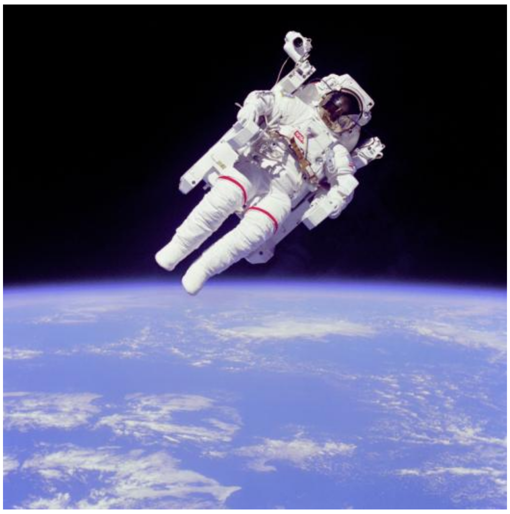


How do we determine the work done by a variable force such as the force of gravitation? We need to solve line integrals.

UNIT 3

INTEGRATION OF VECTOR FUNCTIONS AND LINE INTEGRALS

Structure

STUDY GUIDE

In this unit, you will learn how to integrate vector functions of a scalar variable and solve line integrals. Line integrals are a generalization of ordinary integrals that you have studied in school. In order to learn these concepts better, you should revise integral calculus that you have studied in school. You must also revise the concepts of scalar and vector products, the basic concepts of vector functions of a scalar variable and how to differentiate them, all of which you have studied in Unit 2 of BPHCT-131.

"The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift ..."

Eugene Paul Wigner

3.1 INTRODUCTION

In Unit 2 of BPHCT-131 and Units 1 and 2 of this course, you have studied vector functions, scalar and vector fields, and their properties. You have learnt how to differentiate vector functions and scalar and vector fields. You have studied the concepts of the gradient of a scalar field, and the divergence and curl of vector fields. These are differential operations on scalar and vector fields that find many applications in physics. In this unit, you will learn how to determine the integrals of vector functions, and scalar and vector fields. You will also learn how to evaluate line integrals of vector fields.

There are several problems in physics where we need to calculate the integrals of vector functions and vector fields. For example, we may want to know what path a cricket ball will take after it leaves the bowler's hands with a given acceleration. Finding the path of the cricket ball involves solving a differential equation and integrating vector functions. The actual integration is essentially the same as in ordinary calculus which you have studied as a part of your school curriculum. However, integrals of vector functions and fields are different in the way in which the integrand is handled, as well as in the physical meanings of the quantities obtained. This will become clear as you study this unit.

In Sec. 3.2, you will learn how to integrate a vector function and apply it to solve some simple problems in physics. In this section you will also learn how to integrate the scalar and vector products of vector functions and some applications in physics.

In this unit you will learn how to evaluate line integrals. The line integral is a generalization of an ordinary integral over a single variable. In a line integral the path of integration is not a straight line but an arbitrary curve in space. Line integrals are used extensively in physics. One of the most important applications of the line integral is to determine the work done by a variable force. Suppose an object moves along an arbitrary curve in space, (instead of a straight line) under the action of a force. How would you calculate the work done by the force in moving the object between any two points on this path? The work done is the integral of the scalar product of the force field and an infinitesimal displacement along the path of the object. This is an example of a line integral.

In Sec. 3.3, you will learn how to evaluate line integrals in which the integrand is the scalar product of a vector field and a displacement along an arbitrary path in space. You will also study other types of line integrals of scalar and vector fields. In Sec. 3.4, you will study about conservative vector fields. You will see that line integrals can be used to define conservative force fields, an important concept in physics.

The integrals of vector functions being taken up in this unit involve integration over a single variable. In physics we often need to evaluate integrals over arbitrary surfaces and volumes. These involve integrals over two and three variables. In Unit 4, you will study about surface and volume integrals of a vector field. A brief introduction to integration over two variables is given in

Appendix A2 of this block. You should read Appendix A2 after completing your study of this unit.

Expected Learning Outcomes

After studying this unit, you should be able to:

- evaluate the integral of a vector function with respect to a scalar;
- evaluate the integrals of scalar and vector products of scalar functions; and
- evaluate line integrals of scalar and vector fields.

3.2 INTEGRATION OF A VECTOR FUNCTION

Let us begin our study by asking: **How do we integrate a vector function with respect to a scalar?**

We lay down the basic rules for the integration of a vector function with respect to a scalar. Consider a vector $\vec{\mathbf{a}}$ which is a function of a scalar *t*. Let

$$\vec{\mathbf{a}} = \vec{\mathbf{a}}(t) = a_1(t)\hat{\mathbf{i}} + a_2(t)\hat{\mathbf{j}} + a_3(t)\hat{\mathbf{k}} \tag{3.1a}$$

where $a_1(t)$, $a_2(t)$ and $a_3(t)$ are the *x*, *y* and *z* components of $\vec{\mathbf{a}}(t)$, respectively. If

$$\frac{d\vec{\mathbf{a}}}{dt} = \vec{\mathbf{b}}(t) \tag{3.1b}$$

then the (indefinite) integral of $\vec{\mathbf{b}}(t)$ with respect to *t* is $\vec{\mathbf{a}}(t) + \vec{\mathbf{c}}$, where $\vec{\mathbf{c}}$ is an arbitrary constant vector. Symbolically, we write:

$$\int \vec{\mathbf{b}}(t)\,dt = \vec{\mathbf{a}}(t) + \vec{\mathbf{c}} \tag{3.2}$$

In physics, we deal with quantities that generally have dimensions. Therefore, $\vec{\mathbf{c}}$ is a vector whose dimension is the same as that of $\vec{\mathbf{a}}$. In a physical problem, $\vec{\mathbf{c}}$ can be determined by using given initial conditions.

You have studied integration in school and you know that integration is the reverse process of differentiation. This is also true for the integration of vector functions relative to a scalar.

In order to evaluate the integral of a vector function such as the one in Eq. (3.2), we express the vector $\vec{\mathbf{b}}$ in its component form:

$$\vec{\mathbf{b}}(t) = b_1(t)\hat{\mathbf{i}} + b_2(t)\hat{\mathbf{j}} + b_3(t)\hat{\mathbf{k}} \tag{3.3}$$

where $b_1(t)$, $b_2(t)$ and $b_3(t)$ are the *x*, *y* and *z* components of $\vec{\mathbf{b}}(t)$, respectively. We can now write the integral of the vector function $\vec{\mathbf{b}}(t)$ as:

$$\int \vec{\mathbf{b}}(t)\,dt = \hat{\mathbf{i}}\int b_1(t)\,dt + \hat{\mathbf{j}}\int b_2(t)\,dt + \hat{\mathbf{k}}\int b_3(t)\,dt \tag{3.4}$$

Note that since $\frac{d\vec{\mathbf{a}}}{dt} = \vec{\mathbf{b}}(t)$, we also have:

$$\frac{da_1(t)}{dt} = b_1(t), \quad \frac{da_2(t)}{dt} = b_2(t) \quad \text{and} \quad \frac{da_3(t)}{dt} = b_3(t) \tag{3.5}$$

From our knowledge of calculus, using Eq. (3.2), we can also write,

$$\int b_1(t)dt = a_1(t) + c_1, \quad \int b_2(t)dt = a_2(t) + c_2, \quad \text{and} \quad \int b_3(t)dt = a_3(t) + c_3 \qquad (3.6)$$

where c_1, c_2 and c_3 are the constants of integration.

So to evaluate $\int \vec{\mathbf{b}}(t)dt$, we only need to integrate the scalar functions $b_1(t), b_2(t)$ and $b_3(t)$ with respect to the scalar *t,* as in ordinary calculus. Note that, we leave the unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ outside the integrals as these are constant and do not depend on *t*. In the same way, we can write the expression for the **definite** integral of a vector function in the interval $[t_1, t_2]$ as follows:

$$\int_{t_1}^{t_2} \vec{\mathbf{b}}(t)dt = \hat{\mathbf{i}} \int_{t_1}^{t_2} b_1(t)dt + \hat{\mathbf{j}} \int_{t_1}^{t_2} b_2(t)dt + \hat{\mathbf{k}} \int_{t_1}^{t_2} b_3(t)dt \qquad (3.7)$$

The integration of the two-dimensional vector function with respect to scalar is also carried out in the same way. So, let us now write down the formal definitions of the integral of a vector function $\vec{\mathbf{b}}(t)$ in two and three-dimensions:

INTEGRAL OF A VECTOR FUNCTION

1. For a vector function in three dimensions defined as $\vec{\mathbf{b}}(t) = b_1(t)\hat{\mathbf{i}} + b_2(t)\hat{\mathbf{j}} + b_3(t)\hat{\mathbf{k}}$ where $b_1(t), b_2(t)$ *and* $b_3(t)$ are continuous over the interval $[t_1, t_2]$, the **indefinite** integral of $\vec{\mathbf{b}}(t)$ with respect to *t* is given by:

$$\int \vec{\mathbf{b}}(t)dt = \hat{\mathbf{i}} \int b_1(t)dt + \hat{\mathbf{j}} \int b_2(t)dt + \hat{\mathbf{k}} \int b_3(t)dt \qquad (3.4)$$

The definite integral of $\vec{\mathbf{b}}(t)$ over the interval $[t_1, t_2]$ is:

$$\int_{t_1}^{t_2} \vec{\mathbf{b}}(t)dt = \hat{\mathbf{i}} \int_{t_1}^{t_2} b_1(t)dt + \hat{\mathbf{j}} \int_{t_1}^{t_2} b_2(t)dt + \hat{\mathbf{k}} \int_{t_1}^{t_2} b_3(t)dt \qquad (3.7)$$

2. For a vector function in two dimensions, $\vec{\mathbf{b}}(t) = b_1(t)\hat{\mathbf{i}} + b_2(t)\hat{\mathbf{j}}$ where $b_1(t)$ and $b_2(t)$ are continuous over the interval $[t_1, t_2]$, the **indefinite** integral of $\vec{\mathbf{b}}(t)$ with respect to *t* is given by

$$\int \vec{\mathbf{b}}(t)dt = \hat{\mathbf{i}} \int b_1(t)dt + \hat{\mathbf{j}} \int b_2(t)dt \qquad (3.8)$$

The **definite** integral of $\vec{\mathbf{b}}(t)$ with respect to *t* over the interval $[t_1, t_2]$ is

$$\int \vec{\mathbf{b}}(t)dt = \hat{\mathbf{i}} \int_{t_1}^{t_2} b_1(t)dt + \hat{\mathbf{j}} \int_{t_1}^{t_2} b_2(t)dt \qquad (3.9)$$

We now write down a few properties of the integrals of vector functions.

PROPERTIES OF INTEGRALS OF VECTOR FUNCTIONS

1. For a vector function $\vec{\mathbf{f}}(t)$ and a constant α:

$$\int \alpha \vec{\mathbf{f}}(t)dt = \alpha \int \vec{\mathbf{f}}(t)dt \tag{3.10}$$

2. For any two vector functions $\vec{\mathbf{f}}(t)$ and $\vec{\mathbf{g}}(t)$ and constants α and β:

$$\int [\alpha \vec{\mathbf{f}}(t) + \beta \vec{\mathbf{g}}(t)]dt = \alpha \int \vec{\mathbf{f}}(t)dt + \beta \int \vec{\mathbf{g}}(t)dt \tag{3.11}$$

3. For a vector function $\vec{\mathbf{f}}(t)$ and a constant vector $\vec{\mathbf{a}}$:

$$\int \vec{\mathbf{a}}.\vec{\mathbf{f}}(t)dt = \vec{\mathbf{a}}.\int \vec{\mathbf{f}}(t)dt \tag{3.12}$$

4. For a vector function $\vec{\mathbf{f}}(t)$ and a constant vector $\vec{\mathbf{a}}$:

$$\int \vec{\mathbf{a}} \times \vec{\mathbf{f}}(t)dt = \vec{\mathbf{a}} \times \int \vec{\mathbf{f}}(t)dt \tag{3.13}$$

Let us now work out a simple example on integration of vector functions.

EXAMPLE 3.1: POSITION VECTOR

Determine the position vector of a particle $\vec{\mathbf{r}}(t)$ given that its velocity function is:

$$\vec{\mathbf{v}}(t) = \sin t\,\hat{\mathbf{i}} - \cos t\,\hat{\mathbf{j}} + t^2\hat{\mathbf{k}}$$

and the initial position of the particle (position vector of the particle at $t = 0$) is $\vec{\mathbf{r}}(t=0) = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$

SOLUTION ■ Using the definition of velocity, we can write the position vector of the particle as the integral of its velocity as follows:

$$\vec{\mathbf{v}}(t) = \frac{d\vec{\mathbf{r}}(t)}{dt} \Rightarrow \vec{\mathbf{r}}(t) = \int \vec{\mathbf{v}}(t)dt \tag{3.14}$$

We write the integral in terms of the components of the vector function $\vec{\mathbf{v}}(t)$, as defined in Eq. (3.4):

$$\vec{\mathbf{r}}(t) = \hat{\mathbf{i}}\int \sin t\,dt - \hat{\mathbf{j}}\int \cos t\,dt + \hat{\mathbf{k}}\int t^2\,dt$$

$$= -\cos t\,\hat{\mathbf{i}} - \sin t\,\hat{\mathbf{j}} + \frac{t^3}{3}\hat{\mathbf{k}} + \vec{\mathbf{C}} \tag{i}$$

where $\vec{\mathbf{C}}$ is an arbitrary constant vector.

To determine $\vec{\mathbf{C}}$ we use the given initial condition. Substituting $t = 0$ in Eq. (i) we get

$$\vec{\mathbf{r}}(t=0) = -\,\hat{\mathbf{i}} + \vec{\mathbf{C}} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}} \tag{ii}$$

From this we get: $\vec{\mathbf{C}} = 2\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$ (iii)

Substituting for $\vec{\mathbf{C}}$ in Eq. (i), we can now write the position vector as a function of time as:

$$\vec{\mathbf{r}}(t) = -\cos t\,\hat{\mathbf{i}} - \sin t\,\hat{\mathbf{j}} + \frac{t^3}{3}\hat{\mathbf{k}} + 2\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$$

$$= (2 - \cos t)\,\hat{\mathbf{i}} + (1 - \sin t)\,\hat{\mathbf{j}} + (1 - \frac{t^3}{3})\hat{\mathbf{k}} \qquad \text{(iv)}$$

Before we go further, let us summarize what you have studied so far:

Recap

INTEGRATION OF A VECTOR FUNCTION

- The integral of a vector function is defined as the integral of each scalar component of the function.
- This definition holds for both definite and indefinite integrals of vector functions.

You may now like to work out an SAQ on what you have studied so far.

SAQ 1 **– Integrating a vector function**

a) Evaluate $\int \left[\left(\frac{4}{1+t^2}\right)\hat{\mathbf{i}} + \left(\frac{2t}{1+t^2}\right)\hat{\mathbf{j}}\right] dt$

A table of standard integrals is given at the end of this block.

b) The acceleration of an object is $\vec{\mathbf{a}} = -10\hat{\mathbf{k}}$. Obtain its position as a function of time t if its initial velocity is $\vec{\mathbf{v}}(t=0) = \hat{\mathbf{i}} - \hat{\mathbf{k}}$ and its initial position is $\vec{\mathbf{r}}(t=0) = 2\hat{\mathbf{k}}$.

In Unit 2 of BPHCT-131, you have learnt that many physical quantities can be expressed as the scalar or vector products of vectors. We now study the integrals of scalar and vector products of vector functions.

3.2.1 Integrals involving Scalar and Vector Products of Vectors

Let $\vec{\mathbf{a}}(t)$ and $\vec{\mathbf{b}}(t)$ be two vector functions of a scalar t. Then for evaluating the integrals $I_1 = \int \left[\vec{\mathbf{a}}(t).\vec{\mathbf{b}}(t)\right] dt$ and $I_2 = \int \left[\vec{\mathbf{a}}(t) \times \vec{\mathbf{b}}(t)\right] dt$, we first compute the scalar and vector products in the integrands. Recall from Sec. 1.4 of Unit 1, BPHCT-131 that I_1 will reduce to an integral of a scalar function of t with respect to t. Similarly, I_2 will be the integral of a vector function of t with respect to t. Let us take an example to discuss the evaluation of I_1. After that you can work out another example.

EXAMPLE 3.2: INTEGRAL OF A SCALAR PRODUCT

In free space a transverse electromagnetic (EM) wave propagating in the x-direction has an electric field $\vec{\mathbf{E}} = E_0 \cos\frac{2\pi}{\lambda}(ct - x)\hat{\mathbf{j}}$ and a magnetic field $\vec{\mathbf{B}} = B_0 \cos\frac{2\pi}{\lambda}(ct - x)\hat{\mathbf{k}}$. Here c and λ are, respectively, the velocity and the wavelength of the EM wave and $E_0 = B_0 c$. The energy flowing through a volume V per unit time is given by

$$U = \frac{V}{2}(\vec{\mathbf{E}} \,.\, \vec{\mathbf{D}} + \vec{\mathbf{B}}\,.\,\vec{\mathbf{H}}),$$

where $\vec{\mathbf{D}} = \varepsilon_0 \vec{\mathbf{E}}$ and $\vec{\mathbf{B}} = \mu_0 \vec{\mathbf{H}}$.

Here ε_0 and μ_0 are permittivity and the magnetic permeability, respectively, of free space and $c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$. Compute the total energy flowing through V in one complete cycle of EM wave if its time period is T.

SOLUTION ■ The energy flow during time dt is given by $U\,dt$. So the total energy will be the definite integral of U from $t = 0$ to $t = T$, i.e.

$$U_0 = \int_0^T U\,dt = \frac{V}{2}\int_0^T (\mathbf{E}.\mathbf{D} + \mathbf{B}.\mathbf{H})\,dt = \frac{V}{2}(I_E + I_B) \qquad \text{(i)}$$

where $I_E = \int_0^T \vec{\mathbf{E}}.\vec{\mathbf{D}}\,dt$ and $I_B = \int_0^T \vec{\mathbf{B}}.\vec{\mathbf{H}}\,dt$.

Both I_E and I_B are integrals of the type I_1. So we shall first evaluate the scalar products. Given that

$$\vec{\mathbf{E}} = E_0 \cos\frac{2\pi}{\lambda}(ct - x)\hat{\mathbf{j}} \qquad \text{(ii)}$$

$$\vec{\mathbf{D}} = \varepsilon_0 \vec{\mathbf{E}} = \varepsilon_0 E_0 \cos\frac{2\pi}{\lambda}(ct - x)\,\hat{\mathbf{j}} \qquad \text{(iii)}$$

We get $$\vec{\mathbf{E}}.\vec{\mathbf{D}} = \varepsilon_0 E_0^2 \cos^2\frac{2\pi}{\lambda}(ct - x) \qquad \text{(iv)}$$

Similarly, you can show that

$$\vec{\mathbf{B}}.\vec{\mathbf{H}} = \frac{B_0^2}{\mu_0}\cos^2\frac{2\pi}{\lambda}(ct - x) \qquad \text{(v)}$$

Substituting from Eq. (iv) and Eq. (v) into Eq. (i) we get

$$U_0 = \frac{V}{2}\left(\varepsilon_0 E_0^2 + \frac{B_0^2}{\mu_0}\right) I \qquad \text{(vi)}$$

where (see margin remark) $I = \int_0^T \cos^2\frac{2\pi}{\lambda}(ct - x)\,dt = \frac{T}{2}$

$$\frac{2\pi c}{\lambda} = \frac{2\pi}{T}\quad (\because \lambda = cT)$$

$$\cos^2\frac{2\pi c}{\lambda}(ct - x)$$

$$= \cos^2\left(\frac{2\pi t}{T} - kx\right),$$

where $k = \frac{2\pi}{\lambda}$

$$= \frac{1}{2}\left\{\cos\left[2\left(\frac{2\pi t}{T} - kx\right)\right] + 1\right\}$$

$$\therefore \int_0^T \cos^2\frac{2\pi}{\lambda}(ct - x)dt$$

$$= \frac{1}{2}\int_0^T \cos\left(\frac{4\pi t}{T} - 2kx\right)dt$$

$$+ \frac{1}{2}\int_0^T dt$$

$$= \frac{1}{2}\frac{T}{4\pi}\left|\sin\left(\frac{4\pi t}{T} - 2kx\right)\right|_0^T$$

$$+ \frac{T}{2}$$

$$= \frac{T}{8\pi}[\sin(4\pi - 2kx)$$

$$- \sin(-2kx)] + \frac{T}{2}$$

$$= \frac{T}{8\pi}(-\sin 2kx + \sin 2kx) + \frac{T}{2}$$

$$= \frac{T}{2}$$

$$\therefore \qquad U_0 = \frac{VT}{4}\left(\varepsilon_0 E_0^2 + \frac{B_0^2}{\mu_0}\right) \qquad \text{(vii)}$$

$$\text{Again } B_0^2 = \frac{E_0^2}{c^2} = \varepsilon_0 \mu_0 E_0^2 \left(\because c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}\right)$$

$$\therefore \qquad \frac{B_0^2}{\mu_0} = \varepsilon_0 E_0^2 \qquad \text{(viii)}$$

$$\text{Hence } U_0 = \frac{VT}{2}\varepsilon_0 E_0^2$$

The method will be the same for integrating vector products expressed in their component form.

You may like to solve an SAQ before studying further.

SAQ 2 - Integrals of scalar and vector products

Given two vector functions $\vec{\mathbf{a}}(t) = t\,\hat{\mathbf{i}} + (1-t)\,\hat{\mathbf{j}} + t^2\,\hat{\mathbf{k}}$ and $\vec{\mathbf{b}}(t) = 3t^2\hat{\mathbf{i}} - t\,\hat{\mathbf{j}}$, evaluate the integrals:

a) $\displaystyle\int_0^1 \left[\vec{\mathbf{a}}(t).\,\vec{\mathbf{b}}(t)\right]dt$ and b) $\displaystyle\int_0^1 \left[\vec{\mathbf{a}}(t)\times\vec{\mathbf{b}}(t)\right]dt$

We now discuss line integrals of scalar and vector fields.

3.3 LINE INTEGRAL OF A VECTOR FIELD

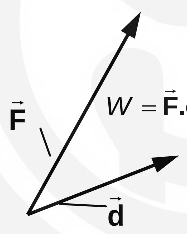


Fig. 3.1: **Work done by a force when the force and displacement are not along the same direction.**

In Unit 2 of BPHCT-131, you have studied that for a constant force, when the displacement is not along the force (Fig. 3.1), the work done is the scalar product of force and displacement:

$$W = \vec{\mathbf{F}}.\vec{\mathbf{d}} = (F\cos\theta)d \qquad (3.15)$$

In your school physics, you have learnt about work done by a constant force and variable force. You may recall that when a variable force *F(x*) is applied on an object along the *x*-axis, the work done in moving the object between any two points x_1 and x_2 is an integral given by

$$W = \int_{x_1}^{x_2} F(x)dx \qquad (3.16)$$

A well-known example of this is the work done in stretching a spring by a length *d.* The spring force is a restoring force: $F(x) = -kx$, where *k* is the spring constant. The work done is:

$$W = \int_0^d (-kx)dx \qquad (3.17)$$

Let us now consider the most general case: a **variable** force applied on an object moving along an **arbitrary path** in space. What is the work done by the force? Refer to Fig. 3.2. A planet is moving around the Sun in an elliptical orbit under the gravitational force. How will you calculate the work done for such systems?

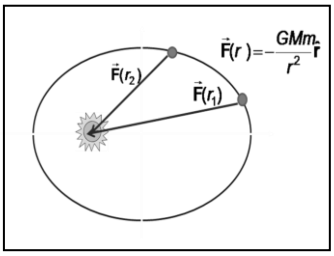


Fig. 3.2: **A planet moves around the Sun in an elliptical orbit. The force of gravitation on the planet is a variable force.**

Consider an object moving along an arbitrary path in space between the points *P* and *Q*. Note that the path is a curve and the force $\vec{\mathbf{F}} = \vec{\mathbf{F}}(x, y, z)$ is a variable force (Fig. 3.3a). Let us calculate the work done by the force in moving the object from *P* to *Q* along the path shown in Fig. 3.3a. We first divide the path *PQ* in *n* tiny segments as shown in Fig. 3.3b. We define the displacement of the object for each of these segments as $\Delta\vec{\mathbf{l}}_1, \Delta\vec{\mathbf{l}}_2, .., \Delta\vec{\mathbf{l}}_i, ... \Delta\vec{\mathbf{l}}_n$, respectively. Let $\Delta\vec{\mathbf{l}}_i$ be the displacement for the i^{th} segment. The magnitude of the displacement for each segment of the curve is almost equal to its length (read the margin remark) (inset of Fig. 3.3b).

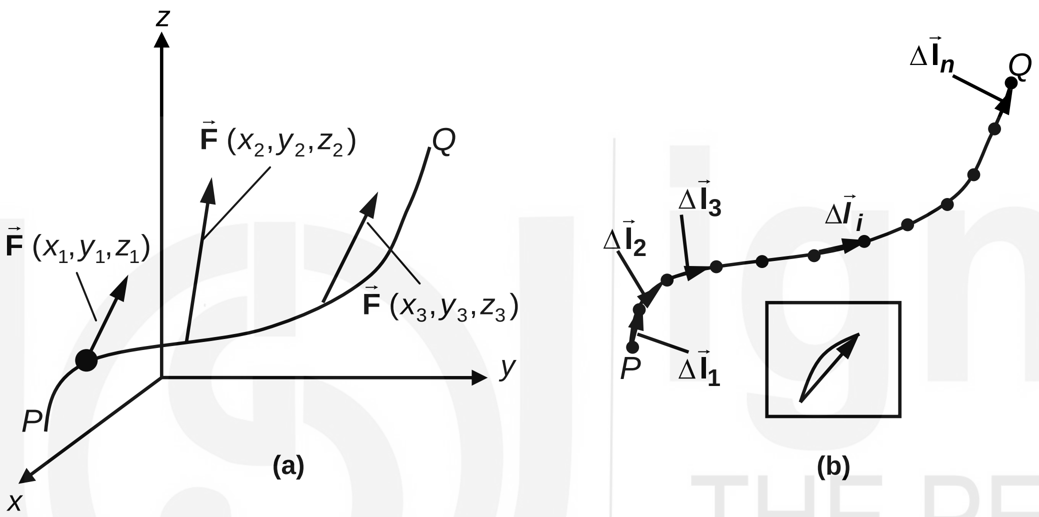


Fig. 3.3: **a) An object moves under a variable force along the path *PQ*. The force is different at different points along the path; b) the path is divided into *n* segments and the displacement is defined for each segment.**

The displacement for each segment of the path has its tail at the starting point of the segment and its head at the final point of the segment as you can see in the inset of Fig. 3.3b.

Although the force is actually different at different points of the path, we **assume** that it is **constant** over each of these segments.

Let the force acting on the object be $\vec{\mathbf{F}}_1$ for the first segment, $\vec{\mathbf{F}}_2$ for the second segment, and so on. Let us consider the *i*th segment. What is the work done by the force $\vec{\mathbf{F}}_i$ for the displacement $\Delta\vec{\mathbf{l}}_i$? From Eq. (3.15), it is $\Delta W_i = \vec{\mathbf{F}}_i . \Delta\vec{\mathbf{l}}_i$. The total work done in moving the object over the entire path is the sum of the work done in moving the object over each segment of the path. We can write it as:

$$W = \vec{\mathbf{F}}_1 . \Delta\vec{\mathbf{l}}_1 + \vec{\mathbf{F}}_2 . \Delta\vec{\mathbf{l}}_2 + ... + \vec{\mathbf{F}}_i . \Delta\vec{\mathbf{l}}_i + ... \vec{\mathbf{F}}_n . \Delta\vec{\mathbf{l}}_n = \sum_{i=1}^{n} \vec{\mathbf{F}}_i . \Delta\vec{\mathbf{l}}_i \quad (3.18a)$$

If the number of segments *n* is large, we can approximate the length of the curve by summing over the magnitude of the displacements.

In the limit as $n \to \infty$, we express the sum in Eq. (3.18a) as an **integral** along the path between *P* and *Q*:

$$W = \int_C \vec{\mathbf{F}} . d\vec{\mathbf{l}} \quad (3.18b)$$

This is an example of a **line integral** along a **path of integration** *C***.** It is the path between the points *P* and *Q* along which the object moves. It should be a

smooth curve. We will explain what is meant by a smooth curve in the next section.

Here we have defined the **line integral** in order to calculate the work done by a force field in moving an object along an arbitrary path. We can define such a line integral for any arbitrary vector field $\vec{\mathbf{A}}$ along a path of integration *C* as $\int_C \vec{\mathbf{A}}.d\vec{\mathbf{l}}$.

The line integral is a generalization of the concept of a definite integral. In a definite integral $\int_a^b f(x)\,dx$, we integrate a function *f(x)* along the *x*- axis between two points, *a* and *b.* The function is defined at every point in the interval [*a, b*]. In a line integral, we integrate along a curve *C* and the integrand ($\vec{\mathbf{F}}.d\vec{\mathbf{l}}$ in Eq. 3.18b) is a function defined at every point on the curve. Note that the path of integration can be any straight line or curve, in space or in a plane.

We now discuss how to calculate this integral. Let us write the force field $\vec{\mathbf{F}}$ in terms of its component functions as $\vec{\mathbf{F}} = F_1(x,y,z)\hat{\mathbf{i}} + F_2(x,y,z)\hat{\mathbf{j}} + F_3(x,y,z)\hat{\mathbf{k}}$, and the displacement along the path as $d\vec{\mathbf{l}} = dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}} + dz\hat{\mathbf{k}}$. The line integral of Eq. (3.18b) is then given by:

$$W = \int_C \vec{\mathbf{F}}.\,d\vec{\mathbf{l}} = \int_C [F_1 dx + F_2 dy + F_3 dz] \qquad (3.19a)$$

If the force field is two-dimensional and the object is moving in the *xy* plane, we can write the line integral as:

$$W = \int_C \vec{\mathbf{F}}.\,d\vec{\mathbf{l}} = \int_C [F_1 dx + F_2 dy] \qquad (3.19b)$$

Note that in general, F_1, F_2 and F_3 are functions of *x*, *y* and *z*. However, the integrals are over either *x* or *y* or *z*. Therefore, **you must express each integral in terms of a single variable.** This means, for example, to evaluate the integral $\int_C F_1(x,y,z)\,dx$, we must express *y* and *z* in terms of *x*, so that F_1 is a function of only *x*.

This is what you will learn about in the next section.

3.3.1 Representation of a Curve

In a plane, a curve can be described by an equation of the form:

$$y = f(x) \qquad (3.20a)$$

To write the equation of the circle in the form of Eq. (3.20a), we write it as:

$$y = \sqrt{a^2 - x^2}$$

For example, $y = 4x^2$ is the equation of a parabola and $x^2 + y^2 = a^2$ is the equation of a circle of radius *a* with its origin at the centre. The coordinates of a point on the curve described by Eq. (3.20a) are given by (*x, f*(*x*)).

In three-dimensional space, we may describe a curve using a set of equations

$$y = f(x); \quad z = g(x) \tag{3.20b}$$

The coordinates of each point on the curve are $(x, f(x), g(x))$. This is also called an **explicit** representation. We may also describe the curve as an intersection of two surfaces:

$$F(x,y,z) = 0; \quad G(x,y,z) = 0 \tag{3.20c}$$

This is called an **implicit** representation. Note that both $F(x,y,z) = 0$ and $G(x,y,z) = 0$ represent surfaces in space.

In the following example, we use the definition of line integral in Eqs. (3.19b) and the representation of a curve in a plane given by Eq. (3.20a) to calculate the work done.

Note that in all the representations of a curve, **there is only one independent variable.** This is important, because the line integral, unlike a double integral or a triple integral, is an integration over one variable.

EXAMPLE 3.3: LINE INTEGRAL OF A VECTOR FIELD IN A PLANE

Calculate the work done by a force field $\vec{\mathbf{F}} = 2xy\hat{\mathbf{i}} - y^2\hat{\mathbf{j}}$ in moving an object along the curve $y = x^2$ in the xy plane from (0,0) to (2,4).

SOLUTION ■ Using Eq. (3.19b) for the work done by a 2-dimensional force field in moving an object in the xy plane with $F_1 = 2xy$ and $F_2 = -y^2$ we can write:

$$W = \int_C \left(2xydx - y^2dy\right) \tag{i}$$

The equation of the curve $y = x^2$ tells us how x and y are related along the path C. Using this in Eq. (i) we get:

$$W = \int_C \left[2x(x^2)dx - y^2dy\right] \tag{ii}$$

Note that each of the integrals in Eq. (ii) is over a single variable.

Since the coordinates of the initial and final points of the path are (0,0) and (2,4) we can write the limits on x and y along the path as:

$$0 \le x \le 2\,; \quad 0 \le y \le 4 \tag{iii}$$

And the integral of Eq. (ii) reduces to:

$$W = \int_0^2 2x^3dx - \int_0^4 y^2dy$$

These can be evaluated as ordinary integrals:

$$\therefore \quad W = \left[\frac{2x^4}{4}\right]_0^2 - \left[\frac{y^3}{3}\right]_0^4 = -\frac{40}{3} \tag{iv}$$

SAQ 3 – Work done by a force

Calculate the line integral of the force field $\vec{\mathbf{F}} = xy\hat{\mathbf{i}} + \left(x^2 + 1\right)\hat{\mathbf{j}}$ from (0,0) to (1,1) along the three paths labeled I, II and III in Fig. 3.4.

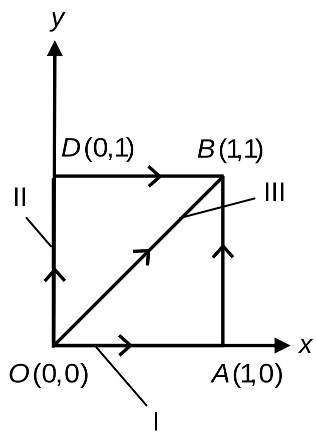


Fig. 3.4

In the next section we discuss another representation of a curve in space which is useful for evaluating line integrals.

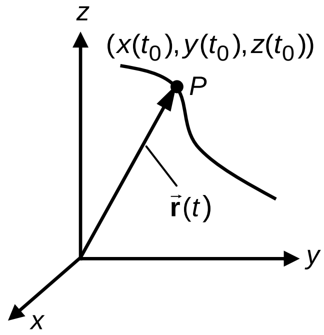


Fig. 3.5: **Parametric representation of a curve. At the point** *P*, **the value of the parameter is** t_0, **the position vector is** $\vec{\mathbf{r}}(t_0)$ **and the coordinates are** $(x(t_0), y(t_0), z(t_0))$.

3.3.2 Parametric Representation

There is yet another representation of the space curve called the **parametric** representation. In a Cartesian coordinate system, we may represent a curve using the position vector function $\vec{\mathbf{r}}(t)$ and a real parameter *t,* as follows:

$$\vec{\mathbf{r}}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} + z(t)\hat{\mathbf{k}} \qquad (3.21a)$$

$\vec{\mathbf{r}}(t)$ is the position vector of a point on the curve, as you can see in Fig. 3.5. As the value of *t* changes, the head of the vector traces out a curve in space.

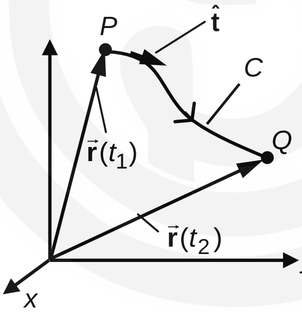
A point on the curve has the coordinates [$x(t)$, $y(t)$, $z(t)$]. The coordinates are functions of the parameter *t* and for each value of *t*, we get a different point on the curve.

Let us now learn how to evaluate line integrals using the parametric representation of the path of integration. Sometimes, it is convenient to use the parametric representation rather than Eqs. (3.19a or 3.19b) as you will see in Example 3.4.

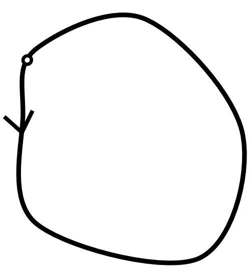
Let us first write down the path of integration in the parametric representation. The parametric representation of the path of integration *C* between two points *P* and *Q* (Fig. 3.6a) is,

$$\vec{\mathbf{r}}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} + z(t)\hat{\mathbf{k}}, \qquad t_1 \le t \le t_2 \qquad (3.21b)$$

where t_1 and t_2 are the values of the parameter *t* at *P* and *Q*, respectively. The coordinates of *P* and *Q* are *P* [$x(t_1)$, $y(t_1)$, $z(t_1)$] and *Q* [$x(t_2)$, $y(t_2)$, $z(t_2)$].



(a)



(b)

Fig. 3.6: **a) Parametric representation of the path of integration; and b) a closed path.**

Remember that we have said earlier in this section that the path of integration in a line integral should be a **smooth** curve. You may now like to know: **When can we say that** *C* **is a smooth curve**? *C* is said to be a **smooth curve** if

- $\vec{\mathbf{r}}(t)$ as defined in Eq. (3.21b) has a continuous derivative $\vec{\mathbf{r}}'(t) = \dfrac{d\vec{\mathbf{r}}(t)}{dt}$ which is not equal to zero anywhere on *C* $(t_1 \le t \le t_2)$, and
- $\vec{\mathbf{r}}'(t)$ is directed along the tangent to the curve at every point (Fig. 3.6a).

The unit tangent vector at each point on the curve is:

$$\hat{\mathbf{t}} = \frac{\vec{\mathbf{r}}'(t)}{|\vec{\mathbf{r}}'(t)|} \qquad (3.22)$$

Since we are integrating **from** *P* **to** *Q*, the path of integration also has a **specific direction** (is **oriented**). We take the direction from *P* to *Q* as the positive direction along the curve (Fig. 3.6a). We mark the positive direction on the curve by an arrow. If the path is such that the initial and final points of the curve coincide, as in Fig. 3.6b, $[\vec{\mathbf{r}}(t_1) = \vec{\mathbf{r}}(t_2)]$, then the curve is a closed curve or closed contour. When the integration is over a closed path *C*, the symbol of integration $\int_C$ is replaced by $\oint_C$.

Before you learn how to evaluate the line integral using the parametric representation, we illustrate the parametric representation of a few simple curves.

EXAMPLE 3.4: PARAMETRIC REPRESENTATION OF CURVES

Write down the parametric representation for the following:

a) A straight line between the points (0,0) and (1,2).

b) The ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

c) The circle $x^2+y^2=a^2$

d) A circular helix

SOLUTION ■ In all four parts, we will express the equations of the curves in terms of a single parameter, say *t*.

a) From school mathematics, you know that the equation of the straight line between any two points (x_1,y_1) and (x_2,y_2) is:

$$y-y_1=\left(\frac{y_2-y_1}{x_2-x_1}\right)(x-x_1)$$

or
$$\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1} \qquad \text{(i)}$$

The LHS of Eq. (i) is a function of only *y* and the RHS is a function of only *x*. We can, therefore, equate this to a parameter *t*. Then

$$\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1}=t$$

or
$$y(t)=y_1+(y_2-y_1)t \quad \text{and} \quad x(t)=x_1+(x_2-x_1)t \qquad \text{(ii)}$$

Eqs. (i) and (ii) are the parametric equations for *x* and *y*. Thus in general

$$\vec{\mathbf{r}}(t)=[x_1+(x_2-x_1)t]\,\hat{\mathbf{i}}+[y_1+(y_2-y_1)t]\,\hat{\mathbf{j}} \qquad (3.23)$$

Using $(x_1,y_1)=(0,0)$ and $(x_2,y_2)=(1,2)$ in Eq. (ii), we get

$$x(t)=t; \qquad y(t)=2t \qquad \text{(iii)}$$

To get the end points of the straight line in terms of *t*, we use Eq. (iii) as follows:

Let $t = t_1$ for the point (0, 0) and $t = t_2$ for the point (1, 2). Then since $x(t) = t$ and $y(t) = 2t$, we get

$$x_1=x(t_1)=t_1=0,\quad y_1=y(t_1)=2t_1=0 \quad \Rightarrow t_1=0$$

and
$$x_2=x(t_2)=t_2=1,\quad y_2=y(t_2)=2t_2=2 \quad \Rightarrow t_2=1$$

Therefore, in terms of the parameter *t*, the initial point of the straight line is $t_1=0$ and the final point is $t_2=1$. The parametric representation of the straight line between (0,0) and (1,2) is:

$$\vec{\mathbf{r}}(t)=t\,\hat{\mathbf{i}}+2t\,\hat{\mathbf{j}};\quad 0\le t\le 1$$

b) Note that for $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, the values of both $\frac{x}{a}$ and $\frac{y}{b}$ should lie between −1 and 1. This suggests (see margin remark) that we can use the identity $\cos^2 t+\sin^2 t=1$ to write the parametric representation:

The values of sin *t* and cos *t* lie between −1 and 1.

$$\frac{x}{a} = \cos t; \qquad \frac{y}{b} = \sin t$$

$$\Rightarrow \qquad x(t) = a\cos t \quad \text{and} \quad y = b\sin t$$

So, an ellipse with its centre at the origin and semi-major and semi-minor axes a and b respectively, has the parametric representation (Fig. 3.7a):

$$\vec{\mathbf{r}}(t) = a\cos t\,\hat{\mathbf{i}} + b\sin t\,\hat{\mathbf{j}} \qquad 0 \le t < 2\pi \qquad (3.24)$$

The parameter t is the angle the position vector $\vec{\mathbf{r}}(t)$ makes with the x-axis. As t changes from 0 to 2π, the tip of the position vector traces the entire ellipse starting from the point A on the x-axis. The coordinate of each point on the ellipse is (a cos$t$, b sin$t$).

Note that if you want to take only a part of the ellipse, you have to choose the range of t accordingly. For example, for the part of ellipse in the first quadrant we write;

$$\vec{\mathbf{r}}(t) = a\cos t\,\hat{\mathbf{i}} + b\sin t\,\hat{\mathbf{j}} \qquad 0 < t < \pi/2$$

c) Substituting $a = b$ in Eq. 3.24, we get the parametric equation of a circle $x^2 + y^2 = a^2$ (Fig. 3.7b):

$$\vec{\mathbf{r}}(t) = a\cos t\,\hat{\mathbf{i}} + a\sin t\,\hat{\mathbf{j}} \qquad 0 \le t < 2\pi \qquad (3.25)$$

The coordinate of each point on the circle is (a cos t, a sin t).

d) The parametric equation for a circular helix (Fig. 3.7c) is:

$$\vec{\mathbf{r}}(t) = a\cos t\,\hat{\mathbf{i}} + a\sin t\,\hat{\mathbf{j}} + bt\,\hat{\mathbf{k}};\ b \ne 0,\ 0 \le t \le 2\pi \qquad (3.26)$$

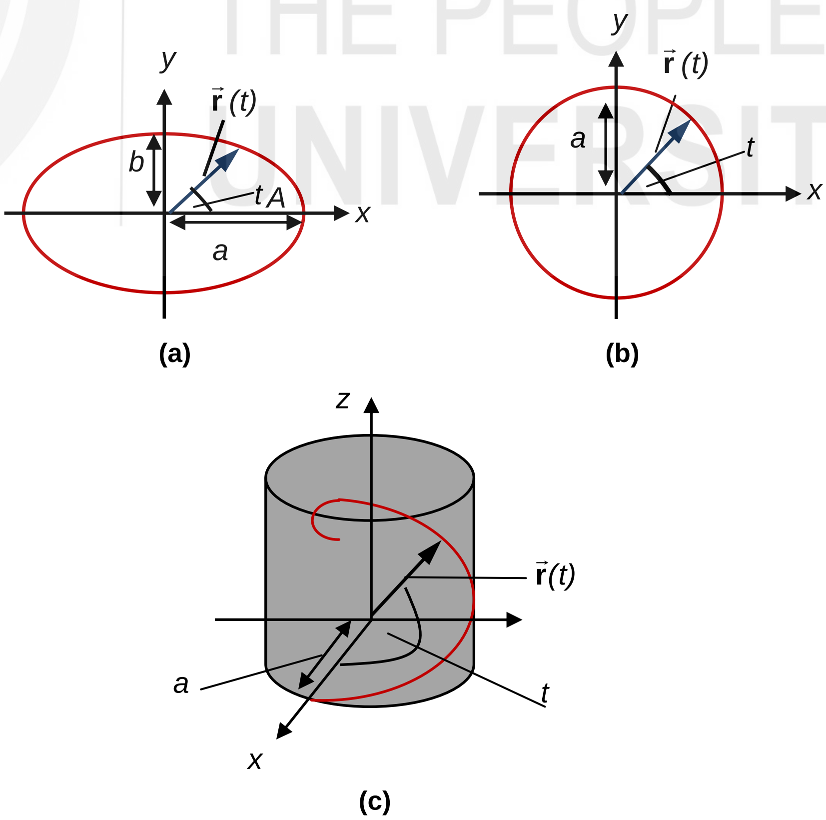


Fig. 3.7: **Parametric representation of the a) ellipse; b) circle; c) right circular helix, in which the curve lies on the cylinder $x^2 + y^2 = a^2$.**

SAQ 4 - Parametric representation of a parabola

Write down the parametric representation for the parabola $y = x^2$ between the points (0,0) and (2,4).

The parametric representation of a curve has several applications. In Mechanics the parameter *t* in Eq. (3.21b) may be used to represent time and we can use the vector function $\vec{\mathbf{r}}(t)$ to determine the velocity and acceleration of an object moving along a curve. We now use the parametric representation of the path of integration to define the line integral of a vector function along the path as:

Usually in Physics we use the symbol $\vec{\mathbf{F}}$ to denote force fields and $d\vec{\mathbf{r}}$ to indicate displacement. Here we use the $d\vec{\mathbf{l}}$ instead merely to highlight that we are talking about an infinitesimal displacement along a curve.

$$W = \int_C \vec{\mathbf{F}}.d\vec{\mathbf{l}} = \int_{t_1}^{t_2} \left[\vec{\mathbf{F}}[\vec{\mathbf{r}}(t)].\frac{d\vec{\mathbf{r}}(t)}{dt}\right] dt \qquad (3.27)$$

$\vec{\mathbf{F}}(\vec{\mathbf{r}}(t))$ is a vector function , $\vec{\mathbf{r}}(t)$ is defined in Eq. (3.21b), t_1 and t_2 are the end points of the path.

This is now the definite integral of a **scalar** function. We can write

$$\frac{d\vec{\mathbf{r}}}{dt} = \frac{d}{dt}\left[x(t)\,\hat{\mathbf{i}} + y(t)\,\hat{\mathbf{j}} + z(t)\,\hat{\mathbf{k}}\right]$$

$$= \frac{dx(t)}{dt}\,\hat{\mathbf{i}} + \frac{dy(t)}{dt}\,\hat{\mathbf{j}} + \frac{dz(t)}{dt}\,\hat{\mathbf{k}} \qquad (3.28)$$

By replacing *x*,*y*,*z* in the vector function $\vec{\mathbf{F}} = F_1(x,y,z)\hat{\mathbf{i}} + F_2(x,y,z)\hat{\mathbf{j}} + F_3(x,y,z)\hat{\mathbf{k}}$ by the parametric functions $x = x(t)$; $y = y(t); z = z(t)$, we can write the vector function as a function of the parameter *t*.

Using $\vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) = F_1(t)\hat{\mathbf{i}} + F_2(t)\,\hat{\mathbf{j}} + F_3(t)\hat{\mathbf{k}}$ (see margin remark) and Eq. (3.27) we get:

$$\int_{t_1}^{t_2}\left[\vec{\mathbf{F}}.\frac{d\vec{\mathbf{r}}}{dt}\right]dt = \int_{t_1}^{t_2}\left[F_1(t)\frac{dx(t)}{dt} + F_2(t)\frac{dy(t)}{dt} + F_3(t)\frac{dz(t)}{dt}\right]dt \qquad (3.29a)$$

For a two-dimensional force field $\vec{\mathbf{F}} = F_1(t)\,\hat{\mathbf{i}} + F_2(t)\,\hat{\mathbf{j}}$, we can write the line integral as:

$$\int_{t_1}^{t_2}\left[\vec{\mathbf{F}}.\frac{d\vec{\mathbf{r}}}{dt}\right]dt = \int_{t_1}^{t_2}\left[F_1(t)\frac{dx(t)}{dt} + F_2(t)\frac{dy(t)}{dt}\right]dt \qquad (3.29b)$$

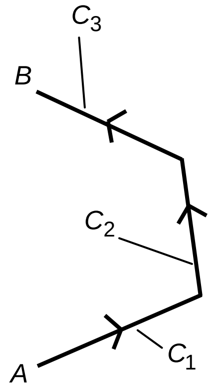


Fig. 3.8: **The curve between** *A* **and** *B* **is piecewise smooth. It is made up of the smooth curves** C_1,C_2 **and** C_3.

Note that the quantity in the bracket in Eq. (3.29b) is a scalar function of a single variable *t*. We can say that the integral is along the *t*-axis, in the direction of increasing *t*. It exists when *C* is a smooth curve or even a piecewise smooth curve. In Fig. 3.8 you can see an example of a curve which is **piecewise smooth**.

Let us now write down a formal definition of the line integral of a vector field using the parametric representation of the path of integration.

LINE INTEGRAL OF A VECTOR FIELD

If a vector field $\vec{\mathbf{F}}$ is continuous on a curve C which has a parametric representation $\vec{\mathbf{r}}(t)$ with $t_1 \le t \le t_2$ and $\vec{\mathbf{r}}(t)$ is differentiable, we define the line integral of the vector field $\vec{\mathbf{F}}$ along the curve C as:

$$W = \int_C \vec{\mathbf{F}}.d\vec{\mathbf{l}} = \int_C \left[\vec{\mathbf{F}}[(t)].\frac{d\vec{\mathbf{r}}(t)}{dt}\right]dt \qquad (3.30)$$

Remember that there can be more than one way of parametrizing a curve.

For example, a circle $x^2 + y^2 = a^2$ can be represented either as

$$\vec{\mathbf{r}}(t) = a\cos t\,\hat{\mathbf{i}} + a\sin t\,\hat{\mathbf{j}} \quad \text{or} \quad \vec{\mathbf{r}}(t) = a\sin t\,\hat{\mathbf{i}} + a\cos t\,\hat{\mathbf{j}}$$

The value of the line integral does not depend on the chosen parametric representation of the path of integration.

In the following example, we calculate the line integral for a two-dimensional vector field.

EXAMPLE 3.5: LINE INTEGRAL OF A VECTOR FIELD

Calculate the line integral of the vector field $\vec{\mathbf{F}}(x,y) = -y\,\hat{\mathbf{i}} + x\,\hat{\mathbf{j}}$ over the curve $\vec{\mathbf{r}}(t) = \cos t\,\hat{\mathbf{i}} + \sin t\,\hat{\mathbf{j}}$ with $0 \le t \le \pi$.

SOLUTION ■ We use Eq. (3.30) to calculate the line integral. Let us write down the steps of this calculation.

Step 1: Calculate $\frac{d\vec{\mathbf{r}}}{dt}$.

$$\frac{d\vec{\mathbf{r}}}{dt} = \frac{d}{dt}[\cos t\,\hat{\mathbf{i}} + \sin t\,\hat{\mathbf{j}}] = -\sin t\,\hat{\mathbf{i}} + \cos t\,\hat{\mathbf{j}} \qquad \text{(i)}$$

Step 2: Write $\vec{\mathbf{F}}[\vec{\mathbf{r}}(t)]$ in terms of the parameter t.

$\vec{\mathbf{F}}$ is the vector field $\vec{\mathbf{F}}(x,y) = -y\,\hat{\mathbf{i}} + x\,\hat{\mathbf{j}}$. We write $\vec{\mathbf{F}}$ in terms of the parameter t by replacing x and y in $\vec{\mathbf{F}}(x,y)$ by $x = x(t) = \cos t$, $y = y(t) = \sin t$.

$$\therefore \quad \vec{\mathbf{F}} = -\sin t\,\hat{\mathbf{i}} + \cos t\,\hat{\mathbf{j}} \qquad \text{(ii)}$$

Step 3: Determine $\vec{\mathbf{F}}.\frac{d\vec{\mathbf{r}}}{dt}$.

Using Eqs. (i) and (ii), we can write :

$$\vec{\mathbf{F}}.\frac{d\vec{\mathbf{r}}}{dt} = \left[-\sin t\,\hat{\mathbf{i}} + \cos t\,\hat{\mathbf{j}}\right].\left[-\sin t\,\hat{\mathbf{i}} + \cos t\,\hat{\mathbf{j}}\right] = \sin^2 t + \cos^2 t = 1 \qquad \text{(iii)}$$

Step 4: Evaluate $\int_{t_1}^{t_2}\left[\vec{\mathbf{F}}.\frac{d\vec{\mathbf{r}}}{dt}\right]dt$.

The limits of integration are the limits of the parameter t for the path of integration. These are given as $t_1 = 0$ and $t_2 = \pi$. So using Eq. (iii), we get:

$$\int_{t_1}^{t_2}\left[\vec{\mathbf{F}}.\frac{d\vec{\mathbf{r}}}{dt}\right]dt = \int_0^{\pi} dt = \pi$$

Let us now work out another example of a line integral of a vector field. We calculate the work done by a three-dimensional force field in moving an object along a given path.

EXAMPLE 3.6: WORK DONE BY A FORCE FIELD

Determine the work done by the force field $\vec{\mathbf{F}}(x,y,z) = xy\,\hat{\mathbf{i}} + yz\,\hat{\mathbf{j}} + zx\hat{\mathbf{k}}$ in moving an object along the curve $\vec{\mathbf{r}}(t) = t\hat{\mathbf{i}} + t^2\hat{\mathbf{j}} + t^3\hat{\mathbf{k}}$ from (0,0,0) to (2,4,8).

SOLUTION ■ We use Eq. (3.29a) to calculate the work done by the force field. Comparing the expression for $\vec{\mathbf{r}}(t)$ with Eq. (3.21b), we can write:

$$x(t) = t, \;\; y(t) = t^2, \;\; z(t) = t^3 \qquad \text{(i)}$$

Note that we have to determine the limits t_1 and t_2 of t for the path of integration as these are not given in the problem. The coordinates of the starting and ending points of the path are (0,0,0) and (2,4,8). Putting these values in the parametric expressions for the coordinates in Eq. (i) we can determine t_1 and t_2 as follows:

$$x(t_1) = t_1 = 0, \quad y(t_1) = t_1^{\,2} = 0, \quad z(t_1) = t_1^{\,3} = 0 \Rightarrow t_1 = 0 \qquad \text{(ii)}$$

and

$$x(t_2) = t_2 = 2, \quad y(t_2) = t_2^{\,2} = 4, \quad z(t_2) = t_2^{\,3} = 8 \Rightarrow t_2 = 2 \qquad \text{(iii)}$$

To calculate the work done we now have to evaluate the line integral

$$W = \int_0^2 \vec{\mathbf{F}}.\frac{d\vec{\mathbf{r}}}{dt}\,dt \qquad \text{(iv)}$$

following the steps outlined in Example 3.5. Here

$$\frac{d\vec{\mathbf{r}}}{dt} = \frac{d}{dt}[t\,\hat{\mathbf{i}} + t^2\,\hat{\mathbf{j}} + t^3\hat{\mathbf{k}}] = \hat{\mathbf{i}} + 2t\,\hat{\mathbf{j}} + 3t^2\,\hat{\mathbf{k}} \qquad \text{(v)}$$

We next write $\vec{\mathbf{F}}$ terms of the parameter t by substituting x, y, z from Eq. (i) to get:

$$\vec{\mathbf{F}}[\vec{\mathbf{r}}(t)] = t^3\,\hat{\mathbf{i}} + t^5\,\hat{\mathbf{j}} + t^4\,\hat{\mathbf{k}} \qquad \text{(vi)}$$

Using Eqs. (v) and (vi), we calculate:

$$\vec{\mathbf{F}}(\vec{\mathbf{r}}(t)).\frac{d\vec{\mathbf{r}}}{dt} = (t^3\,\hat{\mathbf{i}} + t^5\,\hat{\mathbf{j}} + t^4\,\hat{\mathbf{k}}).(\hat{\mathbf{i}} + 2t\,\hat{\mathbf{j}} + 3t^2\,\hat{\mathbf{k}}) = t^3 + 5t^6 \qquad \text{(vii)}$$

The work done is:

$$W = \int_0^2 \left(t^3 + 5t^6\right)dt = \left[\frac{t^4}{4} + 5\frac{t^7}{7}\right]_0^2$$

$$= \frac{668}{7}\text{ units}$$

It is convenient to use the parametric representation when the path of integration is a circle, an ellipse, a helix or a parabola. However, it is not always necessary to use a parametric representation to evaluate a line integral. In Example 3.4 the integral was evaluated using Eq. (3.19b). In some questions, as in SAQ 3, the path of integration may be along the *x, y* or *z*-axes or a combination of all these. In that case, using Eq. (3.19a or b) to evaluate the line integral will be more convenient than using Eq. (3.30).

In evaluating line integrals we can use any of the equations: 3.19a, 3.19b, 3.29a, 3.29b or 3.30.

SAQ 5 **- Line integral of a vector field**

Calculate the line integral of the vector field $\vec{\mathbf{F}} = -\vec{\mathbf{r}}/r^3$ along the curve $\vec{\mathbf{r}}(t) = t\hat{\mathbf{i}} + t\hat{\mathbf{j}} + t\hat{\mathbf{k}}$, with $1 \le t \le 3$.

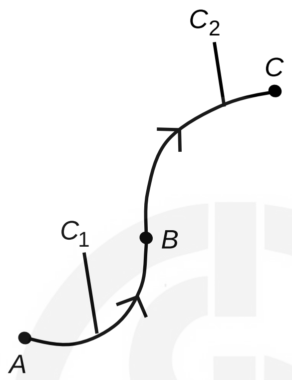


Fig. 3.9: **The curve *C* between points *A* and *C* is made up of the curves C_1 between *A* and *O* and C_2 between *O* and *C*.**

Before you study further, you should learn some properties of line integrals.

PROPERTIES OF LINE INTEGRALS

The line integral of a vector field $\vec{\mathbf{F}}$ along a curve *C* has the following general properties:

1. For a constant α,

$$\int_C \alpha \vec{\mathbf{F}} . d\vec{\mathbf{l}} = \alpha \int_C \vec{\mathbf{F}} . d\vec{\mathbf{l}} \tag{3.31}$$

2. $$\int_C \left[\vec{\mathbf{F}} + \vec{\mathbf{G}}\right] . d\vec{\mathbf{l}} = \int_C \vec{\mathbf{F}} . d\vec{\mathbf{l}} + \int_C \vec{\mathbf{G}} . d\vec{\mathbf{l}} \tag{3.32}$$

   where $\vec{\mathbf{G}}$ is another vector field which is continuous over the curve *C*.

3. If the curve *C* is made up of two curves C_1 and C_2 as shown in Fig. 3.9, we have:

$$\int_C \vec{\mathbf{F}} . d\vec{\mathbf{l}} = \int_{C_1} \vec{\mathbf{F}} . d\vec{\mathbf{l}} + \int_{C_2} \vec{\mathbf{F}} . d\vec{\mathbf{l}} \tag{3.33}$$

   Note that the orientation of the curve is the same in all the three integrals. If the orientation of the path is reversed in any line integral, as in Fig. 3.10, the integral gets multiplied by a negative sign.

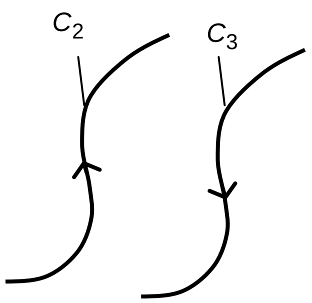


Fig. 3.10: **The line integral over the path C_2 will be the negative of the line integral over the path C_3**

$$\int_{C_2} \vec{\mathbf{F}} . d\vec{\mathbf{l}} = -\int_{C_3} \vec{\mathbf{F}} . d\vec{\mathbf{l}}$$

So far we have discussed line integrals of the form $\int_C \vec{\mathbf{A}} . d\vec{\mathbf{l}}$. There are other types of line integrals. Here we only state these forms.

3.3.3 Other Types of Line Integrals

There are mainly two other types of line integrals that you may need to use. These are:

i) $\int_C f \, dl$

and

ii) $\int_C \vec{\mathbf{A}} \times d\vec{\mathbf{l}}$

where f and $\vec{\mathbf{A}}$ represent a scalar and vector field, respectively. While (i) gives a scalar, (ii) gives a vector.

In the next section we discuss conservative vector fields, which are an important concept in physics. In your mechanics course BPHCT-131 you have studied about central conservative forces which are an example of a conservative vector field.

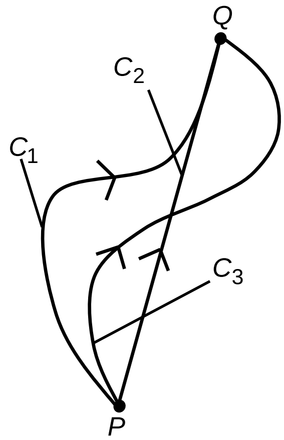


Fig. 3.11: **Three different paths of integration between two points *P* and *Q*, C_1, C_2 and C_3. If the line integral of a vector field $\vec{\mathbf{F}}$ has the same value for all these paths then $\vec{\mathbf{F}}$ is a conservative vector field.**

3.4 CONSERVATIVE VECTOR FIELDS

From the examples you have worked out so far, you have seen that the equation of the path of integration (either in a parametric form or in terms of the Cartesian coordinates) is used to evaluate the line integral. In general, then, the value of the line integral depends on the path (as in SAQ 3). However you will find that in some cases the value of the line integral of a vector field between any two points **does not depend on the path of integration between these points**. This notion of path independence of the line integral of a vector field is used to define a conservative vector field:

A vector field $\vec{\mathbf{F}}$, for which the line integral $\left(\int \vec{\mathbf{F}}.d\vec{\mathbf{l}}\right)$ between any two points *P* and *Q*, has the same value for all paths that begin at the point *P* and end at the point *Q* is called a conservative vector field.

In other words, **the line integral of a conservative force is path independent** (Fig. 3.11).

The force of gravity is an example of a conservative force field. You know that the work done in lifting an object of mass m to a height is the same. Irrespective of the path taken, the work done is $(-mgh)$. Thus, the force of gravity is a conservative force. The electrostatic force field is also conservative, as you have also studied in Unit 10 of BPHCT-131.

If the line integral of $\vec{\mathbf{F}}$ depends on the path between the two points, then it is called a **non-conservative** vector field.

There are **three different ways** of saying that a vector field $\vec{\mathbf{F}}$ is conservative. And ***all of these are equivalent to saying that the line integral of the vector field is path independent***. These are as follows:

1. The vector field can be written as the gradient of a scalar field Φ:

$$\vec{\mathbf{F}} = \vec{\nabla}\Phi \tag{3.34}$$

2. The curl of the vector field is zero or the vector field is **irrotational**:

$$\vec{\nabla} \times \vec{\mathbf{F}} = \vec{\mathbf{0}} \tag{3.35}$$

3. The line integral of the vector field along a closed path is zero:

$$\oint_C \vec{\mathbf{F}}.d\vec{\mathbf{l}} = 0 \tag{3.36}$$

The line integral of a vector field over a closed path is also called a ***closed contour integral*** or a ***loop integral*.** It is denoted by a small circle superimposed on the sign of the integral as shown below:

$$\oint_C \vec{\mathbf{F}}.d\vec{\mathbf{l}} \tag{3.37}$$

For any vector field $\vec{\mathbf{F}}$ the closed contour integral along a curve *C* is also called the **circulation of the vector** $\vec{\mathbf{F}}$ around the path *C*.

SAQ 6 - Circulation of a vector field

Calculate the circulation of a vector field $\vec{\mathbf{A}} = xy\,\hat{\mathbf{i}} + (3x^2 + y)\,\hat{\mathbf{j}}$ around the circle $x^2 + y^2 = 4$.

Note that we can add a constant V_0 to the scalar potential *V*, to find another potential function, $V + V_0$. This is because for any constant V_0, $\vec{\nabla} V_0 = 0$ and therefore we can write $\vec{\mathbf{F}} = -\vec{\nabla}(V + V_0)$. So the scalar potential is arbitrary up to an additive constant.

Let us now introduce another concept which is used very often in physics, that of the scalar potential associated with a conservative force.

3.4.1 Scalar Potential

In mechanics we define the potential energy as the negative of the work done in a process. For example, if we lift a mass *m* to a height *z* the work done by the force of gravity is $W = \Phi = -mgz$. However, the potential energy of the mass increases, and if the potential energy on the surface of the Earth is taken to be zero, the increase in the potential energy $V = mgz$. In other words, the potential energy is the negative of the work done. So,

$$V = -W = -\Phi = -\int_C \vec{\mathbf{F}}.d\vec{\mathbf{l}} \tag{3.38}$$

For every conservative force $\vec{\mathbf{F}}$, we, therefore, define a function *V* which is the scalar potential function $V = -\Phi$ such that $\vec{\mathbf{F}} = -\vec{\nabla} V$.

Let us now work out an example in which we determine the scalar potential for a vector field by evaluating the line integral.

EXAMPLE 3.7: SCALAR POTENTIAL FOR A CONSERVATIVE FORCE FIELD

Determine the scalar potential for an electric field due to a point charge *q* placed at the origin.

SOLUTION ■ The electric field due to a charge *q* placed at the origin of the coordinate system at a point *P* (*x*, *y*, *z*) which is at a distance *r* from the origin is the force on the unit charge placed at that point and is given by:

$\hat{\mathbf{r}}$ is the unit vector along the position vector $\vec{\mathbf{r}}$ from the origin to the point *P*.

$$\vec{\mathbf{E}} = \frac{q}{r^2}\hat{\mathbf{r}} = \frac{q\,\vec{\mathbf{r}}}{r^3} = \frac{q(x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}})}{(x^2 + y^2 + z^2)^{3/2}}$$

We can check that the electric field is conservative by calculating the curl of the the field. Using Eq. (2.7a) for the curl, we get:

$$\vec{\nabla} \times \vec{\mathbf{F}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{x}{(x^2 + y^2 + z^2)^{3/2}} & \dfrac{y}{(x^2 + y^2 + z^2)^{3/2}} & \dfrac{z}{(x^2 + y^2 + z^2)^{3/2}} \end{vmatrix}$$

$$= \hat{\mathbf{i}} \left[\frac{\partial}{\partial y} \left\{ \frac{z}{\left(x^2 + y^2 + z^2\right)^{3/2}} \right\} - \frac{\partial}{\partial z} \left\{ \frac{y}{\left(x^2 + y^2 + z^2\right)^{3/2}} \right\} \right]$$

$$+ \hat{\mathbf{j}} \left[\frac{\partial}{\partial z} \left\{ \frac{x}{\left(x^2 + y^2 + z^2\right)^{3/2}} \right\} - \frac{\partial}{\partial x} \left\{ \frac{z}{\left(x^2 + y^2 + z^2\right)^{3/2}} \right\} \right]$$

$$+ \hat{\mathbf{k}} \left[\frac{\partial}{\partial x} \left\{ \frac{y}{\left(x^2 + y^2 + z^2\right)^{3/2}} \right\} - \frac{\partial}{\partial y} \left\{ \frac{x}{\left(x^2 + y^2 + z^2\right)^{3/2}} \right\} \right] \quad \text{(i)}$$

Calculating the partial derivatives in the first term in Eq. (i) we get:

$$\frac{\partial}{\partial y} \left\{ \frac{z}{\left(x^2 + y^2 + z^2\right)^{3/2}} \right\} = -\frac{3yz}{\left(x^2 + y^2 + z^2\right)^{5/2}}$$

$$\frac{\partial}{\partial z} \left\{ \frac{y}{\left(x^2 + y^2 + z^2\right)^{3/2}} \right\} = -\frac{3yz}{\left(x^2 + y^2 + z^2\right)^{5/2}}$$

$$\therefore \quad \frac{\partial}{\partial y} \left\{ \frac{z}{\left(x^2 + y^2 + z^2\right)^{3/2}} \right\} - \frac{\partial}{\partial y} \left\{ \frac{y}{\left(x^2 + y^2 + z^2\right)^{3/2}} \right\} = 0$$

Similarly, the remaining two terms in Eq. (i) are also zero.

$$\therefore \quad \vec{\nabla} \times \vec{\mathbf{E}} = \vec{\mathbf{0}}$$

To determine the scalar potential associated with the field we calculate the negative of the work done in bringing the unit charge from infinity to the point *P*, which is:

$$V = -\int_{\infty}^{r} \vec{\mathbf{E}} . \, d\vec{\mathbf{r}} = -\int_{\infty}^{r} \frac{q}{r^2} \hat{\mathbf{r}} . \, dr\, \hat{\mathbf{r}} = -\int_{\infty}^{r} \frac{q}{r^2} \, dr$$

$$= \left[\frac{q}{r} \right]_{\infty}^{r} = \frac{q}{r}$$

You will learn about electric potential in detail in Units 8 and 9.

You have seen that when a vector field is irrotational (curl of the vector field is zero), it can be written as the gradient of a scalar function, which we call the scalar potential. What if the vector field were to be solenoidal? This brings us to the concept of a vector potential, which finds many applications in Physics. Let us now study about this.

A vector field with a zero divergence is called a **solenoidal** vector field.

3.4.2 Vector Potentials

Consider a solenoidal vector field $\vec{\mathbf{F}}$. So $\vec{\nabla} . \vec{\mathbf{F}} = 0$. Recall that you have studied in Unit 2 that for any vector field $\vec{\mathbf{A}}$, $\vec{\nabla} . \left(\vec{\nabla} \times \vec{\mathbf{A}} \right) = 0$. Therefore we can write:

$$\vec{\nabla}.\vec{\mathbf{F}}=0 \Rightarrow \vec{\mathbf{F}}=\vec{\nabla}\times\vec{\mathbf{A}} \tag{3.39}$$

$\vec{\mathbf{A}}$ is called the **vector potential** associated with a solenoidal vector field $\vec{\mathbf{F}}$. Just as the scalar potential for a conservative field is not unique and you can add an arbitrary constant to it, similarly the vector potential for a solenoidal field is also not unique. You can add the gradient of an arbitrary function, $\vec{\nabla}f\,(x, y, z)$ to the vector potential, and the result would not change because the curl of a gradient of a scalar field is zero ($\vec{\nabla}\times(\vec{\nabla}f)=0$) .So:

$$\left[\vec{\nabla}\times\left(\vec{\mathbf{A}}+\vec{\nabla}f\right)\right]=\vec{\nabla}\times\vec{\mathbf{A}}=\vec{\mathbf{F}} \tag{3.40}$$

3.5 SUMMARY

Concept	Description
Integral of a vector function	■ For a vector function in three dimensions defined as $\vec{\mathbf{b}}(t)=b_1(t)\hat{\mathbf{i}}+b_2(t)\hat{\mathbf{j}}+b_3(t)\hat{\mathbf{k}}$ the indefinite integral of $\vec{\mathbf{b}}(t)$ is given by: $$\int\vec{\mathbf{b}}(t)\,dt = \hat{\mathbf{i}}\int b_1(t)\,dt + \hat{\mathbf{j}}\int b_2(t)\,dt + \hat{\mathbf{k}}\int b_3(t)\,dt$$ The definite integral of $\vec{\mathbf{b}}(t)$ over the interval $[t_1, t_2]$is: $$\int_{t_1}^{t_2}\vec{\mathbf{b}}(t)dt = \hat{\mathbf{i}}\int_{t_1}^{t_2} b_1(t)dt+\hat{\mathbf{j}}\int_{t_1}^{t_2} b_2(t)dt+\hat{\mathbf{k}}\int_{t_1}^{t_2} b_3(t)\,dt$$ ■ For a vector function in two dimensions defined as $\vec{\mathbf{b}}(t)=b_1(t)\hat{\mathbf{i}}+b_2(t)\hat{\mathbf{j}}$, the indefinite integral of $\vec{\mathbf{b}}(t)$ is given by $$\int\vec{\mathbf{b}}(t)\,dt = \hat{\mathbf{i}}\int b_1(t)\,dt + \hat{\mathbf{j}}\int b_2(t)\,dt$$ The definite integral of $\vec{\mathbf{b}}(t)$ over the interval $[t_1, t_2]$ is $$\int_{t_1}^{t_2}\vec{\mathbf{b}}(t)dt = \hat{\mathbf{i}}\int_{t_1}^{t_2} b_1(t)dt+\hat{\mathbf{j}}\int_{t_1}^{t_2} b_2(t)dt$$
Properties of integrals of vector functions	■ For any two vector functions $\vec{\mathbf{f}}(t)$ and $\vec{\mathbf{g}}(t)$ we can write $$\int\left[\vec{\mathbf{f}}(t)+\vec{\mathbf{g}}(t)\right]dt = \int\vec{\mathbf{f}}(t)dt+\int\vec{\mathbf{g}}(t)\,dt$$ ■ For the product of a vector function $\vec{\mathbf{f}}(t)$ and a constant α we can write $$\int\alpha\vec{\mathbf{f}}(t)\,dt = \alpha\int\vec{\mathbf{f}}(t)\,dt$$ ■ For a vector function $\vec{\mathbf{f}}(t)$ and a constant vector $\vec{\mathbf{a}}$, we can write $$\int\vec{\mathbf{a}}.[\vec{\mathbf{f}}(t)]\,dt = \vec{\mathbf{a}}\int\vec{\mathbf{f}}(t)\,dt$$ $$\int\vec{\mathbf{a}}\times[\vec{\mathbf{f}}(t)(t)]\,dt = \vec{\mathbf{a}}\times\int\vec{\mathbf{f}}(t)(t)\,dt$$

Integrals of the scalar and vector products of vector functions

- For any two vector functions of a scalar t, $\vec{\mathbf{a}}(t)$ and $\vec{\mathbf{b}}(t)$, to evaluate the integrals $I_1 = \int [\vec{\mathbf{a}}(t).\vec{\mathbf{b}}(t)]dt$ and $I_2 = \int [\vec{\mathbf{a}}(t) \times \vec{\mathbf{b}}(t)]dt$, we first compute the scalar and vector products in the integrands. We then integrate the result.

Line integral

- A line integral of a scalar or a vector field is a generalization of the single integral where the path of integration may be any curve in space. It can appear in three forms:

$$\int_C f\, dl, \int_C \vec{\mathbf{A}}.d\vec{\mathbf{l}} \text{ and } \int_C \vec{\mathbf{A}} \times d\vec{\mathbf{l}}$$

Work done by a force field $\vec{\mathbf{F}}$

- The work done by the force field $\vec{\mathbf{F}}$ in moving an object along a path C between the points P and Q is given by the line integral

$$W = \int_C \vec{\mathbf{F}}.d\vec{\mathbf{l}}$$

Line integral in the component form

- The line integral of a three-dimensional force field $\vec{\mathbf{F}} = F_1(x,y,z)\hat{\mathbf{i}} + F_2(x,y,z)\hat{\mathbf{j}} + F_3(x,y,z)\hat{\mathbf{k}}$ along a path C in space can be written in terms of its component functions as:

$$W = \int_C \vec{\mathbf{F}}.d\vec{\mathbf{l}} = \int_C [F_1dx + F_2dy + F_3dz]$$

- The line integral of a two-dimensional force field $\vec{\mathbf{F}} = F_1(x,y)\,\hat{\mathbf{i}} + F_2(x,y)\,\hat{\mathbf{j}}$ along a path C in the xy plane can be written as:

$$W = \int_C \vec{\mathbf{F}}.d\vec{\mathbf{l}} = \int_C [F_1dx + F_2dy]$$

Line integral of a vector field using the parametric representation of the path

- The line integral of the vector field $\vec{\mathbf{F}}$ along the curve C which has a parametric representation $\vec{\mathbf{r}}(t)$ with $t_1 \le t \le t_2$ where $\vec{\mathbf{r}}(t)$ is differentiable is:

$$W = \int_C \vec{\mathbf{F}}.d\vec{\mathbf{l}} = \int_{t_1}^{t_2} \left[\vec{\mathbf{F}}[\vec{\mathbf{r}}(t)].\frac{d\vec{\mathbf{r}}(t)}{dt}\right]dt$$

Properties of the line integral

- For a constant α,

$$\int_C \alpha\vec{\mathbf{F}}.d\vec{\mathbf{l}} = \alpha\int_C \vec{\mathbf{F}}.d\vec{\mathbf{l}}$$

- $\int_C [\vec{\mathbf{F}} + \vec{\mathbf{G}}].d\vec{\mathbf{l}} = \int_C \vec{\mathbf{F}}.d\vec{\mathbf{l}} + \int_C \vec{\mathbf{G}}.d\vec{\mathbf{l}}$ for two vector fields $\vec{\mathbf{G}}$ and $\vec{\mathbf{F}}$.

- If the path of integration C is split into two curves C_1 and C_2

$$\int_C \vec{\mathbf{F}}.d\vec{\mathbf{l}} = \int_{C_1} \vec{\mathbf{F}}.d\vec{\mathbf{l}} + \int_{C_2} \vec{\mathbf{F}}.d\vec{\mathbf{l}}$$

- If the orientation of the path of integration is reversed in any line integral, the integral gets multiplied by a negative sign.

Circulation of a vector field

- For any vector field $\vec{\mathbf{F}}$ the closed contour integral along a curve C $\oint_C \vec{\mathbf{F}}.d\vec{\mathbf{l}}$ is also called the **circulation of the vector $\vec{\mathbf{F}}$** around the path C.

Conservative vector fields

- There are three different ways of saying that a vector field $\vec{\mathbf{F}}$ is conservative or that the line integral of the vector field is path independent:
  - The vector field can be written as the gradient of a scalar field Φ: $\vec{\mathbf{F}} = \vec{\nabla}\Phi$
  - The curl of the vector field is zero: $\vec{\nabla} \times \vec{\mathbf{F}} = \vec{\mathbf{0}}$
  - The **circulation** of the vector field is zero: $\oint_C \vec{\mathbf{F}}.d\vec{\mathbf{l}} = 0$

3.6 TERMINAL QUESTIONS

1. Evaluate the following integrals:

   i) $I = \int_0^{\pi} \left[4\sin t\,\hat{\mathbf{i}} - \cos t\,\hat{\mathbf{j}} + (2-t)\hat{\mathbf{k}}\right] dt$

   ii) $I = \int_1^{2} \left[t^2\hat{\mathbf{i}} + te^t\,\hat{\mathbf{j}} + \ln t\,\hat{\mathbf{k}}\right] dt$

2. Obtain a function $\vec{\mathbf{a}}(t)$ which satisfies the relation $\dfrac{d\vec{\mathbf{a}}(t)}{dt} = \sqrt{t}\,\hat{\mathbf{i}} + (\cos \pi t)\,\hat{\mathbf{j}} + \left(\dfrac{4}{t}\right)\hat{\mathbf{k}}$, given that $\vec{\mathbf{a}}(1) = 2\,\hat{\mathbf{i}} + 3\,\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$.

3. Evaluate $\int_1^2 \left[\vec{\mathbf{a}}(t).\dfrac{d\vec{\mathbf{a}}(t)}{dt}\right] dt$ given that $\vec{\mathbf{a}}(2) = 2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$ and $\vec{\mathbf{a}}(1) = \hat{\mathbf{i}} + \hat{\mathbf{j}} + 5\hat{\mathbf{k}}$.

4. Evaluate $\int_0^1 \left[\vec{\mathbf{a}}(t) \times \dfrac{d^2\vec{\mathbf{a}}(t)}{dt^2}\right] dt$ given that $\vec{\mathbf{a}}(t) = 2t\,\hat{\mathbf{i}} + (1-t)\,\hat{\mathbf{j}} + t^2\,\hat{\mathbf{k}}$.

5. A two-dimensional force field is defined as $\vec{\mathbf{F}} = \dfrac{k\left(x\hat{\mathbf{j}} - y\hat{\mathbf{i}}\right)}{x^2 + y^2}$, where k is a constant. Compute the work done by this force in taking a particle from point $P(1,0)$ to $Q(0, 1)$ along a straight line.

6. Determine the work done by a force $\vec{\mathbf{F}} = (x - 3y)\hat{\mathbf{i}} + (2x - y)\hat{\mathbf{j}}$ in moving a particle along a curve in the *xy* plane given by $x = 2t; y = 3t^2$ from $t = 0$ to $t = 2$.

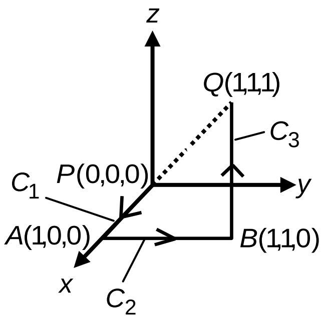


Fig. 3.12: **The path of integration between the points *P* and *Q* for TQ 7.**

7. Calculate the line integral of the vector field $\vec{\mathbf{F}} = (6x^2 + 6y)\,\hat{\mathbf{i}} - 14yz\,\hat{\mathbf{j}} + 10xz^2\,\hat{\mathbf{k}}$ over the path C ($PABQ$) between the points $P(0,0,0)$ and $Q(1,1,1)$ defined by three straight line segments PA, AB and BQ shown in Fig. 3.12.

8. An object of mass m moves along a curve $\vec{\mathbf{r}}(t) = t^2\hat{\mathbf{i}} + \cos t\,\hat{\mathbf{j}} + \sin t\,\hat{\mathbf{k}},\ 0 \le t \le 1.$ Calculate the total force acting on the object and the work done by the force.

9. Show that the line integral of the vector field $\vec{\mathbf{A}} = (2xy + 1)\hat{\mathbf{i}} + (x^2 - 2y)\hat{\mathbf{j}}$ between the points (0, 0) and (2,1) is independent of the path between these points.

10. Calculate the circulation of the vector field $\vec{\mathbf{F}} = y^2\hat{\mathbf{i}} + xy\hat{\mathbf{j}}$ around the closed path along the parabola $y = 2x^2$ from (0,0) to (1,2) and back from (1, 2) to (0, 0) along the straight line $y = 2x$ as shown in Fig. 3.13.

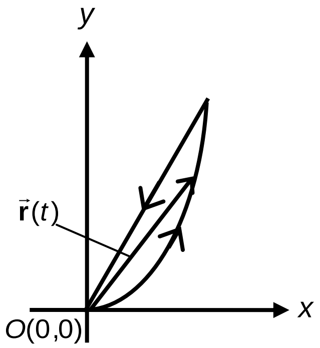


Fig. 3.13: **Figure for TQ 10.**

3.7 SOLUTIONS AND ANSWERS

Self-Assessment Questions

1. a) $I = \hat{\mathbf{i}}\int \frac{4}{1+t^2}\,dt + \hat{\mathbf{j}}\int \frac{2t}{1+t^2}dt$

$$= \left(4\tan^{-1}t\right)\hat{\mathbf{i}} + \ln\left(1+t^2\right)\hat{\mathbf{j}} + \vec{\mathbf{C}}$$

Let $u = 1+t^2$ then

$\frac{du}{dt} = 2t\,dt$

and $\int \frac{2t}{1+t^2}dt = \int \frac{du}{u}$

$\Rightarrow \ln u = \ln(1+t^2)$

b) We use Eq. (3.4) to write down the expression for the velocity of the object as:

$$\vec{\mathbf{v}}(t) = \int \vec{\mathbf{a}}\,dt = -\int 10\hat{\mathbf{k}}\,dt = -10t\,\hat{\mathbf{k}} + \vec{\mathbf{C}}_1 \qquad \text{(i)}$$

To determine $\vec{\mathbf{C}}_1$ (the constant vector) we use the initial condition on the velocity $\vec{\mathbf{v}}(t=0) = \hat{\mathbf{i}} - \hat{\mathbf{k}}$. Substituting $t = 0$ in Eq. (i) we get:

$$\vec{\mathbf{v}}(t=0) = \vec{\mathbf{C}}_1 = \hat{\mathbf{i}} - \hat{\mathbf{k}} \qquad \text{(ii)}$$

Substituting for $\vec{\mathbf{C}}_1$ from Eq. (ii) into Eq. (i) we get

$$\vec{\mathbf{v}}(t) = \hat{\mathbf{i}} - (1+10t)\hat{\mathbf{k}}$$

To determine the position vector $\vec{\mathbf{r}}(t)$ we use Eq. (3.4) to write:

$$\vec{\mathbf{r}}(t) = \int \vec{\mathbf{v}}(t)\,dt = \int \left[\hat{\mathbf{i}} - (1+10t)\hat{\mathbf{k}})\right]dt$$

$$= t\,\hat{\mathbf{i}} - t\hat{\mathbf{k}} - 5t^2\,\hat{\mathbf{k}} + \vec{\mathbf{C}}_2 \qquad \text{(iii)}$$

To evaluate $\vec{\mathbf{C}}_2$ we substitute $t = 0$ in Eq. (iii) and using the given initial position vector $\vec{\mathbf{r}}(t=0) = 2\hat{\mathbf{k}}$ we get:

$$\vec{\mathbf{r}}(t=0) = \vec{\mathbf{C}}_2 = 2\hat{\mathbf{k}} \qquad \text{(iv)}$$

Substituting for $\vec{\mathbf{C}}_2$ from Eq. (iv) into Eq. (iii) we get the position vector of the object:

$$\vec{\mathbf{r}}(t) = t\,\hat{\mathbf{i}} + (2 - t - 5t^2)\hat{\mathbf{k}}$$

2. a) $\vec{\mathbf{a}}(t).\vec{\mathbf{b}}(t) = [t\,\hat{\mathbf{i}} + (1-t)\,\hat{\mathbf{j}} + t^2\,\hat{\mathbf{k}}]\,.\,[3t^2\hat{\mathbf{i}} - t\,\hat{\mathbf{j}}] = 3t^3 - t(1-t) = 3t^3 + t^2 - t$

$$\therefore \int_0^1 \left[\vec{\mathbf{a}}(t).\vec{\mathbf{b}}(t)\right]dt = \int_0^1 \left(3t^3 + t^2 - t\right)dt = \left[\frac{3t^4}{4} + \frac{t^3}{3} - \frac{t^2}{2}\right]_0^1 = \frac{7}{12}$$

$\vec{\mathbf{a}} \times \vec{\mathbf{b}} =$

$$= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ t & (1-t) & t^2 \\ 3t^2 & -t & 0 \end{vmatrix}$$

b) $\vec{\mathbf{a}}(t) \times \vec{\mathbf{b}}(t) = \left[t\,\hat{\mathbf{i}} + (1-t)\,\hat{\mathbf{j}} + t^2\,\hat{\mathbf{k}}\right] \times \left[3t^2\hat{\mathbf{i}} - t\,\hat{\mathbf{j}}\right] = t^3\,\hat{\mathbf{i}} + 3t^4\,\hat{\mathbf{j}} + (3t^3 - 4t^2)\hat{\mathbf{k}}$

$$\therefore \int_0^1 \left[\vec{\mathbf{a}}(t) \times \vec{\mathbf{b}}(t)\right]dt = \int_0^1 \left[t^3\,\hat{\mathbf{i}} + 3t^4\hat{\mathbf{j}} + (3t^3 - 4t^2)\hat{\mathbf{k}}\right]dt$$

$$= \left[\frac{t^4}{4}\hat{\mathbf{i}} + \frac{3t^5}{5}\hat{\mathbf{j}} + \left(\frac{3t^4}{4} - \frac{4t^3}{3}\right)\hat{\mathbf{k}}\right]_0^1 \qquad \text{(i)}$$

or $\int_0^1 \left[\vec{\mathbf{a}}(t)\times\vec{\mathbf{b}}(t)\right]dt = \frac{1}{4}\hat{\mathbf{i}} + \frac{3}{5}\hat{\mathbf{j}} - \frac{7}{12}\hat{\mathbf{k}}$

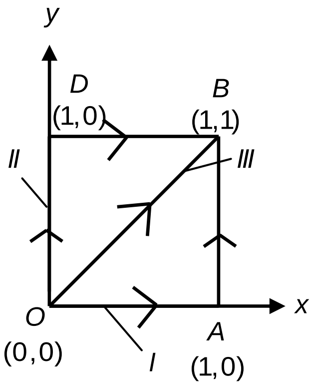


Fig. 3.14: **Path of integration for SAQ 3.**

3. We evaluate these integrals using Eq. (3.19b) with $F_1 = xy$ and $F_2 = x^2 + 1$

Along the path *I* the integral is the sum of the integrals along the straight line segments *OA* and *AB* (see Fig. 3.14):

$$I_I = \int_I \vec{\mathbf{F}}.d\vec{\mathbf{l}} = \int_{OA} \vec{\mathbf{F}}.\, d\vec{\mathbf{l}} + \int_{AB} \vec{\mathbf{F}}.\, d\vec{\mathbf{l}} = \int_{OA} \left[F_x dx + F_y dy\right] + \int_{AB} \left[F_x dx + F_y dy\right]$$

$$= \int_{OA} \left[xydx + (x^2+1)dy\right] + \int_{AB} \left[xydx + (x^2+1)dy\right] \qquad \text{(i)}$$

Along *OA*,

$$0 \le x \le 1; y = 0 \Rightarrow dy = 0 \qquad \text{(ii)}$$

Along *AB*

$$0 \le y \le 1; x = 1 \Rightarrow dx = 0 \qquad \text{(iii)}$$

So substituting from Eqs. (ii) and (iii) into Eq.(i) we get

$$I_I = \int_{AB} \left[(x^2+1)dy\right] = \int_0^1 (1+1)\,dy = [2y]_0^1 = 2$$

Along the path *II* the integral is the sum of the integrals along the straight line segments *OD* and *DB*:

$$I_{II} = \int_{II} \vec{\mathbf{F}}.\, d\vec{\mathbf{l}} = \int_{OD} \vec{\mathbf{F}}.\, d\vec{\mathbf{l}} + \int_{DB} \vec{\mathbf{F}}.\, d\vec{\mathbf{l}} = \int_{OD} \left[F_x dx + F_y dy\right] + \int_{DB} \left[F_x dx + F_y dy\right]$$

$$= \int_{OD} \left[xydx + (x^2+1)dy\right] + \int_{DB} \left[xydx + (x^2+1)dy\right] \qquad \text{(iv)}$$

Along *OD*,

$$0 \le y \le 1; x = 0 \Rightarrow dx = 0 \qquad \text{(v)}$$

Along *DB*,

$$0 \le x \le 1; y = 1 \Rightarrow dy = 0 \qquad \text{(vi)}$$

So substituting from Eqs. (v) and (vi) into Eq.(iv) we get

$$I_{II} = \int_{OD} dy + \int_{DB} xdx = \int_0^1 dy + \int_0^1 xdx = [y]_0^1 + \left[\frac{x^2}{2}\right]_0^1 = \frac{3}{2}.$$

Along the path *III* the integral is the integral along the straight line segment *OB*:

$$I_{III} = \int_{III} \vec{\mathbf{F}}.\, d\vec{\mathbf{l}} = \int_{OB} \vec{\mathbf{F}}.\, d\vec{\mathbf{l}} = \int_{OB} \left[F_x dx + F_y dy\right] = \int_{OB} \left[xydx + (x^2+1)dy\right]$$

(vii)

The equation of the straight line *OB* is *y* = *x*. The limits on *x* and *y* are

$$0 \le x \le 1; \quad 0 \le y \le 1 \qquad \text{(viii)}$$

So substituting from Eqs. (viii) and *y*= *x* into Eq.(vii) and using the methods of Example 3.3 we get:

$$I_{III} = \int_{OB} [xydx + (x^2+1)dy] = \int_0^1 x^2 dx + \int_0^1 (y^2+1)dy$$

On evaluating these integrals we get

$$I_{III} = \left[\frac{x^3}{3}\right]_0^1 + \left[\frac{y^3}{3} + y\right]_0^1 = \frac{1}{3} + \frac{4}{3} = \frac{5}{3}$$

As you can see, the value of the line integral along each of these paths is different.

4. The parametric equation of the parabola $y = x^2$ (Fig. 3.15) is:

$$x(t) = t, y(t) = t^2$$

You can check that this satisfies the equation $y = x^2$. To obtain the end points, we write

$$x(t_1) = t_1 = 0; \quad y(t_1) = t_1^2 = 0 \Rightarrow t_1 = 0$$

and

$$x(t_2) = t_2 = 2; \quad y(t_2) = t_2^2 = 4 \Rightarrow t_2 = 2$$

So the parametric representation is

$$\vec{r}(t) = t\,\hat{\mathbf{i}} + t^2\hat{\mathbf{j}}; \qquad 0 \le t \le 2$$

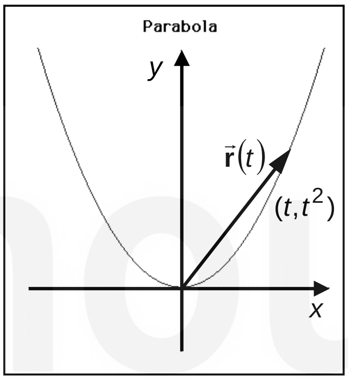


Fig. 3.15

5. We use Eq. (3.27) to evaluate the line integral with:

$$\vec{\mathbf{F}} = -\frac{\vec{\mathbf{r}}}{r^3} = -\frac{x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}}{(x^2+y^2+z^2)^{3/2}}; \vec{\mathbf{r}}(t) = t\,\hat{\mathbf{i}} + t\,\hat{\mathbf{j}} + t\,\hat{\mathbf{k}}; x(t) = y(t) = z(t) = t;$$

and $\quad t_1 = 1; t_2 = 3$

The derivative of $\vec{\mathbf{r}}$ is:

$$\frac{d\vec{\mathbf{r}}}{dt} = \frac{d}{dt}\left[t\,\hat{\mathbf{i}} + t\,\hat{\mathbf{j}} + t\,\hat{\mathbf{k}}\right] = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}} \qquad \text{(i)}$$

In terms of t, we can write $\vec{\mathbf{F}}$ as:

$$\vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) = -\frac{\left[t\,\hat{\mathbf{i}} + t\,\hat{\mathbf{j}} + t\,\hat{\mathbf{k}}\right]}{(t^2+t^2+t^2)^{3/2}} = -\frac{\left[t\,\hat{\mathbf{i}} + t\,\hat{\mathbf{j}} + t\,\hat{\mathbf{k}}\right]}{(3t^2)^{3/2}} == -\frac{1}{3\sqrt{3}\,t^2}\left[\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}\right] \text{(ii)}$$

Using the results of Eqs. (i) and (ii) in Eq. (3.27) we get:

$$I = \int_1^3 \left[\vec{\mathbf{F}}.\frac{d\vec{\mathbf{r}}}{dt}\right] dt = -\int_1^3 \left[\frac{(\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}})}{3\sqrt{3}\,t^2}.(\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}})\right] dt = -\int_1^3 \frac{1}{\sqrt{3}\,t^2} dt = -\frac{2\sqrt{3}}{9}$$

6. Using Eq. (3.25) we write the parametric equation for the circle C $x^2 + y^2 = 4$ as:

$$\vec{\mathbf{r}}(t) = 2\cos t\,\hat{\mathbf{i}} + 2\sin t\,\hat{\mathbf{j}}, \; 0 \le t \le 2\pi \qquad \text{(i)}$$

Writing down $\vec{\mathbf{A}}$ in terms of t using $x(t) = 2\cos t; y(t) = 2\sin t$ we get:

$$\vec{\mathbf{A}}(\vec{\mathbf{r}}(t)) = 4\cos t \sin t\,\hat{\mathbf{i}} + (12\cos^2 t + 2\sin t)\,\hat{\mathbf{j}} \qquad \text{(ii)}$$

$\int_0^{2\pi} \sin^2 t \cos t\, dt$

$= \int_0^0 u^2 du = 0$ (using $u = \sin t$ and $du = \cos t\, dt$)

$\int_0^{2\pi} \cos t\ dt = -\sin t\Big|_0^{2\pi}$

$= 0$

$\int_0^{2\pi} \sin t \cos t\, dt = \dfrac{\sin^2 t}{2}\Bigg|_0^{2\pi}$

$= 0$

Differentiating Eq. (i) w.r.t. t we get

$$\frac{d\vec{\mathbf{r}}}{dt} = -2\sin t\,\hat{\mathbf{i}} + 2\cos t\,\hat{\mathbf{j}}$$

Using Eq. (3.27), with $\vec{\mathbf{F}} = \vec{\mathbf{A}}$, we get the circulation of $\vec{\mathbf{A}}$ as (read the see margin remark):

$$\oint_C \vec{\mathbf{A}}.d\vec{\mathbf{l}} = \int_0^{2\pi}\left[4\cos t\sin t\,\hat{\mathbf{i}} + (12\cos^2 t + 2\sin t)\hat{\mathbf{j}}\right].\left[-2\sin t\,\hat{\mathbf{i}} + 2\cos t\,\hat{\mathbf{j}}\right]dt$$

$$= \int_0^{2\pi}\left[-8\sin^2 t\cos t + 24\cos^3 t + 4\sin t\cos t\right]dt$$

$$= \int_0^{2\pi}\left[-8\sin^2 t\cos t + 24(1-\sin^2 t)\cos t + 4\sin t\cos t\right]dt$$

$$= \int_0^{2\pi}[-32\sin^2 t\cos t + 4\sin t\cos t + 24\cos t]\ dt = 0$$

$$\therefore \qquad \oint_C \vec{\mathbf{A}}.d\vec{\mathbf{l}} = 0$$

The circulation of the vector field is zero.

Terminal Questions

1. i) $I = \hat{\mathbf{i}}\int_0^{\pi} 4\sin t\, dt - \hat{\mathbf{j}}\int_0^{\pi}\cos t\, dt + +\hat{\mathbf{k}}\int_0^{\pi}(2-t)dt$

$$= \hat{\mathbf{i}}[-4\cos t]_0^{\pi} - \hat{\mathbf{j}}[\sin t]_0^{\pi} + \hat{\mathbf{k}}\left[2t - \frac{t^2}{2}\right]_0^{\pi}$$

$$= 8\,\hat{\mathbf{i}} + \left(2\pi - \frac{\pi^2}{2}\right)\hat{\mathbf{k}}$$

ii) $I = \int_1^2 \left[t^2\,\hat{\mathbf{i}} + t e^t\,\hat{\mathbf{j}} + \ln t\,\hat{\mathbf{k}}\right]dt$

$$= \left[\frac{t^3}{3}\right]_1^2\hat{\mathbf{i}} + \left[te^t - e^t\right]_1^2\hat{\mathbf{j}} + [t\ln t - t]_1^2\hat{\mathbf{k}} = \frac{7}{3}\hat{\mathbf{i}} + e^2\hat{\mathbf{j}} + [2\ln 2 - 1]\hat{\mathbf{k}}$$

2. Using Eq. (3.4) with $\vec{\mathbf{b}}(t) = \sqrt{t}\,\hat{\mathbf{i}} + (\cos\pi t)\hat{\mathbf{j}} + \left(\dfrac{4}{t}\right)\hat{\mathbf{k}}$ we can write:

$$\vec{\mathbf{a}}(t) = \int\left[\sqrt{t}\,\hat{\mathbf{i}} + (\cos\pi t)\,\hat{\mathbf{j}} + \left(\frac{4}{t}\right)\hat{\mathbf{k}}\right]dt + \vec{\mathbf{C}}$$

where $\vec{\mathbf{C}}$ is a constant vector. Then

$$\vec{\mathbf{a}}(t) = \frac{2}{3}t^{3/2}\,\hat{\mathbf{i}} + \frac{\sin \pi t}{\pi}\,\hat{\mathbf{j}} + 4\ln t\,\hat{\mathbf{k}} + \vec{\mathbf{C}} \qquad \text{(i)}$$

Substituting $t = 1$ in Eq. (i) and given that $\vec{\mathbf{a}}(1) = 2\,\hat{\mathbf{i}} + 3\,\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$ we get:

$$\vec{\mathbf{a}}(t=1) = \frac{2}{3}\hat{\mathbf{i}} + \vec{\mathbf{C}} \qquad \text{(ii)}$$

$$= 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$$

$$\Rightarrow \qquad \vec{\mathbf{C}} = \frac{4}{3}\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}} \qquad \text{(iii)}$$

Substituting for $\vec{\mathbf{C}}$ in Eq. (i) we get:

$$\vec{\mathbf{a}}(t) = \frac{2}{3}(t^{3/2} + 2)\,\hat{\mathbf{i}} + \left(\frac{\sin \pi t}{\pi} + 3\right)\hat{\mathbf{j}} + (4\ln t + 4)\,\hat{\mathbf{k}}$$

3. For any vector $\vec{\mathbf{a}}(t)$ we can write:

$$\frac{d}{dt}\left[\vec{\mathbf{a}}(t).\vec{\mathbf{a}}(t)\right] = \vec{\mathbf{a}}(t).\frac{d\vec{\mathbf{a}}(t)}{dt} + \frac{d\vec{\mathbf{a}}(t)}{dt}.\vec{\mathbf{a}}(t) = 2\left[\vec{\mathbf{a}}(t).\frac{d\vec{\mathbf{a}}(t)}{dt}\right] \qquad \text{(i)}$$

or

$$\vec{\mathbf{a}}(t).\frac{d\vec{\mathbf{a}}(t)}{dt} = \frac{1}{2}\frac{d}{dt}\left[\vec{\mathbf{a}}(t).\vec{\mathbf{a}}(t)\right] \qquad \text{(ii)}$$

Then we can write:

$$\int_1^2 \left[\vec{\mathbf{a}}(t).\frac{d\vec{\mathbf{a}}(t)}{dt}\right]dt = \int_1^2 \frac{1}{2}\frac{d}{dt}(\vec{\mathbf{a}}(t).\vec{\mathbf{a}}(t))\,dt = \frac{1}{2}\int_1^2 d\left[\vec{\mathbf{a}}(t).\vec{\mathbf{a}}(t)\right] = \frac{1}{2}\left[\vec{\mathbf{a}}(t).\vec{\mathbf{a}}(t)\right]_1^2$$

Using $\vec{\mathbf{a}}(2) = 2\,\hat{\mathbf{i}} - 3\,\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$ and $\vec{\mathbf{a}}(1) = \hat{\mathbf{i}} + \hat{\mathbf{j}} + 5\hat{\mathbf{k}}$, we get:

$$\int_1^2 \left[\vec{\mathbf{a}}(t).\frac{d\vec{\mathbf{a}}(t)}{dt}\right]dt = \frac{1}{2}\left[\vec{\mathbf{a}}(2).\vec{\mathbf{a}}(2) - \vec{\mathbf{a}}(1).\vec{\mathbf{a}}(1)\right] = \frac{1}{2}[29 - 27] = 1$$

4. For any vector $\vec{\mathbf{a}}(t)$ we can write:

$$\frac{d}{dt}\left[\vec{\mathbf{a}}(t)\times\frac{d\vec{\mathbf{a}}(t)}{dt}\right] = \frac{d\vec{\mathbf{a}}(t)}{dt}\times\frac{d\vec{\mathbf{a}}(t)}{dt} + \vec{\mathbf{a}}(t)\times\frac{d^2\vec{\mathbf{a}}(t)}{dt^2} = \vec{\mathbf{a}}(t)\times\frac{d^2\vec{\mathbf{a}}(t)}{dt^2} \qquad \text{(i)}$$

as $\dfrac{d\vec{\mathbf{a}}(t)}{dt}\times\dfrac{d\vec{\mathbf{a}}(t)}{dt} = \vec{\mathbf{0}}$. So we can write:

$$\vec{\mathbf{a}}(t)\times\frac{d^2\vec{\mathbf{a}}(t)}{dt^2} = \frac{d}{dt}\left[\vec{\mathbf{a}}(t)\times\frac{d\vec{\mathbf{a}}(t)}{dt}\right] \qquad \text{(ii)}$$

Therefore,

$$\int_0^1 \left[\vec{\mathbf{a}}(t)\times\frac{d^2\vec{\mathbf{a}}(t)}{dt^2}\right]dt = \int_0^1 \frac{d}{dt}\left[\vec{\mathbf{a}}(t)\times\frac{d\vec{\mathbf{a}}(t)}{dt}\right]dt = \int_0^1 d\left[\vec{\mathbf{a}}(t)\times\frac{d\vec{\mathbf{a}}(t)}{dt}\right] \qquad \text{(iii)}$$

The integral is then:

$$\int_0^1 \left[\vec{\mathbf{a}}(t)\times\frac{d^2\vec{\mathbf{a}}(t)}{dt^2}\right]dt = \left[\vec{\mathbf{a}}(t)\times\frac{d\vec{\mathbf{a}}(t)}{dt}\right]_0^1 \qquad \text{(iv)}$$

Given that $\vec{\mathbf{a}}(t) = 2t\,\hat{\mathbf{i}} + (1-t)\,\hat{\mathbf{j}} + t^2\,\hat{\mathbf{k}}$ we can write:

$$\vec{\mathbf{a}}(t)\times\frac{d\vec{\mathbf{a}}}{dt} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2t & 1-t & t^2 \\ 2 & -1 & 2t \end{vmatrix}$$

$$\frac{d\vec{\mathbf{a}}(t)}{dt} = 2\,\hat{\mathbf{i}} - \hat{\mathbf{j}} + 2t\,\hat{\mathbf{k}}$$

$$\vec{\mathbf{a}}(t)\times\frac{d\vec{\mathbf{a}}(t)}{dt} = \left(2t\,\hat{\mathbf{i}} + (1-t)\,\hat{\mathbf{j}} + t^2\,\hat{\mathbf{k}}\right)\times\left(2\,\hat{\mathbf{i}} - \hat{\mathbf{j}} + 2t\,\hat{\mathbf{k}}\right) = (2t-t^2)\,\hat{\mathbf{i}} - 2t^2\,\hat{\mathbf{j}} - 2\,\hat{\mathbf{k}} \qquad \text{(v)}$$

$$\int_0^1 \left[\vec{\mathbf{a}}(t)\times\frac{d^2\vec{\mathbf{a}}(t)}{dt^2}\right]dt = \left[\vec{\mathbf{a}}(t)\times\frac{d\vec{\mathbf{a}}(t)}{dt}\right]_0^1$$

$$= \hat{\mathbf{i}} - 2\,\hat{\mathbf{j}}$$

5. In order to evaluate the integral we have to express $d\vec{\mathbf{r}}$ and $\vec{\mathbf{F}}$ as a function of the same parameter, say t. The equation of PQ (Fig. 3.16)as explained in Example 3.4 is:

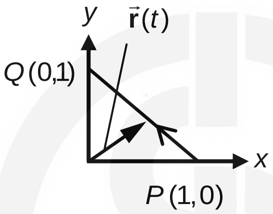


Fig. 3.16: **Figure for TQ 5.**

$$x + y = 1 \Rightarrow y = 1 - x \qquad \text{(i)}$$

This can be expressed in the parametric form as $x(t) = t\,; y(t) = 1 - t$, where t goes from 1 to 0. Following the steps in Example 3.5, we first write the position vector:

$$\vec{\mathbf{r}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} = t\,\hat{\mathbf{i}} + (1-t)\,\hat{\mathbf{j}} \text{ and } \frac{d\vec{\mathbf{r}}}{dt} = \hat{\mathbf{i}} - \hat{\mathbf{j}}$$

Next we write $\vec{\mathbf{F}} = \vec{\mathbf{F}}(t)$

$$\vec{\mathbf{F}} = k\,\frac{t\,\hat{\mathbf{j}} + (t-1)\hat{\mathbf{i}}}{t^2 + (1-t)^2}$$

$$\therefore \qquad \vec{\mathbf{F}}.\frac{d\vec{\mathbf{r}}}{dt} = \frac{k\left[(t-1)\hat{\mathbf{i}} + t\hat{\mathbf{j}}\right].\left[\hat{\mathbf{i}} - \hat{\mathbf{j}}\right]}{t^2 + (1-t)^2} = k\,\frac{(t-1)-t}{2t^2 - 2t + 1} = -\frac{k}{2t^2 - 2t + 1}$$

The work done is calculated using Eq. (3.30) as:

$$W = -k\int_1^0 \frac{dt}{2t^2 - 2t + 1}$$

$$= -\frac{k}{2}\int_1^0 \frac{dt}{t^2 - t + \frac{1}{2}} = -\frac{k}{2}\int_1^0 \frac{dt}{\left(t - \frac{1}{2}\right)^2 + \frac{1}{4}} \qquad \text{(ii)}$$

$$= -\frac{k}{2}(-\pi) = \frac{k\pi}{2} \qquad \text{(read the margin remark)}$$

Let

$$u = t - \frac{1}{2} \Rightarrow du = dt$$

$$\therefore \int_1^0 \frac{dt}{\left(t - \frac{1}{2}\right)^2 + \frac{1}{4}}$$

$$= \int_{1/2}^{-1/2} \frac{du}{u^2 + \frac{1}{4}}$$

$$= \left[2\tan^{-1}(2u)\right]_{1/2}^{-1/2} = -\pi$$

Alternative Method

The integral can be evaluated using Eq. (3.19b) as well, as follows:

$$F_1 = \frac{-y}{x^2+y^2};\ \ F_2 = \frac{x}{x^2+y^2}$$

as : $$\int_{PQ} \vec{\mathbf{F}}.d\vec{\mathbf{r}} = \int_{PQ}\left[-\frac{ky}{x^2+y^2}\right]dx + \int_{PQ}\left[\frac{kx}{x^2+y^2}\right]dy \qquad \text{(iii)}$$

The equation of the straight line PQ is $x + y = 1$

$$\therefore \quad y = 1 - x \text{ and } dy = -dx, \qquad \text{(iv)}$$

$$x^2 + y^2 = x^2 + (1-x)^2 = 2x^2 - 2x + 1 \qquad \text{(v)}$$

Substituting from Eqs. (iv) and (v) into Eq.(iii) we get (see margin remark):

$$\int_{PQ} \vec{\mathbf{F}}.d\vec{\mathbf{r}} = \int_{PQ} \frac{-xdx + (x-1)dx}{2x^2-2x+1} = -k\int_{x=1}^{0} \frac{dx}{2x^2-2x+1} = \frac{k\pi}{2} \quad \text{(vi)}$$

Note that the integral evaluated in Eq. (vi) is the same as the integral you evaluated in Eq. (ii).

6. We use Eq. (3.29b) to evaluate the line integral with:

$$\vec{\mathbf{F}} = (x-3y)\,\hat{\mathbf{i}} + (2x-y)\,\hat{\mathbf{j}},\ x(t) = 2t, y(t) = 3t^2,\ t_1 = 0\ ;\ t_2 = 2 \quad \text{(i)}$$

From Eq. (i) we write:

$$x'(t) = 2,\ y'(t) = 6t \qquad \text{(ii)}$$

In terms of t, we can write the components of $\vec{\mathbf{F}}$ as:

$$F_1 = (x-3y) = 2t - 9t^2,\ F_2 = (2x-y) = 4t - 3t^2 \qquad \text{(iii)}$$

Using the results of Eqs. (i) and (ii) in Eq. (3.29b) we get:

$$I = \int_0^2 (F_1\, x'(t) + F_2 y'(t))dt = \int_0^2 (4t - 18t^2 + 24t^2 - 18t^3)dt$$

$$= \left[2t^2 + 2t^3 - \frac{9t^4}{2}\right]_0^2 = -48$$

7. We calculate the line integral of the vector field using Eq. (3.19a) with: $F_x = (6x^2 + 6y)$, $F_y = -14yz$, $F_z = 10xz^2$. Then

$$I = \int_C \left[(6x^2+6y)dx - (14yz)dy + \left(10xz^2\right)dz\right]$$

We use the path C between P and Q shown in Fig. 3.17. It consists of the straight line C_1 from $P(0,0,0)$ to $A(1,0,0)$, then the straight line C_2 from $A(1,0,0)$ to $B(1,1,0)$ and finally the straight line C_3 from $B(1,1,0)$ to $Q(1,1,1)$. Using the property of the line integral given in Eq. (3.33), we can write the line integral along the path C as:

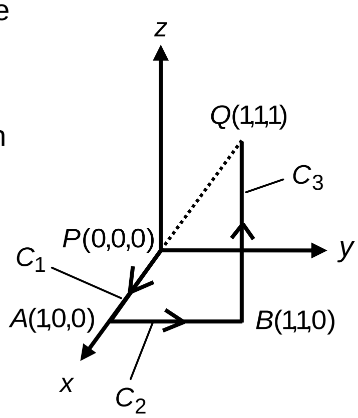


Fig. 3.17: **The path of integration between the points** ***P*** **and** ***Q*** **for TQ 7.**

$$I = I_{PA} + I_{AB} + I_{BQ}$$

$$= \int_{PA}\left[(6x^2+6y)dx - (14yz)dy + \left(10xz^2\right)dz\right]$$

$$+ \int_{AB}\left[(6x^2+6y)dx - (14yz)dy + \left(10xz^2\right)dz\right] \qquad \text{(i)}$$

$$+ \int_{BQ}\left[(6x^2+6y)dx - (14yz)dy + \left(10xz^2\right)dz\right]$$

Along PA, $0 \le x \le 1$, $y = z = 0 \Rightarrow dy = dz = 0$

$$\therefore I_{PA} = \int_{x=0}^{1} 6x^2 dx = \left[\frac{6x^3}{3}\right]_0^1 = 2 \qquad \text{(ii)}$$

Along AB: $0 \le y \le 1,\ x = 1,\ z = 0 \Rightarrow dx = dz = 0$

$$\therefore I_{AB} = -\int_{y=0}^{1} 14yz\,dy = 0 \qquad \text{(iii)}$$

Along $BQ, 0 \le z \le 1,\ x = 1, y = 1 \Rightarrow dx = dy = 0$

And $$I_{BQ} = \int_{z=0}^{1} 10xz^2 dz = \left[\frac{10z^3}{3}\right]_0^1 = \frac{10}{3} \qquad \text{(iv)}$$

$$\therefore I = 2 + 0 + \frac{10}{3} = \frac{16}{3}$$

8. We first derive an expression for the acceleration of the object: $\vec{\mathbf{a}} = \dfrac{d^2\vec{\mathbf{r}}}{dt^2}$

$$\frac{d\vec{\mathbf{r}}}{dt} = \frac{d}{dt}\left[t^2\,\hat{\mathbf{i}} + \cos t\,\hat{\mathbf{j}} + \sin t\,\hat{\mathbf{k}}\right] = 2t\hat{\mathbf{i}} - \sin t\ \hat{\mathbf{j}} + \cos t\ \hat{\mathbf{k}} \qquad \text{(i)}$$

$$\frac{d^2\vec{\mathbf{r}}}{dt^2} = \frac{d}{dt}\left[2t\hat{\mathbf{i}} - \sin t\ \hat{\mathbf{j}} + \cos t\ \hat{\mathbf{k}}\right] = 2\hat{\mathbf{i}} - \cos t\ \hat{\mathbf{j}} - \sin t\ \hat{\mathbf{k}}$$

The force acting on the object is:

$$\vec{\mathbf{F}} = m\vec{\mathbf{a}} = m(2\hat{\mathbf{i}} - \cos t\,\hat{\mathbf{j}} - \sin t\ \hat{\mathbf{k}}) \qquad \text{(ii)}$$

Using Eq. (3.30), the work done is:

$$W = \int_0^1 \left[\vec{\mathbf{F}}.\frac{d\vec{\mathbf{r}}}{dt}\right] dt \qquad \text{(iii)}$$

Using the results of Eqs. (i) and (ii) in Eq.(iii):

$$W = m\int_0^1 \left[2\hat{\mathbf{i}} - \cos t\,\hat{\mathbf{j}} - \sin t\,\hat{\mathbf{k}}\right].\left[2t\,\hat{\mathbf{i}} - \sin t\,\hat{\mathbf{j}} + \cos t\,\hat{\mathbf{k}}\right] dt$$

$$= m\int_0^1 \left[4t + \sin t\cos t - \sin t\cos t\ \right]\ dt = m\int_0^1 [4t]\,dt = m\left[2t^2\right]_0^1 = 2m$$

The equation of a straight line between two points (x_1,y_1) and (x_2,y_2) in the xy plane is:

$$y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$$

For the line AB, we get

$$y = \frac{1}{2}x$$

$(\because x_1 = 0, y_1 = 0, x_2 = 2, y_2 = 1)$

9. Refer to Fig. 3.18. Let us calculate the line integral of the field $\vec{\mathbf{A}}$ between the points $A(0,0)$ and $B(2,1)$, along two different paths: One is the straight line AB and the other is ACB. Let us first consider the path of integration AB. The equation of the straight line AB is $y = \dfrac{x}{2}$ (read the margin remark).

We use Eq. (3.19b) for the line integral along AB with

$$\vec{\mathbf{F}} = \vec{\mathbf{A}} \quad \text{and} \quad F_1 = 2xy + 1;\ F_2 = x^2 - 2y \qquad \text{(i)}$$

We get the integral of $\vec{\mathbf{A}}$ along AB as:

$$I_{AB} = \int_{AB} \vec{\mathbf{A}}.d\vec{\mathbf{l}} = \int_{AB} (2xy+1)\,dx + \int_{AB} (x^2 - 2y)\,dy \qquad \text{(ii)}$$

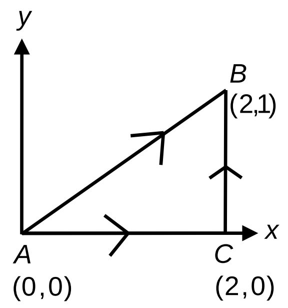


Fig. 3.18: **Paths of integration for TQ 9.**

The limits on x and y are as follows:

$$0 \le x \le 2;\; 0 \le y \le 1 \qquad \text{(iii)}$$

To evaluate the line integral over AB, we need to write each one of the integrals in Eq. (ii) as an integral over one variable. So we write (read the margin remark):

$$I_{AB} = \int_0^2 (2xy+1)\,dx + \int_0^1 (x^2 - 2y)\,dy$$

$$= \int_0^2 (x^2+1)\,dx + \int_0^1 (4y^2 - 2y)\,dy \qquad \text{(iv)}$$

$$= \left[\frac{x^3}{3} + x\right]_0^2 + \left[\frac{4y^3}{3} - y^2\right]_0^1 = 5$$

Note that the integration is along the line AB given by $y = \dfrac{x}{2}$ and not along the x or y axes. Therefore, when we evaluate Eq. (ii), to integrate over x, we must write y in terms of x (i.e $y = \dfrac{x}{2}$) in the integrand. Similarly, when we integrate over y, we write x in terms of y (i.e., $x = 2y$).

Next we evaluate the integral along ACB, which is the sum of the line integrals over AC and CB.

$$\therefore\; I_{ACB} = \int_{ACB} \vec{\mathbf{A}}.d\vec{\mathbf{l}} = \int_{AC} \vec{\mathbf{A}}.d\vec{\mathbf{l}} + \int_{CB} \vec{\mathbf{A}}.d\vec{\mathbf{l}} \qquad \text{(v)}$$

Along AC, the value of y is a constant (y = 0) and therefore dy = 0.

$$\int_{AC} \vec{\mathbf{A}}.d\vec{\mathbf{l}} = \int_0^2 (2xy+1)\,dx = \int_0^1 (2x(0)+1)\,dx = [x]_0^2 = 2 \qquad \text{(vi)}$$

Along CB, the value of x is constant (x = 2), so dx = 0.

$$\therefore\; \int_{CB} \vec{\mathbf{A}}.d\vec{\mathbf{l}} = \int_0^1 (x^2 - 2y)\,dy = \int_0^1 (4-2y)\,dy = \left[4y - y^2\right]_0^1 = 3 \qquad \text{(vii)}$$

Substituting from Eq. (vi) and (vii) into Eq. (v), we get:

$$\therefore\; I_{ACB} = 2 + 3 = 5. \qquad \text{(viii)}$$

Since the value of the integral is same for two different paths AB and ACB, we can say that the line integral is path independent.

10. The closed path of integration C is made up of the curves C_1 and C_2 between the points O(0,0)and A (1,2) (see Fig. 3.14 reproduced here as Fig. 3.19). C_1 is described by the parabola $y = 2x^2$ between the points O and A. C_2 is the straight line $y = 2x$ from A to O, so the circulation of $\vec{\mathbf{F}}$ is:

$$I = \int_C \vec{\mathbf{F}}.d\vec{\mathbf{l}} = \int_{C_1} \vec{\mathbf{F}}.d\vec{\mathbf{l}} + \int_{C_2} \vec{\mathbf{F}}.d\vec{\mathbf{l}}$$

We parameterize the parabola $y = 2x^2$ as :

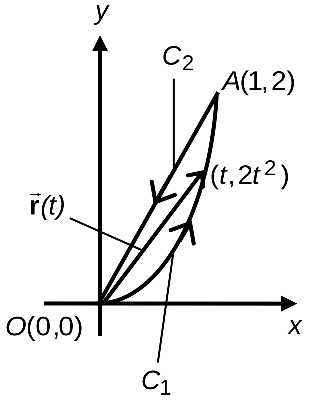


Fig. 3.19: **Figure for TQ 10.**

$$\vec{\mathbf{r}}(t) = t\,\hat{\mathbf{i}} + 2t^2\,\hat{\mathbf{j}};\ x(t) = t\ ;\ y(t) = 2t^2\ ;\ 0 \le t \le 1$$

Therefore $\dfrac{d\hat{\mathbf{r}}}{dt} = \hat{\mathbf{i}} + 4t\,\hat{\mathbf{j}},\ \vec{\mathbf{F}} = y^2\hat{\mathbf{i}} + xy\,\hat{\mathbf{j}} = 4t^4\hat{\mathbf{i}} + 2t^3\,\hat{\mathbf{j}}$ (i)

Using Eq. (3.30) we then get:

$$I_1 = \int_{C_1}\left[\vec{\mathbf{F}}.\frac{d\vec{\mathbf{r}}}{dt}\right]dt = \int_0^1\left[4t^4\hat{\mathbf{i}} + 2t^3\hat{\mathbf{j}}\right].\left[\hat{\mathbf{i}} + 4t\,\hat{\mathbf{j}}\right]dt = \int_0^1\left[4t^4 + 8t^4\right]dt$$

$$= \int_0^1\left[12t^4\right]dt = \left[\frac{12t^5}{5}\right]_0^1 = \frac{12}{5}$$

Here we have used the parametric representation to evaluate the integral along AO. Alternatively we can write, using Eq. (3.19b) and $y = 2x$:

$$I_2 = \int_{C_2}\left(y^2dx + xydy\right)$$

$$= \int_{C_2}\left(4x^2dx + \frac{y^2}{2}dy\right)$$

$$= \int_1^0 4x^2dx + \int_2^0\frac{y^2}{2}dy$$

$$= -\frac{8}{3}$$

We next calculate $I_2 = \int_{C_2}\vec{\mathbf{F}}.d\vec{\mathbf{l}}$. The parametric representation for the straight line C_2 is

$$\vec{\mathbf{r}}(t) = t\,\hat{\mathbf{i}} + 2t\,\hat{\mathbf{j}};\ x(t) = t,\ y(t) = 2t,\ 1 \le t \le 0$$

Then, $\dfrac{d\vec{\mathbf{r}}}{dt} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}},\ \vec{\mathbf{F}} = y^2\hat{\mathbf{i}} + xy\hat{\mathbf{j}} = 4t^2\hat{\mathbf{i}} + 2t^2\hat{\mathbf{j}}$ (ii)

Using Eq. (3.30) we get:

$$I_2 = \int_{C_2}\left[\vec{\mathbf{F}}.\frac{d\vec{\mathbf{r}}}{dt}\right]dt = \int_1^1\left[4t^2\hat{\mathbf{i}} + 2t^2\hat{\mathbf{j}}\right].\left[\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}}\right]dt = \int_1^0\left[4t^2 + 4t^2\right]dt$$

$$= \int_1^0\left[8t^2\right]dt = \left[\frac{8t^3}{3}\right]_1^0 = -\frac{8}{3}$$

Finally, adding I_1 and I_2 we get: $I = I_1 + I_2 = \dfrac{12}{5} - \dfrac{8}{3} = -\dfrac{4}{15}$