UNIFORM ROD

42. *Derive an expression for the moment of inertia of a thin uniform rod about an axis through its centre and perpendicular to its length. Also determine the radius of gyration about the same axis.*

M.I. of a thin uniform rod about a perpendicular axis through its centre. Consider a thin uniform rod AB of length L and mass M, free to rotate about an axis YY' through its centre O and perpendicular to its length.

$\therefore$ Mass per unit length of rod $= \frac{M}{L}$

Consider a small mass element of length dx at a distance x from O.

Mass of the small element $= \frac{M}{L} dx$

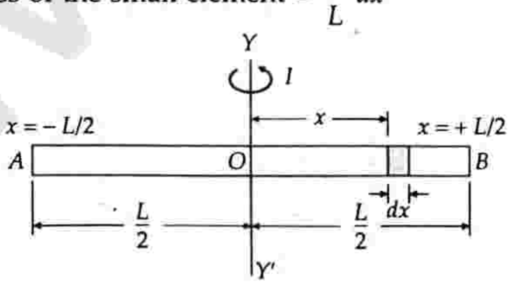


Fig. 7.48 M.I. of a rod about an axis through its centre.

Moment of inertia of the small element about YY'.

$$dI = \text{Mass} \times (\text{distance})^2 = \frac{M}{L} dx \times x^2$$

The moment of inertia of the whole rod about the axis YY' can be obtained by integrating the above expression between the limits $x = -L/2$ and $x = +L/2$.

$$\therefore \quad I = \int dI = \int_{-L/2}^{+L/2} \frac{M}{L} x^2\, dx$$

$$= \frac{M}{L} \int_{-L/2}^{+L/2} x^2\, dx = \frac{M}{L} \left[\frac{x^3}{3} \right]_{-L/2}^{+L/2}$$

$$= \frac{M}{3L} \left[\left(\frac{L}{2} \right)^3 - \left(-\frac{L}{2} \right)^3 \right]$$

$$= \frac{M}{3L} \left[\frac{L^3}{8} + \frac{L^3}{8} \right] = \frac{M}{3L} \times \frac{L^3}{4}$$

or $$I = \frac{ML^2}{12}$$

Radius of gyration. Let k be the radius of gyration of the rod about the axis YY'. Then

$$I = Mk^2$$

$$\therefore \quad Mk^2 = \frac{ML^2}{12} \quad \text{or} \quad k^2 = \frac{L^2}{12}$$

or $$k = \frac{L}{2\sqrt{3}}.$$

Thus, the radius of gyration of a uniform thin *rod rotating about an axis passing through its centre and perpendicular to its length is* $L/2\sqrt{3}$.

43. *Derive an expression for the moment of inertia of a thin uniform rod about an axis passing through its one end and perpendicular to its length. Also determine its radius of gyration about the same axis.*

M.I. of a thin uniform rod about a perpendicular axis through its one end. Consider a thin uniform rod AB of length L and mass M, free to rotate about an axis YY' passing through its one end A and perpendicular to its length, as shown in Fig. 7.49.

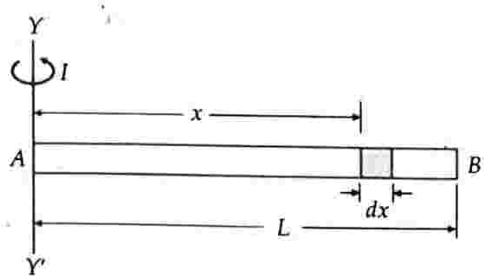


Fig. 7.49 M.I. of a rod about an axis through its one end.

Mass per unit length of the rod $= \frac{M}{L}$

Consider a small element of length dx of the rod at a distance x from the end A.

Mass of the small element $= \frac{M}{L} dx$

Moment of inertia of the small element about the axis YY',

$$dI = \text{Mass} \times (\text{distance})^2 = \frac{M}{L} dx \cdot x^2$$

The moment of inertia of the whole rod about the axis YY' can be obtained by integrating the above expression between the limits $x = 0$ and $x = L$

$$I = \int dI = \int_0^L \frac{M}{L} dx \cdot x^2 = \frac{M}{L} \int_0^L x^2 \, dx$$

$$= \frac{M}{L} \left[\frac{x^3}{3} \right]_0^L = \frac{M}{3L} [x^3]_0^L = \frac{M}{3L} [L^3 - 0] = \frac{ML^3}{3L}$$

or $\quad I = \frac{ML^2}{3}$.

Radius of gyration. Let k be the radius of gyration of the rod about the axis YY'. Then

$$\frac{ML^2}{3} = Mk^2$$

or $\quad k^2 = \frac{L^2}{3}$

or $\quad k = \frac{L}{\sqrt{3}}$

Thus *the radius of gyration of the rod about an axis passing through its one end and perpendicular to its length is* $L/\sqrt{3}$.

7.27 MOMENT OF INERTIA OF A CYLINDER

44. *Write an expression for the moment of inertia of a hollow cylinder of mass M and radius R about its own axis.*

M.I. of a hollow cylinder about its own axis. Consider a hollow cylinder of mass M and radius R. As shown in Fig. 7.50, every element of the cylinder is at the same perpendicular distance R from its axis. Hence the moment of inertia of the hollow cylinder about its own axis is

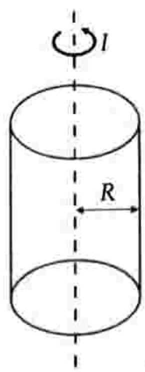


Fig. 7.50 M.I. of a hollow cylinder about its own axis.

$$I = \int r^2 \, dm = \int R^2 \, dm$$

$$= R^2 \int dm = R^2 \times M$$

or $\quad I = MR^2$.

45. *Derive an expression for the moment of inertia of a uniform solid cylinder about its own axis.*

Moment of inertia of uniform solid cylinder about its own axis. Consider a solid cylinder of mass M, radius R and length L. We wish to determine its moment of inertia about its own axis YY'.

Volume of the cylinder

$$= \pi R^2 L$$

Mass per unit volume of the cylinder,

$$\rho = \frac{M}{\pi R^2 L}$$

We can imagine the cylinder to be made of a large number of coaxial cylindrical shells. Consider one such cylindrical shell of inner radius x and outer radius $x + dx$, as shown in Fig. 7.51. The cross-section of the shell is a ring of radius x and thickness dx.

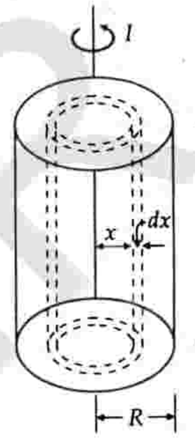


Fig. 7.51 M.I. of a solid cylinder about its own axis.

$\therefore$ Cross-sectional area of the cylindrical shell

$$= \text{Circumference} \times \text{thickness} = 2\pi x \, dx$$

Volume of the cylindrical shell

$$= \text{Cross-sectional area} \times \text{length} = 2\pi x \, dx \times L$$

Mass of the cylindrical shell,

$$m = \text{Volume} \times \rho = 2\pi x L \, dx \times \frac{M}{\pi R^2 L} = \frac{2M}{R^2} x \, dx$$

As the mass of the shell is distributed at the same distance x from its axis, so its moment of inertia about the axis YY' is

$$dI = mx^2 = \frac{2M}{R^2} x \, dx \times x^2 = \frac{2M}{R^2} x^3 \, dx$$

The moment of inertia of the solid cylinder can be obtained by integrating the above expression between the limits $x = 0$ and $x = R$.

$$\therefore \quad I = \int dI = \int_0^R \frac{2M}{R^2} x^3 \, dx$$

$$= \frac{2M}{R^2} \int_0^R x^3 \, dx = \frac{2M}{R^2} \left[\frac{x^4}{4} \right]_0^R = \frac{2M}{R^2} [R^4 - 0]$$

or $\quad I = \frac{1}{2} MR^2$.

Obviously, the moment of inertia of a cylinder about its own axis does not depend on its length.

46. *Derive an expression for the moment of inertia of a uniform solid cylinder about an axis passing through its centre and perpendicular to its length.*

M.I. of a solid cylinder about an axis through its centre and perpendicular to its axis. Consider a uniform solid cylinder of mass M, radius R and length L. We wish to determine its moment of inertia about an axis YY' passing through its centre O and perpendicular to its length.

Mass per unit length $= \frac{M}{L}$

We can imagine the cylinder to be made up of a large number of thin circular discs placed perpendicular to the axis of the cylinder. As shown in Fig. 7.52, consider one such thin disc of thickness dx and at distance x from the centre O.

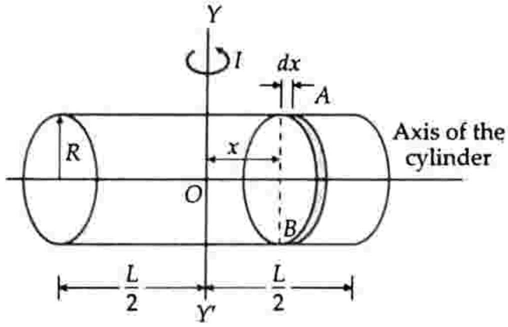


Fig. 7.52 M.I. of a cylinder about an axis through its centre and perpendicular to its axis.

Mass of the elementary disc $= \frac{M}{L} dx$

Radius of the elementary disc $= R^2$

Moment of inertia of the elementary disc about the diameter AB

$$= \frac{1}{4} \text{Mass} \times \text{radius}^2 = \frac{1}{4} \cdot \frac{M}{L} dx \times R^2 = \frac{MR^2}{4L} dx$$

Applying the theorem of parallel axes, the moment of inertia of the elementary disc about the axis YY',

$$dI = \text{M.I. about the diameter } AB + \text{Mass} \times x^2$$

$$= \frac{MR^2}{4L} dx + \frac{M}{L} dx \times x^2 = \frac{M}{L}\left(\frac{R^2}{4} + x^2\right) dx$$

The moment of inertia of the cylinder about the axis YY' can be obtained by integrating the above expression between the limits $x = 0$ and $x = L/2$ and multiplying the result by 2 to cover both halves of the cylinder.

Thus $$I = 2\int dI = 2\int_0^{L/2} \frac{M}{L}\left(\frac{R^2}{4} + x^2\right) dx$$

$$= \frac{2M}{L}\left[\frac{R^2}{4}\int_0^{L/2} dx + \int_0^{L/2} x^2\, dx\right]$$

$$= \frac{2M}{L}\left[\frac{R^2}{4} | x |_0^{L/2} + \left|\frac{x^3}{3}\right|_0^{L/2}\right]$$

$$= \frac{2M}{L}\left[\frac{R^2}{4}\left(\frac{L}{2} - 0\right) + \left(\frac{(L/2)^3}{3} - 0\right)\right]$$

$$= \frac{2M}{L}\left[\frac{R^2}{4} \cdot \frac{L}{2} + \frac{L^3}{24}\right]$$

or $$I = M\left[\frac{R^2}{4} + \frac{L^2}{12}\right].$$

7.28 MOMENT OF INERTIA OF A SOLID SPHERE

47. *Derive an expression for the moment of inertia of a uniform solid sphere about its any diameter. Hence write the expression for its moment of inertia about its tangent.*

Moment of inertia of a solid sphere about its diameter. Consider a uniform solid sphere of mass M and radius R. We wish to determine its moment of inertia about diameter AB.

Volume of the sphere $= \frac{4}{3}\pi R^3$

Mass per unit volume, $\rho = \frac{3M}{4\pi R^3}$

We can imagine the sphere to be made up of a large number of thin slices placed perpendicular to the diameter AB. Consider one such slice of thickness dx placed at distance x from the centre O.

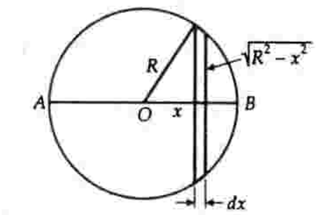


Fig. 7.53 M.I. of a sphere about a diameter.

Radius of the elementary slice $= \sqrt{R^2 - x^2}$

Volume of the elementary slice

$= \text{Area} \times \text{thickness}$

$= \pi(\sqrt{R^2 - x^2})^2 \times dx = \pi(R^2 - x^2)\, dx$

Mass of the elementary slice

$= \text{Volume} \times \rho = \pi(R^2 - x^2)\, dx \times \frac{3M}{4\pi R^3}$

$= \frac{3M(R^2 - x^2)\, dx}{4R^3}$

Moment of inertia of the thin slice about the axis AB passing through its centre and perpendicular to its plane,

$$dI = \frac{1}{2} \text{Mass} \times (\text{radius})^2$$

$$= \frac{1}{2} \cdot \frac{3M(R^2 - x^2)\, dx}{4R^3} \cdot (R^2 - x^2)$$

$$= \frac{3M(R^2 - x^2)^2\, dx}{8R^3}$$

The moment of inertia of the whole sphere about the diameter AB can be obtained by integrating the above expression between the limits $x = 0$ and $x = R$ and multiplying the result by 2 to include both halves of the sphere.

$$\therefore \quad I = 2\int dI = 2\int_0^R \frac{3M(R^2 - x^2)^2\,dx}{8R^3}$$

$$= \frac{2\times 3M}{8R^3}\int_0^R (R^2 - x^2)^2\,dx$$

$$= \frac{3M}{4R^3}\int_0^R (R^4 - 2R^2x^2 - x^4)\,dx$$

$$= \frac{3M}{4R^3}\left[R^4\int_0^R dx - 2R^2\int_0^R x^2\,dx + \int_0^R x^4\,dx\right]$$

$$= \frac{3M}{4R^3}\left[R^4\,|x|_0^R - 2R^2\left|\frac{x^3}{3}\right|_0^R + \left|\frac{x^5}{5}\right|_0^R\right]$$

$$= \frac{3M}{4R^3}\left[R^4(R-0) - 2R^2\left(\frac{R^3}{3} - 0\right) + \left(\frac{R^5}{5} - 0\right)\right]$$

$$= \frac{3M}{4R^3}\left[R^5 - \frac{2}{3}R^5 + \frac{R^5}{5}\right] = \frac{3M}{4R^3}\times\frac{8}{15}R^5$$

or $\quad I = \frac{2}{5}MR^2.$

Moment of inertia of the solid sphere about a tangent. Applying the theorem of parallel axes, the moment of inertia of a solid sphere about a tangent is given by

I_T = M.I. about a diameter + Mass × (radius)^2

$= \frac{2}{5}MR^2 + MR^2$

or $\quad I_T = \frac{7}{5}MR^2.$

Examples based on Moment of Inertia, Radius of Gyration and Rotational K.E.

Formulae Used

1. Moment of inertia of a body about the given axis of rotation,
$$I = m_1 r_1^2 + m_2 r_2^2 + \ldots + m_n r_n^2 = \sum_{i=1}^{n} m_i r_i^2.$$
2. Radius of gyration K is given by
$$I = MK^2 \text{ or } K = \sqrt{\frac{I}{M}}.$$
When all the particles are of same mass,
$$K = \sqrt{\frac{r_1^2 + r_2^2 + \ldots + r_n^2}{n}}$$
3. Theorem of parallel axes : $I_z = I_x + I_y$
4. Theorem of perpendicular axes, $I = I_{CM} + Md^2$
5. M.I. of a circular ring about an axis through its centre and perpendicular to its plane, $I = MR^2$
6. M.I. of a thin ring about any diameter, $I = \frac{1}{2}MR^2$
7. M.I. of a thin ring about any tangent in its plane,
$$I = \frac{3}{2}MR^2$$
8. M.I. of a circular disc about an axis through its centre and perpendicular to its plane, $I = \frac{1}{2}MR^2$
9. M.I. of a circular disc about any diameter,
$$I = \frac{1}{4}MR^2$$
10. M.I. of a circular disc about a tangent in its plane,
$$I = \frac{5}{4}MR^2$$
11. M.I. of a thin rod about an axis through its middle point and perpendicular to rod, $I = \frac{1}{12}ML^2$
12. M.I. of a thin rod about an axis through its one end and perpendicular to rod, $I = \frac{1}{3}ML^2$
13. M.I. of a rectangular lamina of sides l and b about an axis through its centre and perpendicular to its plane,
$$I = M\left(\frac{l^2 + b^2}{12}\right)$$
14. M.I. of a right circular solid cylinder about its symmetry axis,
$$I = \frac{1}{2}MR^2.$$
15. M.I. of a right circular hollow cylinder about its axis
$$I = MR^2.$$
16. M.I. of a solid sphere about an axis through its centre,
$$I = \frac{2}{5}MR^2.$$
17. M.I. of a solid sphere about any tangent, $I = \frac{7}{5}MR^2$
18. M.I. of a hollow sphere about an axis through its centre,
$$I = \frac{2}{3}MR^2$$
19. M.I. of a hollow sphere about any tangent,
$$I = \frac{5}{3}MR^2$$
20. Rotational K.E. $= \frac{1}{2}I\omega^2$
21. Total K.E. = Rotational K.E. + Translational K.E.
$$= \frac{1}{2}I\omega^2 + \frac{1}{2}Mv^2$$

Units Used

Mass M is in kg, radius R in m, moment of inertia I in kg m^2 and radius of gyration K in metre, rotational K.E. in joule and angular velocity ω in rad s^{-1}.

EXAMPLE 22. *A wheel of mass 8 kg and radius of gyration 25 cm is rotating at 300 rpm. What is its moment of inertia ?*

Solution. Here $M = 8$ kg, $K = 25$ cm $= 0.25$ m

$\therefore \quad I = MK^2 = 8 \times (0.25)^2 = \mathbf{0.5\ kgm^2}$.

EXAMPLE 23. *Three mass points m_1, m_2 and m_3 are located at the vertices of an equilateral triangle of length a. What is the moment of inertia of the system about an axis along the altitude of the triangle passing through m_1 ?*

Solution. As shown in Fig. 7.54, the axis of rotation passes through m_1. The distances of m_1, m_2 and m_3 from the axis of rotation are 0, $a/2$ and $a/2$ respectively.

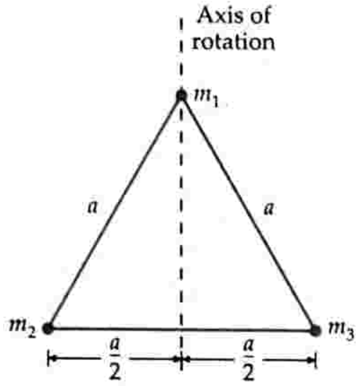


Fig. 7.54

$\therefore$ M.I. of the system about the altitude through m_1 is

$$I = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2$$

$$= m_1 (0)^2 + m_2 \left(\frac{a}{2}\right)^2 + m_3 \left(\frac{a}{2}\right)^2$$

or $$I = \frac{a^2}{4}(m_2 + m_3).$$

EXAMPLE 24. *Three balls of masses 1, 2 and 3 kg respectively are arranged at the corners of an equilateral triangle of side 1 m. What will be the moment of inertia of the system about an axis through the centroid and perpendicular to the plane of the triangle ?*

Solution.

Median $AD = \sqrt{AB^2 - BD^2} = \sqrt{1^2 - (0.5)^2} = \sqrt{0.75}$

$\therefore \quad AG = BG = CG = \frac{2}{3} AD = \frac{2}{3}\sqrt{0.75}$

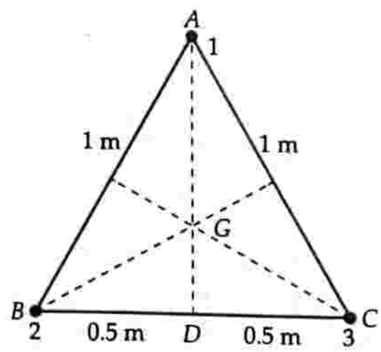


Fig. 7.55

M.I. of the system about an axis through centroid G and perpendicular to the plane of the triangle is

$$I = 1 \times AG^2 + 2 \times BG^2 + 3 \times CG^2$$

$$= (1 + 2 + 3) \times \left(\frac{2}{3}\sqrt{0.75}\right)^2$$

$$= \frac{6 \times 4 \times 0.75}{9} = \mathbf{2\ kg\ m^2}.$$

EXAMPLE 25. *Four particles of masses 4 kg, 2 kg, 3 kg and 5 kg are respectively located at the four corners A, B, C and D of a square of side 1 m, as shown in Fig. 7.56. Calculate the moment of inertia of the system about*

(*i*) *an axis passing through the point of intersection of the diagonals and perpendicular to the plane of the square,*

(*ii*) *the side AB, and* (*iii*) *the diagonal BD.*

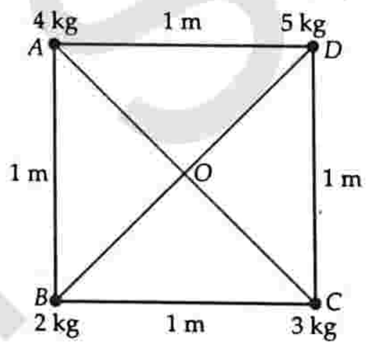


Fig. 7.56

Solution. Here $AB = BC = CD = DA = 1$ m

$$OA = OB = OC = OD = \frac{1}{2}\sqrt{1^2 + 1^2} = \frac{1}{\sqrt{2}}\ \text{m}$$

(*i*) M.I. of the system about an axis through O and perpendicular to the plane of the square,

$$I = 4(OA)^2 + 2(OB^2) + 3(OC)^2 + 5(OD)^2$$

$$= (4 + 2 + 3 + 5) \times \left(\frac{1}{\sqrt{2}}\right)^2 = 14 \times \frac{1}{2} = \mathbf{7\ kg\ m^2}.$$

(*ii*) M.I. of the system about the side AB,

$$I = 3(BC)^2 + 5(AD)^2 = 3 \times 1 + 5 \times 1 = \mathbf{8\ kg\ m^2}.$$

(*iii*) M.I. of the system about the diagonal BD,

$$I = 4(OA)^2 + 3(OC)^2$$

$$= 4 \times \frac{1}{2} + 3 \times \frac{1}{2} = \mathbf{3.5\ kg\ m^2}.$$

EXAMPLE 26. *The moment of inertia of a uniform circular disc about its diameter is 100 g cm². What is its moment of inertia (i) about its tangent (ii) about an axis perpendicular to its plane ?*

Solution. M.I. of disc about its diameter,

$$I_d = \frac{1}{4}MR^2 = 100\ \text{g cm}^2$$

(*i*) By theorem of parallel axes, M.I. about a tangent parallel to the diameter,

$$I = I_d + MR^2 = \frac{1}{4} MR^2 + MR^2 = \frac{5}{4} MR^2$$

$$= 5 \times 100 = \mathbf{500\ g\ cm^2}.$$

(*ii*) By theorem of perpendicular axes, M.I. of the disc about an axis perpendicular to its plane,

I = Sum of the moments of inertia about two perpendicular diameters

$$= I_d + I_d = 2 \times \tfrac{1}{4} MR^2 = 2 \times 100 = \mathbf{200\ g\ cm^2}.$$

Example 27. *Calculate the moment of inertia of a cylinder of length 1.5 m, radius 0.05 m and density 8×10^3 kg m^{-3} about the axis of the cylinder.*

Solution. Here $R = 0.05$ m, $l = 1.5$ m,

$$\rho = 8 \times 10^3 \text{ kg m}^{-3}$$

Mass of cylinder,

$$M = \text{Volume} \times \text{density} = \pi R^2 l . \rho$$

$$= 3.14 \times (0.05)^2 \times 1.5 \times 8 \times 10^3 = 94.2 \text{ kg}$$

M.I. of the cylinder about its own axis,

$$I = \tfrac{1}{2} MR^2 = \tfrac{1}{2} \times 94.2 \times (0.05)^2$$

$$= \mathbf{0.1175\ kg\ m^2}.$$

Example 28. *Calculate the moment of inertia of the earth about its diameter, taking it to be a sphere of 10^{25} kg and diameter 12800 km*

Solution. Here $M = 10^{25}$ kg,

$$R = 6400 \text{ km} = 6.4 \times 10^6 \text{ m}$$

M.I. of the earth about its diameter

$$I = \frac{2}{5} MR^2 = \frac{2}{5} \times 10^{25} \times (6.4 \times 10^6)^2$$

$$= \mathbf{1.64 \times 10^{38}\ kg\ m^2}.$$

Example 29. *Four spheres of diameter 2a and mass M each are placed with their centres on the four corners of a square of side b. Calculate the moment of inertia of the system about one side of the square taken as its axis.*

Solution. The situation is shown in Fig. 7.57. Let us calculate the moment of inertia of the system about the side CD.

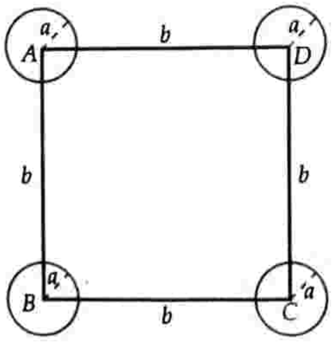


Fig. 7.57

I = M.I. of A about CD + M.I. of B about CD + M.I. of C about CD + M.I. of D about CD

$$= \left(\frac{2}{5} Ma^2 + Mb^2\right) + \left(\frac{2}{5} Ma^2 + Mb^2\right) + \frac{2}{5} Ma^2 + \frac{2}{5} Ma^2$$

$$= \frac{8}{5} Ma^2 + 2\ Mb^2 = \mathbf{\frac{2}{5} M (4a^2 + 5b^2)}.$$

Example 30. *Two point masses of 2 kg and 10 kg are connected by a weightless rod of length 1.2 m. Calculate the M.I. of the system about an axis passing through the centre of mass and perpendicular to the system.*

Solution. Here $m_1 = 2$ kg, $m_2 = 10$ kg,

length of rod = 1.2 m

Suppose the centre of mass lies at distance x from mass m_1. Then

$$m_1 x = m_2 (1.2 - x)$$

or $2x = 10 \times (1.2 - x)$ or $x = 1$ m

As the rod is weightless, its moment of inertia about any axis is zero.

M.I. of m_1 about CM

$$= m_1 x^2 = 2 \times (1)^2 = 2 \text{ kg m}^2$$

M.I. of m_2 about CM

$$= m_2 (1.2 - x)^2 = 10 \times (1.2 - 1)^2 = 0.4 \text{ kgm}^2$$

∴ Total M.I. = 2 + 0.4 = **2.4 kg m²**.

Example 31. *Find the moment of inertia of a rectangular bar magnet about an axis passing through its centre and parallel to its thickness. Mass of the magnet is 100 g, length is 12 cm, breadth is 3 cm and thickness is 2 cm*

Solution. Here $M = 100$ g, $l = 12$ cm, $b = 3$ cm, $t = 2$ cm

M.I. of the bar magnet about the axis through its centre and parallel to its thickness is

$$I = M\left(\frac{l^2 + b^2}{12}\right) = 100\left(\frac{12^2 + 3^2}{12}\right) = \frac{100 \times 153}{12}$$

$$= \mathbf{1275\ g\ cm^2}.$$

Example 32. *Calculate the ratio of radii of gyration of a circular ring and a disc of the same radius about the axis passing through their centres and perpendicular to their planes.*

Solution. Let K_1 and K_2 be the radii of gyration of the ring and the disc about the axis passing through their centres and perpendicular to their planes. Then

M.I. of the ring $= MR^2 = MK_1^2$ or $K_1 = R$

M.I. of the disc $= \frac{1}{2} MR^2 = MK_2^2$ or $K_2 = \frac{1}{\sqrt{2}} R$

$$\therefore \quad \frac{K_1}{K_2} = \frac{R}{R/\sqrt{2}} = \mathbf{\sqrt{2} : 1}.$$

Example 33. *Find (i) the radius of gyration and (ii) the moment of inertia of a rod of mass 100 g and length 100 cm about an axis passing through its centre and perpendicular to its length.* **[Delhi 12]**

Solution. (*i*) Here $M = 100 \text{ g} = 0.1 \text{ kg}$,

$$l = 100 \text{ cm} = 1 \text{ m}, \quad K = ?, \quad I = ?$$

M.I. of the rod about an axis through its centre and perpendicular to its length is

$$I = \frac{Ml^2}{12} = MK^2 \quad \text{or} \quad K^2 = \frac{l^2}{12}$$

$$\therefore \quad K = \frac{l}{\sqrt{12}} = \frac{1 \text{ m}}{3.464} = \mathbf{0.289 \text{ m}}.$$

(*ii*) $I = \frac{Ml^2}{12} = \frac{0.1 \times (1)^2}{12} = \mathbf{0.0083 \text{ kg m}^2}$.

Example 34. *A wheel is rotating at a rate of 1000 rpm and its kinetic energy is 10^6 J. Determine the moment of inertia of the wheel about its axis of rotation.*

Solution. Here $\nu = 1000 \text{ rpm} = \frac{1000}{60} \text{ rps} = \frac{50}{3} \text{ rps}$,

$$\omega = 2\pi\nu = \frac{100\pi}{3} \text{ rad s}^{-1}$$

As rotational K.E. $= \frac{1}{2} I\omega^2$

$$\therefore \quad 10^6 = \frac{1}{2} \times I \times \left(\frac{100\pi}{3}\right)^2$$

or
$$I = \frac{2 \times 10^6 \times 9}{(100)^2 \pi^2} = \frac{200 \times 9}{9.87} = \mathbf{182.4 \text{ kg m}^2}.$$

Example 35. *Calculate the kinetic energy of rotation of a circular disc of mass 1 kg and radius 0.2 m rotating about an axis passing through its centre and perpendicular to its plane. The disc is making $30/\pi$ rotations per minute.*

Solution. Here $M = 1 \text{ kg}, \; R = 0.2 \text{ m}$

$$\nu = \frac{30}{\pi} \text{ rpm} = \frac{30}{\pi \times 60} \text{ rps} = \frac{1}{2\pi} \text{ rps}$$

$$\omega = 2\pi\nu = 2\pi \times \frac{1}{2\pi} = 1 \text{ rad s}^{-1}$$

M.I. of the disc about an axis through its centre and perpendicular to its plane,

$$I = \frac{1}{2} MR^2 = \frac{1}{2} \times 1 \times (0.2)^2 = 0.02 \text{ kgm}^2$$

$\therefore$ Rotational K.E.

$$= \frac{1}{2} I\omega^2 = \frac{1}{2} \times 0.02 \times (1)^2 = \mathbf{0.01 \text{ J}}.$$

Example 36. *Energy of 484 J is spent in increasing the speed of a flywheel from 60 rpm to 360 rpm. Find the moment of inertia of the wheel.*

Solution. Energy spent = 484 J

$$\nu_1 = 60 \text{ rpm} = 1 \text{ rps}, \quad \nu_2 = 360 \text{ rpm} = 6 \text{ rps}$$

$$\omega_1 = 2\pi\nu_1 = 2\pi \text{ rad s}^{-1}, \quad \omega_2 = 2\pi\nu_2 = 12\pi \text{ rad s}^{-1}$$

Let I be the moment of inertia of the wheel.

Initial K.E. of rotation

$$= \frac{1}{2} I\omega_1^2 = \frac{1}{2} I \times (2\pi)^2 = 2\pi^2 I$$

Final K.E. of rotation

$$= \frac{1}{2} I\omega_2^2 = \frac{1}{2} I \times (12\pi)^2 = 72\pi^2 I$$

Increase in K.E. of rotation of wheel

= Energy spent on the wheel

$$\therefore \quad 72\pi^2 I - 2\pi^2 I = 484$$

or
$$I = \frac{484}{70\pi^2} = \frac{484 \times 7 \times 7}{70 \times 22 \times 22} = \mathbf{0.7 \text{ kg m}^2}.$$

Example 37. *Calculate the rotational K.E. of the earth about its own axis. Mass of the earth $= 6 \times 10^{24}$ kg and radius of the earth = 6400 km.*

Solution. Here $R = 6400 \text{ km} = 6.4 \times 10^6 \text{ m}$,

$$M = 6 \times 10^{24} \text{ kg},$$

M.I. of the earth about its own axis,

$$I = \frac{2}{5} MR^2 = \frac{2}{5} \times 6 \times 10^{24} \times (6.4 \times 10^6)^2$$

$$= 9.83 \times 10^{37} \text{ kg m}^2$$

Angular velocity of the earth,

$$\omega = \frac{2\pi}{T} = \frac{2\pi \text{ rad}}{1 \text{ day}} = \frac{2\pi \text{ rad}}{24 \times 60 \times 60 \text{ s}}$$

Rotational K.E. of the earth

$$= \frac{1}{2} I\omega^2 = \frac{1}{2} \times 9.83 \times 10^{37} \times \left(\frac{2\pi}{24 \times 60 \times 60}\right)^2$$

$$= \mathbf{2.6 \times 10^{29} \text{ J}}.$$

Example 38. *A metre scale AB is held vertically with its one end A on the floor and is then allowed to fall. Find the speed of the other end B when it strikes the floor, assuming that the end on the floor does not slip.*

Solution. Let M be the mass and b be the length of the metre scale. When the upper end of the rod strikes the floor, its centre of gravity falls through height $L/2$.

$$\therefore \text{ Loss in P.E.} = Mg \cdot \frac{L}{2}$$

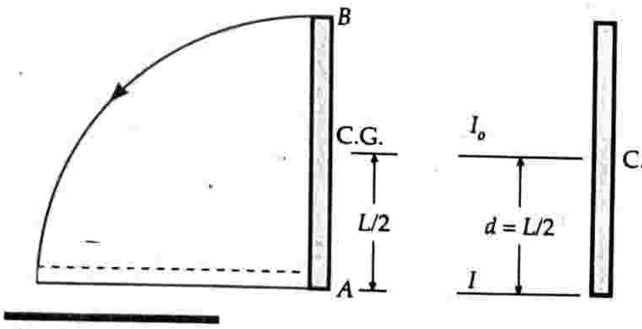


Fig. 7.58

M.I. of the scale about the lower end A,

I = M.I. of the scale about the parallel axis through C.G. + Md^2

$$= I_0 + Md^2 = \frac{ML^2}{12} + \frac{ML^2}{4} = \frac{ML^2}{3} \qquad \left[\because d = \frac{L}{2}\right]$$

Also, $\omega = \frac{v}{r} = \frac{v}{L}$

Gain in rotational K.E.

$$= \frac{1}{2} I\omega^2 = \frac{1}{2} \cdot \frac{ML^2}{3} \cdot \frac{v^2}{L^2} = \frac{Mv^2}{6}$$

Now, Gain in rotational K.E. = Loss in P.E.

$$\frac{Mv^2}{6} = Mg \cdot \frac{L}{2} \quad \text{or} \quad v^2 = 3gl$$

or $\quad v = \sqrt{3gL} = \sqrt{3 \times 9.8 \times 1} = \mathbf{5.4\ ms^{-1}}$.

EXAMPLE 39. *A uniform circular disc of mass m is set rolling on a smooth horizontal table with a uniform linear velocity v. Find the total K.E. of the disc.*

Solution. M.I. of the disc about its own axis,

$$I = \frac{1}{2} mr^2$$

As $\quad v = r\omega \quad \therefore \quad \omega^2 = \frac{v^2}{r^2}$

Rotational K.E. $= \frac{1}{2} I\omega^2 = \frac{1}{2} \times \frac{1}{2} mr^2 \times \frac{v^2}{r^2} = \frac{1}{4} mv^2$

Translational K.E. $= \frac{1}{2} mv^2$

Total K.E. = Rotational K.E. + Translational K.E.

$$= \frac{1}{4} mv^2 + \frac{1}{2} mv^2 = \mathbf{\frac{3}{4} mv^2}.$$

EXAMPLE 40. *A solid sphere is rolling on a frictionless plane surface about its axis of symmetry. Find the rotational energy and the ratio of its rotational energy to its total energy.*

Solution. Suppose the sphere has mass M and rolls with a uniform speed v.

M.I. of the sphere, $I = \frac{2}{5} MR^2$

Angular velocity, $\omega = \frac{v}{R}$

Rotational K.E.

$$= \frac{1}{2} I\omega^2 = \frac{1}{2} \times \frac{2}{5} MR^2 \times \frac{v^2}{R^2} = \mathbf{\frac{1}{5} Mv^2}.$$

Total energy

= Translational K.E. + Rotational K.E.

$$= \frac{1}{2} Mv^2 + \frac{1}{5} Mv^2 = \frac{7}{10} Mv^2$$

$$\frac{\text{Rotational K.E.}}{\text{Total energy}} = \frac{\frac{1}{5} Mv^2}{\frac{7}{10} Mv^2} = \frac{2}{7} = \mathbf{2:7}.$$

EXAMPLE 41. *A wheel of mass 5 kg and radius 0.40 m is rolling on a road without sliding with angular velocity 10 rad s⁻¹. The moment of inertia of the wheel about the axis of rotation is 0.65 kgm². What is the percentage of kinetic energy of rotation in the total kinetic energy of the wheel ?*

Solution. Here $M = 5$ kg, $R = 0.40$ m,

$$\omega = 10 \text{ rad s}^{-1}, \ I = 0.65 \text{ kg m}^2$$

Linear velocity,

$$v = R\omega = 0.40 \times 10 = 4.0 \text{ ms}^{-1}$$

Translational K.E.

$$= \frac{1}{2} Mv^2 = \frac{1}{2} \times 5 \times 16 = 40 \text{ J}$$

Rotational K.E.

$$= \frac{1}{2} I\omega^2 = \frac{1}{2} \times 0.65 \times 100 = 32.5 \text{ J}$$

Total K.E. = Translational K.E. + Rotational K.E.

$$= 40 + 32.5 = 72.5 \text{ J}$$

$$\frac{\text{Rotational K.E.}}{\text{Total K.E.}} = \frac{32.5}{72.5} = 0.448 = \mathbf{44.8\%}.$$

EXAMPLE 42. *A solid cylinder rolls down an inclined plane. Its mass is 2 kg and radius 0.1 m. If the height of the inclined plane is 4 m, what is its rotational K.E. when it reaches the foot of the plane ?* **[NCERT]**

Solution. Here $M = 2$ kg, $R = 0.1$ m

Height of inclined plane, $h = 4$ m

At the top of the inclined plane, the cylinder has P.E. = mgh

At the bottom of the inclined plane, the cylinder has translational K.E. $\left(= \frac{1}{2} Mv^2\right)$ and rotational K.E. $\left(= \frac{1}{2} I\omega^2\right)$

By conservation of energy,

$$\frac{1}{2} Mv^2 + \frac{1}{2} I\omega^2 = Mgh$$

But $\quad v = R\omega$ and $I = \frac{1}{2} MR^2$

$$\therefore \quad \frac{1}{2} M(R\omega)^2 + \frac{1}{2} \times \frac{1}{2} MR^2 \omega^2 = Mgh$$

or $\quad \frac{3}{4} MR^2 \omega^2 = Mgh \quad \text{or} \quad \omega^2 = \frac{4gh}{3R^2}$

$$\therefore \quad \text{Rotational K.E.} = \frac{1}{2} I\omega^2 = \frac{1}{2} \times \frac{1}{2} MR^2 \times \frac{4gh}{3R^2}$$

$$= \frac{Mgh}{3} = \frac{2 \times 9.8 \times 4}{3} = \mathbf{26.13\ J}.$$

EXAMPLE 43. *A bucket of mass 8 kg is supported by a light rope wound around a solid wooden cylinder of mass 12 kg and radius 20 cm free to rotate about its axis. A man holding* *the free end of the rope, with the bucket and the cylinder at rest initially, lets go the bucket freely downwards in a well 50 m deep. Neglecting friction, obtain the speed of the bucket and the angular speed of the cylinder just before the bucket enters water. Take $g = 10\ ms^{-2}$.*

Solution. Mass of bucket, $m_1 = 8$ kg.

Mass of cylinder, $m_2 = 12$ kg.

Radius of the cylinder, $R = 20$ cm $= 0.20$ m.

When the bucket just enters water,

P.E. lost by bucket = Linear K.E. of the bucket + Rotational K.E. of the cylinder

or $$m_1gh = \frac{1}{2} m_1v^2 + \frac{1}{2} I\omega^2$$

$$= \frac{1}{2} m_1v^2 + \frac{1}{2}\left(\frac{1}{2} m_2R^2\right)\frac{v^2}{R^2}$$

$$\left[\because I = \frac{1}{2} m_2R^2, \omega = \frac{v}{R}\right]$$

$$= \frac{1}{2} v^2\left(m_1 + \frac{1}{2} m_2\right) = \frac{1}{2} v^2 (8+6) = 7v^2.$$

or $$v = \sqrt{\frac{m_1gh}{7}} = \sqrt{\frac{8 \times 10 \times 50}{7}}$$

$$= \sqrt{\frac{4000}{7}} = \mathbf{23.9\ ms^{-1}}.$$

The angular speed of cylinder before the bucket touches water,

$$\omega = \frac{v}{R} = \frac{23.9}{0.20} = \mathbf{119.5\ rads^{-1}}.$$

✖ Problems For Practice

1. A body of mass 50 g is revolving about an axis in a circular path. The distance of the centre of mass of the body from the axis of rotation is 50 cm. Find the moment of inertia of the body. (**Ans.** 0.0125 kg m^2)

2. Find the moment of inertia of the hydrogen molecule about an axis passing through its centre of mass and perpendicular to the internuclear axis. Given mass of H-atom = 1.7×10^{-27} kg, interatomic distance = 4×10^{-10} m. (**Ans.** 13.6×10^{-47} kg m^2)

3. Three particles, each of 10 g are located at the corners of an equilateral triangle of side 5 cm. Determine the moment of inertia of this system about an axis passing through one corner of the triangle and perpendicular to the plane of the triangle. (**Ans.** 5×10^{-5} kgm^2)

4. Four point masses of 20 g each are placed at the corners of a square *ABCD* of side 5 cm, as shown in Fig. 7.59. Find the moment of inertia of the system (*i*) about an axis coinciding with the side *BC* and (*ii*) about an axis through *A* and perpendicular to the plane of the square. [**Ans.** (*i*) 1000 g cm^2 (*ii*) 2000 g cm^2]

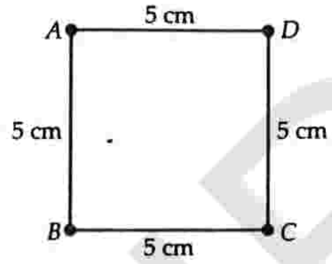


Fig. 7.59

5. The point masses of 0.3 kg, 0.2 kg and 0.1 kg are placed at the corners of a right angled Δ *ABC*, as shown in Fig. 7.60. Find the moment of inertia of the system (*i*) about an axis through *A* and perpendicular to the plane of the diagram and (*ii*) about an axis along *BC*. [**Ans.** (*i*) 0.043 kg m^2 (*ii*) 0.027 kg m^2]

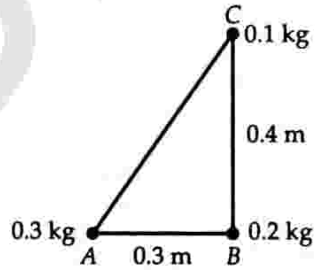


Fig. 7.60

6. Three particles, each of mass *m*, are situated at the vertices of an equilateral Δ *ABC* of side *L*, as shown in Fig. 7.61. Find the moment of inertia of the system about the line *AX* perpendicular to *AB* and in the plane of Δ *ABC*. $\left(\textbf{Ans. } \frac{5\,mL^2}{4}\right)$

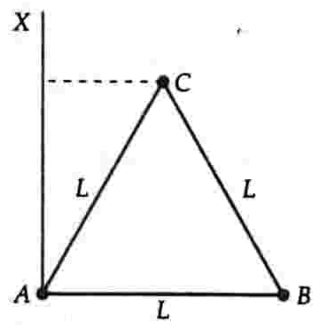


Fig. 7.61

7. Four particles each of mass *m* are kept at the four corners of a square of edge *a*. Find the moment of inertia of the system about an axis perpendicular to the plane of the system and passing through the centre of the square. (**Ans.** 2 ma^2)

8. What is the moment of inertia of a ring of mass 2 kg and radius 50 cm about an axis passing through its centre and perpendicular to its plane ? Also find the moment of inertia about a parallel axis through its edge. (**Ans.** 0.5 kgm^2, 1.0 kgm^2)

9. Calculate the moment of inertia of uniform circular disc of mass 500 g, radius 10 cm about (*i*) diameter of the disc (*ii*) the axis tangent to the disc and parallel to its diameter and (*iii*) the axis through the centre of the disc and perpendicular to its plane.
[**Ans.** (*i*) 12500 g cm^2 (*ii*) 62500 g cm^2 (*iii*) 25000 g cm^2]

10. Calculate moment of inertia of a circular disc of radius 10 cm, thickness 5 mm and uniform density 8 g cm^{-3}, about a transverse axis through the centre of the disc. (**Ans.** 6.28×10^4 gcm^2)

11. The radius of a sphere is 5 cm. Calculate the radius of gyration about (*i*) its diameter and (*ii*) about any tangent. [**Ans.** (*i*) 3.16 cm (*ii*) 5.915 cm]

12. Calculate the radius of gyration of a cylindrical rod of mass M and length L about an axis of rotation perpendicular to its length and passing through its centre. **[MNREC 96]**
(**Ans.** $K = L/\sqrt{12}$)

13. Two masses of 3 kg and 5 kg are placed at 30 cm and 70 cm marks respectively on a light wooden metre scale, as shown in Fig. 7.62. What will the moment of inertia of the system about an axis through (*i*) 0 cm mark and (*ii*) 100 cm mark, and perpendicular to the metre scale ?
[**Ans.** (*i*) 2.72 kg m^2 (*ii*) 1.92 kg m^2]

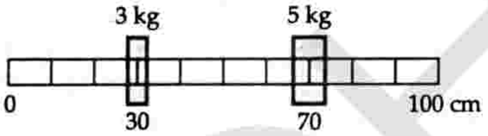


Fig. 7.62

14. Calculate the moment of inertia of a rod of mass 2 kg and length 0.5 m in each of the following cases, as shown in Fig. 7.63.
[**Ans.** (*i*) 0.042 kg m^2 (*ii*) 0.166 kg m^2]

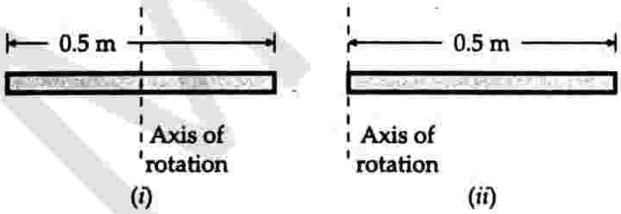


Fig. 7.63

15. A body of mass 2 kg is revolving in a horizontal circle of radius 2 m at the rate of 2 revolutions per second. Determine (*i*) moment of inertia of the body and (*ii*) the rotational kinetic energy of the body.
(**Ans.** 8 kgm^2, 631.65 J)

16. A flywheel of mass 500 kg and diameter 1 m makes 500 rpm. Assuming the mass to be concentrated along the rim, calculate (*i*) angular velocity (*ii*) moment of inertia and (*iii*) rotational K.E. of the flywheel. [**Ans.** (*i*) 52.38 rad s^{-1} (*ii*) 125 kg m^2 (*iii*) 1.715×10^5 J]

17. A rod revolving 60 times in a minute about an axis passing through an end at right angles to the length, has kinetic energy of 400 J. Find the moment of inertia of the rod. (**Ans.** 20.26 kg m^2)

18. A thin metal hoop of radius 0.25 m and mass 2 kg starts from rest and rolls down an inclined plane. If its linear velocity on reaching the foot of the plane is 2 ms^{-1}, what is its rotation K.E. at that instant ?
(**Ans.** 4 J)

19. The earth has a mass of 6×10^{24} kg and a radius of 6.4×10^6 m. Calculate the amount of work in joules that must be done if its rotation were to be slowed down so that the duration of the day becomes 30 hours instead of 24 hours. Moment of inertia of earth $= \frac{2}{5} MR^2$. (**Ans.** 9.36×10^{28} J)

✖ HINTS

1. Here $m = 50\text{ g} = 0.05$ kg, $r = 50\text{ cm} = 0.50$ m

$$I = mr^2 = 0.05 \times (0.50)^2 = \mathbf{0.0125\ kgm^2}.$$

2. Mass of each H-atom, $m = 1.7 \times 10^{-27}$ kg

Distance of each H-atom from the axis of rotation

$$= 2 \times 10^{-10}\text{ m}$$

$$\therefore \quad I = mr^2 + mr^2 = 2mr^2$$
$$= 2 \times 1.7 \times 10^{-27} \times (2 \times 10^{-10})^2$$
$$= \mathbf{13.6 \times 10^{-47}\ kg.}$$

4. (*i*) M.I. about BC is

$$I = 20(AB)^2 + 20(BB)^2 + 20(CC)^2 + 20(DC)^2$$
$$= 20(5)^2 + 20(0)^2 + 20(0)^2 + 20(5)^2$$
$$= \mathbf{1000\ g\ cm^2}.$$

(*ii*) $CA = \sqrt{5^2 + 5^2} = 5\sqrt{2}$ cm

M.I. about an axis through A and perpendicular to the plane of the square is

$$I = 20(AA)^2 + 20(BA)^2 + 20(CA)^2 + 20(DA)^2$$
$$= 20(0)^2 + 20(5)^2 + 20(5\sqrt{2})^2 + 20(5)^2$$
$$= \mathbf{2000\ g\ cm^2}.$$

5. (*i*) $I_A = 0.2(AB)^2 + 0.1(AC)^2$

$$= 0.2 \times (0.3)^2 + 0.1 \times (0.5)^2$$
$$= 0.018 + 0.025 = \mathbf{0.043\ kg\ m^2}.$$

(*ii*) $I_{BC} = 0.3 \times (AB)^2 = 0.3 \times (0.3)^2$
$$= \mathbf{0.027\ kg\ m^2}.$$

7. Distance of each particle from centre $= \frac{a}{\sqrt{2}}$

M.I. of each particle $= m\left(\frac{a}{\sqrt{2}}\right)^2 = \frac{1}{2}ma^2$

Total M.I. of the system $= 4 \times \frac{1}{2}ma^2 = \mathbf{2ma^2}$.

8. Here $M = 2$ kg, $R = 50$ cm $= 0.50$ m

(*i*) M.I. of the ring about an axis through its centre and perpendicular to its plane,

$$I = MR^2 = 2 \times (0.50)^2 = \mathbf{0.5 \ kg \ m^2}.$$

(*ii*) M.I. about a parallel axis through its edge,

$$I' = I + MR^2 = MR^2 + MR^2 = 2MR^2$$
$$= 2 \times 0.5 = \mathbf{1.0 \ kgm^2}.$$

9. (*i*) M.I. of the disc about any diameter,

$$I_d = \frac{1}{4}MR^2 = \frac{1}{4} \times 500 \times (10)^2 = \mathbf{12500 \ g \ cm^2}$$

(*ii*) By theorem of parallel axes, M.I. of the disc about a tangent parallel to the diameter of the disc,

$$I = I_d + MR^2 = \frac{5}{4}MR^2 = \frac{5}{4} \times 500 \times (10)^2$$
$$= \mathbf{62500 \ g \ cm^2}.$$

(*iii*) M.I. of the disc about an axis through its centre and perpendicular to its plane,

$$I = \frac{1}{2}MR^2 = \frac{1}{2} \times 500 \times (10)^2$$
$$= \mathbf{25000 \ g \ cm^2}.$$

10. Radius, $R = 10$ cm,

Thickness, $t = 5$ mm $= 0.5$ cm

Density, $\rho = 8 \text{ g cm}^{-3}$

Mass of disc, M = Area × thickness × density

$$= \pi R^2 t \rho = \frac{22}{7} \times (10)^2 \times 0.5 \times 8 = \frac{8800}{7} \text{ g}$$

M.I. of the disc about a transverse axis through its centre,

$$I = \frac{1}{2}MR^2 = \frac{1}{2} \times \frac{8800}{7} \times (10)^2$$
$$= \mathbf{6.28 \times 10^4 \ g \ cm^2}.$$

11. (*i*) $I_{\text{diameter}} = \frac{2MR^2}{5} = MK^2$

$\therefore \quad K = \sqrt{\frac{2}{5}} . R = \sqrt{0.4} \times 5$

$= 0.632 \times 5 = \mathbf{3.16 \ cm}.$

(*ii*) $I_{\text{tangent}} = \frac{7 \ MR^2}{5} = MK^2$

$\therefore \quad K = \sqrt{\frac{7}{5}} . R$

$= \sqrt{1.4} \ R = 1.183 \times 5 = \mathbf{5.915 \ cm}.$

15. Here $m = 2$ kg, $r = 2$ m, $\nu = 2$ rps

$$\omega = 2\pi\nu = 2\pi \times 2 = 4\pi \text{ rad s}^{-1}$$

(*i*) Moment of inertia, $I = mr^2 = 2 \times (2)^2 = \mathbf{8 \ kg \ m^2}$.

(*ii*) Rotational K.E. $= \frac{1}{2}I\omega^2 = \frac{1}{2} \times 8 \times (4\pi)^2$

$$= \frac{1}{2} \times 8 \times 16 \times 9.87 = \mathbf{631.65 \ J}.$$

16. Here $M = 500$ kg, $R = \frac{1}{2}$ m

$$\nu = 500 \text{ rpm} = \frac{500}{60} \text{ rps} = \frac{25}{3} \text{ rps}$$

(*i*) $\omega = 2\pi\nu = 2 \times 3.14 \times \frac{25}{3} = \mathbf{52.38 \ rad \ s^{-1}}$.

(*ii*) As the mass of flywheel is concentrated at its rim, it can be regarded as a ring.

$\therefore \quad I = MR^2 = 500 \times \left(\frac{1}{2}\right)^2 = \mathbf{125 \ kg \ m^2}.$

(*iii*) Rotational K.E. $= \frac{1}{2}I\omega^2 = \frac{1}{2} \times 125 \times (52.38)^2$

$$= 1.715 \times 10^5 \text{ J}$$

17. Here $\nu = 60$ rpm $= 1$ rps, $\omega = 2\pi\nu = 2\pi$ rad s^{-1}

Rotational K.E. $= \frac{1}{2}I\omega^2$

$\therefore \quad 400 = \frac{1}{2} \times I \times (2\pi)^2$

or $\quad I = \frac{400}{2\pi^2} = \frac{200}{9.87} = \mathbf{20.26 \ kgm^2}.$

18. Here $R = 0.25$ m, $M = 2$ kg, $v = 2 \text{ ms}^{-1}$

Rotational K.E. $= \frac{1}{2}I\omega^2 = \frac{1}{2}MR^2 \times \left(\frac{v}{R}\right)^2$

$$= \frac{1}{2}Mv^2 = \frac{1}{2} \times 2 \times 4 = \mathbf{4 \ J}.$$

19. Here $M = 6 \times 10^{24}$ kg, $R = 6.4 \times 10^6$ m,

$$T_1 = 24 \ h, T_2 = 30 \text{ h}$$

Work done = Increase in rotational K.E.

$$W = K_2 - K_1 = \frac{1}{2}I(\omega_2^2 - \omega_1^2)$$
$$= \frac{1}{2} \times \frac{2}{5}MR^2 \times \left[\left(\frac{2\pi}{T_2}\right)^2 - \left(\frac{2\pi}{T_1}\right)^2\right]$$
$$= \frac{4\pi^2}{5} \times MR^2\left[\frac{1}{T_2^2} - \frac{1}{T_1^2}\right]$$
$$= \frac{4 \times 9.87 \times 6 \times 10^{24} \times (6.4 \times 10^6)^6}{5}$$
$$\times \left[\frac{1}{(36 \times 3600)^2} - \frac{1}{(24 \times 3600)^2}\right]$$
$$= \mathbf{-9.36 \times 10^{28} \ J}$$

7.29 RELATION BETWEEN TORQUE AND MOMENT OF INERTIA

48. *Derive a relation between torque applied and angular acceleration produced in a rigid body and hence define moment of inertia.*

Relation between torque and moment of inertia. When a torque acts on a body capable of rotation about an axis, it produces an angular acceleration in the body. If the angular velocity of each particle is ω, then the angular acceleration, $\alpha = d\omega / dt$ will be same for all particles of the body. The linear acceleration will depend on their distances $r_1, r_2, ..., r_n$ from the axis of rotation.

As shown in Fig. 7.36, consider a particle P of mass m_1 at a distance r_1 from the axis of rotation. Let its linear velocity be v_1.

Linear acceleration of the first particle, $a_1 = r_1 \alpha$

Force acting on the first particle, $F_1 = m_1 r_1 \alpha$

Moment of force F_1 about the axis rotation is

$$\tau_1 = F_1 r_1 = m_1 r_1^2 \alpha$$

Total torque acting on the rigid body is

$$\begin{aligned}\tau &= \tau_1 + \tau_2 + \tau_3 + ... + \tau_n \\ &= m_1 r_1^2 \alpha + m_2 r_2^2 \alpha + m_3 r_3^2 \alpha + ... + m_n r_n^2 \alpha \\ &= (m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + ... + m_n r_n^2) \alpha \\ &= (\Sigma mr^2) \alpha\end{aligned}$$

But $\Sigma mr^2 = I$, moment of inertia of the body about the given axis

$$\therefore \quad \tau = I \alpha$$

Torque = Moment of inertia × Angular acceleration

When $\alpha = 1, \quad \tau = I$

Thus the moment of inertia of a rigid body about an axis of rotation is numerically equal to the external torque required to produce unit angular acceleration in the body about that axis.

7.30 RELATION BETWEEN ANGULAR MOMENTUM AND MOMENT OF INERTIA

49. *Derive a relation between angular momentum, moment of inertia and angular velocity of a rigid body.*

Relation between angular momentum, moment of inertia and angular velocity. As shown in Fig. 7.36, consider a rigid body rotating about a fixed axis with uniform angular velocity ω. The body consists of n particles of masses $m_1, m_2, m_3, ..., m_n$; situated at distance $r_1, r_2, r_3 ..., r_n$ from the axis of rotation. The angular velocity ω of all the n particles will be same but their linear velocities will be different and are given by

$$v_1 = r_1 \omega, \quad v_2 = r_2 \omega, \quad v_3 = r_3 \omega, ..., \quad v_n = r_n \omega$$

Linear momentum of first particle,

$$p_1 = m_1 v_1 = m_1 r_1 \omega$$

Moment of linear momentum of the first particle about the axis YY',

$$L_1 = p_1 r_1 = m_1 r_1^2 \omega$$

The angular momentum of a rigid body about an axis is the sum of moments of linear momenta of all its particles about that axis. Thus

$$\begin{aligned}L &= L_1 + L_2 + L_3 + ... + L_n \\ &= m_1 r_1^2 \omega + m_2 r_2^2 \omega + m_3 r_3^2 \omega + ... + m_n r_n^2 \omega \\ &= (m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + ... + m_n r_n^2) \omega \\ &= (\Sigma mr^2) \omega\end{aligned}$$

But $\Sigma mr^2 = I$, moment of inertia of the body about the given axis

$$\therefore \quad L = I\omega$$

Angular momentum = M.I. × Angular velocity

When $\omega = 1, \quad L = I$

Thus the moment of inertia of a body about an axis is numerically equal to the angular momentum of the rigid body when rotating with unit angular velocity about that axis.

Examples based on
Relations between Torque, Angular momentum and Moment of Inertia

FORMULAE USED

1. Torque = M.I. × angular acceleration
   or $\tau = I\alpha$
2. Work done by a torque, $W = \tau\theta$
3. Angular momentum = M.I. × angular velocity
   or $L = I\omega$

UNITS USED

Torque τ is in Nm, moment of inertia I in kgm^2 and angular momentum L in kgm^2s^{-1}.

EXAMPLE 44. *A torque of 2.0×10^{-4} Nm is applied to produce an angular acceleration of 4 rad s^{-2} in a rotating body. What is the moment of inertia of the body ?*

Solution. Here $\tau = 2.0 \times 10^{-4}$ Nm,

$$\alpha = 4 \text{ rad s}^{-2}, I = ?$$

As $\quad \tau = I \alpha$

$$\therefore \quad I = \frac{\tau}{\alpha} = \frac{2.0 \times 10^{-4}}{4} = \mathbf{0.5 \times 10^{-4}\ kg\ m^2.}$$

EXAMPLE 45. *An automobile moves on a road with a speed of 54 kmh^{-1}. The radius of its wheels is 0.35 m. What is the average negative torque transmitted by its brakes to a wheel* *if the vehicle is brought to rest in 15 s ? The moment of inertia of the wheel about the axis of rotation is 3 kgm^2.* **[NCERT]**

Solution. Here $u = 54\ kmh^{-1} = 15\ ms^{-1}$,

$$R = 0.35\ m,\quad t = 15\ s,\ I = 3\ kg\ m^2$$

$$\omega_0 = \frac{u}{R} = \frac{15}{0.35}\ rad\ s^{-1},\ \omega = 0$$

Average angular acceleration,

$$\alpha = \frac{\omega - \omega_0}{t} = \frac{0 - \frac{15}{0.35}}{15} = -\frac{1}{0.35}\ rad\ s^{-2}$$

Average torque transmitted by the brakes,

$$\tau = I\,.\,\alpha = -3 \times \frac{1}{0.35} = \mathbf{-8.57\ kgm^2\ s^{-2}}.$$

EXAMPLE 46. *A flywheel of mass 25 kg has a radius of 0.2 m. What force should be applied tangentially to the rim of the flywheel so that it acquires an angular acceleration of 2 rad s^{-2} ?*

Solution. Here $M = 25\ kg,\ R = 0.2\ m,\ \alpha = 2\ rad\ s^{-2}$

M.I. of the flywheel about its axis,

$$I = \frac{1}{2}MR^2 = \frac{1}{2} \times 25 \times (0.2)^2 = 0.5\ kg\ m^2$$

As torque, $\tau = F\,.\,R = I\,\alpha$

$$\therefore \text{Force},\quad F = \frac{I\alpha}{R} = \frac{0.5 \times 2}{0.2} = \mathbf{5\ N.}$$

EXAMPLE 47. *A torque of 10 Nm is applied to a flywheel of mass 10 kg and radius of gyration 50 cm. What is the resulting angular acceleration ?*

Solution. Here

$$\tau = 10\ Nm,\ M = 10\ kg,\quad K = 0.50\ m,\quad \alpha = ?$$

As $\quad \tau = I\,\alpha = MK^2\,\alpha$

$$\therefore\quad \alpha = \frac{\tau}{MK^2} = \frac{10}{10 \times (0.50)^2} = \mathbf{4\ rad\ s^{-2}}.$$

EXAMPLE 48. *A grindstone has a moment of inertia of 6 kg m^2. A constant torque is applied and the grindstone is found to have a speed of 150 rpm, 10 seconds after starting from rest. Calculate the torque.* **[Central Schools 07]**

Solution. Here $I = 6\ kg\ m^2,\quad t = 10\ s,\quad \omega_0 = 0$

$$\nu = 150\ rpm = \frac{150}{60}\ rps = \frac{5}{2}\ rps$$

$$\omega = 2\pi\nu = 2\pi \times \frac{5}{2} = 5\pi\ rad\ s^{-1}$$

$$\alpha = \frac{\omega - \omega_0}{t} = \frac{5\pi - 0}{10} = \frac{\pi}{2}\ rad\ s^{-2}$$

Torque, $\quad \tau = I\alpha = 6 \times \frac{\pi}{2} = \mathbf{3\pi\ Nm.}$

EXAMPLE 49. *A flywheel of mass 25 kg has a radius of 0.2 m. It is making 240 r.p.m. What is the torque necessary to bring it to rest in 20 s ? If the torque is due to a force applied tangentially on the rim of the flywheel, what is the magnitude of the force ?* **[NCERT]**

Solution. $M = 25\ kg,\ R = 0.2\ m,\ \nu_0 = 240\ rpm = 4\ rps$

$$\omega_0 = 2\pi\nu_0 = 2\pi \times 4 = 8\pi\ rad\ s^{-1},\ \omega = 0,\ t = 20\ s$$

As $\quad \omega = \omega_0 + \alpha t$

$\therefore \quad 0 = 8\pi + \alpha \times 20$

or $\quad \alpha = -\frac{8\pi}{20} = -\frac{2}{5}\pi\ rad\ s^{-2}$

M.I. of the flywheel about its own axis,

$$I = \frac{1}{2}MR^2 = \frac{1}{2} \times 25 \times (0.2)^2 = \frac{1}{2}\ kg\ m^2$$

Torque acting on the flywheel,

$$\tau = I\,\alpha = -\frac{1}{2} \times \frac{2}{5}\pi = -\frac{\pi}{5}\ \mathbf{Nm}$$

The negative sign indicates that the torque is of retarding nature.

Now $\quad$ Torque = Force × perpendicular distance

i.e., $\quad \tau = F \times R$

$$\therefore\quad F = \frac{\tau}{R} = \frac{\pi}{5 \times 0.2} = \mathbf{\pi\ N.}$$

EXAMPLE 50. *A cord is wound around the circumference of a wheel of diameter 0.3 m. The axis of the wheel is horizontal. A mass of 0.5 kg is attached at the end of the cord and it is allowed to fall from rest. If the weight falls 1.5 m in 4 s, what is the angular acceleration of the wheel ? Also find out the moment of inertia of the wheel.* **[NCERT]**

Solution. Radius of the wheel, $R = \frac{0.3}{2} = 0.15\ m$

For the attached mass :

$$m = 0.5\ kg,\ u = 0,\ s = 1.5\ m,\ t = 4\ s$$

Let a be the linear acceleration of the attached mass.

As $\quad s = ut + \frac{1}{2}at^2$

$\therefore \quad 1.5 = 0 \times 4 + \frac{1}{2}a \times (4)^2$

or $\quad a = \frac{1.5}{8} = \frac{3}{16}\ ms^{-2}$

If α is angular acceleration of the wheel, then

$$a = R\alpha$$

or $\quad \alpha = \frac{a}{R} = \frac{3}{16 \times 0.15} = \mathbf{1.25\ rad\ s^{-2}}$

Torque applied by the attached mass,

$$\tau = F \times R = mgR = 0.5 \times 9.8 \times 0.15\ Nm$$

Now $\quad \tau = I\,\alpha$

$$\therefore\quad I = \frac{\tau}{\alpha} = \frac{0.5 \times 9.8 \times 0.15}{1.25} = \mathbf{0.588\ kg\ m^2}.$$

EXAMPLE 51. *A cord of negligible mass is wound round the rim of a fly wheel of mass 20 kg and radius 20 cm. A steady pull of 25 N is applied on the cord as shown in Fig. 7.64. The flywheel is mounted on a horizontal axle with frictionless bearings.*

(*a*) *Compute the angular acceleration of the wheel.*

(*b*) *Find the work done by the pull, when 2 m of the cord is unwound.*

(*c*) *Find also the kinetic energy of the wheel at this point. Assume that the wheel starts from rest.*

(*d*) *Compare answers to parts (b) and (c).* [NCERT]

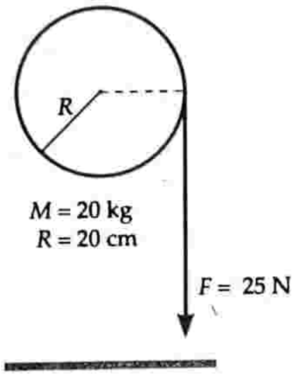


Fig. 7.64

Solution. (*a*) Torque, $\tau = FR = 25\ \text{N} \times 0.20\ \text{m}$

$$= 5.0\ \text{Nm}$$

Moment of inertia of the wheel about its axis,

$$I = \frac{MR^2}{2} = \frac{20 \times (0.20)^2}{2} = 0.4\ \text{kg m}^2$$

As $\tau = I\alpha$

$\therefore$ Angular acceleration,

$$\alpha = \frac{\tau}{I} = \frac{5.0\ \text{Nm}}{0.4\ \text{kg m}^2} = \mathbf{12.5\ rad\ s^{-2}}.$$

(*b*) Work done by the pull unwinding 2 m of the cord

$$= 25\ \text{N} \times 2\ \text{m} = \mathbf{50\ J}.$$

(*c*) Angular displacement of the wheel,

$$\theta = \frac{\text{Length of unwound string}}{\text{Radius of the wheel}}$$

$$= \frac{2\ \text{m}}{0.20\ \text{m}} = 10\ \text{rad}$$

As the wheel starts from rest, $\omega_0 = 0$

Final angular velocity ω is given by

$$\omega^2 = \omega_0^2 + 2\alpha\theta = 0 + 2 \times 12.5 \times 10$$

$$= 250\ (\text{rad s}^{-1})^2$$

$\therefore$ K.E. gained $= \frac{1}{2} I\omega^2 = \frac{1}{2} \times 0.4 \times 250 = \mathbf{50\ J}.$

(*d*) The answers are the same, *i.e.*, the kinetic energy gained by the wheel = work done by the force. There is no loss of energy due to friction.

EXAMPLE 52. *A body whose moment of inertia is 3 kgm², is at rest. It is rotated for 20 s with a moment of force 6 Nm. Find the angular displacement of the body. Also calculate the work done.*

Solution. Here $I = 3\ \text{kg m}^2$, $t = 20\ \text{s}$, $\tau = 6\ \text{Nm}$, $\theta = ?$, $W = ?$

As $\tau = I\alpha$

$$\therefore \quad \alpha = \frac{\tau}{I} = \frac{6}{3} = 2\ \text{rad s}^{-2}$$

Angular displacement in 20 s is

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2 = 0 + \frac{1}{2} \times 2 \times (20)^2$$

$$= \mathbf{400\ rad}$$

Work done,

$$W = \tau\theta = 6 \times 400 = \mathbf{2400\ J}.$$

EXAMPLE 53. *How much tangential force would be needed to stop the earth in one year, if it were rotating with angular velocity of 7.3×10^{-5} rad s^{-1}? Given the moment of inertia of the earth $= 9.3 \times 10^{37}$ kg m^2 and radius of the earth $= 6.4 \times 10^6$ m.*

Solution. Here $I = 9.3 \times 10^{37}\ \text{kg m}^2$, $R = 6.4 \times 10^6\ \text{m}$,

$$\omega_0 = 7.5 \times 10^{-5}\ \text{rad s}^{-1},$$

$$t = 1\ \text{year} = 365 \times 24 \times 3600\ \text{s}$$

As $\omega = \omega_0 + \alpha t$

$$\therefore \quad \alpha = \frac{\omega - \omega_0}{t} = \frac{0 - 7.3 \times 10^{-5}}{365 \times 24 \times 3600}$$

$$= -\frac{7.3 \times 10^{-5}}{365 \times 24 \times 3600}\ \text{rad s}^{-2}$$

Torque, $$\tau = I\alpha = 9.3 \times 10^{37} \times \frac{7.3 \times 10^{-5}}{365 \times 24 \times 3600}\ \text{Nm}$$

[Omitting –ve sign]

Let F be the tangential force needed to stop the earth. Then

$$\tau = FR$$

or $$F = \frac{\tau}{R} = \frac{9.3 \times 10^{37} \times 7.3 \times 10^{-5}}{365 \times 24 \times 3600 \times 6.4 \times 10^6}$$

$$= \mathbf{3.363 \times 10^{19}\ N}.$$

EXAMPLE 54. *The angular momentum of a body is 31.4 Js and its rate of revolution is 10 cycles per second. Calculate the moment of inertia of the body about the axis of rotation.*

Solution. Here $L = 31.4$ Js, $\nu = 10$ rps,

$$\omega = 2\pi\nu = 2 \times 3.14 \times 10\ \text{rad}^{-1}$$

As $L = I\omega$

$$\therefore \quad I = \frac{L}{\omega} = \frac{31.4}{2 \times 3.14 \times 10} = \mathbf{0.5\ kgm^2}.$$

EXAMPLE 55. *A 40 kg flywheel in the form of a uniform circular disc of 1 m radius is making 120 rpm. Calculate the angular momentum.*

Solution. Here $M = 40$ kg, $R = 1$ m,

$$\nu = 120 \text{ rpm} = \frac{120}{60} \text{ rps} = 2 \text{ rps}$$

$$\therefore \quad I = \frac{1}{2} MR^2 = \frac{1}{2} \times 40 \times (1)^2 = 20 \text{ kg m}^2$$

and $\omega = 2\pi\nu = 2\pi \times 2 = 4\pi$ rad s^{-1}

Angular momentum,

$$L = I\omega = 20 \times 4\pi = 80 \times 3.14$$

$$= \mathbf{251.2 \ kg \ m^2 s^{-1}}.$$

EXAMPLE 56. A ring of diameter 0.4 m and of mass 10 kg is rotating about its axis at the rate of 2100 rpm. Find (i) moment of inertia (ii) angular momentum and (iii) rotational K.E. of the ring. **[Delhi 03]**

Solution. Here $R = \frac{0.4}{2} = 0.2$ m, $M = 10$ kg

$$\nu = 2100 \text{ rpm} = \frac{2100}{60} \text{ rps} = 35 \text{ s}^{-1}$$

$$\therefore \quad \omega = 2\pi\nu = 2 \times \frac{22}{7} \times 35 = 220 \text{ rad s}^{-1}$$

(*i*) M.I. of the ring about its axis,

$$I = MR^2 = 10 \times (0.2)^2 = \mathbf{0.4 \ kg \ m^2}.$$

(*ii*) Angular momentum,

$$L = I\omega = 0.4 \times 220 = \mathbf{88 \ kg \ m^2 s^{-1}}.$$

(*iii*) Rotational K.E.

$$= \frac{1}{2} I\omega^2 = \frac{1}{2} \times 0.4 \times (220)^2 = \mathbf{9680 \ J}.$$

EXAMPLE 57. Calculate the angular momentum of the earth rotating about its own axis. Mass of the earth $= 5.98 \times 10^{27}$ kg, mean radius of the earth $= 6.37 \times 10^6$ m, M.I. of the earth $= \frac{2}{5} MR^2$.

Solution. Here $M = 5.98 \times 10^{24}$ kg,

$$R = 6.37 \times 10^6 \text{ m}$$

$$I = \frac{2}{5} MR^2 = \frac{2}{5} \times 5.98 \times 10^{24} \times (6.37 \times 10^6)^2$$

$$= 2.1 \times 10^{38} \text{ kgm}^2$$

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{1 \text{ day}} = \frac{2\pi}{24 \times 60 \times 60} \text{ rad s}^{-1}$$

Angular momentum,

$$L = I\omega = 2.1 \times 10^{38} \times \frac{2\pi}{24 \times 60 \times 60}$$

$$= \mathbf{1.53 \times 10^{34} \ kg \ m^2 \ s^{-1}}.$$

EXAMPLE 58. A cylinder of mass 5 kg and radius 30 cm and free to rotate about its axis, receives an angular impulse of 3 kg m^2 s^{-1} initially followed by a similar impulse after every 4 s. What is the angular speed of the cylinder 30 s after the initial impulse ? The cylinder is at rest initially.

Solution. Here $M = 5$ kg, $R = 30$ cm $= 0.30$ m, $\omega = 0$, $\omega = ?$

Angular impulse = Change in angular momentum

$$\therefore \quad 3 = I(\omega_2 - \omega_1) = \frac{1}{2} MR^2 (\omega - \omega_0)$$

or $$3 = \frac{1}{2} \times 5 \times (0.30)^2 (\omega - 0)$$

or $$\omega = \frac{3 \times 2}{5 \times 0.09} = \frac{40}{3} \text{ rad s}^{-1}$$

Now $$\omega = \omega_0 + \alpha t$$

$$\therefore \quad \frac{40}{3} = 0 + \alpha \times 4 \quad \text{or} \quad \alpha = \frac{10}{3} \text{ rad s}^{-2}$$

The angular impulse is imparted after every 4 seconds. So the pulses are imparted at $t = 0$, 4, 8, 12, 16, 20, 24 and 28 s. But last impulse continues to act upto 32 s, before the next impulse is imparted. So

$$\omega = \omega_0 + \alpha t = 0 + \frac{10}{3} \times 32$$

$$= \mathbf{106.67 \ rad \ s^{-1}}.$$

✖ PROBLEMS FOR PRACTICE

1. The moment of inertia of a flywheel is 4 kg m^2. What angular acceleration will be produced in it by applying a torque of 10 Nm on it ? (**Ans.** 2.5 rad s^{-2})
2. The moment of inertia of a body is 2.5 kg m^2. Calculate the torque required to produce an angular acceleration of 18 rad s^{-2} in the body. (**Ans.** 45 Nm)
3. A cylinder of length 20 cm and radius 10 cm is rotating about its central axis at an angular speed of 100 rad/s. What tangential force will stop the cylinder at a uniform rate in 10 seconds ? The moment of inertia of the cylinder about its axis of rotation is 0.8 kg m^2. **[Delhi 04]** (**Ans.** 80 N)
4. A flywheel of moment of inertia 10^7 g cm^2 is rotating at a speed of 120 rotations per minute. Find the constant breaking torque required to stop the wheel in 5 rotations. (**Ans.** 2.513×10^7 dyne cm)
5. If a constant torque of 500 Nm turns a wheel of moment of inertia 100 kg m^2 about an axis through its centre, find the gain in angular velocity in 2 s. (**Ans.** 10 rad s^{-1})
6. A sphere of mass 2 kg and radius 5 cm is rotating at the rate of 300 rpm. Calculate the torque required to stop it in 6.28 revolutions. Moment of inertia of the sphere about any diameter $= \frac{2}{5} MR^2$. (**Ans.** 2.542×10^{-2} Nm)
7. A body of mass 1.0 kg is rotating on a circular path of diameter 2.0 m at the rate of 10 rotations in 31.4 s. Calculate (*i*) angular momentum of the body and (*ii*) rotational kinetic energy. [**Ans.** (*i*) 2.0 kgm^2 s^{-1} (*ii*) 2.0 J]

8. A circular ring of diameter 40 cm and mass 1 kg is rotating about an axis normal to its plane and passing through the centre with a frequency of 10 rotations per second. Calculate the angular momentum about its axis of rotation.

(**Ans.** $0.8\ \pi\ kgm^2\ s^{-1}$)

✖ HINTS

3. $$\alpha = \frac{\omega - \omega_0}{\tau} = \frac{0-100}{10} = -10\ \text{rad s}^{-2}.$$

$$\tau = F.R = I.\alpha$$

$$\therefore \quad F = \frac{I\alpha}{R} = \frac{0.8\ \text{kg m}^2 \times (-10\ \text{rad s}^{-2})}{0.1\ \text{m}} = \mathbf{-80\ N.}$$

7 Here $m = 1.0\ \text{kg}, r = 1.0\ \text{m}, \nu = \frac{10}{31.4}\ \text{s}$

$$\omega = 2\pi\nu = 2 \times 3.14 \times \frac{10}{31.4} = 2\ \text{rad s}^{-1}$$

$$I = mr^2 = 1.0 \times (1.0)^2 = 1.0\ \text{kg m}^2$$

$$L = I\omega = 1.0 \times 2 = \mathbf{2.0\ kgm^2\ s^{-1}}.$$

$$\text{Rotational K.E.} = \frac{1}{2} I\omega^2 = \frac{1}{2} \times 1.0 \times (2)^2 = \mathbf{2.0\ J.}$$

8. Here $m = 1\ \text{kg}, R = 20\ \text{cm} = 0.2\ \text{m}, \nu = 10\ \text{rps}$

$$\omega = 2\pi\nu = 2\pi \times 10 = 20\ \pi\ \text{rad s}^{-1}$$

$$I = MR^2 = 1 \times (0.2)^2 = 0.04\ \text{kgm}^2$$

$$L = I\omega = 0.04 \times 20\pi = \mathbf{0.8\ \pi\ kg\ m^2\ s^{-1}}.$$

7.31 CONSERVATION OF ANGULAR MOMENTUM

50. *State the law of conservation of angular momentum. Give some illustrations of this law.*

Law of conservation of angular momentum. Suppose the external torque acting on a rigid body due to external forces is zero. Then

$$\tau = \frac{dL}{dt} = 0$$

Hence, L = constant.

So when *the total external torque acting on a rigid body is zero, the total angular momentum of the body is conserved. This is the law of conservation of angular momentum.*

Clearly, when $\tau = 0,\ L = I\,\omega$ = constant

or $$I_1\,\omega_1 = I_2\,\omega_2$$

This means that when no external torque is acting, the angular velocity ω of the body can be increased or decreased by decreasing or increasing the moment of inertia of the body.

Illustrations of the law of conservation of angular momentum :

(*i*) ***Planetary motion.*** The angular velocity of a planet revolving in an elliptical orbit around the sun increases, when it comes closer to the sun because its moment of inertia about the axis through the sun decreases. When it goes far away from the sun, its moment of inertia increases and hence angular velocity decreases so as to conserve angular momentum.

(*ii*) ***A man carrying heavy weights in his hands and standing on a rotating turn-table can change the angular speed of the turn-table.*** As shown in Fig. 7.65, if a person stands on a turn-table with some heavy weights in his hands stretched out and the table is rotated slowly, his angular speed at once increases, as he draws his hands to his chest. The moment of inertia of man and weights taken together decreases, as he draws his arms inward. As moment of inertia decreases, the angular speed increases so as to conserve total angular momentum.

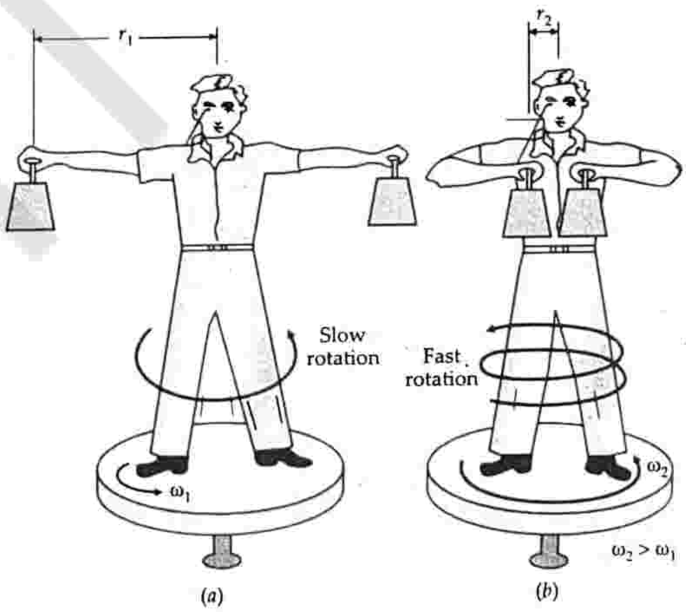


Fig. 7.65 The man begins to rotate faster as he draws his arms inwards.

(*iii*) ***A diver jumping from a spring board exhibits somersaults in air before touching the water surface.*** After leaving the spring board, a diver curls his body by pulling his arms and legs towards the centre of his body. This decreases his moment of inertia and he spins fast in midair. Just before hitting the water surface, he stretches out his arms. This decreases his moment of inertia and the diver enters water at a gentle speed.

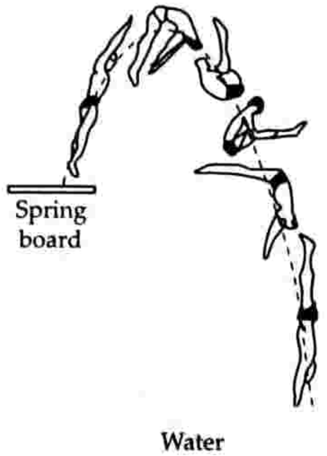


Fig. 7.66 A diver performing somersaults.

(iv) ***An ice-skater or a ballet dancer can increase her angular velocity by folding her arms and bringing the stretched leg close to the other leg.*** When she stretches her hands and a leg outward [Fig. 7.67(*a*)], her moment of inertia increases and hence angular speed decreases to conserve angular momentum. When she folds her arms and brings the stretched leg close to the other leg [Fig. 7.67(*b*)], her moment of inertia decreases and hence angular speed increases.

Fig. 7.67 A ballet dancer increases ω by changing to a posture of smaller M.I.

(v) ***The speed of the inner layers of the whirlwind in a tarnado is alarmingly high.*** The angular velocity of air in a tarnado increases as it goes towards the centre. This is because as the air moves towards the centre, its moment of inertia (I) decreases and to conserve angular momentum ($L = I\omega$), the angular velocity ω increases.

Examples based on Law of Conservation of Angular Momentum

FORMULAE USED

In the absence of any external torque,

$$L = I\,\omega = \text{a constant}$$

$$\text{or} \quad I_1\,\omega_1 = I_2\,\omega_2 \text{ or } I_1 \cdot \frac{2\pi}{T_1} = I_2 \cdot \frac{2\pi}{T_2}$$

UNITS USED

Moment of inertia I is in kg m^2 and angular velocity ω in rad s^{-1}.

EXAMPLE 59. A small block is rotating in a horizontal circle at the end of a thread which passes down through a hole at the centre of table top. If the system is rotating at 2.5 rps in a circle of 30 cm radius, what will be the speed of rotation when the thread is pulled inwards to decrease the radius to 10 cm ? Neglect friction.

Solution. Here $\nu_1 = 2.5$ rps, $r_1 = 30$ cm,

$r_2 = 10$ cm, $\nu_2 = ?$

By law of conservation of angular momentum,

$$L_1 = L_2 \quad \text{or} \quad I_1\,\omega_1 = I_2\,\omega_2$$

$$\text{or} \quad mr_1^2 \cdot 2\pi\nu_1 = mr_2^2 \cdot 2\pi\nu_2$$

$$\therefore \quad \nu_2 = \frac{r_1^2\,\nu_1}{r_2^2} = \frac{30 \times 30 \times 2.5}{10 \times 10} = \mathbf{22.5\ rps.}$$

EXAMPLE 60. A star of mass twice the solar mass and radius 10^6 km rotates about its axis with an angular speed of 10^{-6} rad s^{-1}. What is the angular speed of the star when it collapses (due to inward gravitational force) to a radius of 10^4 km ? Solar mass 1.99×10^{30} kg.

Solution. During collapse, the total angular momentum of an isolated star is conserved, hence

$$I_1\,\omega_1 = I_2\,\omega_2$$

$$\text{or} \quad \frac{2}{5} MR_1^2\,\omega_1 = \frac{2}{5} MR_2^2\,\omega_2 \qquad \left[\because I = \frac{2}{5} MR^2\right]$$

$$\text{or} \quad R_1^2\,\omega_1 = R_2^2\,\omega_2 \quad \therefore \quad \omega_2 = \frac{R_1^2}{R_2^2}\,\omega_1$$

But $R_1 = 10^6$ km, $R_2 = 10^4$ km, $\omega_1 = 10^{-6}$ s^{-1}.

$$\therefore \quad \omega_2 = \frac{(10^6)^2}{(10^4)^2} \times 10^{-6} = \mathbf{0.01\ rad\ s^{-1}.}$$

EXAMPLE 61. If the earth were to suddenly contract to half of its present radius (without any external torque on it), by what duration would the day be decreased ? Assume earth to be a perfect solid sphere of moment of inertia $\frac{2}{5}MR^2$.

Solution. Present radius of the earth, $R_1 = R$

New radius of the earth after contraction,

$R_2 = R/2$

$T_1 = 24$ h, $T_2 = ?$

By conservation of angular momentum,

$$I_1\,\omega_1 = I_2\,\omega_2$$

$$\text{or} \quad \frac{2}{5} MR_1^2 \cdot \frac{2\pi}{T_1} = \frac{2}{5} MR_2^2 \cdot \frac{2\pi}{T_2}$$

$$\text{or} \quad T_2 = \left(\frac{R_2}{R_1}\right)^2 \cdot T_1 = \left(\frac{R/2}{R}\right)^2 \times 24 = \frac{1}{4} \times 24 = 6\text{ h}$$

$\therefore$ Decrease in the duration of the day

$= 24 - 6 = \mathbf{18\ h.}$

EXAMPLE 62. *What will be the duration of the day, if earth suddenly shrinks to 1/64 of its original volume, mass remaining the same ?*

Solution. Original volume of the earth,

$$V = \frac{4}{3}\pi R^3$$

Volume of the earth after shrinking,

$$V' = \frac{V}{64}$$

or $$\frac{4}{3}\pi R'^3 = \frac{1}{64} \times \frac{4}{3}\pi R^3$$

or $$R' = \frac{R}{4}$$

By conservation of angular momentum,

$$I'\omega' = I\omega$$

or $$\frac{2}{5}MR'^2 \times \frac{2\pi}{T'} = \frac{2}{5}MR^2 \times \frac{2\pi}{T}$$

or $$T' = \left(\frac{R'}{R}\right)^2 . T = \left(\frac{R/4}{R}\right)^2 \times 24$$

$$= \frac{1}{16} \times 24 = \mathbf{1.5\ h.}$$

EXAMPLE 63. *The maximum and minimum distances of a comet from the sun are 1.4×10^{12} m and 7×10^{10} m. If its velocity nearest to the sun is $6 \times 10^4\ ms^{-1}$, what is the velocity in the farthest position ? Assume that path of the comet in both the instantaneous positions is circular.*

Solution. At minimum distance, $r_1 = 7 \times 10^{10}$ m; velocity, $v_1 = 6 \times 10^4\ ms^{-1}$

At maximum distance, $r_2 = 1.4 \times 10^{12}$ m;

velocity, $v_2 = ?$

By conservation of angular momentum,

$$I_1\omega_1 = I_2\omega_2$$

or $$mr_1^2 \times \frac{v_1}{r_1} = mr_2^2 . \frac{v_2}{r_2}$$

or $$v_1 r_1 = v_2 r_2$$

or $$v_2 = \frac{v_1 r_1}{r_2} = \frac{6 \times 10^4 \times 7 \times 10^{10}}{1.4 \times 10^{12}} = \mathbf{3000\ ms^{-1}}.$$

EXAMPLE 64. *A horizontal disc rotating about a vertical axis passing through its centre makes 180 rpm. A small piece of wax of mass 10 g falls vertically on the disc and adheres to it at a distance of 8 cm from its axis. If the frequency is thus reduced to 150 rpm, calculate the moment of inertia of the disc.*

Solution. Here $\nu_1 = 180$ rpm $= 3$ rps,

$$\nu_2 = 150 \text{ rpm} = \frac{150}{60} \text{ rps} = \frac{5}{2} \text{ rps}$$

$\therefore$ $$\omega_1 = 2\pi\nu_1 = 2\pi \times 3 = 6\pi \text{ rad s}^{-1},$$

$$\omega_2 = 2\pi \times \frac{5}{2} = 5\pi \text{ rad s}^{-1}$$

Let I be the M.I. of the disc about the given axis and I_2 be the M.I. when mass m sticks to it at distance r. Then

$$I_2 = I + mr^2$$

By conservation of angular momentum,

$$I_1\omega_1 = I_2\omega_2$$

$$I \times 6\pi = (I + mr^2) . 5\pi$$

or $$6I = 5I + 5mr^2$$

or $$I = 5mr^2 = 5 \times 10 \times 10^{-3} \times (8 \times 10^{-2})^2$$

$$= \mathbf{3.2 \times 10^{-8}\ kgm^2}.$$

✖ PROBLEMS FOR PRACTICE

1. An ice skater spins with arms outstretched at 1.9 rps. Her moment of inertia at this instant is 1.33 kg m^2. She pulls in her arms to increase her rate of spin. If the moment of inertia is 0.48 kg m^2 after she pulls in her arms, what is her new rate of rotation ? (**Ans.** 5.26 rps)
2. A mass of 2 kg is rotating on a circular path of radius 0.8 m with angular velocity of 44 rad s^{-1}. If the radius of the path becomes 1.0 m, what will be the value of angular velocity ? (**Ans.** 28.16 rad s^{-1})
3. A ball tied to a string takes 4 s to complete revolution along a horizontal circle. If, by pulling the cord, the radius of the circle is reduced to half of the previous value, then how much time the ball will take in one revolution ? (**Ans.** 1 s)
4. The sun rotates round itself once in 27 days. What will be period of revolution if the sun were to expand to twice its present radius ? Assume the sun to be a sphere of uniform density. (**Ans.** 108 days)
5. If the earth suddenly contracts by one-fourth of its present radius, by how much would the day be shortened ? (**Ans.** 10.5 h)
6. Prove that for an earth satellite, the ratio of its velocity at apogee (when farthest from the earth) to its velocity at perigee (when nearest to the earth) is the inverse ratio of its distances from apogee and perigee.
7. A uniform disc rotating freely about a vertical axis makes 90 revolutions per minute. A small piece of wax of mass m gram falls vertically on the disc and sticks to it at a distance of r cm from the axis. If the number of rotations per minute reduces to 60, find the moment of inertia of the disc. (**Ans.** mr^2 g cm^2)

HINTS

3. By conservation of angular momentum,

$$I_1 \omega_1 = I_2 \omega_2$$

or $$mr^2 \cdot \frac{2\pi}{T_1} = m\left(\frac{r}{2}\right)^2 \cdot \frac{2\pi}{T_2}$$

∴ $$T_2 = \frac{1}{4} T_1$$

$$= \frac{1}{4} \times 4 = \mathbf{1\ s.}$$

4. $I\omega = I'\omega'$

or $$\frac{2}{5} MR^2 \times \frac{2\pi}{T} = \frac{2}{5} M(2R)^2 \times \frac{2\pi}{T'}$$

or $$T' = 4T = 4 \times 27$$

$$= \mathbf{108\ days.}$$

6. By conservation of angular momentum,

Angular momentum of satellite at apogee

= Angular momentum of satellite at perigee

$$m\, v_a\, r_a = m v_p\, r_p \qquad [\because L = mvr]$$

or $$\frac{v_a}{v_p} = \frac{r_p}{r_a}.$$

7.32 ANALOGY BETWEEN TRANSLATIONAL AND ROTATIONAL MOTIONS

51. *Give the analogy between various quantities that describe linear and rotational motions.*

Analogy between translational and rotational motions. Table 7.1 summarises the analogy between the quantities that describe linear motion and the corresponding quantities that describe rotational motion.

Table 7.1 *Analogy between Linear and Rotational Motions.*

Linear motion		***Rotational motion***	
Quantities :			
displacement	s	angular displacement	θ
velocity	v	angular velocity	ω
acceleration	a	angular acceleration	α or a_θ
force	F	torque	τ
mass	m	moment of inertia	I
Expressions :			
velocity	$v = \frac{ds}{dt}$	angular velocity	$\omega = \frac{d\theta}{dt}$
acceleration	$a = \frac{dv}{dt}$	angular acceleration	$\alpha = \frac{d\omega}{dt}$
force	$F = ma = \frac{d}{dt}(mv)$	torque	$\tau = I\alpha = \frac{d}{dt}(I\omega)$
work done	$W = Fs$	work done	$W = \tau\,\theta$
linear K.E.	$E = \frac{1}{2} mv^2$	rotational K.E.	$E = \frac{1}{2} I\omega^2$
power	$P = Fv$	power	$P = \tau\omega$
linear momentum	$p = mv$	angular momentum	$L = I\omega$
impulse	$F\,\Delta t = mv - mu$	angular impulse	$\tau\,\Delta t = I\omega_f - I\omega_i$
Equations of motion :			
(i) $v = u + at$ (ii) $s = ut + \frac{1}{2}at^2$ (iii) $v^2 - u^2 = 2as$		(i) $\omega = \omega_0 + \alpha t$ (ii) $\theta = \theta_0 t + \frac{1}{2}\alpha t^2$ (iii) $\omega^2 - \omega_0^2 = 2\,\alpha\theta$	
Dimensions :			
velocity	$[LT^{-1}]$	angular velocity	$[T^{-1}]$
acceleration	$[LT^{-2}]$	angular acceleration	$[T^{-2}]$
mass	$[M]$	moment of inertia $I = \Sigma\ mr^2$	$[ML^2]$
force	$[MLT^{-2}]$	torque $\tau = Fr$	$[ML^2T^{-2}]$
linear K.E.	$[ML^2T^{-2}]$	rotational K.E.	$[ML^2T^{-2}]$
momentum	$[MLT^{-1}]$	angular momentum	$[ML^2T^{-1}]$
power	$[ML^2T^{-3}]$	power	$[ML^2T^{-3}]$

7.33 ROLLING MOTION

52. What is rolling motion ? Discuss the motion of a disc rolling without slipping on a level surface. Hence find the condition for rolling without slipping.

Ans. Rolling motion. *Rolling motion can be regarded as the combination of pure rotation and pure translation.* The wheels of all vehicles running on a road have rolling motion. Consider a disc of radius R, rolling on a level surface without slipping. This means that at any instant of time the bottom of the disc which is in contact with the surface is at rest.

The rolling motion of the disc has *two* simultaneous motions :

(*i*) ***Translational motion.*** The translational velocity of the disc is the velocity $\vec{v}_{CM}$ of its centre of mass. As the centre of mass of the rolling disc lies at its geometric centre C, so $\vec{v}_{CM}$ is the velocity of C. It is parallel to the level surface as shown in Fig. 7.68.

$\therefore \quad v_{trans} = v_{CM}$

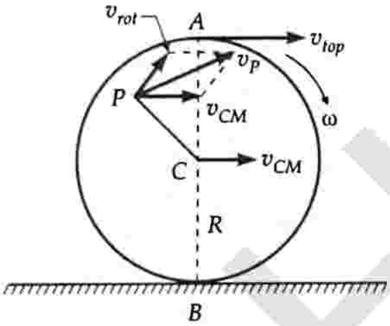


Fig. 7.68 Rolling motion (without slipping) of a disc on a level surface.

(*ii*) ***Rotational motion.*** The disc rotates with angular velocity ω about its symmetry axis through C. The linear velocity of a particle P at distance r from the axis due to the rotational motion is $v_{rot} = r\omega$

The velocity $\vec{v}_{rot}$ is directed perpendicular to the radius vector CP as shown in Fig. 7.68.

The effective linear velocity $\vec{v}$ of particle P is the resultant of the velocities $\vec{v}_{rot}$ and $\vec{v}_{CM}$. It can be shown that $\vec{v}$ is perpendicular to the line AP. Therefore, the line passing through the bottom point B and parallel to the axis through C is called the *instantaneous axis of rotation.*

At the bottom point B, the linear velocity $\vec{v}_{rot}$, due to rotation, is directed exactly opposite to the translational velocity $\vec{v}_{CM}$. Also, $v_{rot} = R\omega$ at the point B. The point B will be instantaneously at rest if $v_{CM} = R\omega$. Hence for the disc the *condition for rolling without slipping is* $v_{CM} = R\omega$

At the top point A of the disc, the linear velocity $R\omega$ due to rotational motion and the translational velocity v_{CM} are in the same direction, parallel to level surface. Therefore,

$$v_{top} = R\omega + v_{CM} = v_{CM} + v_{CM} = 2\,v_{CM}$$

53. Obtain the expression for the linear acceleration of a cylinder rolling down an inclined plane and hence find the condition for the cylinder to roll down without slipping.

Solid cylinder rolling down an inclined plane without slipping. Consider a solid cylinder of mass M and radius R rolling down a plane inclined at an angle θ to the horizontal, as shown in Fig. 7.69. Suppose the *cylinder rolls down without slipping*. The condition for rolling without slipping is that at each instant *the line of contact* of the cylinder with the surface at P *is momentarily at rest* and the cylinder rotates about this line as axis. The centre of mass of the cylinder moves in a straight line parallel to the inclined plane. Notably, it is the friction which prevents slipping.

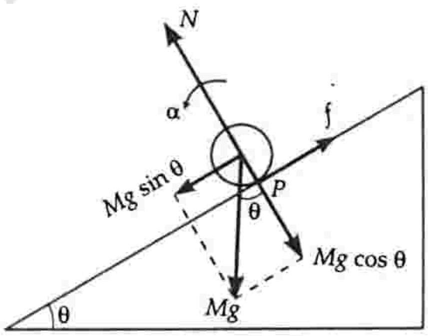


Fig. 7.69 Cylinder rolling without slipping.

The external forces acting on the cylinder are

(*i*) The weight Mg of the cylinder acting vertically downwards through the centre of mass of the cylinder.

(*ii*) The normal reaction N of the inclined plane acting perpendicular to the plane at P.

(*iii*) The frictional force f acting upwards and parallel to the inclined plane.

The weight Mg can be resolved into two rectangular components :

(*i*) $Mg \cos\theta$ perpendicular to the inclined plane.

(*ii*) $Mg \sin\theta$ acting down the inclined plane.

As there is no motion in a direction normal to the inclined plane, so

$$N = Mg \cos\theta$$

Applying Newton's second law to the linear motion of the centre of mass, the net force on the cylinder rolling down the inclined plane is

$$F = Ma = Mg \sin\theta - f \qquad ...(1)$$

It is only the force of friction f which exerts torque τ on the cylinder and makes it rotate with angular acceleration α. It acts tangentially at the point of contact P and has lever arm equal to R.

$\therefore \quad \tau = \text{Force} \times \text{force arm} = f \, . \, R$

Also, $\quad \tau = \text{M.I.} \times \text{angular acceleration} = I\alpha$

$\therefore \quad f R = I\alpha$

or $\quad f = \dfrac{I\alpha}{R} = \dfrac{Ia}{R^2} \qquad \left[\because \alpha = \dfrac{a}{R}\right]$

Putting the value of f in equation (1), we get

$$Ma = Mg \sin\theta - \frac{Ia}{R^2}$$

$$a = g \sin\theta - \frac{Ia}{MR^2}$$

or $\quad a + \dfrac{Ia}{MR^2} = g \sin\theta$

or $\quad a\left[1 + \dfrac{I}{MR^2}\right] = g \sin\theta$

$$a = \frac{g \sin\theta}{1 + \dfrac{I}{MR^2}}$$

Moment of inertia of the solid cylinder about its axis $= \frac{1}{2} MR^2$

$$\therefore \quad a = \frac{g \sin\theta}{1 + \dfrac{\frac{1}{2} MR^2}{MR^2}}$$

or $\quad a = \dfrac{2}{3} g \sin\theta$

Clearly, the linear acceleration a of solid cylinder rolling down an inclined plane is less than the acceleration due to gravity $g \, (a < g)$. The linear acceleration of the cylinder is constant for a given inclined plane (or given θ) and is independent of its mass M and radius R. However, for a hollow cylinder, $I = MR^2$, the value of a would decrease to $\frac{1}{2} g \sin\theta$.

From equation (1), the value of force of friction is

$$f = Mg \sin\theta - Ma$$

$$= Mg \sin\theta - M \, . \, \tfrac{2}{3} g \sin\theta = \tfrac{1}{3} Mg \sin\theta$$

If μ_s is the coefficient of friction between the cylinder and the inclined plane, then

$$\mu_s = \frac{f}{N} = \frac{\frac{1}{3} Mg \sin\theta}{Mg \cos\theta} = \frac{1}{3} \tan\theta$$

To prevent slipping, the coefficient of static friction must be equal to or greater than the above value. That is

$$\mu_s \geq \tfrac{1}{3} \tan\theta \quad \text{or} \quad \tan\theta \leq 3\mu_s.$$

54. *Write an expression for the kinetic energy of a body rolling without slipping.*

Kinetic energy of rolling motion. The kinetic energy of a body rolling without slipping is the sum of kinetic energies of translation and rotation.

K = K.E. of the translational motion of CM
+ K.E. of rotational motion of CM

$$= \frac{1}{2} mv_{CM}^2 + \frac{1}{2} I\omega^2.$$

where v_{CM} is the velocity of CM and I is the moment of inertia about the symmetry axis of the rolling body. If R is the radius and k the radius of gyration of the rolling body, then

$$v_{CM} = R\omega \quad \text{and} \quad I = mk^2$$

$$\therefore \quad K = \frac{1}{2} mv_{CM}^2 + \frac{1}{2} mk^2 \left(\frac{v_{CM}}{R}\right)^2$$

or $\quad K = \dfrac{1}{2} mv_{CM}^2 \left(1 + \dfrac{k^2}{R^2}\right)$

Examples based on
Motion of a Cylinder Rolling without Slipping on an Inclined Plane

FORMULAE USED

For a cylinder of mass M and radius R rolling without slipping down plane inclined at angle θ with the horizontal,

1. Force of friction between the plane and cylinder,
$$f = \frac{1}{3} Mg \sin\theta$$
2. Linear acceleration, $a = \dfrac{2}{3} g \sin\theta$
3. Condition for rolling without slipping is
$$\mu_s > \frac{1}{3} \tan\theta$$

UNITS USED

Accelerations a and g are in ms^{-2} and coefficient of friction μ_s has no units.

EXAMPLE 65. *A cylinder of mass 5 kg and radius 30 cm is rolling down an inclined plane at an angle of 45° with the horizontal. Calculate (i) force of friction, (ii) acceleration with which the cylinder rolls down and (iii) the minimum value of static friction so that cylinder does not slip while rolling down the plane.*

Solution. Here $M = 5$ kg, $R = 30$ cm $= 0.30$ m, $\theta = 45°$

(*i*) Force of friction,

$$f = \frac{1}{3} Mg \sin\theta = \frac{1}{3} \times 5 \times 9.8 \sin 45° = \mathbf{11.55\ N.}$$

(*ii*) Acceleration,

$$a = \frac{2}{3} g \sin\theta = \frac{2}{3} \times 9.8 \sin 45° = \mathbf{4.62\ ms^{-2}}.$$

(*iii*) Minimum value of coefficient of static friction,

$$\mu_s = \frac{1}{3} \tan\theta = \frac{1}{3} \tan 45° = \frac{1}{3}.$$

EXAMPLE 66. *Three bodies, a ring, a solid cylinder and a solid sphere roll down the same inclined plane without slipping. They start from rest. The radii of the bodies are identical. Which of the bodies reaches the ground with maximum velocity ?* **[NCERT]**

Solution. Suppose a body of mass *m* starting from rest rolls down an inclined plane. We assume there is no loss of energy due to friction.

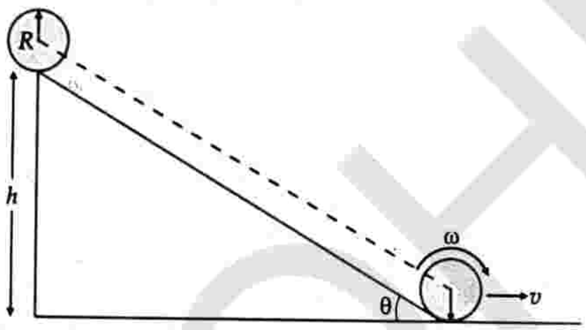


Fig. 7.70 A body rolling down an inclined plane.

By conservation of energy,

P.E. lost by the body in rolling down the inclined plane

= K.E. gained by the body

= Translational K.E. + Rotational K.E.

$$= \frac{1}{2} mv^2 + \frac{1}{2} I\omega^2 = \frac{1}{2} mv^2 + \frac{1}{2} mk^2 . \left(\frac{v}{R}\right)^2$$

or $$mgh = \frac{1}{2} mv^2 \left(1 + \frac{k^2}{R^2}\right)$$

or $$v = \sqrt{\frac{2gh}{1 + k^2 / R^2}}$$

Clearly, the velocity *v* attained by the rolling body at the bottom of the inclined plane is independent of its mass.

For a ring, $k^2 = R^2$

$$\therefore \quad v_{ring} = \sqrt{\frac{2gh}{1+1}} = \sqrt{gh}$$

For a solid cylinder,

$$k^2 = R^2 / 2$$

$$\therefore \quad v_{cylinder} = \sqrt{\frac{2gh}{1 + 1/2}} = \sqrt{\frac{4gh}{3}}$$

For a solid sphere,

$$k^2 = 2R^2 / 5$$

$$\therefore \quad v_{sphere} = \sqrt{\frac{2gh}{1 + 2/5}} = \sqrt{\frac{10gh}{7}}$$

Clearly, among the three bodies the sphere has the greatest and the ring has the least velocity of the centre of mass at the bottom of the inclined plane.

EXAMPLE 67. *A solid cylinder of radius 4 cm and mass 250 g rolls down an inclined plane (1 in 10). Calculate the acceleration and the total energy of the cylinder after 5 s.*

Solution. Here $M = 250$ g $= 0.25$ kg,

$R = 4$ cm $= 0.04$ m, $\sin\theta = \frac{1}{10}$, $t = 5$ s

Acceleration with which the cylinder rolls down,

$$a = \frac{g \sin\theta}{1 + \frac{I}{MR^2}} = \frac{g \sin\theta}{1 + \frac{\frac{1}{2}MR^2}{MR^2}}$$

$$= \frac{2}{3} g \sin\theta = \frac{2}{3} \times 9.8 \times \frac{1}{10} = \mathbf{0.653\ ms^{-2}}.$$

Using first equation of motion,

$$v = u + at = 0 + 0.653 \times 5 = 3.26\ ms^{-1}$$

Total K.E. of the cylinder

= Translational K.E. + Rotational K.E.

$$= \frac{1}{2} Mv^2 + \frac{1}{2} I\omega^2$$

$$= \frac{1}{2} Mv^2 + \frac{1}{2} \times \frac{1}{2} MR^2 \times \frac{v^2}{R^2}$$

$$= \frac{3}{4} Mv^2 = \frac{3}{4} \times 0.25 \times (3.26)^2 = \mathbf{2.0\ J.}$$

PROBLEMS FOR PRACTICE

1. A solid cylinder of mass 10 kg is rolling perfectly on a plane of inclination 30°. Find the force of friction between the cylinder and the surface of the inclined plane. (**Ans.** 16.33 N)
2. A solid cylinder of mass 8 kg and radius 50 cm is rolling down a plane inclined at an angle of 30° with the horizontal. Calculate (*i*) force of friction,

(*ii*) acceleration with which the cylinder rolls down and (*iii*) the minimum value of coefficient of friction so that cylinder does not slip while rolling down the plane.

[**Ans.** (*i*) 13.06 N (*ii*) 3.267 ms^{-2} (*iii*) 0.192]

3. If a sphere rolls (starting from rest) in 5.3 s along a plane 1 m in length of which the upper end is raised 0.01 m above the lower, find the acceleration due to gravity. (**Ans.** 9.97 ms^{-2})

HINTS

3. Here $u = 0,\ t = 5.3\,s,\ s = 1\,m,\ \sin\theta = \frac{1}{100} = 0.01$

As $s = ut + \frac{1}{2}at^2 \quad \therefore \quad 1 = 0 + \frac{1}{2}a \times (5.3)^2$

or $a = \frac{2 \times 1}{5.3 \times 5.3} = 0.0712\ ms^{-2}$

But $a = \frac{g \sin\theta}{1 + \frac{I}{MR^2}} = \frac{g \sin\theta}{1 + \frac{\frac{2}{5}MR^2}{MR^2}} = \frac{5}{7} g \sin\theta$

$\therefore \quad g = \frac{7a}{5 \sin\theta} = \frac{7 \times 0.0712}{5 \times 0.01} = \mathbf{9.97\ ms^{-2}}.$

7.34 MASS POINT ON STRING WOUND ON A CYLINDER

55. *A light string is wound round a cylinder and carries a mass tied to it at the free end. When the mass is released, calculate (a) the linear acceleration of the descending mass and (b) the angular acceleration of the cylinder and (c) the tension in the string. Show that the acceleration of mass is less than `g'.*

Motion of a mass point attached to a string wound on a cylinder. As shown in Fig. 7.71 consider a solid cylinder of mass m and radius R. It is mounted on a frictionless horizontal axle so that it can freely rotate about its axis. A light string is wound round the cylinder and mass m is suspended from it. When the mass m is released from rest, it moves down with acceleration a. Let T be the tension in the string.

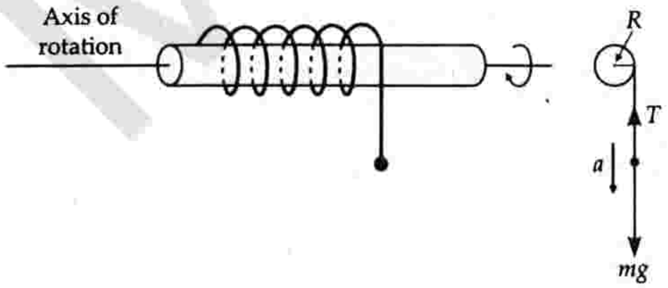


Fig. 7.71 String wound on a cylinder and carrying a point mass.

(*a*) ***Linear acceleration of the point mass.*** The forces acting on the point mass are

(*i*) Its weight mg acting vertically downwards.

(*ii*) Tension T in the string acting upwards.

According to Newton's second law, the net downward force on the point mass is

$$ma = mg - T \quad ...(1)$$

The tension T in the string acts tangentially on the cylinder and produces a torque τ given by

$$\tau = \text{Force} \times \text{lever arm} = T \, . \, R \quad ...(2)$$

If I is the moment of inertia of the cylinder and α, the angular acceleration produced in it, then

$$\tau = I\alpha \quad ...(3)$$

From equations (2) and (3),

$$TR = I\alpha$$

or $$T = \frac{I}{R}\alpha = \frac{Ia}{R^2} \quad \left[\because \alpha = \frac{a}{R}\right] \quad ...(4)$$

From equation (1), we have

$$ma = mg - \frac{Ia}{R^2}$$

or $$ma + \frac{Ia}{R^2} = mg$$

or $$ma\left(1 + \frac{I}{mR^2}\right) = mg$$

or $$a = \frac{g}{1 + \frac{I}{mR^2}} \quad ...(5)$$

This gives the linear downward acceleration of the point mass.

(*b*) ***Angular acceleration of the point mass.*** As I, m and R are positive quantities, so a is always less than 'g'.

Angular acceleration,

$$\alpha = \frac{a}{R} = \frac{g / R}{1 + \frac{I}{mR^2}} \quad ...(6)$$

(*c*) ***Tension in the string.*** From equations (4) and (5), we have

$$T = \frac{Ia}{R^2} = \frac{Ig}{R^2\left(1 + \frac{I}{mR^2}\right)}$$

$$= \frac{Ig}{R^2 \cdot \frac{I}{mR^2}\left(\frac{mR^2}{I} + 1\right)}$$

or $$T = \frac{mg}{1 + \frac{mR^2}{I}} \quad ...(7)$$

Clearly, T is less than the weight mg of the point mass.

EXAMPLE 68. *A body of mass 5 kg is attached to a weightless string wound round a cylinder of mass 8 kg and radius 0.3 m The body is allowed to fall. Calculate (i) tension in the string (ii) acceleration with which the body falls and (iii) the angular acceleration of the cylinder.*

Solution. Here $m = 5$ kg, $M = 8$ kg, $R = 0.3$ m

(*i*) M.I. of the cylinder,

$$I = \frac{1}{2} MR^2$$

∴ Tension in the string,

$$T = \frac{mg}{1 + \frac{mR^2}{I}} = \frac{mg}{1 + \frac{2\,mR^2}{MR^2}} = \frac{mg}{1 + \frac{2m}{M}}$$

$$= \frac{5 \times 9.8}{1 + \frac{2 \times 5}{8}} = \frac{49.0}{2.25} = \mathbf{21.78\ N.}$$

(*ii*) Linear acceleration,

$$a = \frac{g}{1 + \frac{I}{mR^2}} = \frac{g}{1 + \frac{MR^2}{2\,mR^2}}$$

$$= \frac{g}{1 + \frac{M}{2m}} = \frac{9.8}{1 + \frac{8}{2 \times 5}}$$

$$= \frac{9.8}{1.8} = \mathbf{5.44\ ms^{-2}}.$$

(*iii*) Angular acceleration,

$$\alpha = \frac{a}{R} = \frac{5.44}{0.3}$$

$$= \mathbf{18.13\ rad\ s^{-1}}$$

Very Short Answer Conceptual Problems

Problem 1. What is the advantage of the concept of centre of mass ?

Solution. Centre of mass helps us to describe the behaviour of a macroscopic body in terms of the laws developed for the microscopic bodies. If we are not concerned with the internal motion and structure of a system, the gross motion of the system can be analysed by applying Newton's laws of motion to the CM of the system which is a point where the entire mass of the system may be assumed to be concentrated and where all the external forces are assumed to apply.

Problem 2. Should the centre of mass of a body necessarily lie inside the body ? *[Himachal, 01, 03, 04]*

Solution. Not necessarily. For example, the CM of a ring lies in its hollow portion.

Problem 3. Does the centre of mass of a solid necessarily lie within the body ? If not, give an example.

Solution. No. For example, the CM of *L*-shaped rod lies in the region outside the rod.

Problem 4. Is centre of mass a reality ?

Solution. No. The centre of mass of a system is a hypothetical point which acts as a single mass particle of the system for an external force.

Problem 5. Is it correct to say that the centre of mass of a system of *n* particles is always given by the average position vectors of the constituent particles ? If not, when is this statement true ?

Solution. No. This is true when all the particles of the system are of same mass.

Problem 6. If two particles of masses m_1 and m_2 move with velocities v_1 and v_2 towards each other on a smooth horizontal table, what is the velocity of their centre of mass ?

Solution. $v_{CM} = \dfrac{m_1 v_1 - m_2 v_2}{m_1 + m_2}$.

Problem 7. Name the physical quantity that corresponds to the moment of force. On what factors does it depend ?

Solution. The moment of force is called torque. It depends on

(*i*) the magnitude of force.

(*ii*) the perpendicular distance of the line of action of force from the axis of rotation.

Problem 8. What happens to the moment of force about a point, if the line of action of the force moves towards the point ?

Solution. Moment of force

= Force × the perpendicular distance of the line of action of force "from" the axis of rotation.

Hence the moment of force about a point decreases if the line of action of the force moves towards that point.

Problem 9. A body is in rotational motion. Is it necessary that a torque be acting on it ? *[Delhi 12]*

Solution. No, torque is required only for angular acceleration.

Problem 10. Why do we prefer to use a wrench of longer arm ? *[Central Schools 05]*

Solution. The torque applied on the nut by the wrench is equal to the force multiplied by the perpendicular distance from the axis of rotation. Hence to increase torque a wrench of longer arm is preferred.

Problem 11. Why in hand driven grinding machine, handle is put near the circumference of the stone or wheel ?

Solution. For a given force, torque can be increased if the perpendicular distance of the point of application of the force from the axis of rotation is increased. Hence the handle put near the circumference produces maximum torque.

Problem 12. It is difficult to open the door by pushing it or pulling it at the hinge. Why ?

Solution. When the force is applied at the hinges, the line of action of the force passes through the axis of rotation *i.e.*, $r = 0$, so $\tau = rF \sin \theta = 0$. So we cannot open the door by pushing or pulling it at the hinges.

Problem 13. Why a force is applied at right angles to the heavy door at the outer edge while closing or opening it ?

Solution. Torque, $\tau = rF \sin \theta$. For a force applied at right angle to the outer edge of the door both $\sin \theta\,(= \sin 90° = 1)$ and r are maximum. Hence the torque produced is maximum.

Problem 14. A man climbs a tall, old step ladder that has a tendency to sway. He feels much more unstable when standing near the top than when near the bottom. Why ?

Solution. When the man stands at the top of step ladder it increases its distance from the point of contact, the point about which it can slip (rotate). The torque exerted is equal of rF. Hence larger the r, more are the chances to sway away.

Problem 15. Why it is easier to open a tap with two fingers than with one finger ?

Solution. With two fingers, we apply a couple whose moment is equal to the product of the force and perpendicular distance between the two fingers (which is equal to the length of the handle of the tap). When we apply force with one finger, an equal and opposite force of reaction acts at the axis of rotation. This results in a couple of smaller arm and hence lesser moment. So it is easier to open a tap with two fingers than with one finger.

Problem 16. A faulty balance with unequal arms has its beam horizontal. Are the weights of the two pans equal ?

Solution. They are of unequal mass. Their masses are in the inverse ratio of the arms of the balance.

Problem 17. Can the couple acting on a rigid body produce translatory motion ?

Solution. No, the couple acting on a rigid body cannot produce translatory motion ; it can cause only rotatory motion as the resultant force is zero.

Problem 18. A labourer standing near the top of an old wooden step ladder feels unstable. Why ?

Solution. The ladder can rotate about the point of contact of the ladder with the ground. When the labourer is at the top of the ladder, the lever arm of the force is large. Hence the turning effect on the ladder will be large.

Problem 19. Which physical quantities are expressed by the following :

(*i*) the rate of change of angular momentum, and

(*ii*) moment of linear momentum ?

[Himachal 06 ; Central Schools 08]

Solution. (*i*) Torque. (*ii*) Angular momentum.

Problem 20. If no external torque acts on a body, will its angular velocity remain conserved ?

Solution. No. Angular velocity is not conserved but angular momentum is conserved.

Problem 21. When a labourer cuts down a tree, he makes a cut on the side facing the direction in which he wants it to fall. Why ?

Solution. The weight of tree exerts a torque about the point where the cut is made. This causes rotation of the tree about the cut.

Problem 22. Define the term angular impulse.

Solution. The angular impulse acting on a body is the change of angular momentum of the body about a given axis.

Problem 23. Which component of linear momentum does not contribute to angular momentum ?

Solution. The radial component of linear momentum does not contribute to angular momentum.

Problem 24. A particle revolves uniformly along a circular path, on a smooth horizontal table, by means of a string connected to it. Does its angular momentum change, if the string is suddenly cut ?

Solution. No. The angular momentum remains unaltered when the string is cut.

Problem 25. A heavenly body (such as a comet) revolves around a massive star in a highly elliptical orbit. Is its angular momentum constant over the entire orbit ? (Ignore any mass loss of the comet when it comes too close to the star).

Solution. The heavenly body revolves around the massive star under the effect of gravitational force, which is purely radial. The torque exerted by such a force is zero. Hence the angular momentum of the heavenly body remains constant over its entire orbit.

Problem 26. A projectile acquires angular momentum about the point of projection during its flight. Does it violate the conservation of angular momentum ?

Solution. A projectile will not acquire angular momentum, if no external force acts on it. However during its flight, the projectile is acted upon by the force of gravity and acquires angular momentum.

Problem 27. Is a body in circular motion in equilibrium ?

Solution. No. A body in circular motion has a centripetal acceleration $\vec{a}$ directed towards the centre of the circle. Since $\vec{a} \neq 0$, the body is not in equilibrium.

Problem 28. When is a body lying in a gravitation field in stable equilibrium ?

Solution. A body in a gravitation field will be in stable equilibrium if the vertical line through its centre of gravity passes through the base of the body.

Problem 29. Can a body in equilibrium while in motion ? If yes, give an example.

Solution. Yes. A body in motion will be in equilibrium if it has no linear and angular accelerations. Hence a body moving with uniform velocity along a straight line will be in equilibrium.

Problem 30. The bottom of a ship is made heavy. Why ?

Solution. The bottom of a ship is made heavy so that its centre of gravity remains low. This ensures the stability of its equilibrium.

Problem 31. Why does a girl lean towards right while carrying a bag in her left hand ?

Solution. When the girl carries a bag in her left hand, her CG shifts towards left. In order to bring it in the middle (for stability of equilibrium), the girl has to lean towards her right.

Problem 32. Some heavy boxes are to be loaded along with some empty boxes on a cart. Which boxes should be put on the cart first and why ?

Solution. The heavy boxes should be loaded first so that the CG of the loaded cart remains in the lowest position. This ensures stability of equilibrium.

Problem 33. Standing is not allowed in a double decker bus. Why ?

Solution. When the passengers stand in the upper deck, the CG of the loaded bus is raised which makes it less stable.

Problem 34. Why we cannot rise from a chair without bending a little forward ?

Solution. Our weight exerts a torque about our feet. This makes difficult for us to rise from the chair. When we bend forward, the CG of our body comes above our feet. The torque due to our weight becomes zero and we can easily rise from the chair.

Problem 35. A system is in stable equilibrium. What can we say about its potential energy ?

Solution. The potential energy of the system is minimum.

Problem 36. Why is moment of inertia also called rotational inertia ?

Solution. The moment of inertia gives a measure of inertia in rotational motion. So it is also called rotational inertia.

Problem 37. Give the physical significance of moment of inertia.

Solution. The moment of inertia plays the same role in rotatory motion as the mass does in translatory motion. It gives a measure of inertia in rotational motion.

Problem 38. Does moment of inertia of a body change with the change of the axis of rotation ?

Solution. Yes. The moment of inertia of a body changes with the change in position and orientation of the axis of rotation.

Problem 39. Does the moment of inertia of a rigid body change with the speed of rotation ? *[Himachal 08]*

Solution. No, because the moment of inertia depends upon the axis of rotation and the distribution of mass.

Problem 40. About which axis, the moment of inertia of a body is minimum ?

Solution. The moment of inertia of a body is minimum about an axis passing through its centre of mass.

Problem 41. Can the mass of body be taken to be concentrated at its centre of mass for the purpose of calculating its rotational inertia ?

Solution. No. The moment of inertia greatly depends on the distribution of mass about the axis of rotation.

Problem 42. About which axis would a uniform cube have a minimum rotational inertia ?

Solution. About a diagonal, because the mass is more concentrated about a diagonal.

Problem 43. Is radius of gyration a constant quantity ? *[Himachal 01, 04]*

Solution. No. It changes with the change in position of the axis of rotation.

Problem 44. Does the radius of gyration depend upon the speed of rotation of the body ?

Solution. No, it depends only on the distribution of mass of the body.

Problem 45. Two lenses of same mass and same radius are given. One is convex and other is concave. Which one will have greater moment of inertia, when rotating about an axis perpendicular to the plane and passing through the centre ?

Solution. The concave lens will have greater moment of inertia because the concave lens is thin at its centre and its mass is more concentrated at the outer edge.

Problem 46. A disc is recast into a thin walled cylinder of same radius. Which will have large moment of inertia ?

Solution. Hollow cylinder will have larger moment of inertia because most of its mass is located at comparatively larger distance from the axis of rotation.

Problem 47. Two solid spheres of the same mass are made of metals of different densities. Which of them has a larger moment of inertia about the diameter ? Why ? *[Delhi 10]*

Solution. The sphere with small density will have larger radius and hence large moment of inertia.

Problem 48. What is the advantage of the flywheel ?

Solution. In case of a flywheel, the whole mass is practically near the rim which is situated far away from the axis of rotation. This is done to increase the moment of inertia of the wheel, thereby making the motion smooth and less jerky.

Problem 49. Why spokes are provided in a bicycle wheel ?

Solution. By connecting to the rim of wheel to the axle through the spokes, the mass of the wheel gets concentrated at its rim. This increases its moment of inertia. This ensures its uniform speed.

Problem 50. Will two spheres of equal masses, one solid and the other hollow have equal moments of inertia ? Give reason.

Solution. The hollow sphere will have a greater moment of inertia because its entire mass is concentrated at the boundary of the sphere *i.e.*, at maximum distance from the axis of rotation.

Problem 51. Why is it more difficult to revolve a stone tied to a large string than a stone tied to a smaller string ? *[Chandigarh 04]*

Solution. The length of the string increases the distance of rotating mass from the axis of rotation and hence moment of inertia of the system is increased. Now, $\tau = I\alpha$ thus a system with large moment of inertia requires large torque for its rotation.

Problem 52. Two satellites of equal masses, which can be considered as particles are orbiting the earth at different heights ? Will their moments of inertia be same or different ? *[Himachal 06]*

Solution. M.I. of a satellite, $I = Mr^2$ *i.e.,* $I \propto r^2$

Hence the two satellites orbiting the earth at different heights will have different moments of inertia. The satellite orbiting at a larger height will have a larger moment of inertia.

Problem 53. What is the use of flywheel in railway engine ?

Solution. In a flywheel, most of the mass is concentrated at its rim. So it has a large moment of inertia. Any change of angular momentum imparted to the wheel by the piston results in a lesser change of angular velocity ($L = I\omega$ or $\omega = L/I$). Moreover, a flywheel stores a large amount of rotational energy ($\frac{1}{2} I\omega^2$). This helps the wheel fly off the dead point.

Problem 54. There is a stick half of which is wooden and half is of steel. It is pivoted at the wooden end and a force is applied at the steel end at right angles to its length. Next, it is pivoted at the steel end and the same force is applied at the wooden end. In which case is the angular acceleration more and why ?

Solution. The distribution of mass is farther from the axis of rotation in the first case than in the second case. So the moment of inertia is more in first case than in second case, but the applied torque τ is same in both cases. As $\tau = I\alpha$ or $\alpha = \tau / I$, so the angular acceleration α is less in first case than in second case.

Problem 55. Is the angular momentum of a system always conserved ? If not, under what condition is it conserved ?

Solution. No, angular momentum of a system is not always conserved. It is conserved only when no external torque acts on the system.

Problem 56. A flywheel is revolving with a constant angular velocity. A chip of its rim breaks and flies away. What will be the effect on its angular velocity ?

Solution. Due to the decrease in its mass, the moment of inertia of the flywheel will decrease. To conserve angular momentum, the angular velocity of the flywheel will increase.

Problem 57. A cat is able to land on its feet after a fall. Why ? *[Himachal 05, 07C, 08C]*

Solution. When a cat falls to ground from a height, it stretches its body along with the tail so that its moment of inertia becomes high. Since $I\omega$ is to remain constant, the value of angular speed ω decreases and therefore the cat is able to land on the ground gently.

Problem 58. Why there are two propellers in a helicopter ?

Solution. If there were only one propeller in the helicopter then, due to conservation of angular momentum, the helicopter itself would have turned in the opposite direction.

Problem 59. The speed of a whirl wind in a tornado is alarmingly high. Why ?

[Himachal 07C, 08 ; Central Schools 04]

Solution. In a whirl wind, the air from nearby region gets concentrated in a small space thereby decreasing the value of moment of inertia considerably. Since, $I\omega$ = constant, due to decrease in moment of inertia, the angular speed becomes quite high.

Problem 60. If earth contracts to half its radius, what would be the length of the day ? *[Himachal 08]*

Solution. The moment of inertia ($I = \frac{2}{5} MR^2$) of the earth about its own axis will become one fourth and so its angular velocity will become four times ($L = I\omega$ = constant). Hence the time period will reduce to one fourth ($T = 2\pi / \omega$) *i.e.*, 6 hours.

Problem 61. Two boys of the same weight sit at the opposite ends of a diameter of a rotating circular table. What happens to the speed of rotation if they move nearer to the axis of rotation ?

Solution. The moment of inertia of the system (circular table + two boys) decreases. To conserve angular momentum ($L = I\omega$ = constant), the speed of rotation of the circular table increases.

Problem 62. A thin wheel can stay up right on its rim for a considerable length of time when rolled with a considerable velocity, while it falls from its upright position at the slightest disturbance when stationary. Give reason.

Solution. When the wheel is rolling upright, it has angular momentum in the horizontal direction *i.e.*, along the axis of the wheel. Because the angular momentum is to remain conserved, the wheel does not fall from its upright position because that would change the direction of angular momentum. The wheel falls only when it loses its angular velocity due to friction.

Problem 63. A person is standing on a rotating table with metal spheres in his hands. If he withdraws his hands to his chest, what will be the effect on his angular velocity ? *[Himachal 08]*

Solution. When the person withdraws his hands to his chest, his moment of inertia decreases. No external torque is acting on the system. So to conserve angular momentum, the angular velocity increases.

Problem 64. A circular turn table rotates at constant angular velocity about a vertical axis. There is no friction and no driving torque. An ice pan containing ice also rotates with it. The ice melts but none of the water escapes from the pan. Is the velocity now greater, the same or less than the original velocity ? Give reason.

Solution. Due to accumulation of water near the edge, the moment of inertia of the system increases. To conserve angular momentum, the angular velocity of the system decreases.

Problem 65. Many rivers flow towards the equator. What effect does the sediment they carry to the seas have on the rotation of the earth ?

Solution. The rivers carry sediments away from the axis of rotation of the earth. This increases the rotational inertia of the earth. To conserve angular momentum, the rotational speed of the earth decreases.

Problem 66. The moments of inertia of two rotating bodies A and B are I_A and I_B ($I_A > I_B$) and their angular momenta are equal. Which one has a greater kinetic energy ? *[Central Schools 2010]*

Solution. Angular momentum,

$$L = I\omega$$

K.E. of rotation, $K = \frac{1}{2} I\omega^2 = \frac{1}{2}\frac{I^2\omega^2}{I} = \frac{1}{2}\frac{L^2}{I}$

For constant L, $K \propto \frac{1}{I}$

As $I_A > I_B$

$\therefore$ $K_A < K_B$ or $K_B > K_A$.

So body B has a greater rotational K.E.

Problem 67. If angular momentum is conserved in a system whose moment of inertia is decreased, will its rotational kinetic energy be conserved ? *[Himachal 06 ; Chandigarh 08]*

Solution. Here $L = I\omega$ = constant

Rotational K.E. is given by

$$K = \frac{1}{2} I\omega^2 = \frac{1}{2}\frac{I^2\omega^2}{I} = \frac{1}{2}\cdot\frac{L^2}{I}$$

For constant L, $K \propto \frac{1}{I}$.

So when the moment of inertia decreases, the rotational K.E. increases. Hence rotational K.E. is not conserved.

Problem 68. How does an ice-skater, a ballet dancer or an acrobat take advantage of the principle of conservation of angular momentum ? *[Himachal 08]*

Or

How does a ballet dancer vary her angular speed by oustretcing her arms and legs ? *[Central Schools 08]*

Solution. An ice-skater, a ballet dancer or an acrobat is able to change his angular speed during the course of the performance. When the performer stretches out his hands and legs, his moment of inertia increases and the angular speed decreases. On the other hand, when he folds his hands and the legs near his body, the moment of inertia decreases and he is able to increase the angular speed.

Problem 69. If earth were to shrink suddenly, what would happen to the length of the day ?

[Chandigarh 04 ; Central Schools 05]

Solution. When the earth shrinks, the moment of inertia $\left(I = \frac{2}{5}MR^2\right)$ decreases about its own axis due to the decrease in radius R. To conserve angular momentum. ($L = I\omega = I \cdot \frac{2\pi}{T}$), the time period T decreases. That is, the length of the day decreases.

Problem 70. A body A of mass M while falling vertically downwards under gravity breaks into two parts ; a body B of mass $M/3$ and a body C of mass $2M/3$. How does the centre of mass of bodies B and C taken together shift compared to that of A ? *[AIEEE 05]*

Solution. The centre of mass of bodies B and C does not shift compared to body A. It continues to move vertically downwards under the effect of gravity. This is because there is no new external force acting on the system.

Problem 71. Two identical particles move towards each other with velocity 2 v and v respectively. What is the velocity of the centre of mass ? *[AIEEE 02]*

Solution. Here $m_1 = m_2 = m$; $v = 2v$ and $v_2 = -v$

$$\therefore \quad v_{CM} = \frac{m_1 v_1 + m_2 v_2}{m+m} = \frac{m \times 2v + m(-v)}{m+m} = \frac{v}{2}.$$

Problem 72. A particle moves in a circular path with decreasing speed. What happens to its angular momentum ? *[IIT 05 ; Chandigarh 08]*

Solution. The angular momentum of a particle of mass m moving with velocity $\vec{v}$ along a circular path of radius r is given by

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v}$$

When the speed v decreases, the magnitude of angular momentum decreases. But the direction of angular momentum remains unchanged.

Problem 73. A particle performing uniform circular motion has angular momentum L. What will be the new angular momentum, if its angular frequency is doubled and its kinetic energy halved ? *[AIEEE 03]*

Solution. Rotational K.E.,

$$K = \frac{1}{2} I\omega^2 \quad \therefore \quad I = \frac{2K}{\omega^2}$$

Angular momentum,

$$L = I\omega = \frac{2K}{\omega^2} \cdot \omega = \frac{2K}{\omega}$$

When angular frequency is doubled and kinetic energy is halved, the angular momentum becomes,

$$L' = \frac{2(K/2)}{2\omega} = \frac{1}{4} \cdot \frac{2K}{\omega} = \frac{L}{4}.$$

Problem 74. The angular velocity of the earth around the sun increases, when it comes closer to the sun. Why ? *[Himachal 09 ; Delhi 01]*

Solution. When the earth comes closer to the sun, its moment of inertia about the axis through the sun decreases. To conserve angular momentum ($L = I\omega$), the angular velocity of the earth increases.

Problem 75. Why are we not able to rotate a wheel by pulling or pushing along its radius ? *[Central Schools 08]*

Solution. This is because radial component of the applied force cannot produce torque.

$$\tau = rF\sin\theta = 0 \times F\sin\theta = 0$$

Problem 76. Two solid spheres of the same mass are made of metals of different densities. Which of them has larger moment of inertia about its diameter ? *[Central Schools 09]*

Solution. The sphere of metal with smaller density, will have larger size and hence it will have larger moment of inertia than the other sphere.

Problem 77. A planet revolves around a massive star in a highly elliptical orbit. Is the angular momentum constant over the entire orbit ? *[Himachal 06]*

Solution. A planet revolves around a star under the effect of the gravitational force, which is purely radial in nature. As radial component of a force does not contribute to torque, so the angular momentum of the planet remains unaffected.

As $\tau = \frac{dL}{dt} = 0$, so $L = \text{constant}$.

Problem 78. If no external torque acts on a body, will its angular velocity remain constant ? Give reason. *[Himachal 05C, 06]*

Solution. When no external torque acts on a body, its angular momentum remains constant. But

$$L = I\omega$$

Clearly, the angular velocity ω will remain constant only so long as the moment of inertia I of the body remains constant.

Short Answer Conceptual Problems

Problem 1. "Newton's laws of motion are applicable to individual particles". How would you explain the motion of a large body ?

Solution. A large body can be considered as made of a number of mass particles and all mass particles interact with each other. The vector sum of all these internal forces is zero. Therefore for a large body, it can be replaced by a single mass particle whose mass is supposed to be situated at its centre of mass and the Newton's laws can be applied.

Problem 2. If an external force can change the state of motion of CM of a body, how does the internal force of the brakes bring a car to rest ?

Solution. Actually, it is not the external force which brings the car to rest. The internal force of the brakes on the wheel locks the wheel. Now a large frictional force comes into play between the wheels and the ground. This force is external to the system and brings the car to rest.

Problem 3. Two men stand facing each other on two boats floating on still water at a distance apart. A rope is held at its ends by both. The two boats are found to meet always at the same point, whether each man pulls separately or both pull together, why ? Will the time taken be different in the two cases ? Neglect friction.

Solution. The men on the two boats floating on water constitute a single system. So the forces applied by the two men are internal forces. Whether each man pulls separately or both pull together, the centre of mass of the system of boats remains fixed due to the absence of any external force. Consequently, the two boats meet at a fixed point, which is the centre of mass of the system.

Problem 4. Prove that the centre of mass of two particles divides the line joining the particles in the inverse ratio of their masses.

Solution. As shown in Fig. 7.72, consider a system of two particles of masses m_1 and m_2 situated at points A and B respectively. Suppose the origin O of the frame of reference coincides with their centre of mass.

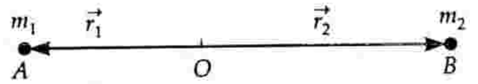


Fig. 7.72

If $\vec{r}_1$ and $\vec{r}_2$ are the position vectors of masses m_1 and m_2 with respect to the centre of mass, then

$$m_1 \vec{r}_1 + m_2 \vec{r}_2 = (m_1 + m_2)\vec{O} = 0$$

or
$$\vec{r}_1 = -\frac{m_2}{m_1}\vec{r}_2$$

or
$$|\vec{r}_1| = \frac{m_2}{m_1}|\vec{r}_2|$$

or
$$\frac{r_1}{r_2} = \frac{m_2}{m_1}$$

Hence the centre of mass of two particles divides the line joining the two particles in the inverse ratio of their masses.

Problem 5. Two balls of mass m each are placed at the two vertices of an equilateral triangle. Another ball of mass $2m$ is placed at the third vertex of the triangle. Locate the centre of mass of the system.

Solution. As shown in Fig. 7.73, the centre of mass of the two balls placed at A and B lies at the midpoint D of the side AB. Now we have the system of two masses, each of mass $2m$ at the points C and D on the perpendicular bisector CD of side AB. Hence the centre of mass of the system lies at the midpoint O of the perpendicular bisector CD.

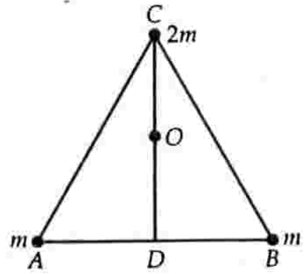


Fig. 7.73

Problem 6. What is the difference between centre of gravity and centre of mass ? *[Himachal 05C]*

Solution. Centre of mass is a point at which whole of the mass of the body may be assumed to be concentrated to describe its motion as a particle. Centre of gravity is a point at which resultant of the gravitational forces on all particles of the body acts. For bodies of normal dimensions, centre of mass and centre of gravity coincide. But centre of mass and centre of gravity relate to two different concepts. Even if the world were devoid of gravitational force, the centre of mass would still have a meaning.

Problem 7. There are 100 passengers in a stationary railway compartment. A physical fight starts between the passengers over some difference of opinion. (*i*) Will the position of CM of the compartment change ? (*ii*) Will the position of CM of system (compartment + 100 passengers) change ? Give reason.

Solution. (*i*) The position of the CM of the compartment will change because the passengers are external bodies for the compartment.

(*ii*) The position of CM of the system will not change as no external force is acting on the system.

Problem 8. Show that moment of a couple does not depend on the point about which you take the moments. *[NCERT]*

Solution. Fig. 7.74 shows a couple acting on a rigid body. The forces $-\vec{F}$ and $\vec{F}$ act at points A and B respectively. Let $\vec{r}_1$ and $\vec{r}_2$ be the position vectors of these points with respect to origin O.

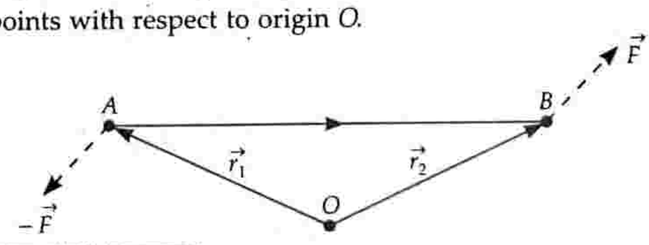


Fig. 7.74

We consider the moments of the forces about the origin O.

Moment of the couple = Sum of the moments of the two forces

$$= \vec{r}_1 \times (-\vec{F}) + \vec{r}_2 \times \vec{F} = \vec{r}_2 \times \vec{F} - \vec{r}_1 \times \vec{F}$$

$$= (\vec{r}_2 - \vec{r}_1) \times \vec{F} = \vec{AB} \times \vec{F} \qquad [\because \vec{r}_1 + \vec{AB} = \vec{r}_2]$$

Clearly, this moment is independent of the point O about which we have considered the moments of the forces.

Problem 9. Show that the angular momentum about any point of a single particle moving with constant velocity remains constant throughout the motion. Is there any external torque on the particle ? *[NCERT]*

Solution. As shown in Fig. 7.75, suppose a particle has velocity $\vec{v}$ at point P at some instant t. Its angular momentum about any arbitrary point O is

$$\vec{l} = \vec{r} \times m\vec{v}$$

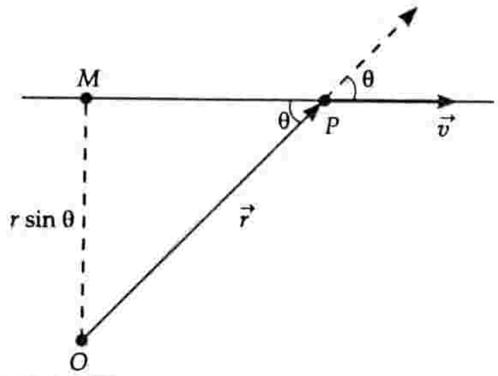


Fig. 7.75

The magnitude of angular momentum is

$$l = mvr \sin \theta$$

where θ is the angle between $\vec{r}$ and $\vec{v}$. Now the particle has a constant velocity. Although the position of the particle changes with time, the line of direction of velocity $\vec{v}$ remains same and hence $OM = r \sin \theta$ is a constant. Consequently, the magnitude l remains constant.

Further, the direction of $\vec{l}$ is perpendicular to the plane of $\vec{r}$ and $\vec{v}$. It is into the plane of the paper. This direction does not change with time.

Thus, $\vec{l}$ remains the same in magnitude and direction and is therefore conserved.

As the velocity of the particle remains constant, so no external force and hence no external torque is acting on the particle.

Problem 10. A rod of weight *W* is supported by two parallel edges *A* and *B* and is in equilibrium in horizontal position. The knives are at a distance *d* from each other. The centre of mass of the rod is at distance *x* from *A*. Find the normal reactions at the knife edges *A* and *B*.

Solution. As shown in Fig. 7.76, let R_A and R_B be the normal reactions at the edges.

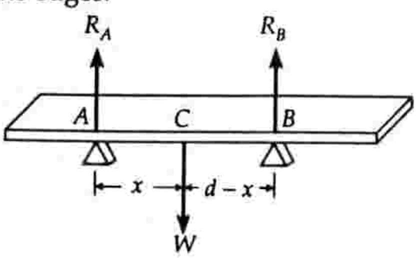


Fig. 7.76

Since the rod is in equilibrium, the sum of moments of the forces about either knife edge must be zero.

Taking moments of forces about point A, we get

$$R_A \times 0 + W \times x - R_B \times d = 0$$

$$\therefore \quad R_B = \frac{x}{d} \cdot W$$

Taking moments of forces about point B, we get

$$R_A \times d - W \times (d - x) + R_B \times 0 = 0$$

$$\therefore \quad R_A = \frac{d - x}{d} \cdot W.$$

Problem 11. Torque and work are both equal to force times distance. Then how do they differ ?

Solution. (*i*) Work is a scalar quantity while torque is a vector quantity.

(*ii*) Work is measured as the product of the applied force and the distance moved by the body in the direction of the force. Torque is measured as the product of the force and its perpendicular distance from the axis of rotation.

Problem 12. When is a rigid body said to be in equilibrium ? State the necessary conditions for a body to be in equilibrium.

Solution. A rigid body is said to be in equilibrium when its linear acceleration and angular acceleration are zero.

(*i*) For translation equilibrium of a rigid body, the sum of all the external forces acting on it must be zero.

$$\Sigma \vec{F}_i = 0$$

(*ii*) For rotational equilibrium of a rigid body, the sum of torques due to all forces acting on it must be zero.

$$\Sigma \vec{\tau}_i = 0$$

Problem 13. Define moment of inertia. On what factors does it depend ?

Solution. The moment of inertia of a rigid body about an axis is the sum of the products of the masses of its various particles and squares of their perpendicular distances from the axis of rotation.

$$I = \Sigma \, m_i r_i^2$$

The moment of inertia of a body depends on

(*i*) Mass of the body.
(*ii*) Size and shape of the body.
(*iii*) Distribution of mass about the axis of rotation.
(*iv*) Position and orientation of the axis of rotation with respect to the body.

Problem 14. How will you distinguish between a hard boiled egg and a raw egg by spinning it on a table top ?

Solution. Hard boiled egg acts just like a rigid body while rotating while it is not the case in a raw egg because of liquid matter present in it. In case of a raw egg, the liquid matter tries to go away from the centre, thereby increasing its moment of inertia. As moment of inertia is more, the raw egg will spin with less angular acceleration provided same torque is applied in both the cases. Thus, hard boiled egg will spin faster.

Problem 15. If two circular discs of the same mass and thickness are made from metals of different densities, which disc will have the larger moment of inertia about its central axis ? Explain.

Solution. $I_1 = \frac{1}{2} m r_1^2$ and $I_2 = \frac{1}{2} m r_2^2$

$$\therefore \quad \frac{I_1}{I_2} = \frac{r_1^2}{r_2^2}$$

Also, $m = \pi r_1^2 t \, \rho_1 = \pi r_2^2 t \, \rho_2$

$$\therefore \quad \frac{r_1^2}{r_2^2} = \frac{\rho_2}{\rho_1}$$

Hence $\frac{I_1}{I_2} = \frac{\rho_2}{\rho_1}$ *i.e.*, $I \propto \frac{1}{\rho}$

Thus the disc with greater density will have less moment of inertia.

Problem 16. Which one is easier to turn—a log or a bench of equal weight and length ? The two have the same coefficient of friction with the ground. Explain.

Solution. In the case of a log, the force of friction is distributed along its entire length and so it is effective midway between the end where the force is applied and the centre of the log, but in the case of a bench, the force of friction is at the end. So to turn the log, the torque required is equal to $\mu \, mg \, l/2$, while to turn the bench, the torque required is equal to $\mu \, mg \, l$. Hence it is easier to turn a log than a bench of equal weight and length.

Problem 17. The Moment of inertia of a disc about an axis passing through its centre and perpendicular to its plane is $1/2MR^2$. Derive the values of moment of inertia of the disc about its diameter and about an axis tangential to the disc lying on its plane ? *[Central Schools 12]*

Solution. Refer answer to Q. 41(*b*) and (*c*) on page 7.29.

Problem 18. What is the moment of inertia of a rod of mass M, length l about an axis perpendicular to it through one end ? Given the moment of inertia about the centre of mass is $\frac{1}{12} Ml^2$ *[NCERT ; Central Schools 03]*

Solution. By using *theorem of parallel axes*,

$$I_{end} = I_{CM} + M\left(\frac{l}{2}\right)^2 = \frac{1}{12} Ml^2 + \frac{1}{4} Ml^2 = \frac{1}{3} Ml^2.$$

Problem 19. What is the moment of inertia of a ring about a tangent to the circle of the ring ? *[NCERT]*

Solution. Refer answer to Q. 40(*c*) on page 7.28.

Problem 20. What is the moment of inertia of a uniform circular disc of radius R and mass M about an axis passing through its centre and normal to the disc ? The moment of inertia of the disc about any of its diameters is given to be $(1/4) MR^2$. *[Himachal 06]*

Solution. M.I. of the disc about any diameter,

$$I_D = \frac{1}{4} MR^2.$$

By the theorem of perpendicular axes,

M.I. of the disc about an axis through the centre and normal to disc = M.I. about any diameter + M.I. about perpendicular diameter

$$I = I_D + I_D = 2 \times \frac{1}{4} MR^2$$

or $$I = \frac{1}{2} MR^2.$$

Problem 21. The moment of inertia of a solid sphere about a tangent is $\frac{5}{3} MR^2$, where M is mass and R is radius of the sphere. Find the M.I. of the sphere about its diameter.

Solution. Here $I_T = \frac{5}{3} MR^2$.

Now diameter of sphere is an axis passing through its centre of mass. By using theorem of parallel axes,

$$I_T = I_{CM} + MR^2$$

$$\therefore \quad I_{CM} = I_T - MR^2 = \frac{5}{3} MR^2 - MR^2 = \frac{2}{3} MR^2.$$

Problem 22. The moment of inertia of a uniform circular disc about a tangent in its own plane is $\frac{5}{4} MR^2$, where M is mass and R is the radius of the disc. Find its moment of inertia about an axis through its centre and perpendicular to its plane.

Solution. If I_D is the M.I. of the disc about its diameter, then from the theorem of parallel axes,

$$I_T = I_D + MR^2$$

or $$I_D = I_T - MR^2 = \frac{5}{4} MR^2 - MR^2 = \frac{1}{4} MR^2$$

By the theorem of perpendicular axes, the M.I. about an axis through the centre and perpendicular to the plane of the disc,

I = Sum of moments of inertia about two perpendicular diameters

$$= I_D + I_D = 2 \times \frac{1}{4} MR^2$$

or $$I = \frac{1}{2} MR^2.$$

Problem 23. Using expressions for power and kinetic energy of rotational motion, derive the relation. $\tau = I\alpha$, where letters have their usual meanings.

Solution. Power in rotational motion, $P = \tau\omega$

Rotational K.E., $K = \frac{1}{2} I\omega^2$

Work done in rotational motion,

$$W = \text{Energy stored as rotational K.E.} = \frac{1}{2} I\omega^2$$

$$\therefore \quad P = \frac{dW}{dt} = \frac{d}{dt}\left(\frac{1}{2} I\omega^2\right) = \frac{1}{2} . I \times 2\omega \frac{d\omega}{dt}$$

or $$\tau\omega = I\omega\alpha \qquad \left[\because \frac{d\omega}{dt} = \alpha\right]$$

$$\therefore \quad \tau = I\alpha.$$

Problem 24. Explain if the ice on the polar caps of the earth melts, how will it affect the duration of the day ?

[Himachal 05, 07C, 08 ; Central Schools 09]

Solution. If the ice on the polar caps of the earth melts, the water so formed will spread on the surface of earth. This increases the moment of inertia (I) of the earth about its own axis (due to change in the distribution of mass of the particles of water going away from the axis of rotation). To conserve angular momentum ($= I\omega$), ω (angular velocity of earth about its own axis) will decrease. As $T = 2\pi/\omega$, hence due to decrease in the value of ω, T *i.e.*, the duration of the day will increase.

Problem 25. Two identical cylinders 'run a race' starting from rest at the top of an inclined plane, one slides without rolling and other rolls without slipping. Assuming that no mechanical energy is dissipated as heat, which one will win ?

Solution. When the cylinder slides without rolling,

$$E = \frac{1}{2} mv'^2 \qquad \therefore \quad v' = \sqrt{\frac{2E}{m}}$$

When the cylinder rolls without slipping,

$$E = \frac{1}{2} mv^2 + \frac{1}{2} I\omega^2 = \frac{1}{2} mv^2 + \frac{1}{2}\left(\frac{1}{2} mr^2\right)\omega^2$$

$$= \frac{1}{2} mv^2 + \frac{1}{4} mv^2 = \frac{3}{4} mv^2 \qquad \therefore \quad v = \sqrt{\frac{4E}{3m}}$$

As $v' > v$, therefore sliding cylinder will win the race.

Problem 26. A uniform circular disc of radius R is rolling on a horizontal surface. Determine the tangential velocity (*i*) at the upper most point, (*ii*) at the centre of mass and (*iii*) at the point of contact.

Solution. ***Rolling can be regarded as the combination of pure rotation and pure translation.*** As shown in Fig. 7.77 (*a*), in case of pure rotation the, velocity of *CM* is zero and the tangential velocity at points *A* and *B* is $v_A = v_B = R\omega$. As shown in Fig. 7.77(*b*), in case of pure translation, $v_A = v_B = v_{CM} = R\omega = v$ (say).

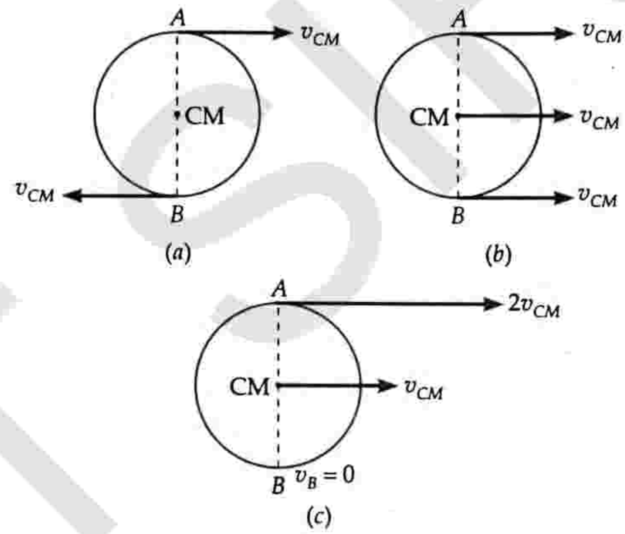


Fig. 7.77 (*a*) Pure rotation. (*b*) Pure translation, (*c*) Rolling.

As shown in Fig. 7.77(*c*), in case of rolling, the tangential velocity at any given point is the vector sum of the velocities in (*a*) and (*b*) at that point.

Hence $\quad v_A = 2v_{CM}, \; v_{CM} = v = R\omega$ and $v_B = 0$.

It may be noted that in case of rolling the instantaneous velocity at the point of contact *B* with surface is zero, because the body is not slipping.

HOTS

Problems on Higher Order Thinking Skills

Problem 1. *An isolated particle of mass m is moving in a horizontal plane (x-y), along the x-axis at a certain height above the ground. It suddenly explodes into two fragments of masses m/4 and 3m/4. At instant later, the smaller fragment is at y = + 15 cm. What is the position of larger fragment at this instant ?* **[IIT 97]**

Solution. As the isolated particle is initially moving along x-axis, so there is no motion along y-axis. The centre of mass should remain stationary along y-axis even after explosion.

Now $$y_{CM} = \frac{m_1 y_1 + m_2 y_2}{m_1 + m_2}$$

But $$y_{CM} = 0$$

$$\therefore \quad m_1 y_1 + m_2 y_2 = 0$$

or $$y_2 = -\frac{m_1}{m_2} \cdot y_1 = -\frac{m/4}{3m/4} . 15 = -5 \text{ cm}.$$

Thus the larger fragment will be at $y = $ **– 5 cm.**

Problem 2. *A small sphere of radius R is held against the inner surface of a larger sphere of radius 6R [Fig. 7.78]. The masses of large and small spheres are 4M and M, respectively. This arrangement is placed on a horizontal table. There is no friction between any surfaces of contact. The small sphere is now released. Find the co-ordinates of the centre of the larger sphere when the smaller sphere reaches the other extreme position.* **[IIT 96]**

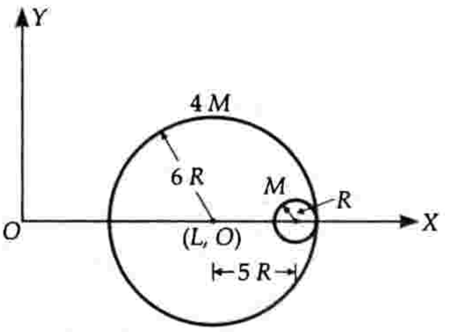


Fig. 7.78

Solution. Initial position of the centre of mass of the system is given by

$$x_{CM} = \frac{4M(L) + M(L+5R)}{5\,M} = L + R$$

When the smaller sphere reaches the other extreme [Fig. 7.79], the larger sphere moves to the right so that

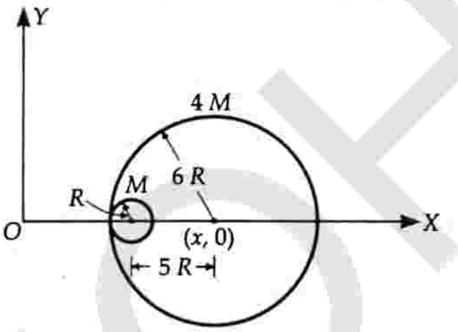


Fig. 7.79

position of CM of the system remains unchanged (as no external force is acting in the system). New position of the CM of the system is

$$x'_{CM} = \frac{4M(x) + M(x-5R)}{5\,M} = x - R$$

But $\quad x_{CM} = x'_{CM}$

or $\quad L + R = x - R$

or $\quad x = L + 2R$

$\therefore$ Centre of larger sphere will lie at **$(L+2R,\ 0)$**.

Problem 3. *A boat of 90 kg is floating in still water. A boy of mass 30 kg walks from the stern to the bow. The Length of the boat is 3 m. Calculate the distance through which the boat will move.* **[REC 92]**

Solution. As shown in Fig. 7.80, let C_1, C_2 and C be the centres of mass of the boy, boat and the system (boy and boat) respectively. Let x_1 and x_2 be the distances of C_1 and C_2 from the shore. Then the centre of mass will be at a distance,

$$x_{CM} = \frac{30x_1 + 90x_2}{30 + 90}$$

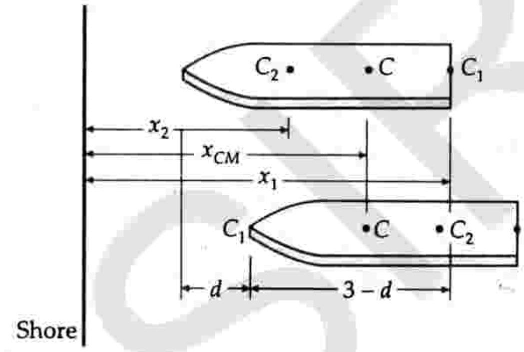


Fig. 7.80

As the boy C_1 moves from the stern to the bow, the boat moves backward through a distance d so that position of the centre of mass of the system remains unchanged.

$$x'_{CM} = \frac{30\,[x_1 - (3-d)] + 90\,(x_2 + d)}{30 + 90}$$

As $\quad x'_{CM} = x_{CM}$

$$\frac{30\,(x_1 - 3 + d) + 90\,(x_2 + d)}{120} = \frac{30\,x_1 + 90\,x_2}{120}$$

or $\quad -90 + 30\,d + 90\,d = 0$

or $\quad d = \mathbf{0.75\ m.}$

Problem 4. *From a uniform circular disc of diameter D, a circular disc or hole of diameter D/6 and having its centre at a distance of D/4 from the centre of the disc is scooped out. Determine the centre of mass of the remaining portion.*

Solution. Let mass per unit area of the disc = m

$\therefore$ Total mass of the disc, $M = \pi\left(\dfrac{D}{2}\right)^2 m = \dfrac{\pi\, mD^2}{4}$

Mass of the scooped out portion of the disc,

$$M' = \pi\left(\frac{D}{12}\right)^2 m = \frac{\pi\, mD^2}{144}$$

We take the centre O of the disc as the origin. The masses M and M' can be supposed to be concentrated at the centres of the disc and scooped out portion respectively. The mass M' of the removed portion is taken negative.

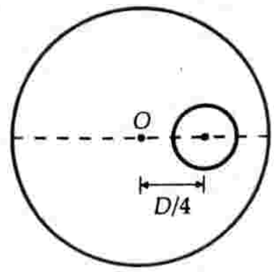


Fig. 7.81

Then x-coordinate of the CM of the remaining portion of the disc will be

$$x_{CM} = \frac{Mx_1 - M'x_2}{M - M'} = \frac{M \times 0 - M'(D/4)}{\frac{\pi mD^2}{4} - \frac{\pi mD^2}{144}}$$

$$= -\frac{M'D}{4} \times \frac{144}{35\pi mD^2}$$

$$= -\frac{\pi mD^2}{144} \times \frac{D}{4} \times \frac{144}{35\pi mD^2} = -\frac{D}{140}.$$

Thus, the centre of mass of the remaining portion lies at a distance of $D/140$ towards the left of the centre O of the disc.

Problem 5. *A tube of length L is filled completely with an incompressible liquid of mass M and closed at both the ends. The tube is then rotated in a horizontal plane about one of its ends with a uniform angular velocity ω. Determine the force exerted by the liquid at the other end.* **[IIT 92]**

Solution. Consider a small element of the liquid of length dx at a distance x from one end.

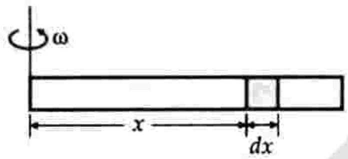


Fig. 7.82

Mass of the small element $= \frac{M}{L} dx$

Centripetal force associated with the element

$$dF = \left(\frac{M}{L} dx\right) x\, \omega^2 \qquad [\because F = mr\omega^2]$$

Force exerted by the liquid = Total centripetal force at the other end

$$F = \int dF = \int_0^L \frac{M}{L} \omega^2 x\, dx = \frac{M}{L}\omega^2 \left[\frac{x^2}{2}\right]_0^L$$

$$= \frac{M}{L}\omega^2 \frac{L^2}{2} = \frac{1}{2} M\omega^2 L.$$

Problem 6. *A particle describes a horizontal circle on the smooth surface of an inverted cone. The height of the plane of the circle above the vertex is 9.8 cm. Find the speed of the particle. Take $g = 98\ ms^{-2}$.* **[MNREC 92]**

Solution. As shown in Fig. 7.83, suppose the particle describes a circle of radius r with speed v. Various forces acting on the particle are

(*i*) its weight mg acting vertically downwards,

(*ii*) normal reaction R of the smooth surface of the cone.

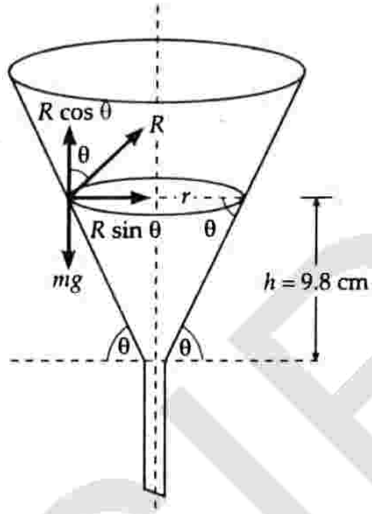


Fig. 7.83

Let the reaction R make angle θ with the vertical. Resolving R into vertical and horizontal components, we get

$$R \cos\theta = mg \qquad ...(i)$$

$$R \sin\theta = \frac{mv^2}{r} \qquad ...(ii)$$

Dividing (*ii*) by (*i*), we get

$$\tan\theta = \frac{v^2}{rg}$$

From Fig. 7.83,

$$\tan\theta = \frac{h}{r}$$

$$\therefore \quad \frac{v^2}{rg} = \frac{h}{r}$$

or $$v = \sqrt{gh} = \sqrt{9.8 \times 9.8 \times 10^{-2}} = 0.98\ \text{ms}^{-1}.$$

Problem 7. *Point masses m_1 and m_2 are placed at the ends of a rigid rod of length L and negligible mass. The rod is to be set rotating about an axis perpendicular to its length. Locate a point on the rod through which the axis of rotation should pass in order that the work required to set the rod rotating with angular velocity ω_0 is minimum.* **[REC 96]**

Solution. As shown in Fig. 7.84, let the axis of rotation be at a distance x from mass m_1. Then moment of inertia of the system about this axis of rotation is

$$I = m_1 x^2 + m_2 (L - x)^2$$

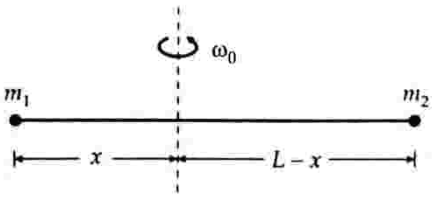


Fig. 7.84

Work done to set the rod rotating with angular velocity ω_0 = Increase in rotational K.E.

$$W = \frac{1}{2} I\omega_0^2 = \frac{1}{2}[m_1 x^2 + m_2 (L-x)^2]\,\omega_0^2$$

For W to be minimum, $\frac{dW}{dx} = 0$

or $\frac{1}{2}[2m_1 x + 2m_2 (L-x)(-1)]\,\omega_0^2 = 0$

or $m_1 x - m_2 (L-x) = 0$ $\quad [\because \omega_0 \neq 0]$

or $(m_1 + m_2)\,x = m_2 L$

or $$x = \frac{m_2 L}{m_1 + m_2}.$$

Problem 8. *A uniform bar of length 6a and mass 8m lies on a smooth horizontal table. Two point-masses m and 2m moving in the same horizontal plane with speeds 2v and v respectively strike the bar as shown in Fig. 7.85, and stick to the bar after collision.*

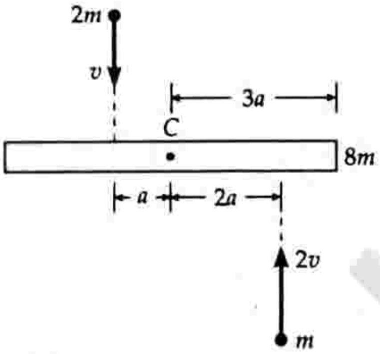


Fig. 7.85

Determine (i) velocity of the centre of mass (ii) angular velocity about centre of mass and (ii) total kinetic energy just after collision. **[IIT 91]**

Solution. (*i*) Let v_c be the velocity of the centre of mass.

By conservation of linear momentum,

$(8m + m + 2m)\,v_c = 2m \times (-v) + m \times 2v + 8m \times 0$

or $v_c = \mathbf{0}$.

(*ii*) M.I. of the bar about the centre of mass C,

$$I_b = \frac{Ml^2}{12} = \frac{8\,m \times (6a)^2}{12} = 24\,ma^2$$

By conservation of angular momentum,

$m_1 v_1 r_1 + m_2 v_2 r_2 = (I_1 + I_2 + I_b)\,\omega$

or $2m \times v \times a + m \times 2v \times 2a$

$= [2m \times a^2 + m \times (2a)^2 + 24\,ma^2]\,\omega$

or $6\,mva = 30\,ma^2\,\omega$

or $$\omega = \frac{6\,mva}{30\,ma^2} = \frac{v}{5a}.$$

(*iii*) As the system has no translatory motion, it has only rotational K.E.

$\therefore$ Total K.E.

= Rotational K.E.

$$= \frac{1}{2} I\omega^2 = \frac{1}{2} \times 30\,ma^2 \times \left(\frac{v}{5a}\right)^2 = \frac{3}{5}\,mv^2.$$

Problem 9. *A carpet of mass M made of inextensible material is rolled along its length in the form of a cylinder of radius R and is kept on a rough floor. The carpet starts unrolling without sliding on the floor when a negligibly small push is given to it. Calculate the horizontal velocity of the axis of the cylindrical part of the carpet when its radius reduces to R/2.* **[IIT 90]**

Solution. Let l be the length of the cylinder and ρ its density.

Mass of carpet, $\quad M = \pi R^2 l \times \rho$

When the carpet unrolls to half the radius, its mass becomes,

$$M' = \pi \left(\frac{R}{2}\right)^2 l \times \rho = \frac{M}{4}$$

Initially, mass is M and CG lies at R.

Finally, mass is $M/4$ and CG lies at $R/2$.

Let v be horizontal velocity of the axis of the cylindrical part of the carpet. Then

Decrease in P.E. = Gain in translational K.E. + Gain in rotational K.E.

$$MgR - \frac{M}{4} \cdot g \cdot \frac{R}{2} = \frac{1}{2} M' v^2 + \frac{1}{2} I\omega^2$$

or $$\frac{7}{8} MgR = \frac{1}{2}\,\frac{M}{4} \cdot v^2 + \frac{1}{2}\left[\frac{1}{2} M'\left(\frac{R}{2}\right)^2\right]\left[\frac{v}{R/2}\right]^2$$

or $$\frac{7}{8}\,MgR = \frac{1}{8} Mv^2 + \frac{1}{4} \cdot \frac{M}{4} \cdot \frac{R^2}{4} \cdot \frac{4v^2}{R^2}$$

or $$\frac{7}{8}\,gR = \frac{v^2}{8} + \frac{v^2}{16} = \frac{3v^2}{16}$$

or $$v = \sqrt{\frac{14}{3}\,gR}.$$

Problem 10. *Consider a body, shown in Fig. 7.86, consisting of two identical balls, each of mass M connected by a light rigid rod. If an impulse J = MV is imparted to the body at one of its ends, what would be its angular velocity ?* **[IIT Screening 03]**

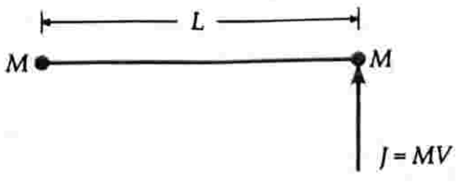


Fig. 7.86

Solution. Applying conservation of angular momentum about the centre of the rod,

$$I_{CM}\,\omega = J\cdot\frac{L}{2}$$

or $$2\times M\left(\frac{L}{2}\right)^2\omega = MV\cdot\frac{L}{2}$$

or $$\omega = \mathbf{V / L}.$$

Problem 11. *A cubical block of side L rests on a rough horizontal surface with coefficient of friction μ. A horizontal force F is applied on the block as shown in Fig. 7.87. If the coefficient of friction is sufficiently high so that the block does not slide before toppling, what is the minimum force required to topple the block ?* **[IIT Screening 2000]**

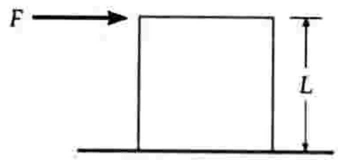


Fig. 7.87

Solution. When F is applied, the normal reaction (N) of the floor moves to the right. The cube topples when N reaches its edge.

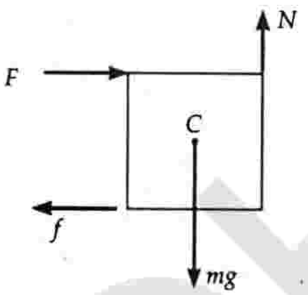


Fig. 7.88

Here, $N = mg$ and the force of friction $= f = F$. Taking torque about the centre C,

$$F\times\frac{L}{2} + f\times\frac{L}{2} = N\times\frac{L}{2}$$

or $$F + f = N$$

or $$F + F = mg$$

or $$F = \mathbf{mg / 2}.$$

Problem 12. *One quarter sector is cut from a uniform circular disc of radius R. This sector has mass M. It is made to rotate about a line perpendicular to its plane and passing through the centre of the original disc. What is its moment of inertia about the axis of rotation ?* **[IIT Screening 2001]**

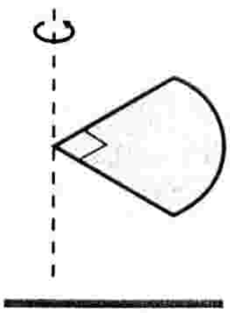


Fig. 7.89

Solution. Mass of the entire disc from which the section has been cut $= 4M$.

Its moment of inertia of the given axis $= \frac{1}{2}(4M)\,R^2$

∴ Moment of inertia of the quarter section about the same axis

$$= \frac{1}{4}\times\frac{1}{2}(4M)\,R^2 = \frac{1}{2}\,\mathbf{MR^2}.$$

Problem 13. *A thin wire of length L and uniform linear mass density ρ is bent into a circular loop with centre at O as shown in Fig. 7.90. What is the moment of inertia of the loop about the axis XX′ ?* **[IIT Screening 2000]**

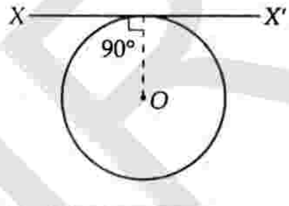


Fig. 7.90

Solution. Let m and r be the mass and radius of the loop.

Then, $m = \rho L$ and $L = 2\pi r$ or $r = \frac{L}{2\pi}$.

Moment of inertia of the loop about an axis through O and perpendicular to its plane $= mr^2$.

By the theorem of perpendicular axes, its moment of inertia about a diameter parallel to $XX' = \frac{1}{2}mr^2$.

By the theorem of parallel axes, its moment of inertia about XX'

$$= \frac{1}{2}mr^2 + mr^2 = \frac{3}{2}mr^2$$

$$= \frac{3}{2}(\rho L)\left(\frac{L}{2\pi}\right)^2 = \mathbf{\frac{3\rho L^3}{8\pi^2}}.$$

Problem 14. *A long horizontal rod has a bead which can slide along its length and is initially placed at a distance L from one end A of the rod. The rod is set in angular motion about A with a constant angular acceleration, α. If the coefficient of friction between the rod and the bead is μ, and gravity is neglected then, find the time after which the bead starts slipping.* **[IIT Screening 2000]**

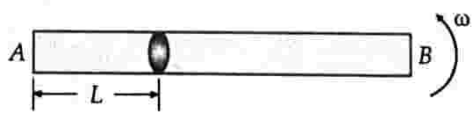


Fig. 7.91

Solution. Linear acceleration of the bead $= a = L\alpha$.

∴ Reaction force on the bead due to the rod

$$= N = ma = mL\alpha.$$

After time t, the angular velocity of the bead

$$= \omega = \alpha t.$$

∴ Centripetal acceleration of the bead

$$= \omega^2 L = \alpha^2 t^2 L$$

Centripetal force on the bead

$$= m\alpha^2 t^2 L$$

$\therefore$ Force of friction at limiting position

$$= \mu N = \mu m L\alpha.$$

$\therefore$ For slipping,

$$\mu m L\alpha = m\alpha^2 t^2 L \quad \text{or} \quad t = \sqrt{\mu / \alpha}.$$

Problem 15. *Initial angular velocity of a circular disc of mass M is ω_1. Then two small spheres of mass m are attached gently to two diametrically opposite points on the edge of the disc. What is the final angular velocity of the disc ?* **[AIEEE 02]**

Solution. Let R be the radius of the circular disc. Then initial angular momentum of the disc is

$$L_1 = I_1\omega_1 = \frac{1}{2} MR^2\omega_1$$

When two small spheres are attached on the edge of the disc, the moment of inertia becomes

$$I_2 = \frac{1}{2} MR^2 + mR^2 + mR^2 = \frac{1}{2} MR^2\left(1 + \frac{4m}{M}\right)$$

If ω_2 is the final angular velocity, then the final angular momentum will be

$$L_2 = I_2\omega_2 = \frac{1}{2} MR^2\left(1 + \frac{4m}{M}\right)\omega_2$$

By conservation of angular momentum in the absence of any external torque,

$$L_2 = L_1$$

or $$\frac{1}{2} MR^2\left(\frac{M+4m}{M}\right)\omega_2 = \frac{1}{2} MR^2\,\omega_1$$

or $$\omega_2 = \left(\frac{\mathbf{M}}{\mathbf{M+4m}}\right)\omega_1.$$

Problem 16. *A T-shaped object with dimensions shown in Fig. 7.92, is lying on a smooth floor. A force $\vec{F}$ is applied at the point P parallel to AB, such that the object has only the translational motion without rotation. Find the location of P with respect to C.* **[AIEEE 05]**

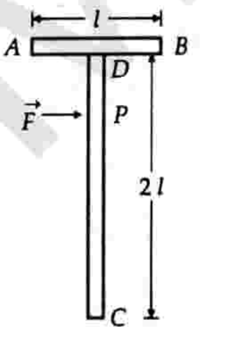


Fig. 7.92

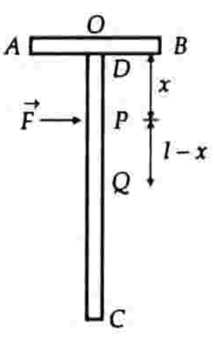


Fig. 7.93

Solution. As shown in Fig. 7.93, let O be the CM of part AB and Q that of CD. Let M be the mass per unit length of the parts AB and CD.

As no rotation is set up about point P, so

Moment of part AB about P = Moment of part CD about P

or $$(ml)\,x = (2\,ml)(l - x)$$

or $$x = 2l - 2x$$

or $$x = 2l/3.$$

Distance of P from the end $C = 2l - 2l/3 = \mathbf{4l/3}.$

Problem 17. *From a circular disc of radius R and mass 9 M, a small disc of radius R / 3 is removed from the disc, as shown in Fig. 7.94. Find the moment of inertia of the remaining disc about an axis perpendicular to the plane of the disc and passing through the point O.* **[IIT 05]**

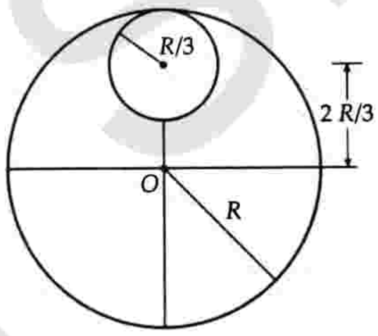


Fig. 7.94

Solution. Total mass of the disc $= 9\,M$

Radius of the disc $= R$

M.I. of the disc about a perpendicular axis through O is

$$I_1 = \frac{1}{2} \times \text{Mass} \times \text{radius}^2$$

$$= \frac{1}{2} \times 9\,M \times R^2 = \frac{9}{2} MR^2$$

Mass of small disc removed

$$= \frac{9\,M}{\pi R^2} \times \pi\left(\frac{R}{3}\right)^2 = M$$

By the theorem of parallel axes, the M.I. of the small disc about the axis through O,

$$I_2 = \frac{1}{2} M\left(\frac{R}{3}\right)^2 + M\left(\frac{2R}{3}\right)^2 = \frac{1}{2} MR^2$$

M.I. of the remaining disc about a perpendicular axis through O,

$$I = I_1 - I_2 = \frac{9}{2} MR^2 - \frac{1}{2} MR^2$$

$$= \mathbf{4\,MR^2}.$$

Problem 18. *A rod of length L and mass M is hinged at point O. A small bullet of mass m hits the rod with velocity v, as shown in Fig. 7.95. The bullet gets embedded in the rod. Find the angular velocity of the system just after the impact.* **[IIT 05]**

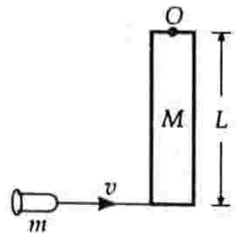


Fig. 7.95

Solution. Before impact, the bullet is moving. The initial angular momentum of the system about O is

$$L_i = mv \times L = mvL$$

After the bullet gets embedded in the rod, suppose the system attains angular velocity ω.

The moment of inertia of the bullet + rod system about the axis through O is

I = (M.I. of bullet + M.I. of rod) about the axis through O

$$= mL^2 + \frac{1}{3}ML^2 = \frac{M+3m}{3}L^2$$

Final angular momentum of the system is

$$L_f = I\omega = \frac{M+3m}{3}L^2\omega$$

By conservation of angular momentum,

or $\quad \dfrac{M+3m}{3}L^2\omega = mvL$

or $\quad \omega = \dfrac{3mv}{(M+3m)L}$.

Guidelines to NCERT Exercises

7.1. *Give the location of the centre of mass of mass of a (i) sphere, (ii) cylinder, (iii) ring, and (iv) cube, each of uniform mass density. Does the centre of mass of a body necessarily lie inside the body ?*

Ans. (*i*) Geometrical centre.

(*ii*) Centre of its axis of symmetry.

(*iii*) Centre of the ring.

(*iv*) Point of intersection of the diagonals.

No, it is not necessary that centre of mass of a body lies inside the body. For example, in a ring, hollow sphere, in a tumbler etc., the centre of mass lies inside their hollow portion.

7.2. *In the HCl molecule, the separation between the nuclei of the two atoms is about 1.27 Å (1 Å = 10^{-10} m). Find the approximate location of the CM of the molecule, given that the chlorine atom is about 35.5 times as massive as a hydrogen atom and nearly all the mass of an atom is concentrated in all its nucleus.*

Ans. As shown in Fig. 7.96, suppose the H nucleus is located at the origin. Then

$$x_1 = 0,\ x_2 = 1.27\ \text{Å},$$
$$m_1 = 1,\ m_2 = 35.5$$

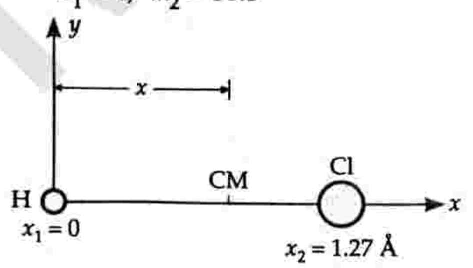


Fig. 7.96

The position of the CM of HCl molecule is

$$x = \frac{m_1x_1 + m_2x_2}{m_1 + m_2} = \frac{1\times 0 + 35.5\times 1.27}{1 + 35.5} = \mathbf{1.235\ \text{Å}}.$$

Thus the CM of HCl is located on the line joining H and Cl nuclei at a distance of 1.235 Å from the H nucleus.

7.3. *A child sits stationary at one end of a long trolley moving uniformly with speed v on a smooth horizontal floor. If the child gets up and runs about on the trolley in any manner, then what is the effect of the speed of the centre of mass of the (trolley + child) system ?*

Ans. The forces involved in the given problem are the internal forces of the system. No external force acts on the system when the child runs. So, there will be no change in the speed of the centre of mass of the (trolley + child) system.

7.4. *Show that the area of the triangle contained between the vectors $\vec{a}$ and $\vec{b}$ is one half of the magnitude of $\vec{a} \times \vec{b}$.*

Ans. Area of a triangle. Suppose two vectors $\vec{a}$ and $\vec{b}$ are represented by the sides OP and OQ of ΔPOQ as shown in Fig. 7.97.

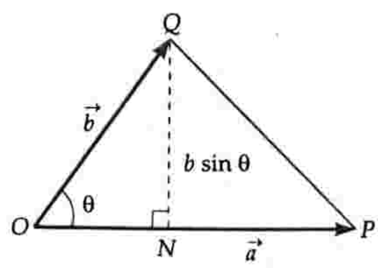


Fig. 7.97

Let $\angle POQ = \theta$. Draw $QN \perp OP$. The magnitude of the vector product $\vec{a} \times \vec{b}$ is

$$|\vec{a} \times \vec{b}| = ab \sin\theta$$
$$= (OP)(OQ)\sin\theta$$
$$= (OP)(QN) \qquad [\because QN = OQ\sin\theta]$$
$$= 2 \times \frac{1}{2}(OP)(QN)$$
$$= 2 \times \text{Area of } \Delta POQ$$

or Area of $\Delta POQ = \frac{1}{2}|\vec{a} \times \vec{b}|$.

Hence the area of the triangle contained between the vectors $\vec{a}$ and $\vec{b}$ is one half of the magnitude of $\vec{a} \times \vec{b}$.

7.5. *Show that $\vec{a} \cdot (\vec{b} \times \vec{c})$ is equal in magnitude to the volume of the parallelopiped formed on the three vectors, $\vec{a}$, $\vec{b}$ and $\vec{c}$.*

Ans. Volume of a parallelopiped. As shown in Fig. 7.98, consider a parallelopiped having the three non-coplanar vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ as edges meeting at a point O. Let $\vec{OA} = \vec{a}$, $\vec{OB} = \vec{b}$ and $\vec{OC} = \vec{c}$.

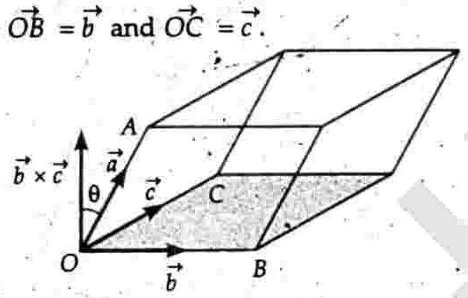


Fig. 7.98 Volume of a parallelopiped.

Then, $\vec{b} \times \vec{c}$ is a vector perpendicular to the plane of $\vec{b}$ and $\vec{c}$. Let θ be the angle between $\vec{a}$ and $\vec{b} \times \vec{c}$. Clearly, $|\vec{a}|\cos\theta$ is the height of the parallelopiped orthogonal to it its base.

Now,

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = |\vec{a}||\vec{b} \times \vec{c}|\cos\theta$$
$$= |\vec{b} \times \vec{c}||\vec{a}|\cos\theta$$

= Base area of the parallelopiped × height of the parallelopiped on this base

= Volume of the parallelopiped having the vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ along edges meeting at a point.

7.6. *Find the components along the x, y, z axes of the angular momentum $\vec{l}$ of a particle, whose position vector is $\vec{r}$ with components x, y, z and momentum is $\vec{p}$ with components p_x, p_y and p_z. Show that if the particle moves only in the x-y plane, the angular momentum has only a z-component.*

Ans. We can write

$$\vec{l} = l_x\hat{i} + l_y\hat{j} + l_z\hat{k}$$
$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

and $$\vec{p} = p_x\hat{i} + p_y\hat{j} + p_z\hat{k}$$

But $$\vec{l} = \vec{r} \times \vec{p}$$
$$= (x\hat{i} + y\hat{j} + z\hat{k}) \times (p_x\hat{i} + p_y\hat{j} + p_z\hat{k})$$
$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix}$$

or $$l_x\hat{i} + l_y\hat{j} + l_z\hat{k} = \hat{i}(yp_z - zp_y) + \hat{j}(zp_x - xp_z) + \hat{k}(xp_y - yp_x)$$

Comparing the coefficients of $\hat{i}$, $\hat{j}$ and $\hat{k}$ on the two sides, we get the components of $\vec{l}$ as follows :

$$l_x = yp_z - zp_y\,;\; l_y = zp_x - xp_z \text{ and } l_z = xp_y - yp_x$$

If the particle is constrained to move only in the x-y plane, then z = 0 and $p_z = 0$. Hence

$$\vec{l} = \hat{k}\,(xp_y - yp_x)$$

As only the unit vector $\hat{k}$ corresponding to z-direction survives, the angular momentum $\vec{l}$ has only a z-component.

7.7. *Two particles, each of mass m and speed v, travel in opposite directions along parallel lines separated by a distance d. Show that the vector angular momentum of the two particle system is the same whatever be the point about which the angular momentum is taken.*

Ans. As shown in Fig. 7.99, suppose the two particles move parallel to the y-axis.

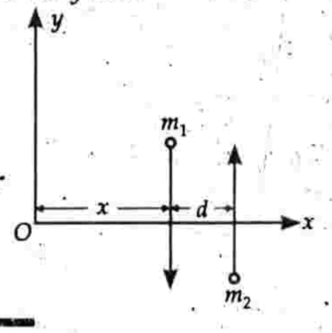


Fig. 7.99

Total angular momentum of the two particle system about O is

$$\vec{l} = \vec{l}_1 + \vec{l}_2 = \vec{r}_1 \times \vec{p}_1 + \vec{r}_2 \times \vec{p}_2$$
$$= x\hat{i} \times (-mv)\hat{j} + (x + d)\hat{i} \times (mv)\hat{j}$$
$$= (-mvx)\hat{i} \times \hat{j} + (mvx + mvd)\hat{i} \times \hat{j}$$
$$= (-mvx + mvx + mvd)\hat{k}$$
$$= mvd\,\hat{k} \qquad [\because \hat{i} \times \hat{j} = \hat{k}]$$

Clearly, $\vec{l}$ does not depend on x and hence on the origin O. Thus the angular momentum of the two particle system is same whatever be point about which the angular momentum is taken.

7.8. *A non-uniform bar of weight W is suspended at rest by two strings of negligible weights as shown in Fig. 7.100. The angles made by the strings with the vertical are 36.9° and 53.1° respectively. The bar is 2 m long. Calculate the distance d of the centre of gravity of the bar from its left end.*

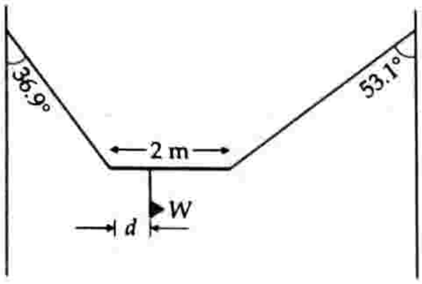


Fig. 7.100

Ans. Let T_1 and T_2 be the tensions in the two strings, as shown in Fig. 7.101.

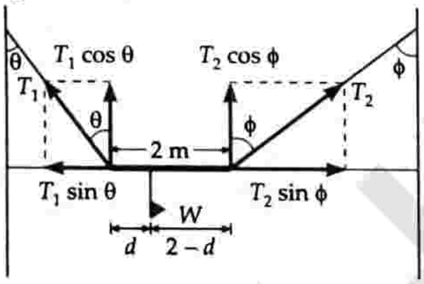


Fig. 7.101

For translational equilibrium, on balancing the vertical components of forces, we get

$$T_1 \cos\theta + T_2 \cos\phi = W \quad ...(1)$$

On balancing the horizontal components, we get

$$T_1 \sin\theta = T_2 \sin\phi \quad ...(2)$$

For rotational equilibrium, we balance the torques about the CG of the bar.

Clockwise torque = Anticlockwise torque

$$T_1 \cos\theta \times d = T_2 \cos\phi \times (2-d) \quad ...(3)$$

Dividing (3) by (2), we get

$$d \cot\theta = (2-d)\cot\phi$$

or $\quad d \cot 36.9° = (2-d)\cot 53.1°$

or $\quad d \cot 36.9° = (2-d)\tan 36.9°$

or $\quad d \times \frac{4}{3} = (2-d) \times \frac{3}{4}$

or $\quad 16d = 18 - 9d$

or $\quad d = 18/25 = 0.72 \text{ m} = \mathbf{72\ cm.}$

7.9. *A car weighs 1800 kg. The distance between its front and back axles is 1.8 m. Its centre of gravity is 1.05 m behind the front axle. Determine the force exerted by the level ground on each front wheel and each back wheel.*

Ans. Let N_B and N_F be the total reaction forces exerted by the level ground on front and back wheels respectively. The situation is shown in Fig. 7.102.

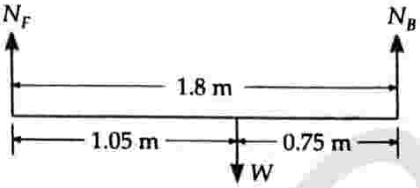


Fig. 7.102

For translational equilibrium of the car,

$$N_F + N_B = W = 1800 \times 9.8 \text{ N}$$

or $\quad N_F + N_B = 17640 \text{ N}$

For rotational equilibrium of the car,

$$1.05\, N_F = 0.75\, N_B$$

or $\quad 1.05\, N_F = 0.75(17640 - N_F)$

or $\quad 1.8\, N_F = 13230$

or $\quad N_F = 13230/1.8 = 7350 \text{ N}$

and $\quad N_B = 17640 - 7350 = 10290 \text{ N}$

Force on each front wheel = 7350 / 2 = **3675 N.**

Force on each back wheel = 10290 / 2 = **5145 N.**

7.10. (*a*) *Find the moment of inertia of a sphere about a tangent to the sphere, given the moment of inertia of the sphere about any of its diameters to be $2\,MR^2/5$, where M is the mass of the sphere and R is the radius of the sphere.* **[Central Schools 11]**

(*b*) *Given the moment of inertia of a disc of mass M and radius R about any of its diameters to be $MR^2/4$, find its moment of inertia about an axis normal to the disc and passing through a point on its edge.*

Ans. (*a*) Here $I_D = \frac{2}{5} MR^2$

The tangent EF is parallel to the diameter CD. By the theorem of *parallel axes*,

$$I_{EF} = I_{CD} + MR^2$$

$$I_T = \frac{2}{5} MR^2 + MR^2$$

$$= \frac{7}{5} \mathbf{MR^2}.$$

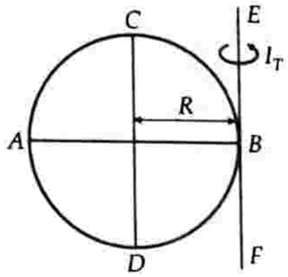


Fig. 7.103

(*b*) By the *theorem of parallel axes*, M.I. about an axis passing through an edge point and normal to disc

= M.I. about central normal axis + MR^2

$$I' = \frac{1}{2} MR^2 + MR^2 = \frac{3}{2} \mathbf{MR^2}.$$

7.11. *Torques of equal magnitude are applied to a hollow cylinder and a solid sphere, both having the same mass and radius. The cylinder is free to rotate about its standard axis of* *symmetry, and the sphere is free to rotate about an axis passing through its centre. Which of the two will acquire a greater angular speed after a given time ?*

Ans. Let M and R be mass and radius of the hollow cylinder and the solid sphere. Then

M.I. of hollow cylinder about its axis of symmetry,

$$I_1 = MR^2$$

M.I. of solid sphere about an axis through its centre,

$$I_2 = \frac{2}{5} MR^2$$

Let α_1 and α_2 be angular accelerations produced in the rotational motion of the cylinder and the sphere on applying a torque τ in each case. Then

$$\alpha_1 = \frac{\tau}{I_1} = \frac{\tau}{MR^2}$$

and $$\alpha_2 = \frac{\tau}{I_2} = \frac{\tau}{\frac{2}{5} MR^2} = 2.5 \frac{\tau}{MR^2} = 2.5\, \alpha_1$$

As $\alpha_2 > \alpha_1$ and $\omega = \omega_0 + \alpha t$, so the solid sphere will acquire a greater angular speed after a given time.

7.12 *A solid cylinder of mass 20 kg rotates about its axis with angular speed 100 rad s^{-1}. The radius of the cylinder is 0.25 m. What is the kinetic energy associated with the rotation of the cylinder ? What is the magnitude of angular momentum of the cylinder about its axis ?*

Ans. Here $M = 20$ kg, $\omega = 100$ rad s^{-1}, $R = 0.25$ m

M.I. of the cylinder about its own axis,

$$I = \frac{1}{2} MR^2 = \frac{1}{2}\, 20 \times (0.25)^2 = 0.625 \text{ kgm}^2$$

Rotational K.E.

$$= \frac{1}{2} I\omega^2 = \frac{1}{2} \times 0.625 \times (100)^2 = \mathbf{3125\ J.}$$

Angular momentum,

$$L = I\omega = 0.625 \times 100 = \mathbf{62.5\ kg\ m^2\ s^{-1}}.$$

7.13 *(i) A child stands at the centre of turntable with his two arms out stretched. The turntable is set rotating with an angular speed of 40 rpm. How much is the angular speed of the child if he folds his hands back and thereby reduces his moment of inertia to 2/3 times the initial value ? Assume that the turntable rotates without friction.*

(ii) Show that the child's new kinetic energy of rotation is more than the initial kinetic energy of rotation. How do you account for this increase in kinetic energy ?

Ans. Here $\omega_1 = 40$ rpm, $I_2 = \frac{2}{5} I_1$

By the principle of conservation of angular momentum,

$$I_1 \omega_1 = I_2 \omega_2 \quad \text{or} \quad I_1 \times 40 = \frac{2}{5} I_1 \omega_2$$

or $\omega_2 = \mathbf{100\ rpm.}$

(*ii*) Initial kinetic energy of rotation

$$= \frac{1}{2} I_1 \omega_1^2 = \frac{1}{2} I_1 (40)^2 = 800\, I_1$$

New kinetic energy of rotation

$$= \frac{1}{2} I_2 \omega_2^2 = \frac{1}{2} \times \frac{2}{3} I_1 \times (100)^2 = 2000\, I_1$$

$$\therefore \quad \frac{\text{New K.E.}}{\text{Initial K.E.}} = \frac{2000\, I_1}{800\, I_1} = \mathbf{2.5.}$$

Thus the child's new kinetic energy of rotation is 2.5 times its initial kinetic energy of rotation. This increase in kinetic energy is due to the internal energy of the child which he uses in folding his hands back from the out stretched position.

7.14. *A rope of negligible mass is wound round a hollow cylinder of mass 3 kg and radius 40 cm. What is angular acceleration of the cylinder if the rope is pulled with a force of 30 N ? What is the linear acceleration of the rope ? Assume that there is no slipping.*

Ans. Here $M = 3$ kg, $R = 40$ cm $= 0.40$ m, $F = 30$ N

Torque, $\tau = F \times R = 30 \times 0.40 = 12$ Nm

M.I. of the hollow cylinder about its own axis,

$$I = MR^2 = 3 \times (0.40)^2 = 0.48 \text{ kg m}^2$$

Angular acceleration,

$$\alpha = \frac{\tau}{I} = \frac{12}{0.48} = \mathbf{25\ rad\ s^{-2}}.$$

Linear acceleration,

$$a = R\alpha = 0.40 \times 25 = \mathbf{10\ ms^{-2}}.$$

7.15. *To maintain a rotor at a uniform angular speed of 200 rad s^{-1}, an engine needs to transmit a torque of 180 Nm. What is the power required by the engine ? Assume that the engine is 100% efficient.*

Ans. Here $\omega = 200$ rad s^{-1}, $\tau = 180$ Nm

$\therefore$ Power, $P = \tau\, \omega = 180 \times 200 = 36{,}000$ W $= \mathbf{36\ kW.}$

7.16. *From a uniform disc of radius R, a circular hole of radius R / 2 is cut out. The centre of the hole is at R / 2 from the centre of the original disc. Locate the centre of gravity of the resulting flat body.*

Ans. The situation is shown in Fig. 7.104. Let O be the CM of the original circular portion, O_1 that of the circular hole cut out and O_2 that of the remaining shaded portion. Let m be the mass per unit area of the disc.

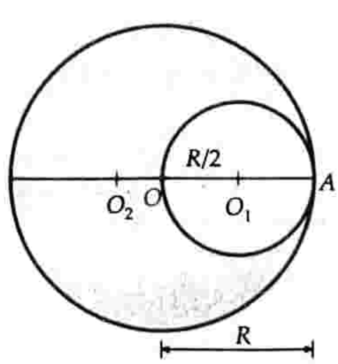


Fig. 7.104

Mass of the original disc,

$$M = \pi R^2 m$$

Mass of the circular hole cut,

$$m_1 = \pi\left(\frac{R}{2}\right)^2 m = \frac{\pi}{4} R^2 m$$

Mass of the remaining portion,

$$m_2 = \pi R^2 m - \frac{\pi}{4} R^2 m = \frac{3}{4}\pi R^2 m$$

Masses m_1 and m_2 may be assumed to be concentrated at O_1 and O_2 respectively and O is their CM.

∴ Moment of m_1 about O

= Moment of m_2 about O

or $m_1 \times O_1O = m_2 \times O_2O$

or $\frac{\pi}{4} R^2 m \times \frac{R}{2} = \frac{3}{4}\pi R^2 m \times O_2O$

or $O_2O = \frac{R}{6}$

Thus the CM of the resulting portion lies at $R/6$ from the centre of the original disc in a direction opposite to the centre of the cut out portion.

7.17. *A metre stick is balanced on a knife edge at its centre. When two coins, each of mass 5 g are put one on top of the other at the 12.0 cm mark, the stick is found to be balanced at 45.0 cm. What is the mass of the metre stick ?*

Ans. As the metre stick balances at 50.0 cm mark, its CG must lie at this mark.

The 45.0 cm mark is the CG of the metre stick + 2 coins system.

Let m be the mass of the metre stick.

Distance between 50.0 cm mark and new CG

= 50.0 – 45.0 = 5.0 cm

Distance between 12.0 cm mark and new CG

= 45.0 – 12.0 = 33.0 cm

From the principle of moments (for equilibrium),

$$mg \times 5.0 = (2 \times 5) \times g \times 33.0$$

or $m = \frac{2 \times 5 \times 33.0}{5.0} = \mathbf{66.0\ g.}$

7.18. *A solid sphere rolls down two different inclined planes of the same heights but different angles of inclination. (a) Will it reach the bottom with the same speed in each case ? (b) Will it take longer to roll down one plane than the other ? (c) If so, which one and why ?*

Ans. Acceleration of the rolling sphere,

$$a = \frac{g \sin\theta}{(1 + k^2/R^2)}$$

Velocity of the sphere at the bottom of the inclined plane,

$$v = \sqrt{\frac{2gh}{(1 + k^2/R^2)}}$$

(*a*) Yes, the sphere will reach the bottom with the same speed v because h is same in both cases.

(*b*) Yes, the sphere will take longer time to roll down one plane than the other.

(*c*) The sphere will take larger time in case of the plane with smaller inclination because the acceleration, $a \propto \sin\theta$.

7.19. *A hoop of radius 2 m weighs 100 kg. It rolls along a horizontal floor so that its centre of mass has a speed of 20 cm/s. How much work has to be done to stop it ?*

Ans. Here $R = 2$ m, $M = 100$ kg,

$$v_{cm} = 20 \text{ cms}^{-1} = 0.20 \text{ ms}^{-1}$$

Work required to stop the hoop

= Total K.E. of the hoop

= Rotational K.E. + Translational K.E.

$$= \frac{1}{2} I\omega^2 + \frac{1}{2} Mv_{cm}^2$$

$$= \frac{1}{2} \times MR^2 \times \left(\frac{v_{cm}}{R}\right)^2 + \frac{1}{2} M(v_{cm})^2$$

$$= Mv_{cm}^2 = 100 \times (0.20)^2 = \mathbf{4\ J.}$$

7.20. *The oxygen molecule has a mass of 5.30×10^{-26} kg and a moment of inertia of 1.94×10^{-46} kg m² about an axis through its centre perpendicular to the line joining the two atoms. Suppose the mean speed of such a molecule in a gas is 500 m/s and that its kinetic energy of rotation is two thirds of its kinetic energy of translation. Find the average angular velocity of the molecule.*

Ans. Rotational K.E. $= \frac{2}{3}$ Translational K.E.

or $\frac{1}{2} I\omega^2 = \frac{2}{3} \cdot \frac{1}{2} mv^2$

or $\omega = v \times \sqrt{\frac{2m}{3I}} = 500 \times \sqrt{\frac{2 \times 5.30 \times 10^{-26}}{3 \times 1.94 \times 10^{-46}}}$

$$= 500 \times \sqrt{1.82 \times 10^{20}}$$

$$= 500 \times 1.35 \times 10^{10} \text{ rad s}^{-1} = \mathbf{6.75 \times 16^{12}\ rad\ s^{-1}.}$$

7.21. *A solid cylinder rolls up an inclined plane of angle of inclination 30°. At the bottom of the inclined plane the centre of mass of the cylinder has a speed of 5 m/s.*

(*a*) *How far will the cylinder go up the plane ?*

(*b*) *How long will it take to return to the bottom ?*

Ans. (*a*) Total initial kinetic energy of the cylinder,

$$K_i = \frac{1}{2} Mv_{CM}^2 + \frac{1}{2} I_{CM}\,\omega^2$$

$$= \frac{1}{2} Mv_{CM}^2 + \frac{1}{2} \times \frac{1}{2} MR^2 \times \frac{v_{CM}^2}{R^2}$$

$$= \frac{1}{2} Mv_{CM}^2 + \frac{1}{4} Mv_{CM}^2 = \frac{3}{4} Mv_{CM}^2.$$

Initial potential energy, $U_i = 0$

Final kinetic energy, $K_f = 0$

Final potential energy,

$$U_f = Mgh = Mgs \sin 30° = \frac{1}{2} Mgs$$

where s is the distance travelled up the incline and h is the vertical height covered above the bottom.

Gain in P.E. = Loss in K.E.

$$\frac{1}{2} Mgs = \frac{3}{4} Mv_{CM}^2$$

$$s = \frac{3v_{CM}^2}{2g} = \frac{3 \times (5)^2}{2 \times 9.8} = \mathbf{3.8\ m.}$$

(*b*) Using equation of motion for the motion up the incline, we get

$$0 = v_{CM} + at \quad \text{or} \quad a = -\frac{v_{CM}}{t}$$

Also, $\quad 0^2 - v_{CM}^2 = 2as$

or $$a = -\frac{v_{CM}^2}{2s}$$

$\therefore$ $$\frac{v_{CM}}{t} = \frac{v_{CM}^2}{2s}$$

or $$t = \frac{2s}{v_{CM}} = \frac{2 \times 3.8}{5} = 1.5\ \text{s}$$

Total time taken in returning to the bottom

$$= 2 \times 1.5 = \mathbf{3.0\ s.}$$

7.22. *As shown in Fig. 7.105, the two sides of a step ladder BA and CA are 1.6 m long and hinged at A. A rope DE, 0.5 m is tied half way up. A weight 40 kg is suspended from a point F, 1.2 m from B along the ladder BA. Assuming the floor to be frictionless and neglecting the weight of the ladder, find the tension in the rope and forces exerted by the floor on the ladder. (Take $g = 9.8\ m/s^2$).*

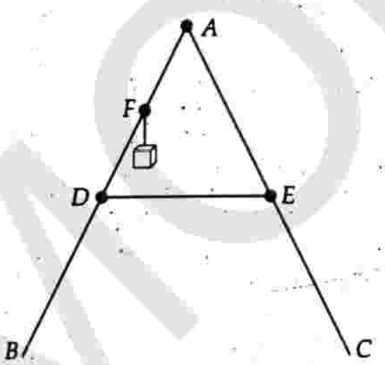


Fig. 7.105

Ans. As shown in Fig. 7.106, let N_B and N_C be the normal reactions of the floor at B and C respectively and T be the tension in the rope DE. Then

$$N_B + N_C = W = 40 \times 9.8\ \text{N}$$

or $$N_B + N_C = 392\ \text{N} \qquad ...(1)$$

Consider the portion AC of the ladder. Balancing torques about A, we get

$$N_C \times LC = T \times AG$$

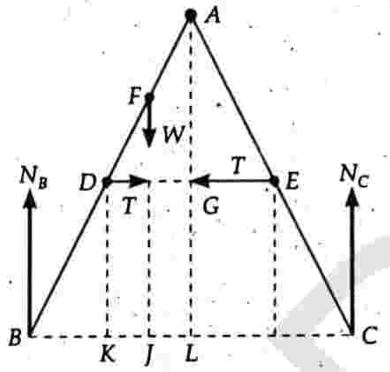


Fig. 7.106

From the geometry of the figure, we get

$$LC = 2GE = DE = 0.5\ \text{m}$$

or $$GE = 0.25\ \text{m}$$

$\therefore$ $$AG = \sqrt{AE^2 - GE^2} = \sqrt{(0.8)^2 - (0.25)^2}$$

$$= \sqrt{0.5775} = 0.76\ \text{m}$$

Hence $N_C \times 0.5 = T \times 0.76$ or $T = 0.66\ N_C$ ...(2)

Now consider the portion AB of the ladder. Balancing the torques about A, we get

$$N_B \times BL - W \times JL = T \times AG$$

But $$JL = \frac{1}{4} DE = \frac{1}{4} \times 0.5 = 0.125\ \text{m},$$

$$BL = DE = 0.5\ \text{m}, \ W = 392\ \text{N}$$

$\therefore$ $$N_B \times 0.5 - 392 \times 0.125 = T \times 0.76$$

or $$N_B \times 0.5 - 392 \times 0.125 = 0.66\ N_C \times 0.76$$

[using (2)]

or $$N_B - 98 = N_C$$

or $$N_B - N_C = 98 \qquad ...(3)$$

On solving (1) and (3), we get

$$N_B = \mathbf{245\ N}, \ N_C = 245 - 98 = \mathbf{147\ N},$$

and $$T = 0.66\ N_C = 0.66 \times 147 = \mathbf{97\ N}.$$

7.23. *A man stands on a rotating platform, with his arms stretched horizontally holding a 5 kg weight in each hand. The angular speed of the platform is 30 revolutions per minute. The man then brings his arms close to his body with the distance of each weight from the axis changing from 90 cm to 20 cm. The moment of inertia of the man together with the platform may be taken to be constant and equal to 7.6 kg m^2.*

(*a*) *What is his new angular speed ? (Neglect friction)*

(*b*) *Is kinetic energy conserved in the process ? If not, from where does the change come about ?*

Ans. (*a*) Total initial moment of inertia,

$$I_1 = \text{M.I. of man and platform} + \text{M.I. of two 5 kg weights}$$

$$= 7.6 + 2 \times 5 \times (0.90)^2 = 7.6 + 8.1$$

$$= 15.7\ \text{kg m}^2$$

Initial angular speed,

$$\omega_1 = 30 \text{ rpm}$$

Total final moment of inertia,

$$I_2 = 7.6 + 2 \times 5 \times (0.20)^2 = 7.6 + 0.4$$
$$= 8.0 \text{ kg m}^2$$

By the principle of conservation of angular momentum,

$$I_1\omega_1 = I_2\omega_2$$

or $$15.7 \times 30 = 8.0 \times \omega_2$$

or $$\omega_2 = \frac{15.7 \times 30}{8.0} = 58.875 \approx \mathbf{59 \text{ rpm}}.$$

(*b*) $$\frac{\text{Final K.E.}}{\text{Initial K.E.}} = \frac{\frac{1}{2} I_1\omega_1^2}{\frac{1}{2} I_2 \omega_2^2} = \frac{8.0 \times (59)^2}{15.7 \times (30)^2} = \mathbf{1.97}.$$

Thus the final K.E. is about twice the initial K.E. *i.e.*, K.E. is not conserved in the process. The increase in K.E. is due to the internal energy the man uses in bringing his arms closer to his body.

7.24. *A bullet of mass 10 g and speed 500 m/s is fired into a door and gets embedded exactly at the centre of the door. The door is 1.0 m wide and weighs 12 kg. It is hinged at one end and rotates about a vertical axis practically without friction. Find the angular speed of the door just after the bullet embeds into it.*

(Hint. *The moment of inertia of the door about the vertical axis at one end is* $ML^2/3$.)

Ans. By the principle of conservation of angular momentum,

Initial angular momentum of the bullet
= Final angular momentum of the door

or $$pr = I\omega$$

or $$mvr = \frac{ML^2}{3} \times \omega \quad \text{or} \quad \omega = \frac{3mvr}{ML^2}$$

Here $m = 10 \text{ g} = 10^{-2} \text{ kg}$, $v = 500 \text{ ms}^{-1}$,

$$r = \frac{1.0}{2} = 0.5 \text{ m}, \quad L = 1.0 \text{ m}, \quad M = 12 \text{ kg}$$

$$\therefore \quad \omega = \frac{3 \times 10^{-2} \times 500 \times 0.5}{12 \times (1.0)^2} = \mathbf{0.625 \text{ rad s}^{-1}}.$$

7.25. *Two discs of moments of inertia* I_1 *and* I_2 *about their respective axes (normal to the disc and passing through the centre), and rotating with angular speed* ω_1 *and* ω_2 *are brought into contact face to face with their axes of rotation coincident. (i) What is the angular speed of the two-disc system ? (ii) Show that the kinetic energy of the combined system is less than the sum of the initial kinetic energies of the two discs. How do you account for this loss in energy ? Take* $\omega_1 \neq \omega_2$.

Ans. (*i*) Let ω be the angular speed of the two-disc system. Then by conservation of angular momentum,

$$(I_1 + I_2)\,\omega = I_1\,\omega_1 + I_2\,\omega_2$$

or $$\omega = \frac{I_1\,\omega_1 + I_2\,\omega_2}{I_1 + I_2}.$$

(*ii*) Initial K.E. of the two discs,

$$K_1 = \frac{1}{2} I_1\,\omega_1^2 + \frac{1}{2} I_2\,\omega_2$$

Final K.E. of the two-disc system,

$$K_2 = \frac{1}{2}(I_1 + I_2)\,\omega^2$$
$$= \frac{1}{2}(I_1 + I_2)\left(\frac{I_1\,\omega_1 + I_2\,\omega_2}{I_1 + I_2}\right)^2$$

Loss in K.E. $= K_1 - K_2$

$$= \frac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2) - \frac{1}{2(I_1 + I_2)}(I_1\omega_1 + I_2\omega_2)^2$$
$$= \frac{1}{2(I_1 + I_2)}[I_1^2\omega_1^2 + I_1I_2\omega_2^2 + I_1I_2\omega_1^2 + I_2^2\omega_2^2 - (I_1^2\,\omega_1^2 + I_2^2\,\omega_2^2 + 2I_1\,I_2\,\omega_1\,\omega_2]$$
$$= \frac{1}{2(I_1 + I_2)}[I_1I_2\omega_2^2 + I_1I_2\omega_1^2 - 2I_1I_2\omega_1\omega_2]$$
$$= \frac{I_1\,I_2}{2(I_1 + I_2)}(\omega_1^2 + \omega_2^2 - 2\omega_1\,\omega_2)$$
$$= \frac{I_1\,I_2}{2(I_1 + I_2)}(\omega_1 - \omega_2)^2 = \text{a positive quantity} \quad [\because\ \omega_1 \neq \omega_2\,]$$

Hence there is a loss of rotational K.E. which appears as heat. When the two discs are brought together, work is done against friction between the two discs.

7.26. (*a*) *Prove the theorem of perpendicular axes.*

(*b*) *Prove the theorem of parallel axes.*

Ans. (*a*) Refer answer to Q.38 on page 7.27.

(*b*) Refer answer to Q.39 on page 7.27.

7.27. *Prove the result that the velocity v of translation of a rolling body (like a ring, disc, cylinder or sphere) at the bottom of an inclined plane of a height h is given by*

$$v^2 = \frac{2gh}{(1 + k^2/R^2)}$$

using dynamical consideration (i.e., by consideration of forces and torques). Note k is the radius of gyration of the body about its symmetry axis, and R is the radius of the body. The body starts from rest at the top of the plane.

Ans. Velocity attained by a body rolling down an inclined plane. Consider a body of mass M and radius R rolling down a plane inclined at an angle θ with the horizontal, as shown in Fig. 7.107. It is only due to friction at the line of contact that body can roll without slipping. The centre of mass of the body moves in a straight line parallel to the inclined plane.

The external forces on the body are

(*i*) The weight Mg acting vertically downwards.

(*ii*) The normal reaction N of the inclined plane.

(*iii*) The force of friction acting up the inclined plane.

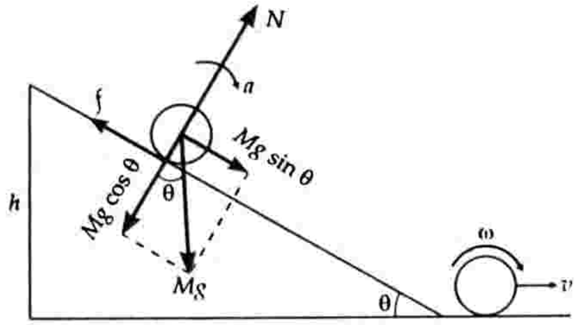


Fig. 7.107 A body rolling without slipping on an inclined plane.

Let a be the downward acceleration of the body. The equations of motion for the body can be written as

$$N - Mg\cos\theta = 0$$

$$F = Ma = Mg\sin\theta - f$$

As the force of friction f provides the necessary torque for rolling, so

$$\tau = f \times R = I\alpha = Mk^2\left(\frac{a}{R}\right)$$

or $$f = M\frac{k^2}{R^2}\cdot a$$

where k is the radius of gyration of the body about its axis of rotation. Clearly

$$Ma = Mg\sin\theta - M\frac{k^2}{R^2}\cdot a$$

or $$a = \frac{g\sin\theta}{(1 + k^2/R^2)}$$

Let h be height of the inclined plane and s the distance travelled by the body down the plane. The velocity v attained by the body at the bottom of the inclined plane can be obtained as follows :

$$v^2 - u^2 = 2as$$

or $$v^2 - 0^2 = 2\cdot\frac{g\sin\theta}{(1 + k^2/R^2)}\cdot s$$

or $$v^2 = \frac{2gh}{1 + k^2/R^2} \qquad \left[\because \frac{h}{s} = \sin\theta\right]$$

or $$v = \sqrt{\frac{2gh}{(1 + k^2/R^2)}}$$

7.28. *A disc rotating about its axis with angular speed ω_0 is placed lightly (without any translational push) on a perfectly frictionless table. The radius of the disc is R. What are the linear velocities of the points A, B and C on the disc shown in Fig. 7.108 ? Will the disc roll in the direction indicated ?*

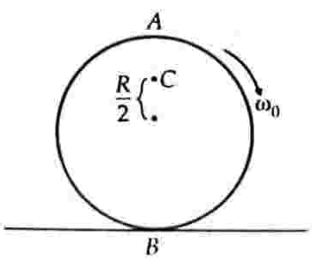


Fig. 7.108

Ans. We know that, $v = R\omega$

For point A : $v_A = R\omega_0$ *(in the direction of the arrow)*

For point B : $v_B = R\omega_0$ *(in the direction opposite to arrow)*

For point C : $v_C = \left(\frac{R}{2}\right)\omega_0$ *(in the direction of the arrow)*

The disc *will not roll* in the direction indicated. It is because the disc is placed on a perfectly frictionless table and without friction, a body cannot roll.

7.29. *Explain why friction is necessary to make the disc in Fig. 7.108 given in previous question roll in the direction indicated.*

(*i*) *Give the direction of frictional force at B, and the sense of frictional torque, before perfect rolling begins.*

(*ii*) *What is the force of friction after perfect rolling begins ?*

Ans. To roll a disc, a torque is required which in turn requires a tangential force to act on it. As the force of friction is the only tangential force acting on the disc, so it is necessarily required for the rolling of a disc.

(*i*) Frictional force at B opposes the velocity of B. Therefore, frictional force is in the same direction as the arrow. The sense of frictional torque is such as to oppose the angular motion. By right hand rule, both $\vec{\omega}_0$ and $\vec{\tau}$ act normal to the plane of paper, $\vec{\omega}_0$ into the plane of paper and $\vec{\tau}$ out of the paper.

(*ii*) Frictional force decreases the velocity of the point of contact B. Perfect rolling begins when this velocity is zero. Once this is, the force of friction is zero.

7.30. *A solid disc and a ring, both of radius 10 cm are placed on a horizontal table simultaneously, with initial angular speed equal to 10π rad s^{-1}. Which of the two will start to roll earlier ? The co-efficient of kinetic friction is $\mu_k = 0.2$.*

Ans. In case of pure rotation without translation, the velocity of centre of mass is zero. The friction reduces the speed at the point of contact and as such accelerates the centre of mass till the velocity of centre of mass becomes equal to $v = R\omega$ and the instantaneous velocity at the contact point becomes zero. Thus the force of friction $\mu_k mg$ produces an acceleration a in the centre of mass. So the equation of motion for CM is

$$\mu_k mg = ma \qquad ...(1)$$

The torque due to force of friction is $\mu_k mg \times R$. It produces angular retardation given by

$$\mu_k mgR = -I\alpha \qquad ...(2)$$

Rolling begins when

$$v = R\omega$$

But $$v = 0 + at = \mu_k gt \qquad ...(3)$$

[From (1), $a = \mu_k g$]

and $\quad \omega = \omega_0 + \alpha t = \omega_0 - \frac{\mu_k mgR}{I} t \quad$ [using (2)]

or $\quad \frac{v}{R} = \omega_0 - \frac{\mu_k mgR}{I} t$

or $\quad \frac{\mu_k gt}{R} = \omega_0 - \frac{\mu_0 mgR}{I} t$

or $\quad \frac{\mu_k gt}{R}\left[1 + \frac{mR^2}{I}\right] = \omega_0$

or $\quad t = \frac{R\omega_0}{\mu_k g\left(1 + \frac{mR^2}{I}\right)}$

For a disc : $I = mR^2/2$

$\therefore \quad t = \frac{R\omega_0}{3\mu_k g} = \frac{0.10 \times 10\pi}{3 \times 0.2 \times 9.8} = \mathbf{0.53\ s.}$

For a ring : $I = mR^2$

$\therefore \quad t = \frac{R\omega_0}{2\mu_k g} = \frac{0.10 \times 10\,\pi}{2 \times 0.2 \times 9.8} = \mathbf{0.80\ s.}$

Thus the disc begins to roll earlier than the ring.

7.31. *A solid cylinder of mass 10 kg and radius 15 cm is rolling perfectly on a plane of inclination 30°. The coefficient of static friction, μ_s = 0.25. (i) Find the force of friction acting on the cylinder. (ii) What is the work done against friction during rolling ? (iii) If the inclination θ of the plane is increased, at what value of θ does the cylinder begin to skid, and not roll perfectly ?*

Ans. Here $M = 10$ kg, $R = 0.15$ m, $\mu_s = 0.25$, $\theta = 30°$

(*i*) Force of friction,

$$f = \frac{1}{3} Mg \sin\theta = \frac{1}{3} \times 10 \times 9.8 \times \sin 30°$$

$$= \frac{1}{3} \times 10 \times 9.8 \times 0.5 = \mathbf{16.33\ N.}$$

(*ii*) Work done against friction during rolling

$= \mathbf{0\ J.}$

(*iii*) Condition for skidding (or no rolling) is

$$\frac{f}{N} \le \mu_s$$

or $\quad \frac{\frac{1}{3} Mg \sin\theta}{Mg\cos\theta} \le \mu_s$

or $\quad \frac{1}{3}\tan\theta \le \mu_s$

Thus the cylinder will start skidding at an angle of inclination θ given by

$$\tan\theta = 3\mu_s = 3 \times 0.25 = 0.75$$

or $\quad \theta = \mathbf{36°52'}.$

7.32. *Read each statement below carefully, and state, with reasons, if it is true or false :*

(*a*) *During rolling, the force of friction acts in the same direction as the direction of motion of the CM of the body.*

(*b*) *The instantaneous speed of the point of contact during rolling is zero.*

(*c*) *The instantaneous acceleration of the point of contact during rolling is zero.*

(*d*) *For perfect rolling motion, work done against friction is zero.*

(*e*) *A wheel moving down a perfectly frictionless inclined plane will undergo slipping (not rolling) motion.*

Ans. (*a*) ***True.*** When a body rolls, the force of friction acts in the same direction as the direction of motion of the CM of the body.

(*b*) ***True.*** A rolling body can be imagined to be rotating about an axis passing through the point of contact of the body and the ground. Hence the instantaneous speed of the point of contact is zero.

(*c*) ***False.*** As the body is rotating, its instantaneous acceleration cannot be zero.

(*d*) ***True.*** Perfect rolling begins when force of friction is zero. So work done against friction is zero.

(*e*) ***True.*** On a perfectly frictionless inclined plane, there is no tangential force of friction. So the wheel cannot roll. It will simply slip under the effect of its own weight.

Text Based Exercises

Type A : Very Short Answer Questions

1 Mark Each

1. Define centre of mass of a system. **[Himachal 08, 09]**
2. On what factors, does the position of centre of mass of a rigid body depend ?
3. Where does the centre of mass of uniform triangular lamina lie ?
4. Where does the centre of mass of a cone lie ?
5. What is an isolated system ? What will be the nature of motion of centre of mass of isolated system ?
6. Where does the centre of mass of a two particle system lie, if one particle is more massive than the other ? **[Delhi 97]**

7. Give an example each for a body, where the centre of mass lies inside the body and outside the body. **[Delhi 99]**
8. Where does the centre of mass of two particles of equal mass lie ? What is its position vector ?
9. Write an expression for the centre of mass of a two particle system.
10. Write an expression for the velocity of the centre of mass of a system of particles.
11. What is a rigid body ? **[Himachal 08 ; Delhi 09]**
12. Name the rotational analogue of force. Give its SI unit.
13. Is torque a scalar or vector ? If vector, which rule is used to determine its direction ?
14. Write the dimensional formula of torque.
15. What is the ratio of SI unit to the CGS unit of torque ?
16. Under what conditions, is the torque zero ?
17. A body is rotating at a steady rate. Is any torque acting on the body ?
18. Which physical quantity corresponds to moment of linear momentum ?
19. Write the dimensional formula of angular momentum. Is it a scalar ?
20. Write the SI unit of angular momentum. **[Delhi 08]**
21. What is the ratio of the SI unit to the CGS unit of angular momentum ?
22. Name the constant whose dimensions are same as that of angular momentum ?
23. Does the magnitude and direction of angular momentum depend on the choice of the origin ?
24. Does the total momentum of a system of particles depend upon the velocity of the centre of mass ?
25. What is the angular momentum of a body of mass m moving in a circular path of radius r with constant speed v ?
26. State right hand rule to find the direction of angular momentum.
27. Name the physical quantity which is equal to the time rate of change of angular momentum.
28. Which physical quantity is conserved when a planet revolves around the sun ?
29. A planet revolves around the sun under the effect of gravitational force exerted by the sun. Why is the torque on the planet due to the gravitational force zero ?
30. A boat is likely to capsize if the persons in the boat stand up. Why ?
31. A body is moving in a circle of radius r centimetre at a constant speed of v cm s^{-1}. What is the angular velocity ?
32. What is the angular velocity of the earth spinning about its own axis ?
33. Is the angular velocity of rotation of hour hand of a watch greater or smaller than the angular velocity of earth's rotation about its own axis ?
34. Two bodies move in two concentric circular paths of radii r_1 and r_2 in the same time. What is the ratio of their angular velocities ?
35. Name the rotational analogue of linear acceleration.
36. What is the analogous of mass in rotational motion ? Write unit of that physical quantity also. **[Delhi 95, 08]**
37. Is moment of inertia a scalar or a vector quantity ? **[Himachal 01]**
38. Write an expression for the moment of inertia of a rod of mass 'M' and length 'L' about an axis perpendicular to it through one end. **[Delhi 2003]**
39. What is the moment of inertia of a solid sphere about its diameter ? **[Delhi 09]**
40. What is the moment of inertia of a hollow sphere about an axis passing through its centre ?
41. What is the moment of inertia of a circular ring about its tangent in its plane ?
42. What is the dimensional formula of radius of gyration ?
43. State the factors on which radius of gyration of a body depends.
44. A circular ring and a circular disc of the same radius have the same moment of inertia about axis passing through their centres and perpendicular to their planes. What is the ratio of their masses ?
45. Where does the centre of mass of a rectangular lamina lie ? **[Himachal 03]**
46. What is torque ? Give its SI unit. **[Himachal 04, 05C, 09]**
47. What are the factors on which the moment of inertia of a body depends ? **[Himachal 04]**
48. Give the physical significance of moment of inertia. **[Himachal 08, 09]**
49. Which physical quantity is represented by the product of moment of inertia and angular velocity ? **[Delhi 04 ; Central Schools 12]**
50. Define angular momentum. **[Himachal 05C, 07C, 08, 09]**
51. Define the term moment of momentum. **[Himachal 07C]**
52. Write two factors on which centre of mass of a body does not depend. **[Chandigarh 09]**

Answers

1. The centre of mass of a system may be defined as the point at which the entire mass of the system may be supposed to be concentrated. The nature of the motion of the system will remain unaffected if all the forces acting on the system were applied directly on the centre of mass of the system.
2. The centre of mass of a rigid body depends on
   (*i*) The geometrical shape of the body.
   (*ii*) The distribution of mass in the body.
3. Centre of mass of a triangular lamina lies at its centroid *i.e.*, at the point of intersection of the three medians.
4. The centre of mass of a cone lies on the line joining the apex to the centre of the base at a distance equal to $\frac{1}{4}$ of the length of this line from the base.
5. An isolated system is one on which no external force acts. The centre of mass of such a system remains either at rest or moves with uniform velocity.
6. The CM will lie on the line joining the two particles, closer to the more massive particle.
7. (*i*) The CM of a solid sphere lies inside the body.
   (*ii*) The CM of a hollow sphere lies outside the body.
8. The centre of mass of two particles of equal masses lies midway between them. Its position vector is the average of the position vectors of the two particles.
9. $R_{CM} = \dfrac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}$.
10. $\vec{v}_{CM} = \dfrac{\Sigma m_i \vec{v}_i}{M}$.
11. A rigid body is one whose constituent particles retain their relative positions even when they move under the action of an external force.
12. Torque. Its SI unit is Nm.
13. Torque is a vector, $\vec{\tau} = \vec{r} \times \vec{F}$. Its direction is perpendicular to the plane of $\vec{r}$ and $\vec{F}$ and is given by right handed screw rule.
14. [Torque] = $[ML^2T^{-2}]$.
15. $\dfrac{\text{SI unit of torque}}{\text{CGS unit of torque}} = 10^7$.
16. As $\tau = rF \sin \theta$, so torque is zero if r is zero or F is zero. It is also zero if $\theta = 0°$ or $180°$.
17. No. A torque produces angular acceleration.
18. Angular momentum.
19. [Angular momentum] = $[ML^2T^{-1}]$.
    Angular momentum is not a scalar but a vector.
20. SI unit of angular momentum = kg m^2 s^{-1}.
21. $\dfrac{\text{SI unit of angular momentum}}{\text{CGS unit of angular momentum}} = 10^7$.
22. Planck's constant (h) has the same dimensions as angular momentum.
23. Yes.
24. Yes.
    Total momentum = Mv_{CM}.
25. Angular momentum
    = Linear momentum × moment arm
    or $L = p \times r = mvr$.
26. Curl the fingers of the right hand in the direction of rotation, then the thumb points in the direction of angular momentum.
27. Torque.
28. Angular momentum of the planet.
29. The gravitational force acts along the line joining the planet to the sun. Vectors $\vec{r}$ and $\vec{F}$ are always parallel.
    $\therefore \quad \tau = rF \sin 0° = 0$.
30. If the passengers stand, the CG of the system (boat + passengers) is raised and so it is likely to capsize.
31. $\omega = \dfrac{v}{r}$ rad s^{-1}.
32. Angular velocity of spinning of earth
    $= \dfrac{2\pi}{86400}$ rad s^{-1}.
33. The hour hand completes one rotation in 12 hours while the earth completes one rotation in 24 hours. So, angular velocity of hour hand is double the angular velocity of earth, because $\omega = 2\pi / T$.
34. Angular velocity, $\omega = 2\pi / T$ is same for both bodies. So the ratio is 1 : 1.
35. Angular acceleration is the rotational analogue of linear acceleration.
36. Moment of inertia is the rotational analogue of mass. Its SI unit is kg m^2.
37. M.I. is a scalar quantity.
38. $I = \dfrac{ML^2}{3}$.

39. $I = \frac{2}{5} MR^2$, where M is the mass and R is the radius of the solid sphere.

40. $I = \frac{2}{3} MR^2$, where M is the mass and R is the radius of the hollow sphere.

41. $I = \frac{3}{2} MR^2$, where M is the mass and R is the radius of the ring.

42. The dimensional formula of radius of gyration is $[M^0L^1T^0]$.

43. The radius of gyration of a body depends on
    (*i*) the position and orientation of the axis of rotation.
    (*ii*) the distribution of mass of the body about the axis of rotation.

44. $I = M_{ring} R^2 = \frac{1}{2} M_{disc} R^2$

$\therefore \quad M_{ring} : M_{disc} = 1 : 2$

45. The CM of a rectangle lies at the point of intersection of the diagonals of the rectangle.

46. Refer to point 8 of Glimpses.

47. Moment of inertia of a body depends on :
    (*i*) Mass of the body.
    (*ii*) Distribution of mass about the axis of rotation.

48. Moment of inertia of a body is the rotational inertia of the body. It is the rotational analogue of mass in linear motion.

49. Angular momentum ($L = I\omega$).

50. Angular momentum of a particle is the moment of its linear momentum about the axis of rotation.

51. It is equal to the product of linear momentum and the perpendicular distance of its line of action from the axis of rotation.

52. The centre of mass of a system does not depend on (*i*) choice of the coordinate system, and (*ii*) force acting on the particles.

Type B : Short Answer Questions

2 or 3 Marks Each

1. What is meant by a particle, a system and internal and external forces ?

2. Define centre of mass of a system. How does it differ from the centre of gravity ?

3. Write an expression for the centre of mass of a two particle system. What will be the location of centre of mass if the two particles have equal masses ?

4. Show that the centre of mass of two particles is on the line joining them at a point whose distance from each particle is inversely proportional to the mass of that particle.

5. Write an expression for the centre of mass of n-particle system. Also write the equations of motion which govern the motion of the centre of mass.

6. Show that in the absence of any external force, the velocity of the centre of mass remains constant.

7. Show that the total linear momentum of a system of particles is conserved in the absence of any external force.

8. Show that the total linear momentum of a system of particles is equal to the product of the total mass of the system and the velocity of its centre of mass.

9. What are binary stars ? Discuss their motion in respect of their centre of mass.

10. Briefly describe the motion of CM of a diatomic molecule like O_2.

11. Briefly explain the motion of the centre of mass of the earth-moon system.

12. Discuss the motion of the centre of mass of a fire cracker that explodes in air.

13. Define a rigid body. State the factors on which the centre of mass a rigid body depends.

14. Define the term torque or moment of force. Give its units and dimensions. **[Himachal 01C, 04, 08]**

15. Name the physical quantity corresponding to force in rotational motion. How is it related to force ? Give its units. **[Central Schools 08]**

16. State and explain the principle of moments.

17. Define a couple. Show that the moment of a couple is same irrespective of the point of rotation of a body.

18. Derive an expression for the work done by a torque. Hence write the expression for the power delivered by a torque.

19. How torque is expressed as the vector product of two vectors ? State the rule for determining the direction of torque.

20. What is torque ? Give its unit. Show that it is equal to the product of force and the perpendicular distance of its line of action from the axis of rotation. **[Chandigarh 07 ; Himachal 07C]**

21. Show that the magnitude of torque = magnitude of force × moment arm. Also show that only the angular component of the force is responsible for producing torque.
22. Define angular momentum of a particle. State its SI unit.
23. How angular momentum can be expressed as the vector product of two vectors ? How is its direction determined ?
24. Show that the angular momentum of a particle is the product of its linear momentum and moment arm. Also show that the angular momentum is produced only by the angular component of linear momentum. What is the physical meaning of angular momentum ?
25. Define angular momentum. Prove that angular momentum of a particle is equal to twice the product of its mass and areal velocity. **[Himachal 05C, 06]**
26. Write the SI unit of torque and angular momentum. Derive the relation between angular momentum and torque. **[Himachal 05, 09 ; Delhi 10]**
27. Prove that the time rate of change of the angular momentum of a particle is equal to the torque acting on it. **[Delhi 97, 05]**
28. Define a rigid body. Name *two* kinds of motion which a rigid body can execute. What is meant by the term equilibrium ? For the equilibrium of a body, *two* conditions need to be satisfied. State them. **[Delhi 2003]**
29. Distinguish between the stable, unstable and neutral equilibria of a body.
30. Define moment of inertia of a body. Give its units and dimensions. What is the physical significance of moment of inertia ? **[Himachal 05C, 06]**
31. State the factors on which the moment of inertia of a body depends. **[Himachal 06 ; Delhi 03C]**
32. Establish the relation between kinetic energy and moment of inertia for a rigid body. **[Himachal 07]**
33. Show that the moment of inertia of a body about the given axis of rotation is equal to twice the kinetic energy of rotation of the body rotating with unit angular velocity. **[Himachal 07C]**
34. Derive an expression for the rotational kinetic energy of a body and hence define moment of inertia. **[Himachal 07, 09C ; Delhi 08]**
35. Define radius of gyration. What is its physical significance ? **[Himachal 01]**
36. State and prove the theorem of perpendicular axes on moment of inertia. **[Himachal 02, 03, 04]**
37. State the theorem of parallel axes on moment of inertia. **[Delhi 12]**
38. Derive an expression for moment of inertia of a circular disc about an axis passing through its centre and perpendicular to its plane. **[Delhi 95, 96 ; Himachal 05, 08]**
39. What is the moment of inertia of a rod of mass M and length L about an axis perpendicular to it through one end ? Given the moment of inertia about the centre of mass is $\frac{1}{12} ML^2$. **[Central Schools 03, 08, 09]**
40. Deduce an expression for the moment of inertia of a hollow cylinder of mass M and radius R about its own axis.
41. Establish the relation between torque and angular acceleration. Hence define moment of inertia. **[Delhi 99 ; Himachal 03]**
42. Establish the relation between moment of inertia and torque on a rigid body. **[Himachal 07, 08, 09C]**
43. Establish the relation between angular momentum and moment of inertia for a rigid body. **[Himachal 07]**
44. Show that angular momentum = moment of inertia × angular acceleration and hence define moment of inertia. **[Chandigarh 03]**
45. State the law of conservation of angular momentum and illustrate it with the example of planetary motion. **[Delhi, 96, 04 ; Himachal 05]**
46. State and prove the principle of conservation of angular momentum. **[Himachal 03, 05, 08]**
47. Define moment of inertia. Write any two factors on which it depends. When the diver leaves the diving board, why he brings his hand and feet closer together in order to make a somersault. **[Delhi 05]**
48. Derive an expression for moment of inertia of a thin circular ring about an axis passing through the centre and perpendicular to the plane of the ring. **[Himachal 07 ; Central Schools 05]**
49. (*a*) State theorem of parallel axes for the moment of inertia of a body.
    (*b*) Determine the moment of inertia of a thin ring about a tangent to the circle in the plane of the ring. **[Delhi 09]**

50. State perpendicular axis theorem for calculation of moment of inertia using appropriate diagram. Also calculate moment of inertia about a diameter if that of an axis perpendicular to the plane of a disc and passing through its centre is given by $\frac{1}{2}MR^2$. **[Central Schools 08, 09]**

51. Derive an expression for angular momentum of a rigid body rotating at an angular speed of ω. Give its geometrical meaning and hence derive Kepler's second law of planetary motion. **[Delhi 06]**

52. Give the analogy between various quantities that describe linear and rotational motions.

53. Prove that the acceleration of a solid cylinder rolling without slipping down an inclined plane is $2/3\, g \sin\theta$. **[Central Schools 03]**

Answers

1. Refer answer to Q. 1 on page 7.1.
2. Refer answer to Q. 2 on page 7.1.
3. Refer answer to Q. 3 on page 7.2.
4. Refer to solution of Problem 4 on page 7.59.
5. Refer answer to Q. 5 on page 7.3.
6. Refer answer to Q. 6 on page 7.4.
7. Refer answer to Q. 7 on page 7.5.
8. Refer answer to Q. 7 on page 7.5.
9. Refer answer to Q. 8(*i*) on page 7.11.
10. Refer answer to Q. 8(*ii*) on page 7.11.
11. Refer answer to Q. 8(*iii*) on page 7.11.
12. Refer answer to Q. 9 on page 7.11.
13. Refer answer to Q. 10 and Q. 11 on page 7.12.
14. Refer answer to Q. 16 on page 7.15.
15. The analogue of force in rotational motion is torque.

Torque, $\vec{\tau} = \vec{r} \times \vec{F}$

S.I. unit of torque = Nm.

16. Refer answer to Q. 17 on page 7.15.
17. Refer answer to Q. 18 on page 7.16.
18. Refer answer to Q. 19 on page 7.16.
19. Refer answer to Q. 20 on page 7.16.
20. Refer answer to Q.21 on page 7.17.
21. Refer answer to Q. 21 on page 7.17.
22. Refer answer to Q. 23 on page 7.18.
23. Refer answer to Q. 24 on page 7.18.
24. Refer answer to Q. 25 on page 7.18.
25. Refer answer to Q.28 on page 7.20.
26. Refer answer to Q.27 on page 7.19.
27. Refer answer to Q. 27 on page 7.19.
28. Refer answer to Q. 30 on page 7.23.
29. Refer answer to Q. 32 on page 7.25.
30. Refer answer to Q. 33 on page 7.25.
31. Refer answer to Q. 34 on page 7.25.
32. Refer answer to Q.36 on page 7.26.
33. Refer answer to Q.36 on page 7.26.
34. Refer answer to Q. 36 on page 7.26.
35. Refer answer to Q. 37 on page 7.26.
36. Refer answer to Q. 38 on page 7.27.
37. Refer answer to Q. 39 on page 7.27.
38. Refer answer to Q. 41(*a*) on page 7.29.
39. By using theorem of parallel axes,

$$I_{end} = I_{CM} + M\left(\frac{L}{2}\right)^2$$

$$= \frac{1}{12}ML^2 + \frac{1}{4}ML^2 = \frac{1}{3}ML^2.$$

40. Refer answer to Q. 44 on page 7.31.
41. Refer answer to Q. 48 on page 7.41.
42. Refer answer to Q.48 on page 7.41.
43. Refer answer to Q.49 on page 7.41.
44. Refer answer to Q. 49 on page 7.41.
45. Refer answer to Q. 50 on page 7.45.
46. Refer answer to Q.50 on page 7.45.
47. Refer answer to Q. 50(*iii*) on page 7.45.
48. Refer answer to Q. 41 on page 7.29.
49. (*a*) Refer answer to Q.39 on page 7.27.
(*b*) Refer answer to Q.40(*c*) on page 7.28.
50. Refer answer to Q.39 on page 7.27 and Q.41(*b*) on page 7.29.
51. Refer answer to Q. 28 on page 7.20.
52. Refer answer to Q. 51 on page 7.48.
53. Refer answer to Q. 52 on page 7.49.

Type C : Long Answer Questions — 5 Marks Each

1. Define centre of mass of a system. Derive an expression for the centre of mass of a two particle system from *ab*-initio. **[Chandigarh 03]**
2. Define rotational motion of a body. Derive the following equations of rotational motion under constant angular acceleration.

   (*i*) $\omega = \omega_0 + \alpha t$ (*ii*) $\theta = \omega_0 t + \frac{1}{2}\alpha t^2$

   (*iii*) $\omega^2 - \omega_0^2 = 2\alpha\theta$.
3. Prove that the angular momentum of a particle is twice the product of its mass and areal velocity. How does it lead to the Kepler's second law of planetary motion ?
4. Prove that the rate of change of total angular momentum of a system of particles about a reference point is equal to the total torque acting on the system.
5. Derive an expression for the total work done on a rigid body executing both translational and rotational motions. Hence deduce the condition for the equilibrium of the rigid body.
6. Derive an expression for the moment of inertia of a thin uniform circular ring about an axis through its centre and perpendicular to its plane. Hence write the expressions for its moment of inertia

   (*i*) about its diameter and

   (*ii*) about a tangent in its plane. **[Himachal 05]**
7. Define the term moment of inertia. Derive an expression for moment of inertia of a circular disc about an axis passing through its centre and perpendicular to its plane. **[Himachal 07 ; Delhi 03C ; Chandigarh 04]**
8. Derive an expression for the moment of inertia of a thin uniform rod about an axis through its centre and perpendicular to its length. Also determine the radius of gyration about the same axis.
9. Derive an expression for the moment of inertia of a thin uniform rod about an axis passing through its one end and perpendicular to its length. Also determine the radius of gyration about the same axis.
10. Derive an expression for the moment of inertia of a uniform solid cylinder about its own axis.
11. Derive an expression for the moment of inertia of a uniform cylinder about an axis passing through its centre and perpendicular to its length.
12. Derive an expression for the moment of inertia of a uniform solid sphere about its any diameter. Hence write the expression for its moment of inertia about its tangent.
13. Obtain an expression for the linear acceleration of a cylinder rolling down an inclined plane and hence find the condition for the cylinder to roll down without slipping.
14. A light string is wound round a cylinder and carries a mass tied to it at the free end. When the mass is released, calculate

    (*i*) the linear acceleration of the descending mass,

    (*ii*) the angular acceleration of the cylinder and

    (*iii*) the tension in the string.

    Show that the acceleration of the mass is less than g.

Answers

1. Refer answer to Q. 4 on page 7.2.
2. Refer answer to Q. 13 on page 7.12 and Q. 14 on page 7.13.
3. Refer answer to Q. 28 on page 7.20.
4. Refer answer to Q. 28 on page 7.20.
5. Refer answer to Q. 31 on page 7.24.
6. Refer answer to Q. 41 on page 7.29.
7. Refer answer to Q. 41 on page 7.29.
8. Refer answer to Q. 42 on page 7.30.
9. Refer answer to Q. 43 on page 7.30.
10. Refer answer to Q. 45 on page 7.31.
11. Refer answer to Q. 46 on page 7.31.
12. Refer answer to Q. 47 on page 7.32.
13. Refer answer to Q. 52 on page 7.49.
14. Refer answer to Q. 55 on page 7.52.

Competition Section

Systems of Particles & Rotational Motion

GLIMPSES

1. **Particle.** A particle is an object whose mass is finite but whose size and internal structure can be neglected.
2. **System.** A system is a collection of a very large number of particles which mutually interact with one another.
3. **Centre of mass.** It is the point at which entire mass of a system may be supposed to be concentrated. The nature of motion of the system remains unaffected when all the forces acting on the system are applied directly on the centre of mass of the system.

   If $\vec{r_1}$ and $\vec{r_2}$ are the position vectors of two particles of masses m_1 and m_2, then the position vector of their centre of mass is given by

$$\vec{R}_{CM} = \frac{m_1 \vec{r_1} + m_2 \vec{r_2}}{m_1 + m_2}$$

   For a system of N-particles, the centre of mass is given by

$$\vec{R}_{CM} = \frac{m_1 \vec{r_1} + m_2 \vec{r_2} + \dots + m_N \vec{r_N}}{m_1 + m_2 + \dots + m_N} = \frac{\sum_{i=1}^{N} m_i \vec{r_i}}{M}$$

   For a continuous mass distribution, the centre of mass is given by

$$\vec{R}_{CM} = \frac{1}{M} \int \vec{r}\, dm$$

   where dm is the mass of small element located at position $\vec{r}$

   The velocity of the centre of mass of a two-particle system is given by

$$\vec{v}_{CM} = \frac{m_1 \vec{v_1} + m_2 \vec{v_2}}{m_1 + m_2}$$

4. **Properties of centre of mass :**
   (*i*) The location of the centre of mass is the weighted average of the locations of the particles of the system.
   (*ii*) Centre of mass moves as if the external force acts on the entire mass concentrated at this point.
   (*iii*) In the absence of any external force, the centre of mass moves with a constant velocity.
   (*iv*) For bodies of normal dimensions, centre of mass and centre of gravity coincide.
5. **Rigid body.** A rigid body is one whose constituent particles retain their relative positions even when they move under the action of an external force.
6. **Rotational motion of a rigid body.** A body is said to possess rotational motion if all its particles move along circles in parallel planes. The centres of these circles lie on a fixed line perpendicular to the parallel planes and is called the axis of rotation.
7. **Equations of rotational motion.** For a body moving in circle, let ω_0 be the initial angular velocity, ω the final angular velocity, α the angular acceleration and θ be the angular displacement after the time t. Then equations of rotational motion can be written as

   (*i*) $\omega = \omega_0 + \alpha t$ (*ii*) $\theta = \omega_0 t + \frac{1}{2} \alpha t^2$

   (*iii*) $\omega^2 - \omega_0^2 = 2\alpha\theta$.

8. **Torque** ($\vec{\tau}$). The turning effect of a force about the axis of rotation is called moment of force or torque due to the force.

Torque = Force × its perpendicular distance from the axis of rotation

or τ = Force × moment arm = Fd

If a force $\vec{F}$ acts at a point whose position vector is $\vec{r}$ with respect to the axis of rotation, then torque is

$$\vec{\tau} = \vec{r} \times \vec{F}$$

(*i*) In Cartesian coordinates : $\tau_z = x\,F_y - y\,F_x$

(*ii*) In polar coordinates : $\tau = rF \sin\theta$

where θ is the angle between force vector $\vec{F}$ and position vector $\vec{r}$.

SI Unit of torque is Nm.

9. **Principle of moments of rotational equilibrium.** When a body is in rotational equilibrium, the sum of the clockwise moments about any point is equal to the sum of the anticlockwise moments about that point. Or, the algebraic sum of moments about any point is zero. In rotational equilibrium,

Clockwise moment = Anticlockwise moment

or $F_1 \times d_1 = F_2 \times d_2$

or Load × load arm = Effort × effort arm

This is sometimes called the lever principle.

10. **Couple.** A pair of equal and opposite forces acting on a body along two different lines of action constitute a couple. In a couple, total external force is zero but torque is non-zero. So a couple has a turning effect but cannot produce translational motion.

11. **Moment of a couple.** The moment of a couple is equal to the product of either of the forces and the perpendicular distance, called the arm of the couple, between their lines of action. It is independent of the choice of the fulcrum or the point of rotation.

Moment of couple = Force × arm of the couple

or $\tau = Fd$

A couple can only be balanced by an equal and opposite couple.

12. **Work done by a torque and power of a torque.** If a torque τ applied on a body rotates it through an angle $\Delta\theta$, the work done by the torque is

$$\Delta W = \tau\Delta\theta$$

or Work done = Torque × Angular displacement

Power of a torque is given by

$$P = \frac{\Delta W}{\Delta t} = \frac{\tau\Delta\theta}{\Delta t} = \tau\,\omega$$

i.e., Power of a torque = Torque × Angular velocity

13. **Angular momentum** ($\vec{L}$). It is the moment of linear momentum of a particle about the axis of rotation.

Angular momentum = Linear momentum × its perpendicular distance from the axis of rotation.

or $L = p\,d$

If $\vec{p}$ is the linear momentum of a particle and $\vec{r}$ its position vector, then its angular momentum is

$$\vec{L} = \vec{r} \times \vec{p}$$

(*i*) In Cartesian coordinates : $L_z = xp_y - yp_x$

(*ii*) In polar coordinates : $L = rp \sin\theta$

where θ is the angle between the linear momentum vector $\vec{p}$ and the position vector $\vec{r}$.

If a particle of mass m moves with uniform speed v along a circle of radius r, then its angular moment is $L = mvr$

SI unit of angular momentum is kgm^2 s^{-1}.

The linear momentum $\vec{p}$ and velocity vector $\vec{v}$ are always parallel to each other. But the angular momentum $\vec{L}$ and the angular velocity $\vec{\omega}$ are not necessarily parallel vectors.

14. **Geometrical meaning of angular momentum.** Geometrically, the angular momentum of a particle is equal to twice the product of its mass and areal velocity. Equivalently, the areal velocity of a particle is just half its angular momentum per unit mass.

$$\vec{L} = 2m\frac{\Delta\vec{A}}{\Delta t} \quad \text{or} \quad \frac{\Delta\vec{A}}{\Delta t} = \frac{\vec{L}}{2m}.$$

15. **Relation between torque and angular momentum.** The rate of change of angular momentum of a system of particles about a fixed point is equal to the total external torque acting on the system about that point.

$$\vec{\tau}^{\,tot} = \frac{d\vec{L}}{dt}$$

16. **Equilibrium of rigid bodies.** A rigid body is said to be in equilibrium if both the linear momentum and angular momentum of the rigid body remain constant with time. It must possess the following two equilibria simultaneously :

(*i*) **Translational equilibrium.** The resultant of all the external forces acting on the body must be zero.

$$\Sigma\,\vec{F}_i^{\,ext} = 0$$

(*ii*) **Rotational equilibrium.** The resultant of all the torques due to all the forces acting on the body about any point must be zero.

$$\Sigma\,\vec{\tau}_i^{\,ext} = \Sigma\,\vec{r}_i \times \vec{F}_i^{\,ext} = 0$$

17. **Moment of inertia.** The moment of inertia of a rigid body about an axis of rotation is the sum of the products of masses of the various particles and squares of their perpendicular distances from the axis of rotation. Mathematically

$$I = m_1 r_1^2 + m_2 r_2^2 + ... + m_n r_n^2 = \sum_{i=1}^{n} m_i r_i^2$$

SI unit of moment of inertia = kg m^2.

The moment of inertia of a body can be regarded as the rotational inertia of a body. It is the rotational analogue of mass in linear motion.

18. **Factors on which the moment of inertia depends.** The moment of inertia of a body depends on the following factors :
   (*i*) Mass of the body.
   (*ii*) Size and shape of the body.
   (*iii*) Distribution of mass about the axis of rotation.
   (*iv*) Position and orientation of the axis of rotation with respect to the body.

19. **Radius of gyration.** It may be defined as the distance from the axis of rotation at which if whole mass of the body were supposed to be concentrated, the moment of inertia, would be same as with the actual distribution of mass. The relation between moment of inertia I and radius of gyration K is

$$I = MK^2 \quad \text{or} \quad K = \sqrt{\frac{I}{M}}$$

For a body composed of particles of equal masses,

$$K = \sqrt{\frac{r_1^2 + r_2^2 + ... + r_n^2}{n}}$$

i.e. radius of gyration is equal to the root mean square distance of the particles from the axis of rotation.

SI unit of radius of gyration = m.

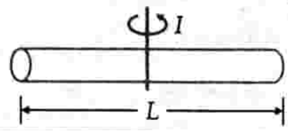
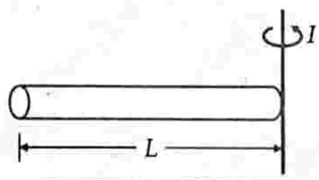
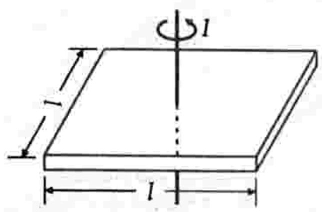
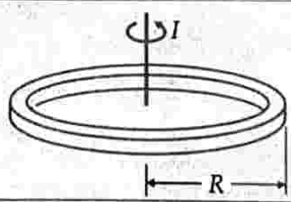
20. **Theorem of perpendicular axes.** It states that the moment of inertia of a plane lamina about an axis perpendicular to its plane is equal to the sum of the moments of inertia of the lamina about any two mutually perpendicular axes in its plane and intersecting each other at the point, where the perpendicular axes pass through the lamina. Mathematically,

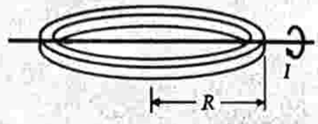
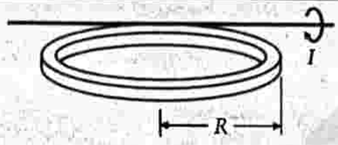
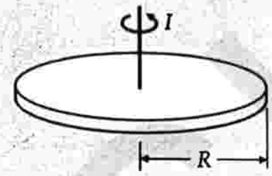
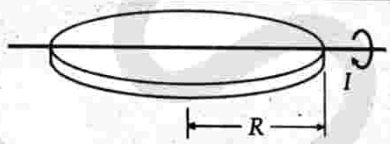
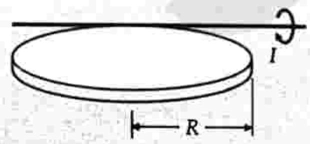
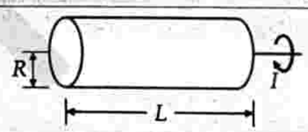
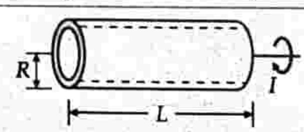
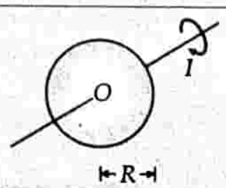
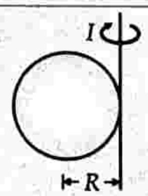
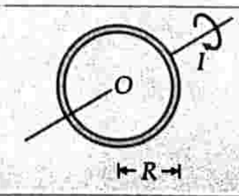
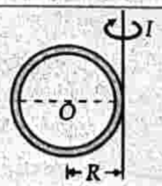
$$I_z = I_x + I_y$$

where X- and Y-axes lie in the plane of the lamina and Z-axis is perpendicular to its plane and passes through the point of intersection of X- and Y-axes.

21. **Theorem of parallel axes.** It states that the moment of inertia of a rigid body about any axis is equal to the moment of inertia of the body about a parallel axis through its centre of mass plus the product of mass of the body and the square of the perpendicular distance between the parallel axes. Mathematically, $I = I_{CM} + Md^2$

Table Moments of Inertia of some Bodies of Regular Shape

Body	Axis	Figure	Moment of Inertia
Thin rod of length L	Passing through centre and perpendicular to the rod		$I = \frac{1}{12} ML^2$
Thin rod of length L	Through its one end and perpendicular to its length		$I = \frac{1}{3} ML^2$
Rectangular lamina of length l and breadth b	Through its centre and perpendicular to its plane		$I = M\left(\frac{l^2 + b^2}{12}\right)$
Circular ring of radius R	Passing through its centre and perpendicular to its plane		$I = MR^2$

Body	Axis	Figure	Moment of Inertia
Circular ring of radius R	Diameter		$I = \frac{1}{2} MR^2$
Circular ring of radius R	Tangent in its plane		$I = \frac{3}{2} MR^2$
Circular disc of radius R	Passing through its centre and perpendicular to its plane		$I = \frac{1}{2} MR^2$
Circular disc of radius R	Diameter		$I = \frac{1}{4} MR^2$
Circular disc of radius R	Tangent in its plane		$I = \frac{5}{4} MR^2$
Right circular solid cylinder of radius R and length L	Symmetry axis		$I = \frac{1}{2} MR^2$
Hollow cylinder of radius R and length L	Symmetry axis		$I = MR^2$
Solid sphere of radius R	Diameter		$I = \frac{2}{5} MR^2$
Solid sphere of radius R	Any tangent		$I = \frac{7}{5} MR^2$
Hollow sphere of radius R	Symmetry axis		$I = \frac{2}{3} MR^2$
Hollow sphere of radius R	Any tangent		$I = \frac{5}{3} MR^2$

22. **Rotational K.E.** If a body of mass M and moment of inertia I rotates about an axis of rotation with an angular velocity ω, then

$$\text{Rotational K.E.} = \frac{1}{2} I\omega^2.$$

23. **Total K.E. of a rolling body.** The centre of mass of a rolling body moves along a straight line, so it possesses translational K.E. in addition to rotational K.E.

∴ Total K.E. of a rolling body

$$= \text{Translational K.E.} + \text{Rotational K.E.}$$
$$= \frac{1}{2} Mv^2 + \frac{1}{2} I\omega^2$$

24. **Relation between M.I. and angular momentum.**

Angular momentum = M.I. × Angular velocity

i.e., $L = I\omega$

25. **Relation between M.I. and torque.**

Torque = M.I. × Angular acceleration

i.e., $\tau = I\alpha$.

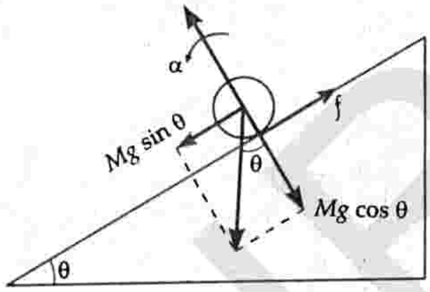
26. **Law of conservation of angular momentum.** If no external torque acts on a system, total angular momentum of the system remains unchanged.

In the absence of any external torque,

$$L = I\omega = \text{constant}$$

or $I_1 \omega_1 = I_2 \omega_2$ or $I_1 \cdot \frac{2\pi}{T_1} = I_2 \cdot \frac{2\pi}{T_2}$.

27. **Motion of a cylinder rolling down an inclined plane without slipping.** As shown in figure, consider a solid cylinder of mass M and radius R rolling down an inclined plane of inclination θ without slipping.



(*i*) Linear acceleration of the cylinder down the inclined plane,

$$a = \frac{g \sin\theta}{1 + \frac{I}{MR^2}} = \frac{2}{3} g \sin\theta$$

(*ii*) Force of friction between inclined plane and cylinder,

$$f = \frac{1}{3} Mg \sin\theta$$

(*iii*) Condition for rolling of cylinder without slipping is

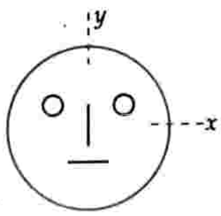
$$\frac{1}{3} \tan\theta \le \mu_s$$

where μ_s is the coefficient of static friction.

IIT Entrance Exam

Multiple Choice Questions with One Correct Answer

1. Look at the drawing given in the figure which has been drawn with ink of uniform line-thickness. The mass of ink used to draw each of the two inner circles, and each of the two line segments is m. The mass of the ink used to draw the outer circle is $6m$. The coordinates of the centres of the different parts are : *outer circle* (0, 0), *left inner circle* $(-a, a)$, *right inner circle* (a, a), *vertical line* (0, 0) and *horizontal line* $(0, -a)$. The y-coordinate of the centre of mass of the ink in this drawing is



(*a*) $\frac{a}{10}$ (*b*) $\frac{a}{8}$
(*c*) $\frac{a}{12}$ (*d*) $\frac{a}{3}$ [IIT 09]

2. Two particles A and B, initially at rest, move towards each other under mutual force of attraction. At the instant when the speed of A is v and the speed of B is $2v$, the speed of the centre of mass of the system is

(*a*) $3v$ (*b*) v
(*c*) $1.5v$ (*d*) zero. [IIT 82]

3. Two blocks of masses 10 kg and 4 kg are connected by a spring of negligible mass and placed on a frictionless horizontal surface. An impulse gives a velocity of 14 m/s to the heavier block in the direction of the lighter block.

The velocity of the centre of mass is

(*a*) 30 m/s (*b*) 20 m/s
(*c*) 10 m/s (*d*) 5 m/s [IIT 02]

4. An isolated particle of mass m is moving in horizontal plane (x-y) along the x-axis at a certain height above the ground. It suddenly explodes into two fragments of masses $m/4$ and $3m/4$. An instant later, the smaller fragment is at $y = +15$ cm. The larger fragment at this instant is at

(*a*) $y = -5$ cm (*b*) $y = +20$ cm
(*c*) $y = +5$ cm (*d*) $y = -20$ cm. [IIT 97]

5. A smooth sphere A is moving on a frictionless horizontal plane with angular speed ω and centre of mass velocity u. It collides elastically and head on with an identical sphere B at rest. Neglect friction everywhere. After the collision, their angular speeds are ω_A and ω_B respectively. Then

(a) $\omega_A < \omega_B$ (b) $\omega_A = \omega_B$
(c) $\omega_A = \omega$ (d) $\omega_B = \omega$ [IIT 99]

6. A mass m is moving with a constant velocity along a line parallel to the x-axis, away from the origin. Its angular momentum with respect to the origin

(a) is zero (b) remains constant
(c) goes on increasing (d) goes on decreasing [IIT 97]

7. A particle of mass m is projected with a velocity v making an angle of 45° with the horizontal. The magnitude of the angular momentum of the projectile about the point of projection, when the particle is at its maximum height h is

(a) zero (b) $\frac{mv^3}{4\sqrt{2}g}$
(c) $\frac{mv^2}{\sqrt{2}g}$ (d) $m\sqrt{2gh^3}$ [IIT 90]

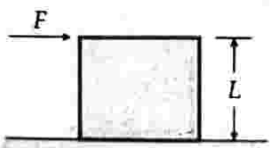
8. A particle undergoes uniform circular motion. About which point on the plane of the circle, will the angular momentum of the particle remain conserved ?

(a) centre of the circle
(b) on the circumference of the circle
(c) inside the circle
(d) outside the circle [IIT 03]

9. A particle is confined to rotate in a circular path with decreasing linear speed. Then which of the following is correct ?

(a) $\vec{L}$ (angular momentum) is conserved about the centre
(b) only direction of angular momentum $\vec{L}$ is conserved
(c) it spirals towards the centre
(d) its acceleration is towards the centre. [IIT 05]

10. A cubical block of side L rests on a rough horizontal surface with coefficient of friction μ. A horizontal face F is applied on the block as shown. If the coefficient of friction is sufficiently high so that the block does not slide before toppling, the minimum force required to topple the block is

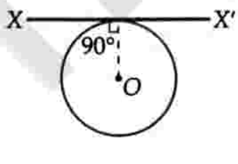


(a) infinitesimal (b) $mg/4$
(c) $mg/2$ (d) $mg(1-\mu)$ [IIT 2K]

11. A disc is rolling without slipping with angular velocity ω. P and Q are two points equidistant from the centre C. The order of magnitude of velocity is

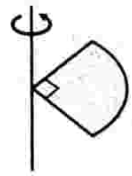
(a) $v_Q > v_C > v_l$ (b) $v_P > v_C > v_Q$
(c) $v_P = v_C$, $v_Q = v_C/2$ (d) $v_P < v_C > v_Q$ [IIT 04]

12. A thin wire of length L and uniform linear mass density ρ is bent into a circular loop with centre at O as shown. The moment of inertia of the loop about the axis XX' is



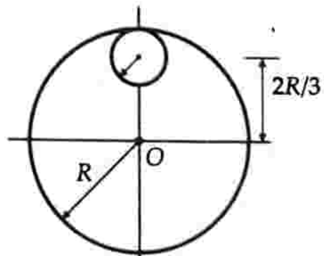
(a) $\frac{\rho L^3}{8\pi^2}$ (b) $\frac{\rho L^3}{16\pi^2}$
(c) $\frac{5\rho L^3}{16\pi^2}$ (d) $\frac{3\rho L^3}{8\pi^2}$ [IIT 2K]

13. One quarter sector is cut from a uniform circular disc of radius R. This sector has mass M. It is made to rotate about a line perpendicular to its plane and passing through the centre of the original disc. Its moment of inertia about the axis of rotation is



(a) $\frac{1}{2}MR^2$ (b) $\frac{1}{4}MR^2$
(c) $\frac{1}{8}MR^2$ (d) $\sqrt{2}MR^2$ [IIT 01]

14. From a circular disc of radius R and mass $9M$, a small disc of radius $R/3$ is removed from the disc. The moment of inertia of the remaining disc about an axis perpendicular to the plane of the disc and passing through O is



(a) $4MR^2$
(b) $\frac{40}{9}MR^2$
(c) $10MR^2$
(d) $\frac{37}{9}MR^2$ [IIT 05]

15. A solid sphere of mass M and radius R having moment of inertia I about its diameter is recast into a solid disc of radius r and thickness t. The moment of inertia of the disc about an axis passing the edge and perpendicular to the plane remains I. Then R and r are related as

(a) $r = \sqrt{\frac{2}{15}}R$ (b) $r = \frac{2}{\sqrt{15}}R$

(c) $r = \frac{2}{15}R$ (d) $r = \frac{\sqrt{2}}{15}R$ [IIT 06]

16. Two point masses of 0.3 kg and 0.7 kg are fixed at the ends of a rod of length 1.4 m and of negligible mass. The rod is set rotating about an axis perpendicular to its length with a uniform angular speed. The point on the rod through which the axis should pass in order that the work required for rotation of the rod is minimum, is located at a distance of

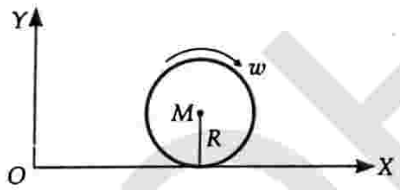
(a) 0.42 m from mass of 0.3 kg

(b) 0.70 m from mass of 0.7 kg

(c) 0.98 m from mass of 0.3 kg

(d) 0.98 m from mass of 0.7 kg [IIT 95]

17. A disc of mass M and radius R is rolling with angular speed ω on a horizontal plane as shown in figure.

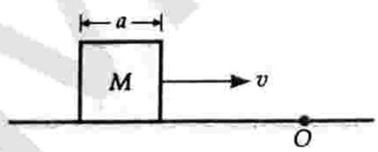


The magnitude of angular momentum of the disc about the origin O is

(a) $(1/2)MR^2\omega$ (b) $MR^2\omega$

(c) $(3/2)MR^2\omega$ (d) $2MR^2\omega$ [IIT 99]

18. A cubical block of side a is moving with velocity v on a horizontal smooth plane as shown in figure. It

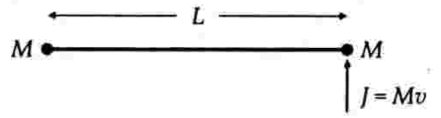


hits a ridge at point O. The angular speed of the block after it hits O is

(a) $3v/4a$ (b) $3v/2a$

(c) $3v/\sqrt{2}a$ (d) zero. [IIT 99]

19. Consider a body, shown in figure, consisting of two identical balls, each of mass M connected by a light rigid rod. If an impulse $J = Mv$ is imparted to the body at one of its ends, what would be its angular velocity ?



(a) v/L (b) $2v/L$

(c) $v/3L$ (d) $v/4L$ [IIT 03]

20. A cylinder rolls up an inclined plane, reaches some height, and then rolls down (without slipping throughout these motions). The directions of the frictional force acting on the cylinder are

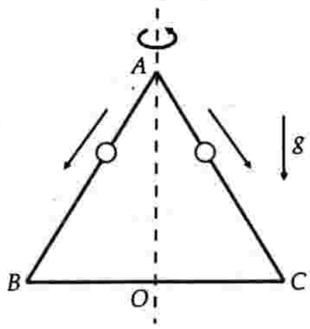
(a) up the incline, while ascending and down the incline, while descending.

(b) up the incline, while ascending as well as descending.

(c) down the incline, while ascending and up the incline, while descending.

(d) down the incline, while ascending as well as descending. [IIT 02]

21. An equilateral triangle ABC formed from a uniform wire has two small identical beads initially located at A. The triangle is set rotating about the vertical axis AO. Then the beads are released from rest simultaneously and allowed to slide down, one along AB and the other along AC as shown. Neglecting frictional effects, the quantities that are conserved as the beads slide down are



(a) angular velocity and total energy (kinetic and potential)

(b) total angular momentum and total energy.

(c) angular velocity and moment of inertia about the axis of rotation.

(d) total angular momentum and moment of inertia about the axis of rotation. [IIT 2K]

22. A horizontal circular plate is rotating about a vertical axis passing through its centre with an angular velocity ω_0. A man sitting at the centre having two blocks in his hands stretches out his hands so that the moment of inertia of the system doubles. If the kinetic energy of the system is K initially, its final kinetic energy will be

(a) $2K$ (b) $K/2$

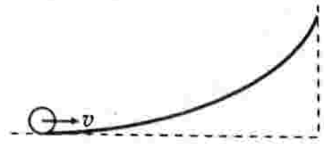
(c) K (d) $K/4$ [IIT 04]

23. A child is standing with folded hands at the centre of a platform rotating about its central axis. The kinetic energy of the system is K. The child now stretches his arms so that the moment of inertia of the system doubles. The kinetic energy of the system now is

(a) $2K$ (b) $K/2$

(c) $K/4$ (d) $4K$ [IIT 04]

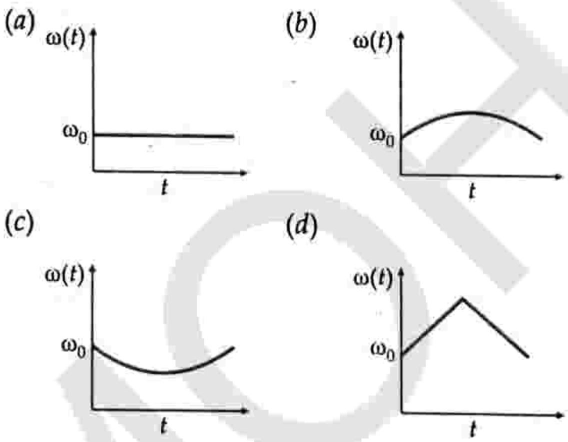
24. A small object of uniform density rolls up a curved surface with an initial velocity v. It reaches up to a maximum height of $\frac{3v^2}{4g}$ with respect to the initial position. The object is



(a) ring (b) solid sphere

(c) hollow sphere (d) disc. [IIT 07]

25. A circular platform is free to rotate in horizontal plane about a vertical axis passing through its centre. A tortoise is sitting at the edge of the platform. Now, the platform is given an angular velocity ω_0. When the tortoise moves along a chord of the platform with a constant velocity (with respect to the platform), the angular velocity of the platform $\omega(t)$ will vary with time t as [IIT 02]



26. A piece of wire is bent in the shape of a parabola $y = kx^2$ (y-axis vertical) with a bead of mass m on it. The bead can slide on the wire without friction. It stays at the lowest point of the parabola when the wire is at rest. The wire is now accelerated parallel to the x-axis with a constant acceleration a. The distance of the new equilibrium position of the bead, where the bead can stay at rest w.r.t. the wire, from the y-axis is

(a) $\frac{a}{gk}$ (b) $\frac{a}{2gk}$

(c) $\frac{2a}{gk}$ (d) $\frac{a}{4gk}$ [IIT 09]

Multiple Choice Questions with One or More Than One Correct Answer

27. A ball hits the floor and rebounds after an inelastic collision. In this case

(a) the momentum of the ball just after the collision is the same as that just before the collision

(b) the mechanical energy of the ball remains the same in the collision

(c) the total momentum of the ball and the earth is conserved

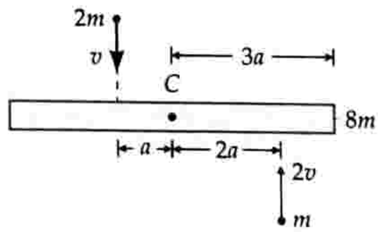
(d) the total energy of the ball and the earth is conserved. [IIT 86]

28. A shell is fired from a cannon with a velocity v (m/sec) at an angle θ with the horizontal direction. At the highest point in its path it explodes into two pieces of equal mass. One of the pieces retraces its path to the cannon and the speed (in m/sec) of the other piece immediately after the explosion is

(a) $3v\cos\theta$ (b) $2v\cos\theta$

(c) $\frac{3}{2}v\cos\theta$ (d) $\sqrt{\frac{3}{2}}\,v\cos\theta$ [IIT 86]

29. A uniform bar of length $6a$ and mass $8m$ lies on a smooth horizontal table. Two point masses m and $2m$ moving in the same horizontal plane with speed $2v$ and v respectively, strike the bar [as shown in figure]



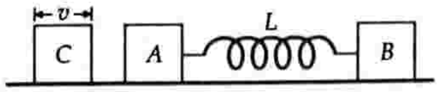
and stick to the bar after collision. Denoting angular velocity (about the centre of mass), total energy and centre of mass velocity by ω, E and v_C respectively, we have after collision

(a) $v_C = 0$ (b) $\omega = \frac{3v}{5a}$

(c) $\omega = \frac{v}{5a}$ (d) $E = \frac{3mv^2}{5}$. [IIT 91]

30. Two blocks A and B, each of mass m, are connected by a massless spring of natural length L and spring constant k. The blocks are initially resting on a smooth horizontal floor with the spring at its natural length, as shown in the figure. A third identical block

C, also of mass m, moves on the floor with a speed v along the line joining A and B, and collides elastically with A. Then



(a) the kinetic energy of the A-B system, at maximum compression of the spring, is zero.

(b) the kinetic energy of the A-B system, at maximum compression of the spring, is $mv^2/4$

(c) the maximum compression of the spring is $v\sqrt{(m/k)}$

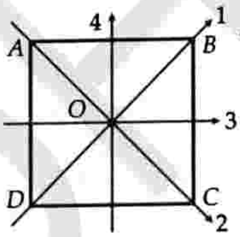
(d) the maximum compression of the spring is $v\sqrt{(m/2k)}$. [IIT 93]

31. A tube of length L is filled completely with an incompressible liquid of mass M and closed at both the ends. The tube is then rotated in a horizontal plane about one of its ends with a uniform angular velocity ω. The force exerted by the liquid at the other end is

(a) $\dfrac{M\omega^2 L}{2}$ (b) $M\omega^2 L$

(c) $\dfrac{M\omega^2 L}{4}$ (d) $\dfrac{M\omega^2 L^2}{2}$ [IIT 92]

32. The moment of inertia of a thin square plate ABCD, as shown in the figure, of uniform thickness about an axis passing through the centre O and perpendicular to the plane of the plate is



(a) $I_1 + I_2$

(b) $I_3 + I_4$

(c) $I_1 + I_3$

(d) $I_1 + I_2 + I_3 + I_4$

where I_1, I_2, I_3 and I_4 are respectively the moments of inertia about axis 1, 2, 3 and 4 which are in the plane of the plate. [IIT 92]

33. Let I be the moment of inertia of a uniform square plate about an axis AB that passes through its centre and is parallel to two of its sides. CD is a line in the plane of the plate that passes through the centre of the plate and makes an angle θ with AB. The moment of inertia of the plate about the axis CD is then equal to

(a) I (b) $I\sin^2\theta$

(c) $I\cos^2\theta$ (d) $I\cos^2\theta/2$ [IIT 98]

34. A solid cylinder is rolling down a rough inclined plane of inclination θ. Then

(a) the friction force is dissipative

(b) the friction force is necessarily changing

(c) the friction force will aid rotation but hinder translation.

(d) the friction force is reduced if θ is reduced. [IIT]

35. If the resultant of all the external forces acting on a system of particles is zero, then from an inertial frame, one can surely say that

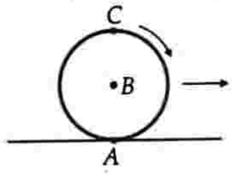
(a) linear momentum of the system does not change in time

(b) kinetic energy of the system does not change in time

(c) angular momentum of the system does not change in time

(d) potential energy of the system does not change in time. [IIT 09]

36. A sphere is rolling without slipping on a fixed horizontal plane surface. In the figure, A is the point of contact, B is the centre of the sphere and C is its topmost point. Then



(a) $\vec{V}_C - \vec{V}_A = 2(\vec{V}_B - \vec{V}_C)$

(b) $\vec{V}_C - \vec{V}_B = \vec{V}_B - \vec{V}_A$

(c) $|\vec{V}_C - \vec{V}_A| = 2|\vec{V}_B - \vec{V}_C|$

(d) $|\vec{V}_C - \vec{V}_A| = 4|\vec{V}_B|$ [IIT 09]

37. A point mass of 1 kg collides elastically with a stationary point mass of 5 kg. After their collision, the 1 kg mass reverses its direction and moves with a speed of 2 ms^{-1}. Which of the following statement(s) is (are) correct for the system of these two masses ?

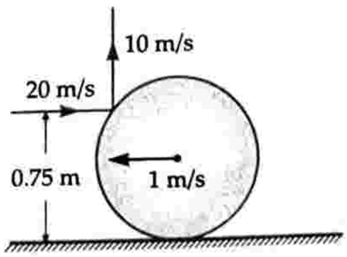
(a) Total momentum of the system is 3 kg ms^{-1}

(b) Momentum of 5 kg mass after collision is 4 kg ms^{-1}

(c) Kinetic energy of the centre of mass is 0.75 J

(d) Total kinetic energy of the system is 4 J [IIT 2010]

38. A thin ring of mass 2 kg and radius 0.5 m is rolling without slipping on a horizontal plane with



velocity 1 m/s. A small ball of mass 0.1 kg, moving with velocity 20 m/s in the opposite direction, hits the ring at a height of 0.75 m and goes vertically up with velocity 10 m/s. Immediately after the collision

(*a*) the ring has pure rotation about its stationary CM

(*b*) the ring comes to a complete stop

(*c*) friction between the ring and the ground is to the left

(*d*) there is no friction between the ring and the ground. [IIT 2011]

Reasoning Type

Instructions. *This question contains statement – 1 (assertion) and statement – 2 (reason). Of these statements, mark correct choice if*

(*a*) statements – 1 and 2 are true and statement – 2 is a correct explanation for statement – 1.

(*b*) statements – 1 and 2 are true and statement – 2 is not a correct explanation for statement – 1.

(*c*) statement – 1 is true, statement – 2 is false.

(*d*) statement – 1 is false, statement – 2 is true.

39. **Statement – 1 :** If there is no external torque on a body about its centre of mass, then the velocity of the centre of mass remains constant.

Statement – 2. The linear momentum of an isolated system remains constant. [IIT 07]

40. **Statement – 1 :** Two cylinders, one hollow (metal) and the other solid (wood) with the same mass and identical dimensions are simultaneously allowed to roll without slipping down an inclined plane from the same height. The hollow cylinder will reach the bottom of the inclined plane first.

Statement – 2 : By the principle of conservation of energy, the total kinetic energies of both the cylinders are identical when they reach the bottom of the incline.

Comprehension Based Questions

Paragraph for Question Nos. 41 to 43

Two discs A and B are mounted coaxially on a vertical axle. The discs have moments of inertia I and $2I$ respectively about the common axis. Disc A is imparted an initial angular velocity 2ω using the entire potential energy of a spring compressed by a distance x_1. Disc B is imparted an angular velocity ω by a spring having the same spring constant and compressed by a distance x_2. Both the discs rotate in the clockwise direction.

Read the passage given above and answer the following questions

41. The ratio x_1 / x_2 is

(*a*) 2 (*b*) 1/2

(*c*) $\sqrt{2}$ (*d*) $1/\sqrt{2}$ [IIT 07]

42. When disc B is brought in contact with disc A, they acquire a common angular velocity in time t. The average frictional torque on the disc by the other during this period is

(*a*) $\frac{2I\omega}{3t}$ (*b*) $\frac{9I\omega}{2t}$

(*c*) $\frac{9I\omega}{4t}$ (*d*) $\frac{3I\omega}{2t}$ [IIT 07]

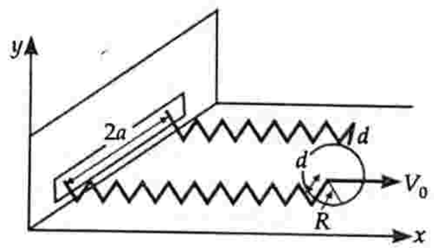
43. The loss of kinetic energy during the above process is

(*a*) $\frac{I\omega^2}{2}$ (*b*) $\frac{I\omega^2}{3}$

(*c*) $\frac{I\omega^2}{4}$ (*d*) $\frac{I\omega^2}{6}$ [IIT 07]

Paragraph for Question Nos. 44 to 46

A uniform thin cylindrical disk of mass M and radius R is attached to two identical massless springs of spring constant k which are fixed to the wall as shown in the figure. The springs are attached to the axle of the disk symmetrically on either side at a distance d from its centre. The axle is massless and both the springs and the axle are in a horizontal plane. The unstretched length of each spring is L. The disk is initially at its equilibrium position with its centre of mass (CM) at a distance L from the wall. The disk rolls without slipping with velocity $\vec{V}_0 = V_0 \hat{i}$. The coefficient of friction is μ.



Read the passage given above and answer the following questions

44. The net external force acting on the disk when its centre of mass is at displacement x with respect to its equilibrium position is

(*a*) $-kx$ (*b*) $-2kx$

(*c*) $-\frac{2kx}{3}$ (*d*) $-\frac{4kx}{3}$ [IIT 09]

45. The centre of mass of the disk undergoes simple harmonic motion with angular frequency ω equal to

(a) $\sqrt{\frac{k}{M}}$ (b) $\sqrt{\frac{2k}{M}}$

(c) $\sqrt{\frac{2k}{3M}}$ (d) $\sqrt{\frac{4k}{3M}}$ [IIT 09]

46. The maximum value of V_0 for which the disk will roll without slipping is

(a) $\mu g\sqrt{\frac{M}{k}}$ (b) $\mu g\sqrt{\frac{M}{2k}}$

(c) $\mu g\sqrt{\frac{3M}{k}}$ (d) $\mu g\sqrt{\frac{5M}{2k}}$ [IIT 09]

Integer Answer Type

47. Four solid spheres each of diameter $\sqrt{5}$ cm and mass 0.5 kg are placed with their centres at the corners of a square of side 4 cm. The moment of inertia of the system about the diagonal of the square is $N\times10^{-4}$ kgm^2. Find N. [IIT 2011]

48. A binary star consists of two stars A(Mass 2.2 M_s) and B(Mass 11 M_s), where M_s is the mass of the sun. They are separated by distance d and are rotating about their centre of mass, which is stationary. Find the ratio of the total angular momentum of the binary star to the angular momentum of star B about the centre of mass. [IIT 2010]

Answers and Explanations

1. (a) The positions of different masses are $m(-a,a)$, $m(a,a)$, $m(0,0)$, $m(0,-a)$ and $6m(0,0)$.

$$\therefore \quad y_{CM}=\frac{m(a)+m(a)+m(0)+m(-a)+6m(0)}{m+m+m+m+6m}=\frac{a}{10}$$

2. (d) No external force is acting on the centre of mass. It remains at rest. The speed of CM is zero.

3. (c) $v_{CM}=\frac{m_1v_1+m_2v_2}{m_1+m_2}=\frac{10\times14+4\times0}{4+10}=\mathbf{10\,ms^{-1}}$

4. (a) As the isolated particle is initially moving along x-axis, so there is no motion along y-axis. CM should remain stationary along y-axis even after explosion.

Now $\quad y_{CM}=\frac{m_1y_1+m_2y_2}{m_1+m_2}$

But $\quad y_{CM}=0$

$\therefore \quad m_1y_1+m_2y_2=0$

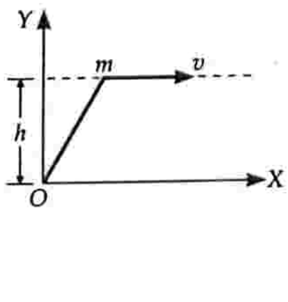
or $\quad y_2=-\frac{m_1}{m_2}\cdot y_1=-\frac{m/4}{3m/4}\times15=-5$ cm.

Thus the larger fragment will be at $y=-\mathbf{5}$ **cm**.

5. (c) Only the linear velocities are exchanged. The two spheres cannot exert torques on each other, as their surfaces are frictionless, and so that angular velocities of the spheres do not change.

$\therefore \quad \omega_A=\omega$ and $\omega_B=0.$

6. (b)



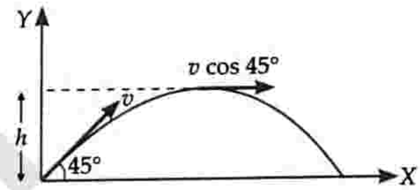
Angular momentum

= Mement of momentum

$L=mv\times h=$ constant

As the particle moves, m, v and h all remain unchanged.

7. (b)



When the particle is at its maximum height,

$L=$ Horizontal component of momentum × maximum height

$$=mv\cos45°\times\frac{v^2\sin^2 45°}{2g}$$

$$=mv\times\frac{1}{\sqrt{2}}\times\frac{v^2}{2g}\times\frac{1}{2}=\frac{mv^3}{4\sqrt{2}g}.$$

8. (a) In uniform circular motion, centripetal force acts towards the centre. Torque due to such a force about the centre is zero. Hence angular momentum is conserved about the centre of the circle.

9. (b) $\vec{L}=\vec{r}\times\vec{p}=m(\vec{r}\times\vec{v})$

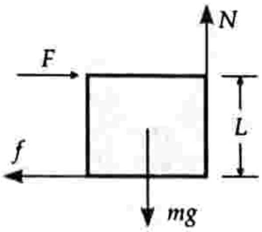
As the particle has decreasing linear speed ($\vec{v}$), so $|\vec{L}|$ also decreases *i.e.*, $\vec{L}$ is not conserved. Option (a) is not correct.

Direction of $(\vec{r}\times\vec{v})$, hence the direction of $\vec{L}$ remains the same. Option (b) is correct.

As the particle is confined to rotate in a circular path, it cannot spiral towards the centre. Option (c) is incorrect.

As the particle has a decreasing speed, it must have a transverse acceleration in addition to the centripetal acceleration. Option (d) is in correct.

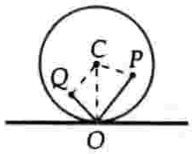
10. (*c*) When force F is applied, the normal reaction N of the floor shifts to the right. The cube topples when N reaches the edge. Consider torque about O.



$$F \times L = mg \times \frac{L}{2}$$

or $$F = \frac{mg}{2}.$$

11. (*b*) In pure rolling, instan- taneous velocity at the lowermost point O is zero. In the figure shown, $CP = CQ$. Velocity of any point on the disc, $v = r\omega$, where r is the distance of the point from O.



Now
$$OP > OC > OQ$$
or $$r_P > r_C > r_Q$$
∴ $$v_P > v_C > v_Q$$

12. (*d*) Mass of the ring,
$$M = \rho L$$
If R is the radius of the ring, then
$$L = 2\pi R$$
or $$R = \frac{L}{2\pi}$$
M.I. of the ring about a diameter through O and parallel to XX',
$$I_O = \frac{1}{2}MR^2$$
By parallel axes theorem, moment of inertia about XX' will be
$$I_{XX'} = I_O + MR^2 = \frac{1}{2}MR^2 + MR^2$$
$$= \frac{3}{2}MR^2 = \frac{3}{2} \times \rho L \times \left(\frac{L}{2\pi}\right)^2 = \frac{3\rho L^3}{8\pi^2}.$$

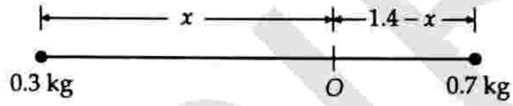
13. (*a*) Mass of complete disc $= 4M$

M.I. of whole disc about the given axis,
$$I = \frac{1}{2}(4M)R^2 = 2\,MR^2$$
M.I. of quarter section of the disc,
$$I' = \frac{I}{4} = \frac{1}{2}MR^2$$

14. (*a*) Refer to the solution of Problem 17 on page 7.67.

15. (*b*) M.I. of the solid sphere about a diameter,
$$I = \frac{2}{5}MR^2$$
M.I. of the disc about an axis through its edge and perpendicular to its plane (from parallel axes theorem) is
$$I = \frac{Mr^2}{2} + Mr^2$$
∴ $$\frac{2}{5}MR^2 = \frac{Mr^2}{2} + Mr^2$$
or $$r = \frac{2}{\sqrt{15}}R.$$

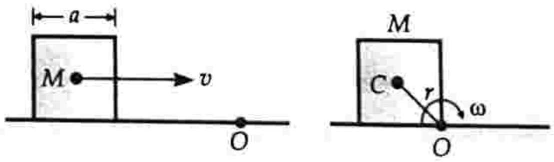
16. (*c*) Let the required point be located at a distance x from 0.3 kg mass.



M.I. about an axis through O,
$$I = 0.3x^2 + 0.7(1.4 - x)^2$$
Work done,
$$W = \frac{1}{2}I\omega^2$$
W will be minimum when I is minimum.
∴ $$\frac{dI}{dx} = 0$$
$$2 \times 0.3x - 2 \times 0.7(1.4 - x) = 0$$
$$0.3x + 0.7x = 0.7 \times 1.4$$
$$x = \mathbf{0.98\ m}.$$

17. (*c*) $$L_{tot} = L_{tran} + L_{rot}$$
$$= MvR + I\omega$$
$$= M(R\omega)R + \frac{1}{2}MR^2\omega$$
$$= \frac{3}{2}MR^2\omega.$$

18. (*a*) Clearly, $(2r)^2 = a^2 + a^2$
or $$r^2 = \frac{a^2}{2}$$



As net torque about O is zero, angular momentum is conserved,
$$L_i = L_f$$
or $$Mv\left(\frac{a}{2}\right) = I_0\omega = (I_C + Mr^2)\omega$$
$$= \left[\frac{Ma^2}{6} + M\left(\frac{a^2}{2}\right)\right]\omega = \frac{2}{3}Ma^2\omega$$
∴ $$\omega = \frac{3v}{4a}.$$

19. (*a*) Applying conservation of angular momentum about the centre of the rod,

$$I_{CM}\omega = J\cdot\frac{L}{2}$$

or $$2\times M\left(\frac{L}{2}\right)^2\omega = Mv\cdot\frac{L}{2}$$

or $$\omega = \frac{v}{L}.$$

20. (*b*) Whether the cylinder rolls up or down, the CM and hence the point of contact of the cylinder has an acceleration $g\sin\theta$ in the downward direction. Hence in both cases, the force of friction acts up the inclined plane.

21. (*b*) No external torque acts on system. Therefore, angular momentum is conserved. Again, forces acting on the system are conservative. Therefore, total mechanical energy of the system is conserved.

22. (*b*) Initial K.E., $K = \frac{1}{2}I_0\omega_0^2$

By conservation of angular momentum ($L_0 = I_0\omega_0$), when M.I. is doubled, angular velocity is halved. So, the final K.E. is

$$K' = \frac{1}{2}(2I_0)\left(\frac{\omega_0}{2}\right)^2 = \frac{1}{2}\cdot\frac{1}{2}I_0\omega_0^2 = \frac{K}{2}.$$

23. (*b*) Same reasoning as in the above problem.

24. (*d*) Loss in K.E. = Gain in P.E.

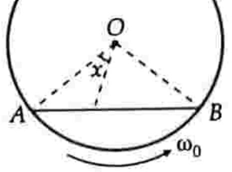
$$\frac{1}{2}Mv_{CM}^2 + \frac{1}{2}I_{CM}\omega^2 = Mgh$$

$$\frac{1}{2}Mv_{CM}^2 + \frac{1}{2}I_{CM}\left(\frac{v}{r}\right)^2 = Mg\frac{3v^2}{gh}$$

$$\therefore \quad I_{CM} = \frac{1}{2}MR^2$$

Clearly, the small object is a disc.

25. (*b*) Suppose initially the tortoise is at *A*. Its distance from the centre = *OA* = radius of the platform. As the tortoise moves along the chord *AB*, its distance from the axis of rotation varies. Let it be *x* at any instant *t*.



By conservation of angular momentum,

$$m(OA)^2\omega_0 = m\cdot x^2\omega$$

$$\omega = \frac{(OA)^2}{x^2}\cdot\omega_0$$

$$\therefore \quad \omega \propto \frac{1}{x^2}$$

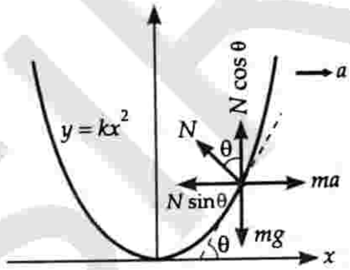
Thus the variation of ω with *x* is non-linear. Moreover, then value of *x* first decreases, takes a minimum value and then increases to maximum value (equals to radius). Consequently, ω first increases non-linearly, takes a maximum value (at *C*) and again falls to original value (at *B*). Hence option (*b*) is correct.

26. (*b*) In equilibrium,

$$N\cos\theta = mg$$

and $$N\sin\theta = ma$$

$$\therefore \quad \tan\theta = \frac{a}{g}$$



Given :

$$y = kx^2$$

$$\therefore \quad \frac{dy}{dx} = 2kx$$

But $$\frac{dy}{dx} = \text{Slope of parabolic curve} = \tan\theta$$

$$\therefore \quad 2kx = \frac{a}{g} \quad \text{or} \quad x = \frac{a}{2gk}.$$

27. (*c*), (*d*) Option (*a*) is incorrect because after the collision, momentum of the ball changes both in magnitude and direction. Option (*b*) is also incorrect because during collision some mechanical energy gets converted into heat and sound.

Options (*c*) and (*d*) are correct.

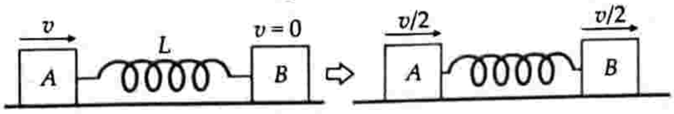
28. (*a*) The shell follows a parabolic path. At the highest point, it has horizontal velocity $v\cos\theta$. After explosion, one piece retraces its path with velocity $-v\cos\theta$. Let velocity of other piece be v'. Then conservation of momentum,

$$2m\times v\cos\theta = m(-v\cos\theta) + m\cdot v'$$

or $$v' = 3v\cos\theta.$$

29. (*a*), (*c*), (*d*) Refer to the solution of Problem 8 on page 7.65.

30. (*b*), (*d*) As a result of head-on collision between *C* and *A*, *C* stops and *A* begins to move with speed *v*. It compresses the spring *L* which pushes the block *B* towards right. At maximum compression, *A* and *B* both have same speed $v/2$ (from conservation of momentum).



Let x be the maximum compression of the spring. Then the K.E. of A-B system at maximum compression,

$$K = \frac{1}{2}m\left(\frac{v}{2}\right)^2 + \frac{1}{2}m\left(\frac{v}{2}\right)^2 = \frac{mv^2}{4}$$

Again by conservation of mechanical energy,

$$\frac{1}{2}mv^2 = \frac{1}{4}mv^4 + \frac{1}{2}kx^2$$

or $$\frac{1}{2}kx^2 = \frac{1}{4}mv^2$$

or $$x = v\sqrt{\frac{m}{2k}}.$$

31. (*a*) Refer to the solution of Problem 5 on page 7.64.

32. (*a*), (*b*), (*c*) By symmetry,

$$I_1 = I_2 \text{ and } I_3 = I_4$$

By perpendicular axes theorem,

$$I_O = I_1 + I_2$$

and $$I_O = I_3 + I_4$$

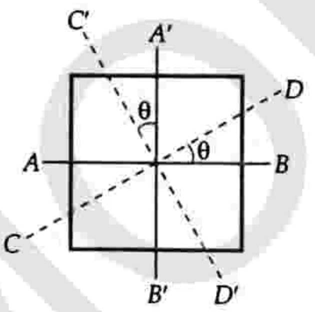
On adding,

$$2I_O = (I_1 + I_2) + (I_3 + I_4)$$
$$= (I_1 + I_1) + (I_3 + I_3)$$
$$= 2(I_1 + I_3)$$

or $$I_O = I_1 + I_3$$

Hence options (*a*), (*b*) and (*c*) are correct.

33. (*a*) Let $A'B' \perp AB$ and $C'D' \perp CD$.



By symmetry,

$$I_{AB} = I_{A'B'} \text{ and } I_{CD} = I_{C'D'}$$

By perpendicular axes theorem, the M.I. about O, perpendicular to the plane of the plate,

$$I_O = I_{AB} + I_{A'B'} = I_{CD} + I_{C'D'}$$

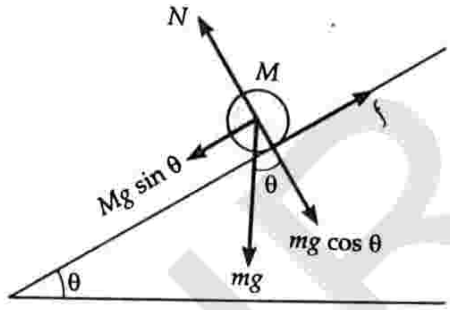
or $$2I_{AB} = 2I_{CD}$$

But $$I_{AB} = I$$

$\therefore$ $$I_{CD} = I$$

This is independent of θ.

34. (*c*), (*d*) The component $Mg\sin\theta$ tends to slide the point of contact down the inclined plane. The sliding friction acts in its opposite direction. The cylinder rolls because of friction. Thus frictional force aids rotation but hinders translation.



Hence option (*c*) is correct. If a_c is the acceleration of CM of the cylinder, then

$$Ma_c = Mg\sin\theta - f$$

or $$f = Mg\sin\theta - Ma_c \quad ...(i)$$

Also, $$a_c = \frac{g\sin\theta}{1+\frac{I_C}{MR^2}} = \frac{g\sin\theta}{1+\frac{\frac{1}{2}MR^2}{MR^2}}$$

$$= \frac{2}{3}g\sin\theta$$

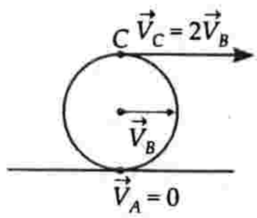
$\therefore$ $$f = Mg\sin\theta - M\times\frac{2}{3}g\sin\theta$$

or $$f = \frac{1}{3}Mg\sin\theta$$

Clearly, if θ is reduced, f also reduces. Hence option (*d*) is correct.

35. (*a*) According to law of conservation of linear momentum if $\vec{F}_{ext} = 0$, then the linear momentum of the system does not change in time. There may be external forces acting due to which K.E. or P.E. or both may change. Also, net force is zero does not mean net torque is zero. So, angular momentum may change. Hence only option (*a*) is correct.

36. (*b*), (*c*) The velocities at points A, B and C are as shown



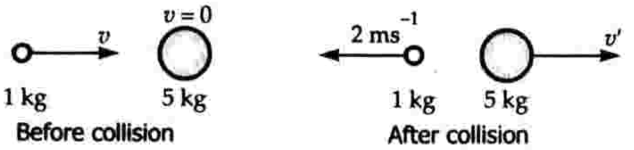
$$\vec{V}_C - \vec{V}_A = 2V_B$$

$$\vec{V}_B - \vec{V}_C = -\vec{V}_B$$

$$\vec{V}_B - \vec{V}_A = \vec{V}_B$$

Clearly, only options (*b*) and (*c*) are correct.

37. (*a*), (*c*)



By conservation of linear momentum,

$$1\times v+5\times 0=1\times(-2)+5v'$$

or $$v=-2+5v' \quad ...(i)$$

As the collision is elastic,

$$-\frac{v_1-v_2}{u_1-u_2}=1$$

or $$-\frac{2-v'}{v-0}=1$$

or $$2+v'=v \quad ...(ii)$$

On solving (*i*) and (*ii*), $v = 3\,\text{ms}^{-1}$, $v' = 1\,\text{ms}^{-1}$

Total momentum of the system

$$=1\times 3+5\times 0=3\,\text{kgms}^{-1}$$

$$v_{CM}=\frac{1\times 3+5\times 0}{1+5}=\frac{1}{2}\text{ms}^{-1}$$

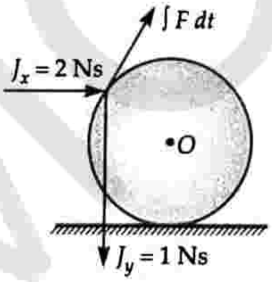
∴ K.E. of the centre of mass

$$=\frac{1}{2}\times(1+5)^2\times\left(\frac{1}{2}\right)^2=\mathbf{0.75\ J}$$

Total K.E. $=\frac{1}{2}\times 1\times 3^2=\mathbf{4.5\ J}$

Hence the correct options are (*a*) and (*c*).

38. (*c*) During collision, friction is impulsive. The angular impulse created by the impulsive forces tends to decrease the angular speed of the ring about *O* but the ring continues to rotate anticlockwise. Hence the friction between the ring and the ground at the point of contact will act towards left.



39. (*d*) The absence of external torque does not ensure the absence of external force. If external force is present, then the velocity of CM will not remain constant. Thus statement – 1 is false.

The linear momentum of an isolated system remains constant because no external force acts on such a system. Thus statement – 2 is true.

Hence option (*d*) is correct.

40. (*d*) From energy conservation

$$1/2\,mv_c^2+1/2\,I_c\omega^2=mgh$$

$$\omega=v_c/R$$

$$(I_c)_{\text{solid}}<(I_c)_{\text{hollow}}$$

Hence $(v_c)_{\text{solid}}>(v_c)_{\text{hollow}}$

Hence solid cylinder will reach the bottom first.

41. (*c*) Clearly, $\frac{1}{2}kx_1^2=\frac{1}{2}I(2\omega)^2$

and $$\frac{1}{2}kx_2^2=\frac{1}{2}(2I)\omega^2$$

∴ $$\frac{x_1}{x_2}=\sqrt{2}.$$

42. (*a*) By conservation of angular momentum,

$$(I+2I)\omega'=I.2\omega+2I.\omega$$

or $$\omega'=\frac{4I\omega}{3I}=\frac{4\omega}{3} \quad ...(i)$$

Angular acceleration of disc *B*,

$$\alpha=\frac{\text{Torque}}{\text{M.I.}}=\frac{\tau}{2I}$$

∴ $$\omega'=\omega+\alpha t=\omega+\frac{\tau}{2I}t \quad ...(ii)$$

From (*i*) and (*ii*),

$$\omega+\frac{\tau}{2I}t=\frac{4\omega}{3}$$

or $$\tau=\frac{2I\omega}{3t}.$$

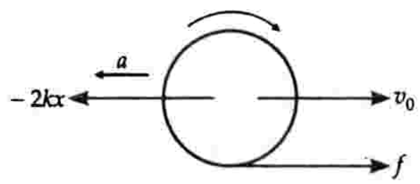
43. (*b*) Loss in K.E.

$$=K_i-K_f$$

$$=\left[\frac{1}{2}I(2\omega)^2+\frac{1}{2}(2I)\omega^2\right]-\frac{1}{2}(I+2I)\left(\frac{4\omega}{3}\right)^2$$

$$=\frac{I\omega^3}{3}.$$

44. (*d*)



$$F_{net}=2kx-f=ma$$

Torque, $fR=I_C\alpha$

$$a=\alpha R$$

$\therefore \quad f = I_C \dfrac{\alpha}{R}$

$= \dfrac{1}{2} mR^2 . \dfrac{a}{R^2} = \dfrac{1}{2} ma$

$F_{net} = 2kx - \dfrac{1}{2} ma = ma$

$2kx = \dfrac{3}{2} ma$

$ma = \dfrac{4kx}{3}$

$\vec{F}_{net} = -\dfrac{4}{2} k\vec{x}.$

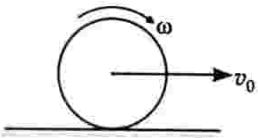
45. (*d*) $-(2kx)R = I_P \alpha$

$2k . R\theta . R = \dfrac{3}{2} MR^2 \alpha$

$\alpha = -\dfrac{4k}{3M}\theta = -\omega^2 \theta$

$\therefore \quad \omega = \sqrt{\dfrac{4k}{3M}}$

46. (*a*)



$f \le \mu N$

$\dfrac{I_c . \alpha}{R} \le \mu N$

$\alpha \le \dfrac{\mu N R}{I_c}$

$\alpha \le \dfrac{\mu mgR}{mR^2/2}$

$\dfrac{a_c}{R} = \alpha \le \dfrac{2\mu g}{R}$

$\dfrac{4}{3} \dfrac{kA}{M} = a_c \le 2\mu g$

$A \le \dfrac{2\mu g M \times 3}{4k} = \dfrac{3}{2} \dfrac{\mu g M}{k}$

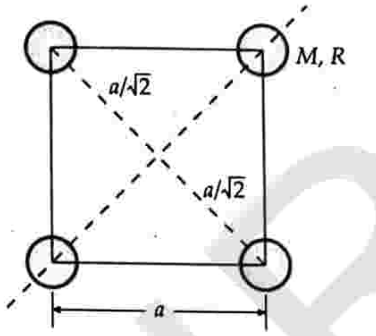
$v \le \omega A = \sqrt{\dfrac{4k}{3M}} \times \dfrac{3}{2} \dfrac{\mu g M}{k}$

$v \le \mu g \sqrt{\dfrac{M}{k}}$

$v_0 = \mu g \sqrt{\dfrac{M}{k}}.$

47.

0	0	0	9



$I = 2\left(\dfrac{2}{5} MR^2\right) + 2\left[\dfrac{2}{5} MR^2 + M\left(\dfrac{a}{\sqrt{2}}\right)^2\right]$

$= \dfrac{8}{5} MR^2 + Ma^2$

$= \dfrac{8}{5} \times 0.5 \times \left(\dfrac{\sqrt{5}}{2} \times 10^{-2}\right)^2 + 0.5 \times (4 \times 10^{-2})^2$

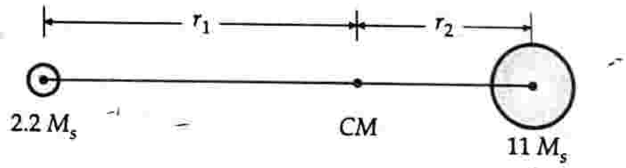
$= 1 \times 10^{-4} + 8 \times 10^{-4} = 9 \times 10^{-4}$ kgm^2

But $I = N \times 10^{-4}$ kgm^2

Hence, $N = \mathbf{9}$

48.

0	0	0	6



Clearly $\quad 2.2 r_1 = 11 r_2$

$\Rightarrow \quad r_1 = 5r_2$

But $\quad r_1 + r_2 = d$

$\therefore \quad r_1 = \dfrac{5d}{6}$ and $r_2 = \dfrac{d}{6}$

Now $\quad L_1 = I_1\omega$ and $L_2 = I_2\omega$

$\therefore \quad \dfrac{L_1}{L_2} = \dfrac{I_1}{I_2}$

$= \dfrac{2.2 M_s(5d/6)^2 + 11 M_s(d/6)^2}{11 M_s(d/6)^2}$

$= \dfrac{\dfrac{55}{36} + \dfrac{11}{36}}{\dfrac{11}{36}} = \mathbf{6}$