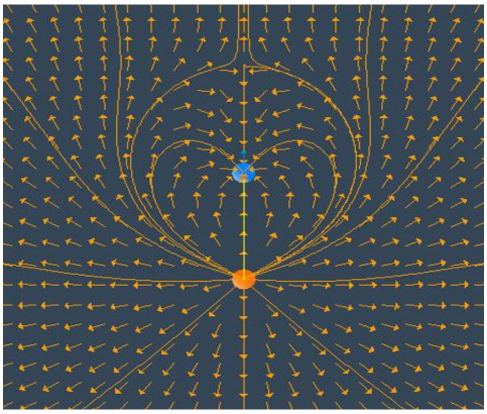


Field lines around an electric dipole. How will you find out if the field is spreading out or not?

UNIT 2

VECTOR FIELDS, DIVERGENCE AND CURL

Structure

STUDY GUIDE

In this unit, you will study two new concepts of vector differential calculus, namely, **divergence** and **curl of vector fields**. The concept of vector field may be new for you. So study it carefully. For calculating the divergence and curl of vector fields, you will need to use partial derivatives. These are discussed in the Appendix of Unit 1. You have to be well versed with these. You will also be using scalar and vector products. Therefore, you should revise scalar and vector products in the algebraic notation from Unit 2 of your Physics Elective BPHCT-131 as also in the Appendix A1 of this Block. You will learn how to apply the del operator more than once. So revise Sec. 1.3 of Unit 1. Practice will make you learn the concepts of this unit better. So you must work through all the examples, SAQs and Terminal Questions.

"Mathematics is the tool especially suited for dealing with abstract concepts of any kind and there is no limit to its power in this field."

P.A.M. Dirac

2.1 INTRODUCTION

In Unit 1, you have studied the concept of a scalar field and learnt how to calculate the **gradient** of a scalar field, which is related to its **directional derivative**. You now know that the directional derivative of a scalar field tells us how the scalar field changes in a particular direction. In this unit, you will learn the concept of a **vector field and operations on vector fields**. Vector fields are quite common in physics. One of the most common examples of a vector field is the velocity field. The gravitational force and the electrostatic force are also familiar examples of vector fields. In the previous unit you have also seen that the gradient of a scalar field is a vector function of position, and is, therefore, a **vector field**.

We begin this unit by giving a formal definition of a **vector field** in Sec. 2.2. In Sec. 1.3, you have learnt about the **del** operator and its operation on a scalar field. In Secs. 2.3 and 2.4, you will learn two ways in which the del operator can operate on a vector field. These give us the **divergence** and **curl** of a vector field. The divergence and curl of vector fields are used extensively in physics. In this course you will learn that Maxwell's equations in electromagnetic theory can be written in a compact form using the del operator. In Sec. 2.5, you will learn about successive applications of the del operator and product rules involving the del operator.

Expected Learning Outcomes

After studying this unit, you should be able to:

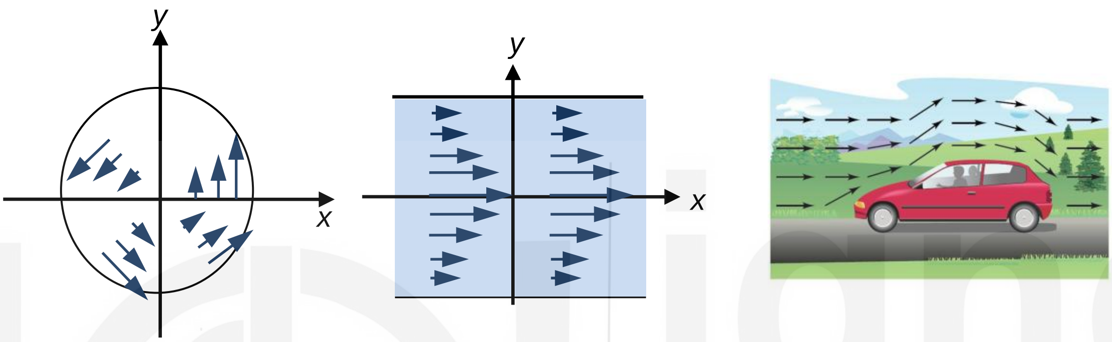
- ❖ explain the concept of a vector field and identify vector fields;
- ❖ calculate the divergence and curl of vector fields; and
- ❖ solve problems based on successive applications of the del operator and product rules involving the del operator.

2.2 VECTOR FIELD

Have you been to a riverside on a calm sunny day and observed the flow of water? Did you observe a leaf floating near the river bank? You may have noticed that it moves very slowly since the water is almost at rest there. Suppose the leaf were dropped in the middle of the river. It would flow faster. Now suppose you want to describe the flow of water in the river. In principle, you could describe the motion of each water particle using Newton's laws. But it would be a cumbersome task since the number of water particles in the river is very large.

Another way to study the flow of water is to specify the velocity at each point in the river. That is, we describe the velocity that a small floating object (e.g., a leaf) would have at each point. Water particles at different points in a flowing river have different velocities. Note that the velocities could change with time. This is an example of a **velocity field**. Since velocity is a vector, the velocity field is a vector field. Velocity fields are used to describe the motion of a system of particles in space.

Refer to Fig. 2.1, where we have shown some more examples of velocity fields. From Fig. 2.1a, you can see the velocity field for a wheel rotating on its axle. The vectors represent the velocity at different points on the wheel and the length of the vector at each point represents its magnitude at that point. As you may recall from your school physics courses or from the course BPHCT-131 entitled Mechanics, the farther away we move from the centre of the wheel, the greater is the velocity. In Fig. 2.1b, we have shown the velocity field for water flowing through a pipe. You may note that the velocity of water is maximum at the centre of the pipe and minimum at its sides. In Fig. 2.1c, you can see the velocity field for air around a moving car. (Car manufacturing companies constantly strive to improve the design of their vehicle so as to increase efficiency through improved aerodynamics.)



(a) (b) (c)

Fig. 2.1: **Velocity field for a) a wheel rotating on its axle; b) water flowing through a pipe; (c) air around a moving car**.

You may recall from Sec. 1.2 of Unit 1 that a scalar field is a scalar function, which associates a scalar to every point in a specified region of space. Similarly, a vector field associates a vector to every point in a specified region, as you have seen in the examples of velocity fields. Let us now define a vector field.

2.2.1 Definition of a Vector Field

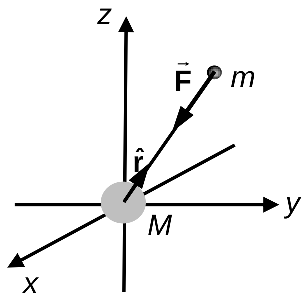
We can define the vector field as follows: A **vector field** $\vec{\mathbf{F}}$ over a region **R** in space is a function which assigns a unique vector $\vec{\mathbf{F}}(x, y, z)$ to every point in **R**. Sometimes we refer to vector fields as **vector field functions**.

In a Cartesian coordinate system, we write a vector field in terms of the unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ as

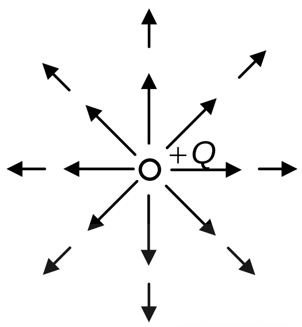
$$\vec{\mathbf{F}}(x, y, z) = F_1(x, y, z)\,\hat{\mathbf{i}} + F_2(x, y, z)\,\hat{\mathbf{j}} + F_3(x, y, z)\,\hat{\mathbf{k}} \qquad (2.1)$$

where $F_1(x, y, z)$, $F_2(x, y, z)$ and $F_3(x, y, z)$ are the component functions of $\vec{\mathbf{F}}$. Note that **each component function is actually a scalar field** defined over the same region of space as the vector field $\vec{\mathbf{F}}$.

In Fig. 2.1, we have given some examples of a velocity field. Other vector fields that we come across in physics are force fields, and the electric and magnetic fields. Let us consider a few examples.



(a)



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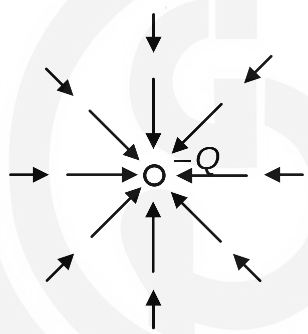


(c)

Fig. 2.2: **a) The gravitational force field due a mass *M* located at the origin of the Cartesian coordinate system on a mass *m* located at a distance *r* from it; b) The electric field due to a positive charge located at the origin. The magnitude of the field is greater at a point nearer the charge; c) The electric field due to a negative charge located at the origin.**

EXAMPLE 2.1 : EXAMPLES OF VECTOR FIELDS

a) **Gravitational Force Field**

Consider a particle of mass *M* located at the origin of the 3D Cartesian coordinate system (Fig. 2.2a). The gravitational force $\vec{\mathbf{F}}$ due to this particle on a particle of mass *m* located at the point (x, y, z) is directed along the line joining the point (x, y, z) to the origin and is given by

$$\vec{\mathbf{F}}(x, y, z) = -\frac{GMm}{(x^2 + y^2 + z^2)}\,\hat{\mathbf{r}} \qquad \text{(i)}$$

where $\hat{\mathbf{r}}$ is the unit vector along the line joining the origin to the point (x, y, z) pointing away from the origin. Note that $\vec{\mathbf{F}}(x,y,z)$ is a vector field. The negative sign in Eq. (i) means that $\vec{\mathbf{F}}(x,y,z)$ is an attractive force field. Since

$$\hat{\mathbf{r}} = \frac{\vec{\mathbf{r}}}{r} = \frac{x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}}{(x^2 + y^2 + z^2)^{1/2}} \qquad \text{(ii)}$$

We can rewrite Eq. (i) as

$$\vec{\mathbf{F}}(x, y, z) = -\frac{GMm}{(x^2 + y^2 + z^2)^{3/2}}(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}) \qquad \text{(iii)}$$

The component functions for the gravitational force field of Eq. (iii) are

$$F_1(x, y, z) = -\frac{GMmx}{(x^2 + y^2 + z^2)^{3/2}} \qquad \text{(iv)}$$

$$F_2\,(x, y, z) = -\frac{GMmy}{(x^2 + y^2 + z^2)^{3/2}} \qquad \text{(v)}$$

$$F_3\,(x, y, z) = -\frac{GMmz}{(x^2 + y^2 + z^2)^{3/2}} \qquad \text{(vi)}$$

b) **Electric Fields**

In your school physics, you have learnt about the **electrostatic force** (also called the Coulomb force) between charged particles **at rest**. Consider a charge *Q* located at the origin of the Cartesian coordinate system (Fig. 2.2b). The electrostatic force on a charge *q* located at a point (*x*, *y*, *z*) at a distance $\vec{\mathbf{r}}$ from *Q* is

$$\vec{\mathbf{F}}(x, y, z) = k\frac{Qq}{r^2}\,\hat{\mathbf{r}} = \frac{kqQ}{(x^2 + y^2 + z^2)^{3/2}}(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}) \qquad \text{(vii)}$$

where $\hat{\mathbf{r}}$ is the unit vector given by Eq. (ii). It points from the origin towards (*x*, *y*, *z*). The electrostatic force on charge *q* is directed towards the origin if *Q* and *q* are unlike charges (since in this case the force is attractive). It points away from the origin if *Q* and *q* are like charges (because in this case the force is repulsive). $\vec{\mathbf{F}}$ is a force field. The **electric field** due to the charge *Q* at the point (*x*, *y*, *z*) is defined as

$$\vec{\mathbf{E}}(x, y, z) = k\frac{Q}{r^2}\,\hat{\mathbf{r}} = \frac{kQ}{(x^2 + y^2 + z^2)^{3/2}}(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}) \qquad \text{(viii)}$$

The electric field due to Q is also a vector field, which points away from Q if Q is a positive charge and towards Q if Q is a negative charge (Figs. 2.2b and c, respectively).

c) Magnetic Fields

A magnetic field, such as the magnetic field due to a bar magnet or a current carrying wire, is another example of a vector field. You may have traced the lines of force for a bar magnet in your school physics laboratory using a compass needle. The alignment of the compass needle defines the direction of the magnetic field, as shown in Fig. 2.3.

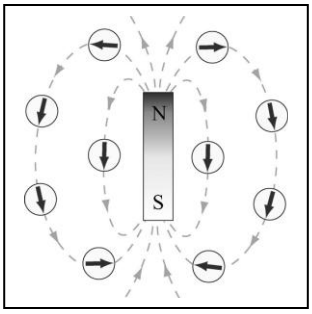


Fig. 2.3: **The magnetic field of a bar magnet as traced by a compass needle.**

Let us now summarize the concept of a vector field.

Recap

VECTOR FIELD

- A vector field is a function that assigns a unique vector to every point of a given region in space.
- A three-dimensional vector field $\vec{\mathbf{F}}$ can be written as

$$\vec{\mathbf{F}}(x, y, z) = F_1(x, y, z)\,\hat{\mathbf{i}} + F_2(x, y, z)\,\hat{\mathbf{j}} + F_3(x, y, z)\hat{\mathbf{k}} \qquad (2.1)$$

- The components of the vector field $\vec{\mathbf{F}}(x, y, z)$, viz., $F_1(x, y, z)$, $F_2(x, y, z)$ and $F_3(x, y, z)$ are **scalar fields**. These are defined over the same region as the vector field.
- A vector field $\vec{\mathbf{F}}$ in two-dimensions can be written as

$$\vec{\mathbf{F}}(x, y) = F_1(x, y)\,\hat{\mathbf{i}} + F_2(x, y)\,\hat{\mathbf{j}} \qquad (2.2)$$

In the previous unit you have learnt how scalar fields are represented by contour lines and contour maps. We now discuss how to represent a vector field.

2.2.2 Representation of a Vector Field

We know that a scalar field gives us the magnitude of a scalar function at every point in a region of space. However, a vector field gives both the magnitude and the direction of a vector function at every point in a specified region of space. So when you represent the vector field in a diagram, you must show both the magnitude and direction of the vector field at every point in the region. This can be done in two different ways: we can either use the vector field representation or the field lines representation. You will learn it now.

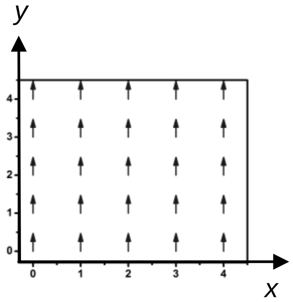
In some textbooks you may observe that the vectors are all of the same length but are colour coded to show relative magnitudes.

In the vector field representation, we draw arrows to represent the vector at each point in the region in which the vector field is defined. We have shown this representation for the velocity fields in Fig. 2.1. The length of the arrow represents the magnitude of the field at a point and the sense of the arrow gives the direction of the field at that point. If we draw the vectors at a sufficient number of points in the region, we can visualize the vector field better.

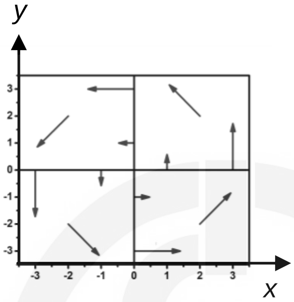
Now go back to Fig. 2.2a. The gravitational force field is represented by the vectors pointing toward the origin. Note that the force becomes weaker as we move away from the origin. That is why the arrows in the figure are longer at points closer to the origin and become shorter as we move away from it.

Figs. 2.2b and c show the vector field representation for the electric field due to a charge Q located at the origin.

In the example given below, we draw the vector field representation of some simple vector fields.



(a)



(b)

Fig. 2.4: **The vector fields**
a) $\vec{\mathbf{F}} = 2\hat{\mathbf{j}}$;
b) $\vec{\mathbf{F}}(x, y) = -y\hat{\mathbf{i}} + x\hat{\mathbf{j}}$

Faraday observed that iron filings arranged themselves around a bar magnet. He called these curved paths, the lines of force. He visualized a similar pattern of lines around positively and negatively charged bodies. For a bar magnet, the lines appeared to originate on its North pole and terminate on the South pole. So he imagined that the lines of force of an electric field would originate on a positive charge and end on a negative charge.

EXAMPLE 2.2 : REPRESENTING A VECTOR FIELD

Represent the following vector fields diagramatically:

a) $\vec{\mathbf{F}} = 2\hat{\mathbf{j}}$ b) $\vec{\mathbf{F}} = -y\hat{\mathbf{i}} + x\hat{\mathbf{j}}$

SOLUTION ■ We draw the vectors at different points in space for both fields.

a) Note that $\vec{\mathbf{F}} = 2\hat{\mathbf{j}}$ is a constant vector field. At each point of the vector field, we just have to draw the vector $2\hat{\mathbf{j}}$. This vector field is shown in Fig. 2.4a for the first quadrant of the *xy* plane. You can see that all vectors are of the same length.

b) First let us write down the vectors at some representative points in the *xy* plane corresponding to the vector field $\vec{\mathbf{F}}(x, y) = -y\hat{\mathbf{i}} + x\hat{\mathbf{j}}$:

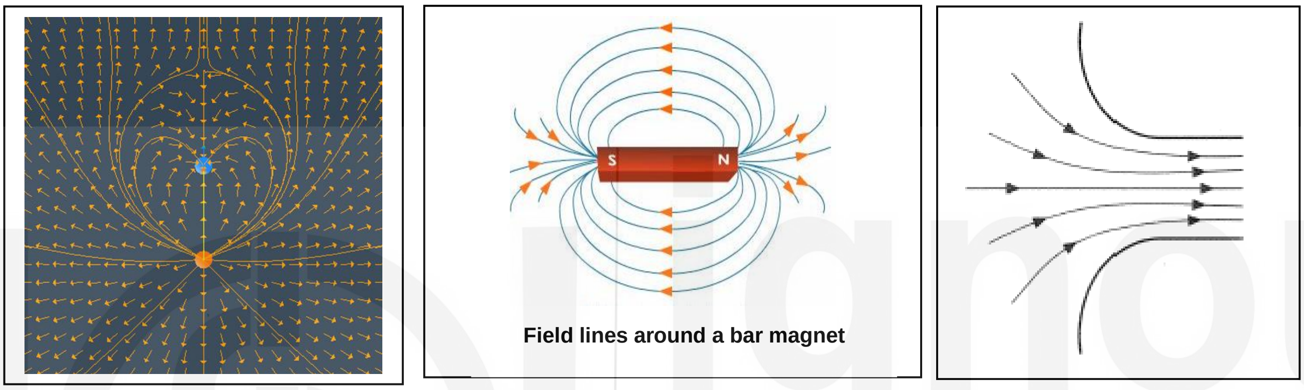
x	y	$\vec{\mathbf{F}}(x, y)$	x	y	$\vec{\mathbf{F}}(x, y)$
0	1	$-\hat{\mathbf{i}}$	0	2	$-2\hat{\mathbf{i}}$
1	0	$\hat{\mathbf{j}}$	2	0	$2\hat{\mathbf{j}}$
1	1	$-\hat{\mathbf{i}} + \hat{\mathbf{j}}$	0	–2	$2\hat{\mathbf{i}}$
0	–1	$\hat{\mathbf{i}}$	–2	0	$-2\hat{\mathbf{j}}$
–1	0	$-\hat{\mathbf{j}}$	1	–1	$\hat{\mathbf{i}} + \hat{\mathbf{j}}$
–1	–1	$\hat{\mathbf{i}} - \hat{\mathbf{j}}$	–1	1	$-\hat{\mathbf{i}} - \hat{\mathbf{j}}$

This field is shown in Fig. 2.4b for some values of *x* and *y*.

In physics, we also use vector field lines to depict vector fields, particularly in electricity and magnetism. The concept of field lines was introduced in physics by Michael Faraday, in the context of electromagnetic induction. He called these **lines of force** (read the margin remark).

A **vector field line** is a line such that the tangent drawn to it at any point gives the direction of the vector field at that point. How can you draw such a field line? Choose any point in the region in which the vector field is defined as a starting point. Walk a small distance in the direction of the vector field at that point and draw a line as you walk. From the new point, walk a short distance in the direction of the vector field and draw a line again. As you continue this process, the tangent to the line at any point will give the direction of the vector field at that point. By choosing different starting points, you can generate a set of lines that represents the vector field. In Figs. 2.5a and b, we have shown the field lines for the electric field around a **pair of charges** and the magnetic field around a bar magnet. For electric and magnetic fields, the field lines are also called **the lines of force**. You should remember that these figures give us only a 2-dimensional view of the vector fields that actually exist in 3-dimensional space.

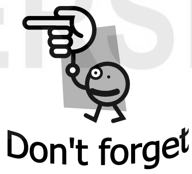
Field lines are also used to represent velocity fields. The field lines for velocity fields are called **streamlines** (Fig. 2.5c). They represent the path followed by a particle whose velocity is given by the velocity field.



(a) (b) (c)

Fig. 2.5: **a) The field lines around an electric dipole; b) the magnetic field lines around a bar magnet; and c) streamlines for velocity field of water flowing into a pipe.**

1. The tangent drawn to the field line at any point gives the direction of the vector field at that point.
2. The field lines never cross each other. If the field lines were to cross, it would mean that the vector field points in two different directions at the point of intersection. This has no physical meaning.



To be sure that you have understood the concept of a vector field and learnt how to represent it, you should answer the following SAQ.

SAQ 1 - Vector fields

All particles of a fluid flow in one direction with a constant speed. What is the velocity field of the fluid? Draw the vector field lines representing this field.

So far you have learnt the concept of vector fields and how to depict them. We now proceed to learn about another property of vector fields, namely, the presence of sources and sinks. It is useful for understanding the behaviour of several vector fields in physics.

2.2.3 Sources and Sinks of a Vector Field

Refer to Fig. 2.5a again. It shows field lines for the electric field due to an electric dipole. The points *A* and *B* mark the location of the positive and negative charges, respectively. Note that all field lines diverge from point *A* and converge to point *B*. The point *A* is called the **"source"** of the vector field and *B* is called the **"sink".**

In the field line representation of the vector field, some authors prefer to show sources by **full circles** and sinks by **open circles**.

Similarly, in fluid flow, a source in the velocity field is the point at which fluid enters the region, whereas sink is the point where fluid leaves the region. That is, particles flow **out** from a source and hence a source is a point of diverging flow. A sink is a point of converging flow because particles flow **into** it.

In Example 2.1 you have seen that in electric fields, field lines diverge from a positive charge and converge on a negative charge. Hence for electric fields, a positive charge acts as a source and the negative charge as a sink.

In Sec. 1.3 of Unit 1, you have learnt the concept of the gradient of a scalar field. It is the vector differential operator $\left(\vec{\nabla} = \hat{\mathbf{i}}\,\frac{\partial}{\partial x} + \hat{\mathbf{j}}\,\frac{\partial}{\partial y} + \hat{\mathbf{k}}\,\frac{\partial}{\partial z}\right)$ applied to a scalar field *f* and defined as:

$$\vec{\nabla} f = \hat{\mathbf{i}}\,\frac{\partial f}{\partial x} + \hat{\mathbf{j}}\,\frac{\partial f}{\partial y} + \hat{\mathbf{k}}\,\frac{\partial f}{\partial z}$$

You now know that physically, the gradient of a scalar field defines the direction of its maximum rate of change. You may also like to know: How rapidly (at what rate) does a vector field change in a given region? Can we extend the analysis of Sec. 1.3.2 as such? The answer is: We cannot. But why? To answer this question, you may recall from Eq. (2.1) that each component of a vector field is a scalar field. There are two different ways in which the **del** operator can act upon the vector field. Each of these ways defines a type of derivative of the vector field. One of these involves the rate of change of a vector field component in its own direction such as $\partial F_1 / \partial x,\ \partial F_2 / \partial y,\ \partial F_3 / \partial z$ and is called the **divergence** of the vector field. The other type of derivative is called the **curl** of the vector field. It involves the rate of change of the vector field components in directions other than their own, e.g., $\partial F_1 / \partial y,\ \partial F_1 / \partial z,\ \partial F_2 / \partial x$, and so on. We now discuss the divergence of vector fields.

2.3 DIVERGENCE OF A VECTOR FIELD

Consider a three-dimensional vector field function
$\vec{\mathbf{F}}(x, y, z) = F_1(x, y, z)\,\hat{\mathbf{i}} + F_2(x, y, z)\,\hat{\mathbf{j}} + F_3(x, y, z)\,\hat{\mathbf{k}}$
where $F_1(x, y, z), F_2(x, y, z)$ and $F_3(x, y, z)$ are the component scalar functions. Its **divergence** is defined as:

$$\boxed{\vec{\nabla}\,.\,\vec{\mathbf{F}}(x, y, z) = \text{div}\,\vec{\mathbf{F}}(x, y, z) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}} \qquad (2.3)$$

In Sec. 1.3 you saw that the gradient of a scalar field is a vector field.

This expression "$\vec{\nabla}\,.$" is read as "divergence of" or "del dot". Note that the divergence of a vector field is a scalar field (read also the margin remark). This suggests that we can construct a scalar field from a vector field.

For a two-dimensional vector field $\vec{\mathbf{F}} = F_1(x,y)\,\hat{\mathbf{i}} + F_2(x,y)\,\hat{\mathbf{j}}$, the divergence is

$$\boxed{\vec{\nabla}.\vec{\mathbf{F}}(x,y) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}} \qquad (2.4)$$

The meaning of the divergence of a vector field is contained in its name itself. Divergence of the vector field $\vec{\mathbf{F}}$ at a point is a measure of its spread from the point. To appreciate this, go through the following example carefully.

EXAMPLE 2.3 : DIVERGENCE OF A VECTOR FIELD

Calculate the divergence of the following vector fields:

(i) $\vec{\mathbf{F}} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}$ (ii) $\vec{\mathbf{F}} = -x\,\hat{\mathbf{i}} - y\,\hat{\mathbf{j}}$ (iii) $\vec{\mathbf{F}} = \hat{\mathbf{i}} + \hat{\mathbf{j}}$

SOLUTION ■ We use Eq. (2.4) to calculate the divergence of these two-dimensional vector fields.

(i) For $\vec{\mathbf{F}} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}$ from Eq. (2.4), we get

$$\vec{\nabla}.\vec{\mathbf{F}} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} = 1 + 1 = 2$$

(ii) For $\vec{\mathbf{F}} = -x\,\hat{\mathbf{i}} - y\,\hat{\mathbf{j}}$ from Eq. (2.4), we get

$$\vec{\nabla}.\vec{\mathbf{F}}(x,\, y) = \frac{\partial(-x)}{\partial x} + \frac{\partial(-y)}{\partial y} = -1 - 1 = -2$$

(iii) For $\vec{\mathbf{F}} = \hat{\mathbf{i}} + \hat{\mathbf{j}}$, $\quad \vec{\nabla}.\vec{\mathbf{F}}(x,\, y) = \dfrac{\partial(1)}{\partial x} + \dfrac{\partial(1)}{\partial y} = 0$

While going through the solution of Example 2.3, you must have noted that the divergence of the vector field $\vec{\mathbf{F}} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}$ is positive, the divergence of $\vec{\mathbf{F}} = -x\,\hat{\mathbf{i}} - y\,\hat{\mathbf{j}}$ is negative whereas $\vec{\mathbf{F}} = \hat{\mathbf{i}} + \hat{\mathbf{j}}$ has zero divergence. Plots of the two-dimensional vector fields of Example 2.3 are shown in Fig. 2.6. These plots suggest that the vector field $\vec{\mathbf{F}} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}$ has a source at the origin (Fig. 2.6a), $\vec{\mathbf{F}} = -x\,\hat{\mathbf{i}} - y\,\hat{\mathbf{j}}$ has a sink at the origin (Fig. 2.6b) and the field $\vec{\mathbf{F}} = \hat{\mathbf{i}} + \hat{\mathbf{j}}$ has neither a source nor a sink (Fig. 2.6c). **In general, a point of positive divergence is a source and point of negative divergence is a sink.**

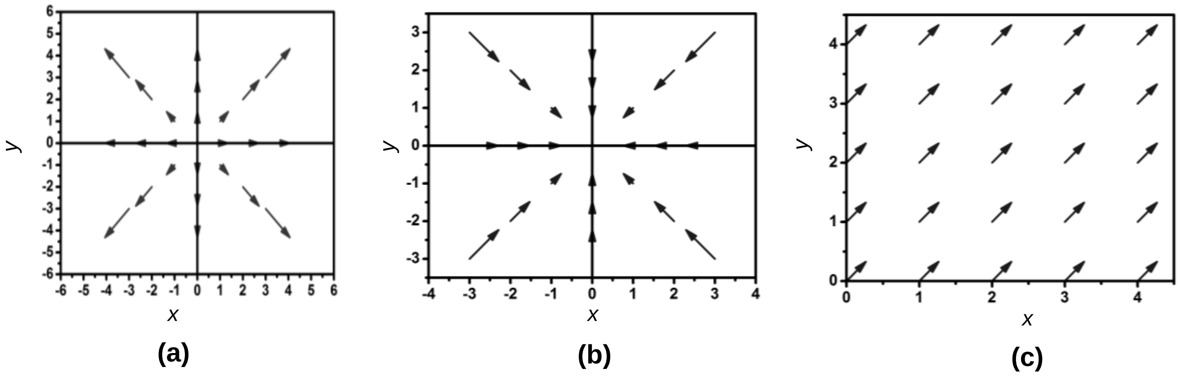


Fig. 2.6: **Plots of the vector fields a)** $\vec{\mathbf{F}} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}$ **; b)** $\vec{\mathbf{F}} = -x\,\hat{\mathbf{i}} - y\,\hat{\mathbf{j}}$ **; c)** $\vec{\mathbf{F}} = \hat{\mathbf{i}} + \hat{\mathbf{j}}$**.**

A vector field $\vec{\mathbf{F}}$ is called **divergence-free** or **solenoidal** in a given region, if for all points in that region

$$\vec{\nabla} \cdot \vec{\mathbf{F}} = 0 \qquad (2.5)$$

Solenoidal comes from a greek word meaning a tube.

The vector field $\vec{\mathbf{F}} = \hat{\mathbf{i}} + \hat{\mathbf{j}}$ shown in Fig. 2.6c is solenoidal at all points in the xy plane. The magnetic field is an example of a solenoidal vector field:

$$\vec{\nabla} \cdot \vec{\mathbf{B}} = 0$$

Let us now summarize what you have learnt about the concept of divergence.

Recap

DIVERGENCE OF A VECTOR FIELD

- The **divergence** of a three-dimensional vector field $\vec{\mathbf{F}}(x, y, z) = F_1(x, y, z)\hat{\mathbf{i}} + F_2(x, y, z)\hat{\mathbf{j}} + F_3(x, y, z)\hat{\mathbf{k}}$ is defined as

$$\text{div}\,\vec{\mathbf{F}}(x, y, z) = \vec{\nabla} \cdot \vec{\mathbf{F}}(x, y, z) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \qquad (2.3)$$

- The divergence of a two-dimensional vector field $\vec{\mathbf{F}}(x, y) = F_1(x, y)\hat{\mathbf{i}} + F_2(x, y)\hat{\mathbf{j}}$ is defined as

$$\text{div}\,\vec{\mathbf{F}}(x, y) = \vec{\nabla} \cdot \vec{\mathbf{F}}(x, y) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} \qquad (2.4)$$

- A non-zero value of the divergence at any point in a vector field signifies the presence of a source or a sink at that point: $\vec{\nabla}.\vec{\mathbf{F}} > 0$ for a **source** and $\vec{\nabla}.\vec{\mathbf{F}} < 0$ for a **sink**.

- A vector field is called **"divergence-free"** or **"solenoidal"** if its divergence is zero: $\vec{\nabla}.\vec{\mathbf{F}} = 0$ (2.5)

Let us now calculate the divergence of a 3D vector field.

EXAMPLE 2.4 : DIVERGENCE OF A 3D VECTOR FIELD

Calculate the divergence of the vector field $\vec{\mathbf{E}} = \dfrac{\vec{\mathbf{r}}}{r^3}$.

SOLUTION ■ We define the given vector field $\vec{\mathbf{E}} = \dfrac{\vec{\mathbf{r}}}{r^3} = \dfrac{x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}}{(x^2 + y^2 + z^2)^{3/2}}$.

Then, using Eq. (2.3) with $\vec{\mathbf{F}} = \vec{\mathbf{E}}$, we can write:

$$\vec{\nabla}.\vec{\mathbf{E}} = \vec{\nabla}.\left\{\frac{x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}}{(x^2 + y^2 + z^2)^{3/2}}\right\}$$

$$= \frac{\partial}{\partial x}\left[\frac{x}{(x^2 + y^2 + z^2)^{3/2}}\right] + \frac{\partial}{\partial y}\left[\frac{y}{(x^2 + y^2 + z^2)^{3/2}}\right]$$

$$+ \frac{\partial}{\partial z}\left[\frac{z}{(x^2 + y^2 + z^2)^{3/2}}\right] \qquad \text{(i)}$$

We evaluate each derivative separately:

$$\frac{\partial}{\partial x}\left\{\frac{x}{(x^2+y^2+z^2)^{3/2}}\right\} = \frac{\partial}{\partial x}(x).\left[\frac{1}{(x^2+y^2+z^2)^{3/2}}\right]$$

$$+x.\frac{\partial}{\partial x}\left[\frac{1}{(x^2+y^2+z^2)^{3/2}}\right]$$

$$=\frac{1}{(x^2+y^2+z^2)^{3/2}} - \frac{3x^2}{(x^2+y^2+z^2)^{5/2}} = \frac{y^2+z^2-2x^2}{(x^2+y^2+z^2)^{5/2}} \quad \text{(ii)}$$

Similarly

$$\frac{\partial}{\partial y}\left\{\frac{y}{(x^2+y^2+z^2)^{3/2}}\right\} = \frac{x^2+z^2-2y^2}{(x^2+y^2+z^2)^{5/2}} \quad \text{(iii)}$$

and $$\frac{\partial}{\partial z}\left\{\frac{z}{(x^2+y^2+z^2)^{3/2}}\right\} = \frac{x^2+y^2-2z^2}{(x^2+y^2+z^2)^{5/2}} \quad \text{(iv)}$$

Substituting from Eqs. (ii), (iii) and (iv) in Eq. (i) we get

$$\vec{\nabla}.\vec{\mathbf{E}} = \frac{2(x^2+y^2+z^2)-2(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{5/2}} = 0$$

Before we discuss the physical meaning of the divergence of a vector field, you may like to work out a few problems.

SAQ 2 - Divergence of a vector field

a) Determine the divergence of the following vector fields:

(i) $(x^2-y^2)\hat{\mathbf{i}}+(y^2-z^2)\hat{\mathbf{j}}+(z^2-x^2)\hat{\mathbf{k}}$

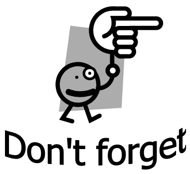
(ii) $y^2z\,\hat{\mathbf{i}}+xy^3\,\hat{\mathbf{j}}-z^2\,\hat{\mathbf{k}}$

b) Calculate the value of the constant a such that the vector field $\vec{\mathbf{u}}=(x+3y)\hat{\mathbf{i}}+(y+2z)\hat{\mathbf{j}}+(x+az)\hat{\mathbf{k}}$ is solenoidal.

You may now like to know: **What is the physical significance of the divergence of a vector field?**

The divergence of a vector field can be related to **the flux of that vector field.** You will study about the flux of a vector field in Unit 4.

It is important that we understand another aspect of the divergence at this point. We have written down the expression for the divergence in the rectangular Cartesian coordinate system. However, since the divergence of a vector field can be interpreted as the flux of the vector field per unit volume, it means that:



The value of the divergence of the vector field at any point is independent of the coordinate system.

In your physics courses you will often encounter quantities which involve the sum and products of vector fields or products of scalar fields and vector fields. You will be required to calculate the divergence of these quantities and for this you need to know the rules governing these operations.

IDENTITIES INVOLVING THE DIVERGENCE OF A VECTOR FIELD

The proof of these identities is beyond the scope of this syllabus.

We can write down the following rules for the divergence of the sum and product of vector fields $\vec{\mathbf{F}}$ and $\vec{\mathbf{G}}$, and the product of a scalar field $f = f(x, y, z)$ with $\vec{\mathbf{F}}$:

$$\vec{\nabla} . (\vec{\mathbf{F}} \pm \vec{\mathbf{G}}) = \vec{\nabla} . \vec{\mathbf{F}} \pm \vec{\nabla} . \vec{\mathbf{G}} \qquad (2.6a)$$

$$\vec{\nabla} . (k\vec{\mathbf{F}}) = k\vec{\nabla} . \vec{\mathbf{F}} \quad \text{where } k \text{ is a constant} \qquad (2.6b)$$

$$\vec{\nabla} . (f\vec{\mathbf{F}}) = f(\vec{\nabla} . \vec{\mathbf{F}}) + \vec{\mathbf{F}} . (\vec{\nabla} f) \qquad (2.6c)$$

You may now like to work out an SAQ.

SAQ 3 **– Identities on the divergence of a vector field**

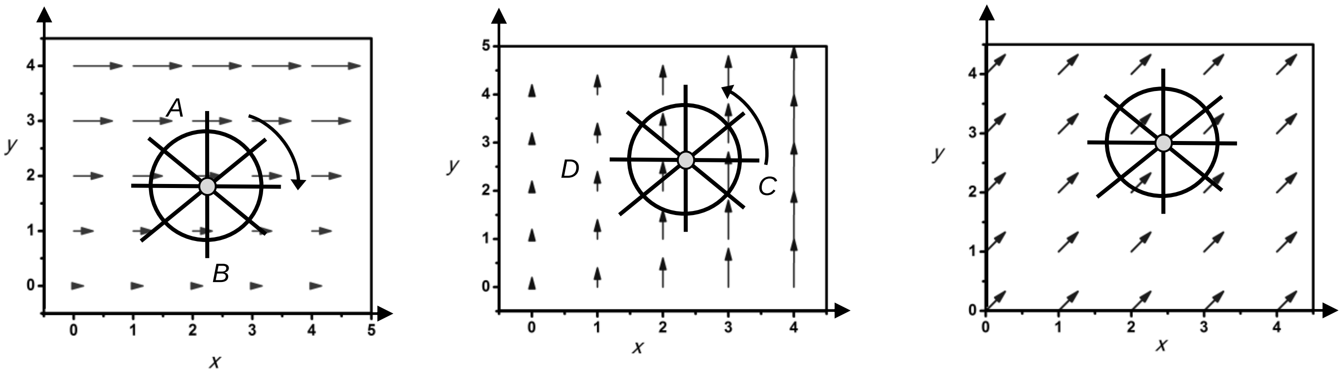
For a scalar field *f*, show that $\vec{\nabla} . (f\vec{\mathbf{r}}) = 3f + \vec{\mathbf{r}} . (\vec{\nabla} f)$, where $\vec{\mathbf{r}} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$.

Let us now study the concept of the curl of a vector field.

2.4 CURL OF VECTOR FIELD

Suppose that you are standing near a pond. Float a small cork with toothpicks (or needles) sticking out radially from it or a paddle wheel in the pond. You may observe that sometimes the cork or paddle wheel rotates. Does this tell us anything about the velocity vector field of the water on the surface of the pond? We can describe this observation in terms of the curl of the velocity vector field. If the cork rotates, it means that the velocity vector field has a non-zero curl at that point. Let us first try to understand, intuitively: **What is the curl of a vector field?**

Let us consider the velocity vector field of water particles on the pond's surface. To keep things simple, we consider a two-dimensional flow in the *xy* plane. Imagine that we put a paddle wheel in the pond with its axis in the z-direction (Fig. 2.7).



(a) (b) (c)

Fig. 2.7: **A paddle wheel in the vector field a)** $\vec{\mathbf{v}} = (y+1)\,\hat{\mathbf{i}}$; **b)** $\vec{\mathbf{v}} = (x+1)\,\hat{\mathbf{j}}$; **c)** $\vec{\mathbf{v}} = \hat{\mathbf{i}} + \hat{\mathbf{j}}$.

Let us first consider the velocity vector field given by $\vec{\mathbf{v}} = (y+1)\,\hat{\mathbf{i}}$ shown in Fig. 2.7a. Note that this velocity vector field is directed along the x-axis and its magnitude increases with y. Since the velocity of water is greater at the top end of the paddle wheel (A) than at the bottom (B), the wheel will have a tendency to rotate in the ***clockwise*** direction, as shown in Fig. 2.7a.

> The vector field in Fig. 2.7a is a **particular example** of a vector field which is along the x-direction but the magnitude of the field increases with y.

Next, let us consider the velocity vector field $\vec{\mathbf{v}} = (x+1)\,\hat{\mathbf{j}}$. Here the field is directed along the y-axis and its magnitude increases with x. Since the velocity of water is greater at the right end of the paddle wheel (C) than at the left (D), the wheel will have a tendency to rotate ***counterclockwise*** as shown in Fig. 2.7b. For a constant velocity field $\vec{\mathbf{v}} = \hat{\mathbf{i}} + \hat{\mathbf{j}}$, shown in Fig. 2.7c, the paddle wheel ***will not rotate*** at all.

We use the concept of curl of a vector field to describe these three observations mathematically.

The curl of a vector field is defined as follows:

CURL OF A VECTOR FIELD

The curl of the vector field

$\vec{\mathbf{F}}(x, y, z) = F_1(x, y, z)\,\hat{\mathbf{i}} + F_2(x, y, z)\,\hat{\mathbf{j}} + F_3(x, y, z)\,\hat{\mathbf{k}}$ is given by

$$\vec{\nabla} \times \vec{\mathbf{F}} = \text{curl } \vec{\mathbf{F}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \qquad (2.7a)$$

$$\vec{\nabla} \times \vec{\mathbf{F}} = \text{curl } \vec{\mathbf{F}} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{\mathbf{k}} \qquad (2.7b)$$

> For three-dimensional vector fields the curl gives the net rotation of the field which would be about some axis. The axis may not be so easy to visualize.

The expression "curl $\vec{\mathbf{F}}$" is pronounced as curl $\vec{\mathbf{F}}$ and "$\vec{\nabla} \times \vec{\mathbf{F}}$" as "del cross F".

The curl of a **two-dimensional vector field** $\vec{\mathbf{F}}(x, y) = F_1(x, y)\,\hat{\mathbf{i}} + F_2(x, y)\,\hat{\mathbf{j}}$ is:

$$\vec{\nabla}\times\vec{\mathbf{F}} = \text{curl } \vec{\mathbf{F}} = \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{\mathbf{k}} \qquad (2.8)$$

Note that the curl of a two-dimensional field is normal to the field. If the field is in the *xy* plane, the curl of the field is along the *z*-direction. You can use Eq. (2.8) to obtain the curl of the vector fields depicted in Fig. 2.7.

For $\vec{\mathbf{v}} = (y+1)\,\hat{\mathbf{i}}$, $\vec{\nabla}\times\vec{\mathbf{v}} = -\hat{\mathbf{k}}$,

for $\vec{\mathbf{v}} = (x+1)\,\hat{\mathbf{j}}$, $\vec{\nabla}\times\vec{\mathbf{v}} = \hat{\mathbf{k}}$ and

for $\vec{\mathbf{v}} = \hat{\mathbf{i}} + \hat{\mathbf{j}}$, $\vec{\nabla}\times\vec{\mathbf{v}} = \vec{\mathbf{0}}$.

So the velocity vector fields in Figs. 2.7a and b have a finite non-zero curl and the vector field in Fig. 2.7c has a zero curl. Note that the paddle wheel turns anticlockwise if the curl is positive and clockwise if the curl is negative. So in the two-dimensional *xy* plane the curl of a vector field $\vec{\mathbf{F}}$ is a measure of the tendency of the vector field to produce a rotation about the *z*-axis. You will see later in Example 2.6 that the angular velocity of rotation of a rigid body is proportional to the curl of the velocity vector field.

A vector field with zero curl at every point is said to be an **irrotational vector field.** The gravitational force field and electric fields (Example 2.1, Fig. 2.2) are examples of irrotational fields.

Although the expression for the vector product and the curl of a vector look similar, there are some important differences:

1. $\vec{\nabla}\times\vec{\mathbf{F}}$ is not necessarily orthogonal to $\vec{\mathbf{F}}$. In general, it can lie at any angle to $\vec{\mathbf{F}}$ or even be parallel to $\vec{\mathbf{F}}$. **For any two-dimensional vector field, however, the curl of the vector field is always normal to the vector field.**
2. $\vec{\nabla}$ is a vector differential operator. It means that $\vec{\nabla}\times\vec{\mathbf{F}}$ is not the same as $\vec{\mathbf{F}}\times\vec{\nabla}$ and $\vec{\mathbf{F}}\times\vec{\nabla} \neq -\vec{\nabla}\times\vec{\mathbf{F}}$.

In the following example we calculate the curl of three vector fields.

EXAMPLE 2.5 : CURL OF A VECTOR FIELD

Calculate the curl of the following vector fields:

(i) $\vec{\mathbf{F}} = -y\,\hat{\mathbf{i}} + x\,\hat{\mathbf{j}}$

(ii) $\vec{\mathbf{F}} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}$

(iii) $\vec{\mathbf{F}} = xe^{y}\,\hat{\mathbf{i}} + ye^{z}\,\hat{\mathbf{j}} + ze^{x}\,\hat{\mathbf{k}}$

SOLUTION ■ (i) Substituting $F_1 = -y$, and $F_2 = x$ in Eq. (2.8), we get:

$$\vec{\nabla}\times\vec{\mathbf{F}} = \left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y}\right)\hat{\mathbf{k}} = 2\hat{\mathbf{k}}$$

The curl of this two-dimensional vector field is a constant vector in the z-direction.

ii) Substituting $F_1 = x$ and $F_2 = y$ in Eq. (2.8) we get:

$$\vec{\nabla} \times \vec{\mathbf{F}} = \left[\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right]\hat{\mathbf{k}} = \vec{\mathbf{0}}$$

The curl of this two dimensional vector field is a null vector so the vector field is **irrotational**.

ii) When we use Eq. (2.7a) with $F_1 = xe^y; F_2 = ye^z$ and $F_3 = ze^x$, we get

$$\vec{\nabla} \times \vec{\mathbf{F}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xe^y & ye^z & ze^x \end{vmatrix}$$

$$= \hat{\mathbf{i}}\left[\frac{\partial}{\partial y}(ze^x) - \frac{\partial}{\partial z}(ye^z)\right] + \hat{\mathbf{j}}\left[\frac{\partial}{\partial z}(xe^y) - \frac{\partial}{\partial x}(ze^x)\right] + \hat{\mathbf{k}}\left[\frac{\partial}{\partial x}(ye^z) - \frac{\partial}{\partial y}(xe^y)\right]$$

$$= \hat{\mathbf{i}}[-ye^z] + \hat{\mathbf{j}}[-ze^x] + \hat{\mathbf{k}}[-xe^y] \quad = -ye^z\,\hat{\mathbf{i}} - ze^x\,\hat{\mathbf{j}} - xe^y\,\hat{\mathbf{k}}$$

The vector field of Example 2.5(i) is shown in Fig. 2.4b. It has a **positive curl.** As you can see in the figure, the field has the appearance of a *whirlpool* rotating anticlockwise. What is the divergence of this vector field? Let us calculate it using Eq. (2.4):

$$\vec{\nabla} . \vec{\mathbf{F}} = \vec{\nabla} . (-y\,\hat{\mathbf{i}} + x\,\hat{\mathbf{k}}) = \frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}(x) = 0$$

So $\vec{\mathbf{F}} = -y\,\hat{\mathbf{i}} + x\,\hat{\mathbf{k}}$ is an example of a vector field which has **zero divergence and a finite (positive) curl**. This is an example of a **circulating field**, which has no sources or sinks.

On the other hand, the vector field of Example 2.5(ii) which is shown in Fig. 2.6a has zero curl and is an **irrotational** vector field. From the figure, it appears that this vector field has a source.

What is the divergence of this vector field? Using Eq. (2.4) we can write

$$\vec{\nabla} . \vec{\mathbf{F}} = \vec{\nabla} . (x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{k}}) = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) = 2$$

So $\vec{\mathbf{F}} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}$ is an example of a vector field which has **finite (positive) divergence and zero curl.** This is an example of a **diverging field** that has no rotation. We also say that it has no **circulation**.

Let us now summarise what you have learnt about the curl of a vector field.

Recap

CURL OF A VECTOR FIELD

- The curl of a three-dimensional vector field $\vec{\mathbf{F}}(x, y, z) = F_1(x, y, z)\,\hat{\mathbf{i}} + F_2(x, y, z)\,\hat{\mathbf{j}} + F_3(x, y, z)\,\hat{\mathbf{k}}$ is defined as

$$\text{curl}\,\vec{\mathbf{F}} = \vec{\nabla} \times \vec{\mathbf{F}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \qquad (2.7a)$$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{\mathbf{k}} \qquad (2.7b)$$

- The curl of a two-dimensional vector field $\vec{\mathbf{F}}(x, y) = F_1(x, y)\,\hat{\mathbf{i}} + F_2(x, y)\,\hat{\mathbf{j}}$ is defined as

$$\text{curl}\,\vec{\mathbf{F}} = \vec{\nabla} \times \vec{\mathbf{F}} = \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{\mathbf{k}} \qquad (2.8)$$

- If the curl of the vector field is zero, i.e.,

$$\vec{\nabla} \times \vec{\mathbf{F}} = \vec{\mathbf{0}}$$

the vector field is said to be "**irrotational**".

In the next section, we establish some identities for the curl of vector fields. But before studying further, you may like to work out the following SAQ.

SAQ 4 – Curl of a vector field

Determine the curl of the following vector fields:

a) $\vec{\mathbf{F}} = (2x - y)\,\hat{\mathbf{i}} - 2yz^2\,\hat{\mathbf{j}} + -2zy^2\,\hat{\mathbf{k}}$

b) $\vec{\mathbf{F}} = z\cos x\,\hat{\mathbf{i}} + (y + \sin x)\,\hat{\mathbf{j}} + xyz\,\hat{\mathbf{k}}$

We now write down important identities for the curl of the sum and product of vector fields and the product of a scalar field and a vector field.

IDENTITIES INVOLVING THE CURL OF A VECTOR FIELD

For the vector fields $\vec{\mathbf{F}}$ and $\vec{\mathbf{G}}$, and the scalar field $f = f(x, y, z)$:

$$\vec{\nabla} \times (\vec{\mathbf{F}} \pm \vec{\mathbf{G}}) = \vec{\nabla} \times \vec{\mathbf{F}} \pm \vec{\nabla} \times \vec{\mathbf{G}} \qquad (2.9a)$$

$$\vec{\nabla} \times (k\vec{\mathbf{F}}) = k\vec{\nabla} \times \vec{\mathbf{F}} \qquad \text{where } k \text{ is a constant} \qquad (2.9b)$$

$$\vec{\nabla} \times (f\,\vec{\mathbf{F}}) = f\,(\vec{\nabla} \times \vec{\mathbf{F}}) - \vec{\mathbf{F}} \times (\vec{\nabla} f) \qquad (2.9c)$$

$$\vec{\nabla} \times (\vec{\mathbf{F}} \times \vec{\mathbf{G}}) = \vec{\mathbf{F}}(\vec{\nabla} . \vec{\mathbf{G}}) - \vec{\mathbf{G}}(\vec{\nabla} . \vec{\mathbf{F}}) - (\vec{\mathbf{F}} . \vec{\nabla})\vec{\mathbf{G}} + (\vec{\mathbf{G}} . \vec{\nabla})\vec{\mathbf{F}} \qquad (2.9d)$$

The proof of these identities is beyond the scope of this course.

We also add the following rule for the divergence of the vector product of the two vector fields because it also involves the curl of a vector field:

$$\vec{\nabla} . (\vec{\mathbf{F}} \times \vec{\mathbf{G}}) = \vec{\mathbf{G}} . (\vec{\nabla} \times \vec{\mathbf{F}}) - \vec{\mathbf{F}} . (\vec{\nabla} \times \vec{\mathbf{G}}) \qquad (2.9e)$$

Let us now study some simple applications of these product rules.

EXAMPLE 2.6 : ROTATION OF A RIGID BODY

Consider a rigid body rotating about a fixed axis with a constant angular velocity $\vec{\boldsymbol{\omega}}$, directed along the axis of rotation (Fig. 2.8). The velocity $\vec{\mathbf{v}}$ of a particle on the rigid body is $\vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}}$. Here $\vec{\mathbf{r}}$ is the position vector of the particle relative to the origin of the coordinate system located at some point on the axis of rotation. Calculate $\vec{\nabla} \times \vec{\mathbf{v}}$.

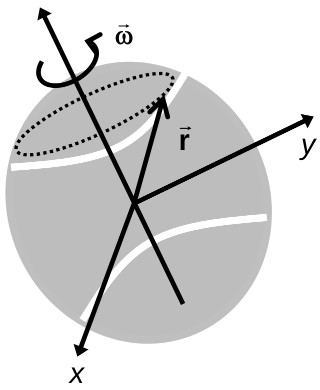


Fig. 2.8: **A rigid body rotating about an axis.**

SOLUTION ■ Since velocity $\vec{\mathbf{v}}$ of a particle on the rigid body is given by $\vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}}$, we can write $\vec{\nabla} \times \vec{\mathbf{v}} = \vec{\nabla} \times (\vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}})$. To obtain the desired expression, we use Eq. (2.9d) for the curl of the cross product of two vector fields. Substituting $\vec{\mathbf{F}}$ by $\vec{\boldsymbol{\omega}}$ and $\vec{\mathbf{G}}$ by $\vec{\mathbf{r}}$ in Eq. (2.9d) we get

$$\vec{\nabla} \times (\vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}}) = \vec{\boldsymbol{\omega}}(\vec{\nabla} . \vec{\mathbf{r}}) - \vec{\mathbf{r}}\,(\vec{\nabla} . \vec{\boldsymbol{\omega}}) - (\vec{\boldsymbol{\omega}} . \vec{\nabla})\vec{\mathbf{r}} + (\vec{\mathbf{r}} . \vec{\nabla})\vec{\boldsymbol{\omega}} \qquad \text{(i)}$$

The position vector is $\vec{\mathbf{r}} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$ and the angular velocity is $\vec{\boldsymbol{\omega}} = \omega_x\hat{\mathbf{i}} + \omega_y\hat{\mathbf{j}} + \omega_z\hat{\mathbf{k}}$. As the angular velocity ω is a constant, all terms in Eq. (i) which involve the derivatives of ω_x, ω_y and ω_z are zero. For example:

$$(\vec{\mathbf{r}} \,.\, \vec{\nabla})\vec{\boldsymbol{\omega}} = \left[(x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}) \,.\, \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right)\right]\vec{\boldsymbol{\omega}}$$

$$= \left[x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right]\vec{\boldsymbol{\omega}} = \vec{\mathbf{0}} \qquad \text{(ii)}$$

Also $\vec{\mathbf{r}}\,(\vec{\nabla} . \vec{\boldsymbol{\omega}}) = \vec{\mathbf{0}}$ (iii)

On combining Eqs. (i), (ii) and (iii), we get

$$\vec{\nabla} \times (\vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}}) = \vec{\boldsymbol{\omega}}(\vec{\nabla} \,.\, \vec{\mathbf{r}}) - (\vec{\boldsymbol{\omega}} \,.\, \vec{\nabla})\vec{\mathbf{r}} \qquad \text{(iv)}$$

Now $$\vec{\boldsymbol{\omega}}(\vec{\nabla} \,.\, \vec{\mathbf{r}}) = \vec{\boldsymbol{\omega}}\left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}\right) = 3\vec{\boldsymbol{\omega}} \qquad \text{(v)}$$

and $$(\vec{\boldsymbol{\omega}} \,.\, \vec{\nabla})\vec{\mathbf{r}} = \left(\omega_x\frac{\partial}{\partial x} + \omega_y\frac{\partial}{\partial y} + \omega_z\frac{\partial}{\partial z}\right)(x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}})$$

$$= \omega_x\hat{\mathbf{i}} + \omega_y\hat{\mathbf{j}} + \omega_z\hat{\mathbf{k}} = \vec{\boldsymbol{\omega}} \qquad \text{(vi)}$$

The final expression for $\vec{\nabla} \times \vec{\mathbf{v}}$ is obtained by substituting Eqs. (v) and (vi) into Eq. (iv):

$$\vec{\nabla} \times \vec{\mathbf{v}} = 3\vec{\boldsymbol{\omega}} - \vec{\boldsymbol{\omega}} = 2\vec{\boldsymbol{\omega}} \qquad \text{(vii)}$$

So we can now write the angular velocity of the rigid body as the curl of the velocity as:

$$\vec{\boldsymbol{\omega}} = \frac{1}{2}(\vec{\nabla} \times \vec{\mathbf{v}}) \qquad \text{(viii)}$$

Therefore, $\vec{\nabla} \times \vec{\mathbf{v}}$ describes the rate at which the body rotates about the axis of rotation.

If $\vec{\mathbf{v}}$ is a two dimensional velocity field describing fluid flow, instead of the velocity of the particles on a rigid body, we can say that $\vec{\nabla} \times \vec{\mathbf{v}}$ at any point (x, y) in the field, is **twice the angular velocity of an infinitesimal paddle wheel placed at the point** (x, y).

Let us study another important example in physics.

EXAMPLE 2.7: CURL OF A CENTRAL FORCE FIELD

A central force field is a force field of the form $\vec{\mathbf{F}} = f(r)\vec{\mathbf{r}}$. Determine $\vec{\nabla} \times \vec{\mathbf{F}}$.

SOLUTION ■ In Eq. (2.9c), we replace *f* by $f(r)$ and $\vec{\mathbf{F}}$ by $\vec{\mathbf{r}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ to write:

$$\vec{\nabla} \times (f(r)\vec{\mathbf{r}}) = f(r)\,(\vec{\nabla} \times \vec{\mathbf{r}}) - \vec{\mathbf{r}} \times (\vec{\nabla} f(r)) \qquad \text{(i)}$$

From Example 1.2 of Unit 1, you may recall that

$$\vec{\nabla} f(r) = \frac{df}{dr}\hat{\mathbf{r}} \quad \text{and} \quad \vec{\mathbf{r}} = r\,\hat{\mathbf{r}} \qquad \text{(ii)}$$

So we can write $\vec{\mathbf{r}} \times (\vec{\nabla} f(r)) = (r\,\hat{\mathbf{r}}) \times \left(\frac{df}{dr}\hat{\mathbf{r}}\right) = \vec{\mathbf{0}}$ (iii)

since $\hat{\mathbf{r}} \times \hat{\mathbf{r}} = \vec{\mathbf{0}}$. Further, you can show that

$$\vec{\nabla} \times \vec{\mathbf{r}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \vec{\mathbf{0}} \qquad \text{(iv)}$$

Using Eqs. (iii) and (iv) in Eq. (i), we get

$$\vec{\nabla} \times (f(r)\vec{\mathbf{r}}) = \vec{\mathbf{0}} \qquad \text{(v)}$$

Thus, a central force field $\vec{\mathbf{F}}$ of the form $\vec{\mathbf{F}} = f(r)\vec{\mathbf{r}}$ is **irrotational**.

Before you study the next section you may like to work out an SAQ.

SAQ 5 - Identities of the Curl Operator

a) If $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are both irrotational, show that $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$ is solenoidal.

b) A vector function $\vec{\mathbf{f}}(x, y, z)$ is not irrotational but its product with a scalar function *g* (*x*, *y*, *z*) is irrotational. Show that $\vec{\mathbf{f}} \cdot (\vec{\nabla} \times \vec{\mathbf{f}}) = 0$.

2.5 SUCCESSIVE APPLICATIONS OF THE DEL OPERATOR

We now write down five basic identities involving repeated applications of the del operator. These are commonly used in physics: In Poisson's equation and Laplace's equation in electrostatics, electromagnetic wave equation and to describe conservative force fields.

We have presented the repeated applications of del operator in the light of their applications.

Since $\vec{\nabla} f$ is a vector field, we can take its divergence and curl:

i) Divergence of $\vec{\nabla} f : \vec{\nabla} \cdot (\vec{\nabla} f)$

ii) Curl of $\vec{\nabla} f : \vec{\nabla} \times (\vec{\nabla} f)$

Since $\vec{\nabla} \cdot \vec{\mathbf{F}}$ is a scalar field, we obtain its gradient as

iii) Gradient of $\vec{\nabla} \cdot \vec{\mathbf{F}}$: $\vec{\nabla}(\vec{\nabla} \cdot \vec{\mathbf{F}})$

Since $\vec{\nabla} \times \vec{\mathbf{F}}$ is vector field we can take its divergence and curl:

iv) Divergence of $\vec{\nabla} \times \vec{\mathbf{F}}$: $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{\mathbf{F}})$

v) Curl of $\vec{\nabla} \times \vec{\mathbf{F}}$: $\vec{\nabla} \times (\vec{\nabla} \times \vec{\mathbf{F}})$

Thus, we can construct five different second order derivatives of scalar and vector fields. Let us consider them one at a time with examples.

i) Divergence of $\vec{\nabla} f$

Using Eqs. (1.11a) and (2.3) we can write

$$\vec{\nabla} \cdot (\vec{\nabla} f) = \left(\hat{\mathbf{i}} \frac{\partial}{\partial x} + \hat{\mathbf{j}} \frac{\partial}{\partial y} + \hat{\mathbf{k}} \frac{\partial}{\partial z} \right) \cdot \left(\hat{\mathbf{i}} \frac{\partial f}{\partial x} + \hat{\mathbf{j}} \frac{\partial f}{\partial y} + \hat{\mathbf{k}} \frac{\partial f}{\partial z} \right)$$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

or

$$\vec{\nabla} \cdot (\vec{\nabla} f) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) f \equiv \nabla^2 f \qquad (2.10a)$$

The operator $\nabla^2 (\equiv \vec{\nabla} \cdot \vec{\nabla})$ is called the **Laplace operator** and $\nabla^2 f$ is called the **Laplacian** of *f*. Notice that $\nabla^2 f$ is a scalar field. The Laplace operator plays an extremely important role in determining the charge density ρ (*x, y, z*) of a charge distribution which gives rise to an electrostatic potential ϕ. This is done by solving the following equation, known as **Poisson's equation**:

$$\nabla^2 \phi = -\frac{\rho}{\varepsilon_0}$$

Conversely, given ρ (*x, y, z*), we can obtain ϕ from this equation but the method of solving for ϕ is beyond the scope of this course. You will learn to do so in a course on partial differential equations.

To obtain ϕ in a charge-free region we solve **Laplace's equation**:

$$\nabla^2 \phi = 0$$

In electromagnetic theory, you will come across the Laplacian of a vector field: $\nabla^2 \vec{\mathbf{F}}$. This means that $\nabla^2 \vec{\mathbf{F}}$ is a vector quantity whose *x, y* and z components are the Laplacians $\nabla^2 F_x$, $\nabla^2 F_y$ and $\nabla^2 F_z$, respectively, i.e.

$$\nabla^2 \vec{\mathbf{F}} = (\nabla^2 F_x)\,\hat{\mathbf{i}} + (\nabla^2 F_y)\,\hat{\mathbf{j}} + (\nabla^2 F_z)\,\hat{\mathbf{k}} \qquad (2.10b)$$

ii) Curl of $\vec{\nabla} f$

We can show that **curl** $\vec{\nabla} f$ is always zero:

$$\vec{\nabla} \times (\vec{\nabla} f) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z} \end{vmatrix}$$

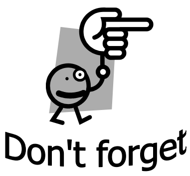
$$= \hat{\mathbf{i}}\left(\frac{\partial}{\partial y}\frac{\partial f}{\partial z} - \frac{\partial}{\partial z}\frac{\partial f}{\partial y}\right) + \hat{\mathbf{j}}\left(\frac{\partial}{\partial z}\frac{\partial f}{\partial x} - \frac{\partial}{\partial x}\frac{\partial f}{\partial z}\right) + \hat{\mathbf{k}}\left(\frac{\partial}{\partial x}\frac{\partial f}{\partial y} - \frac{\partial}{\partial y}\frac{\partial f}{\partial x}\right)$$

From the theory of partial derivatives we know that $\dfrac{\partial}{\partial y}\dfrac{\partial f}{\partial x} = \dfrac{\partial}{\partial x}\dfrac{\partial f}{\partial y}$ $\dfrac{\partial}{\partial z}\dfrac{\partial f}{\partial x} = \dfrac{\partial}{\partial x}\dfrac{\partial f}{\partial z}$, and so on. Therefore, it follows that

$$\vec{\nabla} \times (\vec{\nabla} f) = 0 \qquad (2.10c)$$

You may think that $\vec{\nabla} \times (\vec{\nabla} f) = 0$ is an obvious result: Isn't it just $(\vec{\nabla} \times \vec{\nabla}) f$, and the cross product of a vector with itself is zero. This reasoning is **not correct**. This is because $\vec{\nabla}$ is an operator and does not multiply in the usual way. The proof of Eq. (2.10c), in fact, depends on the relation $\dfrac{\partial}{\partial y}\dfrac{\partial f}{\partial x} = \dfrac{\partial}{\partial x}\dfrac{\partial f}{\partial y}$, etc.

There are some vector fields like the inverse square force field $\vec{\mathbf{F}}$ (e.g., the gravitational force field or the electrostatic force field) which can be expressed as the gradient of the scalar field $\phi = k(x^2 + y^2 + z^2)^{-1/2}$. For such fields, using the identity (2.10c) you can see that $\vec{\nabla} \times \vec{\mathbf{F}} = \vec{\mathbf{0}}$. **Such vector fields with zero curl can be expressed as gradients of scalar fields and are called conservative fields.**



> For a conservative vector field $\vec{\mathbf{F}}$, $\vec{\nabla} \times \vec{\mathbf{F}} = \vec{\mathbf{0}}$ everywhere.

iii) Gradient of $\vec{\nabla}.\vec{\mathbf{F}}$

Using the definitions of $\vec{\nabla}$ and the divergence, we can write

$$\vec{\nabla}(\vec{\nabla}.\,\vec{\mathbf{F}}) = \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right)\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right)$$

$$= \hat{\mathbf{i}}\left(\frac{\partial^2 F_x}{\partial x^2} + \frac{\partial}{\partial x}\frac{\partial F_y}{\partial y} + \frac{\partial}{\partial x}\frac{\partial F_z}{\partial z}\right) + \hat{\mathbf{j}}\left(\frac{\partial}{\partial y}\frac{\partial F_x}{\partial x} + \frac{\partial^2 F_y}{\partial y^2} + \frac{\partial}{\partial y}\frac{\partial F_z}{\partial z}\right)$$

$$+ \hat{\mathbf{k}}\left(\frac{\partial}{\partial z}\frac{\partial F_x}{\partial x} + \frac{\partial}{\partial z}\frac{\partial F_y}{\partial y} + \frac{\partial^2 F_z}{\partial z^2}\right) \qquad (2.10d)$$

This operator has no name of its own and is called the gradient of the divergence. It appears in the wave equation of an electromagnetic wave $\vec{\mathbf{E}}$:

$$\nabla^2\vec{\mathbf{E}} - \vec{\nabla}\,(\vec{\nabla}.\,\vec{\mathbf{E}}) = -\,\mu_0\,\varepsilon_0\,\frac{\partial^2\vec{\mathbf{E}}}{\partial t^2} - \mu_0\,\frac{\partial\vec{\mathbf{J}}}{\partial t}$$

where μ_0 and ε_0 are the permeability and permittivity of free space.

Remember that $\vec{\nabla}\,(\vec{\nabla}\,.\,\vec{\mathbf{F}})$ is not the same as $\nabla^2 f$:

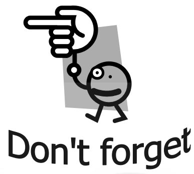
$$\nabla^2 f = \vec{\nabla}.\,(\vec{\nabla} f) \neq \vec{\nabla}(\vec{\nabla}\,.\,\vec{\mathbf{F}}) \qquad (2.10e)$$

iv) Divergence of $(\vec{\nabla}\times\vec{\mathbf{F}})$

We can show that $\vec{\nabla}.(\vec{\nabla}\times\vec{\mathbf{F}})$ is zero. (2.10f)

You can try this out for yourself in SAQ 6.

> Do not equate $\vec{\nabla}.(\vec{\nabla}\times\vec{\mathbf{F}}) = 0$ with the property $\vec{\mathbf{A}}\,.\,(\vec{\mathbf{A}}\times\vec{\mathbf{B}}) = 0$ for vectors, as $\vec{\nabla}$ is a differential operator.



v) Curl of $(\vec{\nabla}\times\vec{\mathbf{F}})$

This can be expressed as the following:

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{\mathbf{F}}) = \vec{\nabla}(\vec{\nabla}\,.\,\vec{\mathbf{F}}) - \nabla^2\vec{\mathbf{F}} \qquad (2.10g)$$

Of course, $\nabla^2\vec{\mathbf{F}}$ has the meaning as explained in this section before Eq. (2.10b).

Using Eq. (2.10g) you can express the electromagnetic wave equation as

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{\mathbf{E}}) = -\mu_0\varepsilon_0\frac{\partial^2\vec{\mathbf{E}}}{\partial t^2} - \mu_0\,\frac{\partial\vec{\mathbf{J}}}{\partial t}$$

We can apply the del operator once more to get a few more identities but this is beyond the scope of this course.

Let us understand some physical implications of what you have learnt in this section.

> You have seen in Eq. (2.10c) that the curl of the gradient of a scalar field is zero. This means that if **a vector field is irrotational or has a zero curl, you may write it as the gradient of a scalar field**. In other words, an irrotational vector field may be generated from **a scalar field alone**.
>
> Similarly, if a vector field is solenoidal, its divergence is zero. It can then be written as the curl of a vector field as you can see from Eq. (2.10f). Therefore, a solenoidal vector field can be generated from a **vector field** alone.
>
> The most general vector field, which has both a non-zero divergence and a non-zero curl, can therefore be written as the sum of a solenoidal field and an irrotational field (see margin remark).

This is also called the Helmholtz Theorem.

The magnetic field $\vec{\mathbf{B}}$ is an example of a solenoidal vector field. Since $\nabla.\vec{\mathbf{B}} = 0$ we can write, using Eq. (2.10f)

$$\vec{\mathbf{B}} = \vec{\nabla} \times \vec{\mathbf{A}} \tag{2.10h}$$

The vector field $\vec{\mathbf{A}}$ associated with the magnetic field is also called the vector potential.

You may now like to work out an SAQ on the repeated applications of the del operator.

SAQ 6 **- Successive applications of the Del Operator**

a) Verify Eq. (2.10f).

b) For a function $\phi = xyz^2$, show that $\vec{\nabla}.(\vec{\nabla}\phi) = \nabla^2\phi$.

We now summarise what you have learnt in this unit.

2.6 SUMMARY

Concept	Description
Vector field	■ A vector field is a function that assigns a vector to every point of a given region in space. A three-dimensional vector field $\vec{\mathbf{F}}$ can be written as follows: $\vec{\mathbf{F}}(x, y, z) = F_1(x, y, z)\hat{\mathbf{i}} + F_2(x, y, z)\hat{\mathbf{j}} + F_3(x, y, z)\hat{\mathbf{k}}$ The components of the vector field $\vec{\mathbf{F}}(x, y, z)$ namely $F_1(x, y, z)$, $F_2(x, y, z)$ and $F_3(x, y, z)$ are **scalar fields** defined over the same region as the vector field. A vector field $\vec{\mathbf{F}}$ in two-dimensions can be written as follows: $\vec{\mathbf{F}}(x, y) = F_1(x, y)\hat{\mathbf{i}} + F_2(x, y)\hat{\mathbf{j}}$
Divergence of a vector field	■ The divergence of a two-dimensional vector field $\vec{\mathbf{F}}(x, y) = F_1(x, y)\hat{\mathbf{i}} + F_2(x, y)\hat{\mathbf{j}}$ is defined as $\text{div } \vec{\mathbf{F}}(x, y) = \vec{\nabla} . \vec{\mathbf{F}}(x, y) = \dfrac{\partial F_1}{\partial x} + \dfrac{\partial F_2}{\partial y}$ The divergence of a three-dimensional vector field $\vec{\mathbf{F}}(x, y, z) = F_1(x, y, z)\hat{\mathbf{i}} + F_2(x, y, z)\hat{\mathbf{j}} + F_3(x, y, z)\hat{\mathbf{k}}$ is defined as $\text{div } \vec{\mathbf{F}}(x, y) = \vec{\nabla}.\vec{\mathbf{F}}(x, y) = \dfrac{\partial F_1}{\partial x} + \dfrac{\partial F_2}{\partial y} + \dfrac{\partial F_3}{\partial z}$ The divergence of a vector field gives the extent to which the vector field flow behaves like source or a sink at a given point. A non-zero value of the divergence at any point in a vector field signifies the presence of a source or a sink: $\vec{\nabla}.\vec{\mathbf{F}} > 0$ for a source and $\vec{\nabla}.\vec{\mathbf{F}} < 0$ for a sink. If the divergence of the vector field is zero, the vector field is called "divergence-free" or "solenoidal".

Curl of a vector field

- The curl of a two-dimensional vector field $\vec{\mathbf{F}}(x, y) = F_1(x, y)\,\hat{\mathbf{i}} + F_2(x, y)\,\hat{\mathbf{j}}$ is defined as

$$\vec{\nabla}\times\vec{\mathbf{F}} = \text{curl }\vec{\mathbf{F}} = \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{\mathbf{k}}$$

The curl of a three-dimensional vector field $\vec{\mathbf{F}}(x, y, z) = F_1(x, y, z)\,\hat{\mathbf{i}} + F_2(x, y, z)\,\hat{\mathbf{j}} + F_3(x, y, z)\,\hat{\mathbf{k}}$ is defined as

$$\text{curl }\vec{\mathbf{F}} = \vec{\nabla}\times\vec{\mathbf{F}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{\mathbf{k}}$$

If the curl of the vector field is zero, the vector field is called **irrotational**.

Identities involving the divergence and curl of a vector field

- For arbitrary vector fields $\vec{\mathbf{F}}$ and $\vec{\mathbf{G}}$, and a scalar field $f = f(x,y,z)$:

$$\vec{\nabla}\,.(\vec{\mathbf{F}} \pm \vec{\mathbf{G}}) = \vec{\nabla}\,.\,\vec{\mathbf{F}} \pm \vec{\nabla}\,.\,\vec{\mathbf{G}}$$

$$\vec{\nabla}\,.(k\vec{\mathbf{F}}) = k\vec{\nabla}\,.\,\vec{\mathbf{F}} \qquad \text{where } k \text{ is a constant}$$

$$\vec{\nabla}\,.(f\,\vec{\mathbf{F}}) = f\,(\vec{\nabla}\,.\,\vec{\mathbf{F}}) + \vec{\mathbf{F}}\,.\,(\vec{\nabla}f)$$

$$\vec{\nabla}\times(\vec{\mathbf{F}} \pm \vec{\mathbf{G}}) = \vec{\nabla}\times\vec{\mathbf{F}} \pm \vec{\nabla}\times\vec{\mathbf{G}}$$

$$\vec{\nabla}\times(k\vec{\mathbf{F}}) = k\,\vec{\nabla}\times\vec{\mathbf{F}} \qquad \text{where } k \text{ is a constant}$$

$$\vec{\nabla}\times(f\,\vec{\mathbf{F}}) = f\,(\vec{\nabla}\times\vec{\mathbf{F}}) - \vec{\mathbf{F}}\times(\vec{\nabla}f)$$

$$\vec{\nabla}\times(\vec{\mathbf{F}}\times\vec{\mathbf{G}}) = (\vec{\mathbf{G}}\,.\,\nabla)\vec{\mathbf{F}}) - (\vec{\mathbf{F}}\,.\,\nabla)\vec{\mathbf{G}} + \vec{\mathbf{F}}(\vec{\nabla}\,.\,\vec{\mathbf{G}}) - \vec{\mathbf{G}}(\vec{\nabla}\,.\,\vec{\mathbf{F}})$$

Successive application of the Del operator

- For an arbitrary vector field $\vec{\mathbf{F}}$ and a scalar field f

$$\vec{\nabla}.(\vec{\nabla}f) = \nabla^2 f = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)f$$

$$\vec{\nabla}\times(\vec{\nabla}f) = 0$$

$$\vec{\nabla}.(\vec{\nabla}\times\vec{\mathbf{F}}) = 0$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{\mathbf{F}}) = \vec{\nabla}(\vec{\nabla}\,.\,\vec{\mathbf{F}}) - \nabla^2\vec{\mathbf{F}}$$

2.7 TERMINAL QUESTIONS

1. Determine the divergence and curl of the following vector fields

   (i) $\vec{\mathbf{u}} = x^2y^2\hat{\mathbf{i}} - x^2y\hat{\mathbf{j}}$

   (ii) $\vec{\mathbf{u}} = \ln(x)\,\hat{\mathbf{i}} + \ln(xy)\hat{\mathbf{j}} + \ln(xyz)\hat{\mathbf{k}}$

2. Calculate $\vec{\nabla}.\dfrac{\vec{\mathbf{r}}}{r}$, given $\vec{\mathbf{r}} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$, $r = \sqrt{x^2 + y^2 + z^2}$.

3. Determine whether the vector field $\vec{\mathbf{F}} = \dfrac{\vec{\mathbf{r}}}{r^2}$ is: (i) irrotational, (ii) solenoidal.

4. Given $\vec{\mathbf{u}} = 2y\hat{\mathbf{i}} + 4z\hat{\mathbf{j}} + x^2z^2\,\hat{\mathbf{k}}$, calculate $\vec{\nabla}.\vec{\mathbf{u}}$ and $\vec{\nabla}\times\vec{\mathbf{u}}$ at the point (0, 1, 2).

5. If $\vec{\mathbf{a}}$ is a constant vector show that $\vec{\nabla}\times(\vec{\mathbf{a}}\times\vec{\mathbf{r}}) = 2\vec{\mathbf{a}}$

6, Determine the value of the constant *k* for which curl of the vector field

$$\vec{\mathbf{F}} = -\frac{y}{(x^2+y^2)^{k/2}}\hat{\mathbf{i}} + \frac{x}{(x^2+y^2)^{k/2}}\hat{\mathbf{j}}$$

is (i) positive (ii) negative and (iii) a null vector.

7. If $\vec{\mathbf{A}} = 2yz\hat{\mathbf{i}} - x^2y\hat{\mathbf{j}} + xz^2\hat{\mathbf{k}}$ and $f = xyz$ show that $(\vec{\mathbf{A}}\,.\,\vec{\nabla})f = \vec{\mathbf{A}}\,.\left(\vec{\nabla}f\right)$.

8. Determine the values of *a*, *b* and *c* such that the vector field $\vec{\mathbf{A}} = (3x - y + az)\,\hat{\mathbf{i}} + (bx + 2y + z)\,\hat{\mathbf{j}} + (x + cy - 2z)\hat{\mathbf{k}}$ is irrotational.

9. Prove that $\vec{\nabla}\,.\,(\vec{\nabla}f\times\vec{\nabla}g) = 0$.

10. Determine $\nabla^2\phi$ for (i) $\phi = \ln(x^2 + y^2)$ and (ii) $\phi = xyz(x^2 - y^2 + z^2)$

2.8 SOLUTIONS AND ANSWERS

Self-Assessment Questions

1. We choose a constant velocity in the *y*-direction. Then the velocity vector at every point in the field is the same and given by $\vec{\mathbf{v}} = a\ \hat{\mathbf{j}}$. We sketch the field in Fig. 2.9.

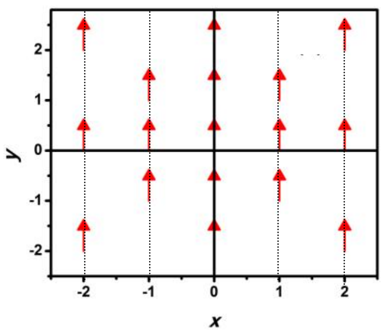


Fig. 2.9: **The vector field** $\vec{\mathbf{v}} = a\ \hat{\mathbf{j}}$

2. a) We use Eq. (2.3) to evaluate the divergence of a three dimensional field.

i) $\vec{\nabla}.\,[(x^2 - y^2)\,\hat{\mathbf{i}} + (y^2 - z^2)\,\hat{\mathbf{j}} + (z^2 - x^2)\hat{\mathbf{k}}]$

$$= \frac{\partial}{\partial x}(x^2 - y^2) + \frac{\partial}{\partial y}(y^2 - z^2) + \frac{\partial}{\partial z}(z^2 - x^2)$$

$$= 2x + 2y + 2z$$

ii) $\vec{\nabla}.(y^2 z\hat{\mathbf{i}} + xy^3\hat{\mathbf{j}} - z^2\hat{\mathbf{k}})$

$$= \frac{\partial}{\partial x}(y^2 z) + \frac{\partial}{\partial y}(xy^3) - \frac{\partial}{\partial z}(z^2)$$

$$= 3y^2 x - 2z$$

b) For a vector field to be solenoidal, its divergence has to be zero. Imposing this condition on the given vector field we can write

$$\vec{\nabla}.[(x+3y)\hat{\mathbf{i}} + (y+2z)\hat{\mathbf{j}} + (x+az)\hat{\mathbf{k}}] = 0$$

$$\Rightarrow \frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y+2z) + \frac{\partial}{\partial z}(x+az) = 0$$

$$\Rightarrow 1+1+a = 0 \quad \text{or} \quad a = -2.$$

For the value of $a = -2$ the divergence of the vector field is zero and the field is solenoidal.

3. We use Eq. (2.6c) with $\vec{\mathbf{F}} = \vec{\mathbf{r}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ to obtain the result.

$$\vec{\nabla}.(f\,\vec{\mathbf{r}}) = f\,(\vec{\nabla}.\vec{\mathbf{r}}) + \vec{\mathbf{r}}.(\vec{\nabla}f) \qquad \text{(i)}$$

$$\vec{\nabla}.\vec{\mathbf{r}} = \vec{\nabla}.(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})$$

$$= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1+1+1 = 3 \qquad \text{(ii)}$$

Substituting Eq. (ii) in Eq. (i) we get

$$\vec{\nabla}.(f\vec{\mathbf{r}}) = 3f + \vec{\mathbf{r}}.\vec{\nabla}f$$

4. a) Using Eq. (2.7a) with $F_1 = 2x - y$, $F_2 = -2yz^2$ and $F_3 = -2zy^2$, we get :

$$\vec{\nabla} \times \vec{\mathbf{F}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x - y & -2yz^2 & -2zy^2 \end{vmatrix}$$

$$= \hat{\mathbf{i}}\,(-4yz + 4yz) + \hat{\mathbf{j}}\,(0-0) + \hat{\mathbf{k}}\,(1) = \hat{\mathbf{k}}$$

b) We use Eq. (2.7a) with $F_1 = z\cos x, F_2 = y + \sin x$ and $F_3 = xyz$.

$$\therefore \qquad \vec{\nabla} \times \vec{\mathbf{F}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z\cos x & y + \sin x & xyz \end{vmatrix}$$

$$= \hat{\mathbf{i}}[xz-0]+\hat{\mathbf{j}}[\cos x - yz] + \hat{\mathbf{k}}[\cos x - 0]$$

$$= xz\hat{\mathbf{i}} + (\cos x - yz)\hat{\mathbf{j}} + \cos x\,\hat{\mathbf{k}}$$

5. a) To show that the vector field $\vec{\mathbf{u}}\times\vec{\mathbf{v}}$ is solenoidal we must prove that the divergence of $\vec{\mathbf{u}}\times\vec{\mathbf{v}}$ is zero or $\vec{\nabla}.(\vec{\mathbf{u}}\times\vec{\mathbf{v}})=0$.

   It is given that the vector fields $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are irrotational, so $\vec{\nabla}\times\vec{\mathbf{u}}=\vec{\mathbf{0}}$ and $\vec{\nabla}\times\vec{\mathbf{v}}=\vec{\mathbf{0}}$. Using Eq. (2.9e) with $\vec{\mathbf{F}}=\vec{\mathbf{u}}$ and $\vec{\mathbf{G}}=\vec{\mathbf{v}}$ we get:

$$\vec{\nabla}.(\vec{\mathbf{u}}\times\vec{\mathbf{v}})=\vec{\mathbf{v}}.(\vec{\nabla}\times\vec{\mathbf{u}})-\vec{\mathbf{u}}.(\vec{\nabla}\times\vec{\mathbf{v}})=0 \text{ since } \vec{\nabla}\times\vec{\mathbf{u}}=\vec{\mathbf{0}} \text{ and } \vec{\nabla}\times\vec{\mathbf{v}}=\vec{\mathbf{0}}$$

   b) Since $g\vec{\mathbf{f}}$ is irrotational $\vec{\nabla}\times(g\vec{\mathbf{f}})=\vec{\mathbf{0}}$. So, using Eq. (2.9c)

$$\vec{\nabla}\times(g\vec{\mathbf{f}})=g(\vec{\nabla}\times\vec{\mathbf{f}})-\vec{\mathbf{f}}\times(\vec{\nabla}g)=\vec{\mathbf{0}}$$

$$\Rightarrow\quad g(\vec{\nabla}\times\vec{\mathbf{f}})=\vec{\mathbf{f}}\times(\vec{\nabla}g) \qquad \text{(i)}$$

   Taking the scalar product of Eq. (i) with $\vec{\mathbf{f}}$, we have

$$g\vec{\mathbf{f}}.(\vec{\nabla}\times\vec{\mathbf{f}})=\vec{\mathbf{f}}.[\vec{\mathbf{f}}\times\vec{\nabla}g] \qquad \text{(ii)}$$

   Since a scalar triple product of the kind $\vec{\mathbf{a}}.(\vec{\mathbf{a}}\times\vec{\mathbf{b}})$ is always zero, the RHS of Eq. (ii) is zero. Hence $\vec{\mathbf{f}}.(\vec{\nabla}\times\vec{\mathbf{f}})=0$

6. a) With $\vec{\mathbf{F}}=F_1\,\hat{\mathbf{i}}+F_2\,\hat{\mathbf{j}}+F_3\,\hat{\mathbf{k}}$, we can write using Eq. (2.7b)

$$\vec{\nabla}\times\vec{\mathrm{F}} = \left(\frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z}\right)\hat{\mathbf{i}}+\left(\frac{\partial F_1}{\partial z}-\frac{\partial F_3}{\partial x}\right)\hat{\mathbf{j}}+\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)\hat{\mathbf{k}}$$

$$\therefore\ \vec{\nabla}\cdot(\vec{\nabla}\times\vec{\mathrm{F}})=\frac{\partial}{\partial x}\left(\frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z}\right)+\frac{\partial}{\partial y}\left(\frac{\partial F_1}{\partial z}-\frac{\partial F_3}{\partial x}\right)+\frac{\partial}{\partial z}\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)$$

$$=\frac{\partial^2 F_3}{\partial x\partial y}-\frac{\partial^2 F_2}{\partial x\partial z}+\frac{\partial^2 F_1}{\partial y\partial z}-\frac{\partial^2 F_3}{\partial y\partial x}+\frac{\partial^2 F_2}{\partial z\partial x}-\frac{\partial^2 F_1}{\partial z\partial y}=0$$

$$\because\ \frac{\partial^2 F_3}{\partial x\partial y}=\frac{\partial^2 F_3}{\partial y\partial x},\ \frac{\partial^2 F_2}{\partial x\partial z}=\frac{\partial^2 F_2}{\partial z\partial x} \text{ and } \frac{\partial^2 F_1}{\partial y\partial z}=\frac{\partial^2 F_1}{\partial z\partial y}$$

   b) We first determine $\vec{\nabla}\phi=\vec{\mathbf{F}}$. Using Eq. (1.11a) for the gradient of a scalar function with $f=\phi=xyz^2$ we can write:

$$\vec{\mathbf{F}}=\vec{\nabla}\phi=\frac{\partial}{\partial x}(xyz^2)\,\hat{\mathbf{i}}+\frac{\partial}{\partial y}(xyz^2)\hat{\mathbf{j}}+\frac{\partial}{\partial z}(xyz^2)\hat{\mathbf{k}}$$

$$=yz^2\hat{\mathbf{i}}+xz^2\hat{\mathbf{j}}+2xyz\hat{\mathbf{k}}$$

   Next we find $\vec{\nabla}.\vec{\mathbf{F}}$ using Eq. (2.3):

$$\vec{\nabla}.\vec{\mathbf{F}}=\vec{\nabla}.(\vec{\nabla}\phi)=\frac{\partial}{\partial x}(yz^2)+\frac{\partial}{\partial y}(xz^2)+\frac{\partial}{\partial z}(2xyz)=2xy \qquad \text{(i)}$$

We next determine $\nabla^2\phi$:

$$\nabla^2\phi = \frac{\partial}{\partial x^2}(xyz^2) + \frac{\partial^2}{\partial y^2}(xyz^2) + \frac{\partial^2}{\partial z^2}(xyz^2) \qquad \text{(ii)}$$

We now calculate the following partial derivatives:

$$\frac{\partial}{\partial x}(xyz^2) = yz^2;\ \frac{\partial^2}{\partial x^2}(xyz^2) = \frac{\partial}{\partial x}(yz^2) = 0 \qquad \text{(iii)}$$

$$\frac{\partial}{\partial y}(xyz^2) = xz^2;\ \frac{\partial^2}{\partial y^2}(xyz^2) = \frac{\partial}{\partial y}(xz^2) = 0. \qquad \text{(iv)}$$

and
$$\frac{\partial}{\partial z}(xyz^2) = 2zxy; \frac{\partial^2}{\partial z^2}(xyz^2) = \frac{\partial}{\partial z}(2xyz) = 2xy \qquad \text{(v)}$$

Substituting from Eqs. (iii), (iv) and (v) in Eq. (ii) we get

$$\nabla^2\phi = 0 + 0 + 2xy = 2xy \qquad \text{(vi)}$$

Comparing Eqs. (i) and (vi), we can see that

$$\vec{\nabla}.(\vec{\nabla}\phi) = \nabla^2\phi = 2xy$$

Terminal Questions

1. i) We use Eq. (2.4) with $F_1 = x^2y^2,\ F_2 = -x^2y$

$$\vec{\nabla}.\ \vec{\mathbf{F}} = \frac{\partial}{\partial x}(x^2y^2) + \frac{\partial}{\partial y}(-x^2y) = 2xy^2 - x^2$$

Using Eq. (2.8) for the curl we get:

$$\vec{\nabla} \times \vec{\mathbf{F}} = \hat{\mathbf{k}}\left[\frac{\partial}{\partial x}(-x^2y) - \frac{\partial}{\partial y}(x^2y^2)\right] = (-2xy - 2x^2y)\ \hat{\mathbf{k}}$$

ln $(x\,y)$ = ln x + ln y

ln (xyz) = ln x + ln y + ln z

$$\frac{\partial}{\partial x}(\ln x) = \frac{1}{x}$$

ii) We use Eq. (2.3) with $F_1 = \ln x, F_2 = \ln xy$ and $F_3 = \ln xyz$ to get

$$\vec{\nabla}.\ \vec{\mathbf{F}} = \frac{\partial}{\partial x}(\ln x) + \frac{\partial}{\partial y}(\ln x + \ln y) + \frac{\partial}{\partial z}(\ln x + \ln y + \ln z) = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}.$$

Using Eq. (2.7a) for the curl we get:

$$\vec{\nabla} \times \vec{\mathbf{F}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \ln x & \ln x + \ln y & \ln x + \ln y + \ln z \end{vmatrix}$$

$$= \hat{\mathbf{i}}\left(\frac{1}{y}\right) + \hat{\mathbf{j}}\left(-\frac{1}{x}\right) + \hat{\mathbf{k}}\left(\frac{1}{x}\right)$$

2. Here the given field is $\vec{\mathbf{F}} = \dfrac{x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}}{r}$ where $r = (x^2 + y^2 + z^2)^{1/2}$

$$\therefore \vec{\nabla}.\vec{\mathbf{F}} = \vec{\nabla}.\left[\frac{x}{(x^2+y^2+z^2)^{1/2}}\hat{\mathbf{i}} + \frac{y}{(x^2+y^2+z^2)^{1/2}}\hat{\mathbf{j}} + \frac{z}{(x^2+y^2+z^2)^{1/2}}\hat{\mathbf{k}}\right]$$

$$= \frac{\partial}{\partial x}\left[\frac{x}{(x^2+y^2+z^2)^{1/2}}\right] + \frac{\partial}{\partial y}\left[\frac{y}{(x^2+y^2+z^2)^{1/2}}\right] + \frac{\partial}{\partial z}\left[\frac{z}{x^2+y^2+z^2)^{1/2}}\right] \quad \text{(i)}$$

Differentiate this as the product of two functions, as you have learnt to do for ordinary derivatives in your school calculus course. If you are still not clear about partial derivatives, please read the Appendix of Unit 1.

Let us evaluate each of the three partial derivatives separately.

$$\frac{\partial}{\partial x}\left[\frac{x}{(x^2+y^2+z^2)^{1/2}}\right] = \frac{1}{(x^2+y^2+z^2)^{1/2}} - \frac{x^2}{(x^2+y^2+z^2)^{3/2}} \quad \text{(See MR)}$$

$$= \frac{(y^2+z^2)}{(x^2+y^2+z^2)^{3/2}} \quad \text{(ii)}$$

Similarly,

$$\frac{\partial}{\partial y}\left[\frac{y}{(x^2+y^2+z^2)^{1/2}}\right] = \frac{(x^2+z^2)}{(x^2+y^2+z^2)^{3/2}} \quad \text{(iii)}$$

and

$$\frac{\partial}{\partial z}\left[\frac{z}{(z^2+y^2+z^2)^{1/2}}\right] = \left[\frac{(x^2+y^2)}{(x^2+y^2+z^2)^{3/2}}\right] \quad \text{(iv)}$$

Substituting from Eqs. (ii), (iii) and (iv) in Eq. (i) we get

$$\vec{\nabla}.\vec{\mathbf{F}} = \frac{2(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{3/2}} = \frac{2}{(x^2+y^2+{}^2)^{1/2}} = \frac{2}{r}$$

3. The field $\vec{\mathbf{F}}$ is written in Cartesian coordinates as

$$\vec{\mathbf{F}} = \frac{x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}}{(x^2+y^2+z^2)}$$

i) For an irrotational field $\vec{\nabla} \times \vec{\mathbf{F}} = \vec{\mathbf{0}}$. Using Eq. (2.7a) we evaluate $\vec{\nabla} \times \vec{\mathbf{F}}$ as:

$$\vec{\nabla} \times \vec{\mathbf{F}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{x}{(x^2+y^2+z^2)} & \dfrac{y}{(x^2+y^2+z^2)} & \dfrac{z}{(x^2+y^2+z^2)} \end{vmatrix}$$

$$= \hat{\mathbf{i}}\left[\frac{\partial}{\partial y}\left(\frac{z}{x^2+y^2+z^2}\right) - \frac{\partial}{\partial z}\left(\frac{y}{x^2+y^2+z^2}\right)\right]$$

$$+ \hat{\mathbf{j}}\left[\frac{\partial}{\partial z}\left(\frac{x}{x^2+y^2+z^2}\right) - \frac{\partial}{\partial x}\left(\frac{z}{x^2+y^2+z^2}\right)\right]$$

$$+ \hat{\mathbf{k}}\left[\frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2+z^2}\right) - \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2+z^2}\right)\right]$$

$$= \hat{\mathbf{i}}\left[\frac{-2yz}{(x^2+y^2+z^2)^2}+\frac{2yz}{(x^2+y^2+z^2)^2}\right]$$

$$+\hat{\mathbf{j}}\left[\frac{-2xz}{(x^2+y^2+z^2)^2}+\frac{2xz}{(x^2+y^2+z^2)^2}\right]$$

$$+\hat{\mathbf{k}}\left[\frac{-2xy}{(x^2+y^2+z^2)^2}+\frac{2xy}{(x^2+y^2+z^2)^2}\right]$$

$$= \vec{\mathbf{0}}$$

So, the vector filed $\vec{\mathbf{F}}$ is irrotational at all points (because $\vec{\nabla}\times\vec{\mathbf{F}}$ is always zero), **except at the origin**. The field is **not defined** at the origin.

ii) To find whether the field is solenoidal, we must calculate $\vec{\nabla}.\vec{\mathbf{F}}$. Using Eq. (2.3) we get:

$$\vec{\nabla}.\vec{\mathbf{F}} = \frac{\partial}{\partial x}\left[\frac{x}{x^2+y^2+z^2}\right]+\frac{\partial}{\partial y}\left[\frac{y}{x^2+y^2+z^2}\right]+\frac{\partial}{\partial z}\left[\frac{z}{x^2+y^2+z^2}\right]$$

$$=\left[\frac{1}{(x^2+y^2+z^2)}-\frac{2x^2}{(x^2+y^2+z^2)^2}\right]$$

$$+\left[\frac{1}{(x^2+y^2+z^2)}-\frac{2y^2}{(x^2+y^2+z^2)^2}\right]$$

$$+\left[\frac{1}{(x^2+y^2+z^2)}-\frac{2z^2}{(x^2+y^2+z^2)^2}\right]$$

$$=\frac{3}{(x^2+y^2+z^2)}-\frac{2(x^2+y^2+z^2)}{(x^2+y^2+z^2)^2}$$

$$=\frac{3}{\left(x^2+y^2+z^2\right)}-\frac{2}{\left(x^2+y^2+z^2\right)}$$

$$=\frac{1}{(x^2+y^2+z^2)}$$

Since the value of $\vec{\nabla}.\vec{\mathbf{F}}$ is not zero, the field is not solenoidal.

4. To evaluate the divergence of the vector field $\vec{\mathbf{u}} = 2y\,\hat{\mathbf{i}} + 4z\,\hat{\mathbf{j}} + x^2z^2\,\hat{\mathbf{k}}$ we use Eq. (2.3)

$$\vec{\nabla}.\,\vec{\mathbf{u}} = \frac{\partial}{\partial x}(2y)+\frac{\partial}{\partial y}(4z)+\frac{\partial}{\partial z}\left(x^2z^2\right)$$

$$= 0+0+2x^2z = 2x^2z$$

At point (0, 1, 2), $\vec{\nabla}.\,\vec{\mathbf{u}} = 0$.

To evaluate $\vec{\nabla}\times\vec{\mathbf{u}}$ we use Eq. (2.7a) as follows:

$$\vec{\nabla}\times\vec{\mathrm{u}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y & 4z & x^2z^2 \end{vmatrix} = \hat{\mathbf{i}}(-4)+\hat{\mathbf{j}}\left(-2xz^2\right)+\hat{\mathbf{k}}(-2)$$

$$= -4\hat{\mathbf{i}} - 2xz^2\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$$

At point (0, 1, 2)

$$\vec{\nabla}\times\vec{\mathbf{u}} = -4\,\hat{\mathbf{i}} - 2\hat{\mathbf{k}}$$

5. We can write the vector $\vec{\mathbf{a}}$ as $\vec{\mathbf{a}} = a_1\,\hat{\mathbf{i}} + a_2\,\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$ and the position vector $\vec{\mathbf{r}}$ as $\vec{\mathbf{r}} = x\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$. We first calculate the cross product $\vec{\mathbf{a}}\times\vec{\mathbf{r}}$. Using Eq. (2.21b) from Unit 2, BPHCT-131, we get,

$$\vec{\mathbf{a}}\times\vec{\mathbf{r}} = \hat{\mathbf{i}}\left(a_2 z - a_3 y\right) + \hat{\mathbf{j}}\left(a_3 x - a_1 z\right) + \hat{\mathbf{k}}\left(a_1 y - a_2 x\right)$$

Using Eq. (2.7a) for the curl we get:

$$\vec{\nabla}\times(\vec{\mathbf{a}}\times\vec{\mathbf{r}}) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ a_2 z - a_3 y & a_3 x - a_1 z & a_1 y - a_2 x \end{vmatrix}$$

$$= 2a_1\hat{\mathbf{i}} + 2a_2\hat{\mathbf{j}} + 2a_3\hat{\mathbf{k}} = 2\vec{\mathbf{a}}$$

6. We use Eq. (2.8) with:

$$F_1 = \frac{-y}{(x^2+y^2)^{k/2}} \quad ; \; F_2 = \frac{x}{(x^2+y^2)^{k/2}}$$

$$\therefore\; \vec{\nabla}\times\vec{\mathbf{F}} = \hat{\mathbf{k}}\left[\frac{\partial}{\partial x}\left(\frac{x}{(x^2+y^2)^{k/2}}\right) + \frac{\partial}{\partial y}\left(\frac{y}{(x^2+y^2)^{k/2}}\right)\right]$$

$$= \hat{\mathbf{k}}\left[\frac{1}{(x^2+y^2)^{k/2}} - \frac{kx^2}{(x^2+y^2)^{(k/2)+1}} + \frac{1}{(x^2+y^2)^{k/2}} - \frac{ky^2}{(x^2+y^2)^{(k/2)+1}}\right]$$

$$= \hat{\mathbf{k}}\left[\frac{2}{(x^2+y^2)^{k/2}} - \frac{k(x^2+y^2)}{(x^2+y^2)^{(k/2)+1}}\right] = \frac{(2-k)}{(x^2+y^2)^{k/2}}\hat{\mathbf{k}}$$

Now, let us find the values of *k* for which

(i) Curl $\hat{\mathbf{F}}$ is positive: $\vec{\nabla}\times\vec{\mathbf{F}} > 0 \Rightarrow 2-k > 0 \;\Rightarrow k < 2$

(ii) Curl $\vec{\mathbf{F}}$ is negative: $\vec{\nabla}\times\vec{\mathbf{F}} < 0 \Rightarrow 2-k < 0 \;\Rightarrow k > 2$

(iii) Curl $\vec{\mathbf{F}}$ is zero: $\nabla\times\vec{\mathbf{F}} = 0 \Rightarrow 2-k = 0 \Rightarrow k = 2$

7. Let us write an expression for $(\vec{\mathbf{A}}.\vec{\nabla})$, using the rules of the scalar product.

$$\vec{\mathbf{A}}.\vec{\nabla} = \left[2yz\,\hat{\mathbf{i}} - x^2y\hat{\mathbf{j}} + xz^2\hat{\mathrm{k}}\right]\cdot\left[\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right]$$

$$= \left[2yz\frac{\partial}{\partial x} - x^2y\frac{\partial}{\partial y} + xz^2\frac{\partial}{\partial z}\right]$$

This is now an operator and can act of on the scalar field *f*.

$$(\vec{\mathbf{A}}.\vec{\nabla})f = \left[2yz\frac{\partial}{\partial x} - x^2y\frac{\partial}{\partial y} + xz^2\frac{\partial}{\partial z}\right]\;[xyz]$$

$$= \; 2yz\frac{\partial}{\partial x}(xyz) - x^2y\frac{\partial}{\partial y}(xyz) + xz^2\frac{\partial}{\partial z}(xyz)$$

$$= 2yz(yz) - x^2y\,(xz) + xz^2(xy)$$

$$= 2y^2z^2 - x^3yz + x^2yz^2 \qquad \text{(i)}$$

Next we evaluate $\vec{\mathbf{A}}.\left(\vec{\nabla} f\right)$:

We first determine $\vec{\nabla} f$ using Eq. (1.11a).

$$\vec{\nabla} f = \hat{\mathbf{i}}\frac{\partial}{\partial x}(xyz) + \hat{\mathbf{j}}\frac{\partial}{\partial y}(xyz) + \hat{\mathbf{k}}\frac{\partial}{\partial z}(xyz)$$

$$= yz\,\hat{\mathbf{i}} + xz\hat{\mathbf{j}} + xy\hat{\mathbf{k}}.$$

$\vec{\nabla} f$ is a vector. So $\vec{\mathbf{A}}.(\vec{\nabla} f)$ is evaluated as a scalar product as follows:

$$\vec{\mathbf{A}}.(\vec{\nabla} f) = (2yz\hat{\mathbf{i}} - x^2y\hat{\mathbf{j}} + xz^2\hat{\mathbf{k}}).(yz\hat{\mathbf{i}} + xz\hat{\mathbf{j}} + xy\hat{\mathbf{k}}) \qquad \text{(ii)}$$

$$= 2y^2z^2 - x^3yz + x^2yz^2.$$

Comparing Eqs. (i) and (ii) we can see that :

$$(\vec{\mathbf{A}}.\vec{\nabla})f = \vec{\mathbf{A}}.(\vec{\nabla} f)$$

8. Let us first find $\vec{\nabla} \times \vec{\mathbf{A}}$ using Eq. (2.7a):

$$\vec{\nabla} \times \vec{\mathbf{A}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 3x - y + az & bx + 2y + z & x + cy - 2z \end{vmatrix}$$

$$= (c-1)\,\hat{\mathbf{i}} + (a-1)\,\hat{\mathbf{j}} + (b+1)\,\hat{\mathbf{k}} \qquad \text{(i)}$$

For $\vec{\mathbf{A}}$ to be irrotational, $\vec{\nabla} \times \vec{\mathbf{A}} = \vec{\mathbf{0}}$. In other words, each component of $\vec{\nabla} \times \vec{\mathbf{A}}$ is zero. So from Eq. (i) we can write:

$$(c-1) = 0,\ (a-1) = 0 \text{ and } (b+1) = 0$$

which gives us the values for a, b, c as:

$$a = 1, b = -1 \text{ and } c = 1$$

9. Let us write $\vec{\mathbf{A}} = \vec{\nabla} f$ and $\vec{\mathbf{B}} = \vec{\nabla} g$.

Then $\vec{\nabla}.(\vec{\nabla} f \times \vec{\nabla} g) = \vec{\nabla}.(\vec{\mathbf{A}} \times \vec{\mathbf{B}})$

Using Eq. (2.9e) with $\vec{\mathbf{F}} = \vec{\mathbf{A}}$ and $\vec{\mathbf{G}} = \vec{\mathbf{B}}$ we get,

$$\vec{\nabla}.(\vec{\mathbf{A}} \times \vec{\mathbf{B}}) = \vec{\mathbf{B}}.(\nabla \times \vec{\mathbf{A}}) - \vec{\mathbf{A}}.(\nabla \times \vec{\mathbf{B}}) \qquad \text{(i)}$$

Replacing $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ by $\vec{\nabla} f$ and $\vec{\nabla} g$, respectively, in Eq. (i) we get

$$\vec{\nabla}.\,(\vec{\nabla} f \times \vec{\nabla} g) = \vec{\nabla} g.\,[\vec{\nabla} \times (\vec{\nabla} f)] - \vec{\nabla} f.[\vec{\nabla} \times (\vec{\nabla} g)] \qquad \text{(ii)}$$

But we already know that the curl of the gradient of scalar field is zero. So

$$\vec{\nabla} \times (\vec{\nabla} f) = \vec{\nabla} \times (\vec{\nabla} g) = 0 \qquad \text{(iii)}$$

Replacing from Eq. (iii) in Eq. (ii) we get: $\vec{\nabla}.(\vec{\nabla} f \times \vec{\nabla} g) = 0.$

10. i) We have defined the operator ∇^2 in Eq. (2.10a). Using that we can write:

$$\nabla^2\phi = \frac{\partial^2}{\partial x^2}\left[\ln(x^2+y^2)\right] + \frac{\partial^2}{\partial y^2}\left[\ln(x^2+y^2)\right] \qquad \text{(i)}$$

We first evaluate all the partial derivatives in Eq. (i).

$$\frac{\partial}{\partial x}\left[\ln(x^2+y^2)\right] = \frac{2x}{x^2+y^2}$$

$$\frac{\partial^2}{\partial x^2}\left[\ln(x^2+y^2)\right] = \frac{\partial}{\partial x}\left[\frac{2x}{x^2+y^2}\right] = \frac{2}{(x^2+y^2)} - \frac{4x^2}{(x^2+y^2)^2} \qquad \text{(ii)}$$

Similarly,

$$\frac{\partial}{\partial y}\left[\ln(x^2+y^2)\right] = \frac{2y}{x^2+y^2}$$

$$\frac{\partial^2}{\partial y^2}\left[\ln(x^2+y^2)\right] = \frac{\partial}{\partial y}\left[\frac{2y}{x^2+y^2}\right] = \frac{2}{(x^2+y^2)} - \frac{4y^2}{(x^2+y^2)^2} \qquad \text{(iii)}$$

Substituting from Eqs. (ii) and (iii) into Eq. (i) we get:

$$\nabla^2\phi = \frac{4}{(x^2+y^2)} - \frac{4(x^2+y^2)}{(x^2+y^2)^2}$$

$$= \frac{4}{(x^2+y^2)} - \frac{4}{(x^2+y^2)} = 0$$

ii) We first rewrite the function ϕ as

$$\phi = x^3yz - xy^3z + xyz^3$$

Then

$$\nabla^2\phi = \frac{\partial^2}{\partial x^2}\left[x^3yz - xy^3z + xyz^3\right] + \frac{\partial^2}{\partial y^2}\left[x^3yz - xy^3z + xyz^3\right]$$

$$+ \frac{\partial^2}{\partial z^2}\left[x^3yz - xy^3z + xyz^3\right] \qquad \text{(i)}$$

Next we evaluate all the partial derivative of ϕ in Eq. (i).

$$\frac{\partial\phi}{\partial x} = 3x^2yz - y^3z + yz^3, \quad \frac{\partial^2\phi}{\partial x^2} = 6xyz \qquad \text{(ii)}$$

$$\frac{\partial\phi}{\partial y} = x^3z - 3y^2xz + xz^3, \frac{\partial^2\phi}{\partial y^2} = -6xyz \qquad \text{(iii)}$$

$$\frac{\partial\phi}{\partial z} = x^3y - xy^3 + 3xyz^2, \frac{\partial^2\phi}{\partial z^2} = 6xyz \qquad \text{(iv)}$$

Substituting the second order partial derivatives from Eqs. (ii), (iii) and (iv) in Eq. (i), we get:

$$\therefore \quad \nabla^2\phi = 6xyz - 6xyz + 6xyz = 6xyz$$