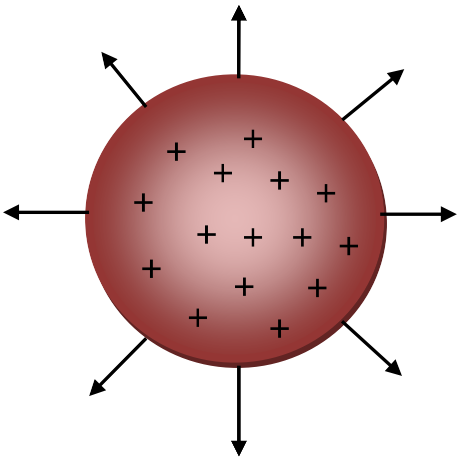

How do we calculate the electric field of a spherical charge distribution? We need to solve a volume integral for this.

UNIT 4

SURFACE AND VOLUME INTEGRALS

Structure

STUDY GUIDE

In this unit, you will study surface integrals and volume integrals. You should study Appendix A2 of this block thoroughly before you start studying this unit so that you understand the methods of evaluating double integrals. Surface integrals are evaluated by reducing them to double integrals. Volume integrals are integrations over three variables. Line integrals are used in this unit in the applications of Stokes' theorem. Therefore, revise how to evaluate line integrals from Unit 3.

"Everyone now agrees that a physics lacking all connection with mathematics .... would only be an historical amusement, fitter for entertaining the idle, than occupying the mind of a philosopher."

Franz Karl Achard

4.1 INTRODUCTION

The real world is three-dimensional and as such, most physical functions depend on all the three spatial variables (x,y,z), as you have seen in Units 1 and 2. You have already studied how to integrate vector functions and fields with respect to one variable in Unit 3. However, in physics you often have to integrate functions of two and three variables, over planes and arbitrary surfaces and volumes in space. Such integrals are called multiple integrals. In this unit you will study multiple integrals and their applications in physics. You will also study two important theorems of vector integral calculus, namely, Stokes' theorem and Gauss's divergence theorem.

In Appendix A2 of this block, you have learnt how to evaluate double integrals which are integration of functions of two variables and the regions of integration are on the coordinate planes. At the beginning of this unit in Sec. 4.2, we discuss some applications of double integrals in physics, like determining the volume of objects and their centre of mass, etc.

In Unit 3, you have studied line integrals. Recall that in a line integral, the integration is over a single independent variable but the path may be an arbitrary curve in space. In Sec. 4.3 of this unit, you will study the surface integral of a vector field, in which the integration is over a two-dimensional surface in space. Surface integrals are a generalisation of double integrals. You will learn how to evaluate a special type of surface integral which is the **flux** of a vector field across a surface. This is used extensively in physics, e.g., in electromagnetic theory. You will learn about some other types of surface integrals as well. In Sec. 4.4, you will study Stokes' theorem and its applications. Stokes' theorem tells us how to transform a line integral into a surface integral and vice versa.

In Sec. 4.5, you will learn how to evaluate a volume integral in which the integrand is a function of three variables. This is the same as triple integral. In Sec. 4.6 you will study Gauss's divergence theorem and its application. The divergence theorem tells us how to transform a surface integral into a volume integral and vice versa.

With this unit we will complete our study of Vector Analysis. In the remaining blocks of the course you will study the basic principles of electricity, magnetism and electromagnetic theory, where you will use the concepts and techniques of vector analysis covered in this block.

Expected Learning Outcomes

After studying this unit, you should be able to:

- ❖ use double integrals to evaluate physical quantities;
- ❖ calculate the flux of a vector field;
- ❖ evaluate volume integrals of scalar and vector fields;
- ❖ state Stokes' theorem and Gauss's divergence theorem and write them in a mathematical form; and
- ❖ solve problems based on these theorems and apply them to simple physical situations.

4.2 APPLICATIONS OF DOUBLE INTEGRALS

In Appendix A2 you have studied that a double integral can be used to determine the area of a region and volume of a solid. In the example below, you will use the techniques for evaluating double integrals explained in A2.2 and A2.3 to calculate area and volume.

EXAMPLE 4.1 : AREA AND VOLUME USING DOUBLE INTEGRALS

i) Determine the area of the region *R* on *xy* plane bounded by the curves $y = x + 2$ and $y = x^2$ by evaluating a double integral.

ii) Calculate the volume of the solid below the surface defined by the function $f(x, y) = 4 + \cos x + \cos y$, above the region *R* on the *xy* plane (z = 0), bounded by the curves $x = 0, x = \pi, y = 0$ and $y = \pi$ by evaluating a double integral.

SOLUTION ■ i) To determine the area of the region *R*, we have to evaluate $\iint_R dx\, dy$ where *R* is the region bounded by the curves $y = x + 2$ and $y = x^2$ (Eq. A2.7). To carry out the double integration we first obtain the limits of integration for the variables *x* and *y* in the region *R.*

To obtain the bounds (limits) on *x*, we solve the system of equations $y = x^2$ and $y = x + 2$, to get

$$x^2 = x + 2 \Rightarrow x = -1, 2$$

The region of integration *R* is then defined by the conditions $x^2 \le y \le x + 2,\ -1 \le x \le 2$ (read the margin remark) and we write

$$\text{Area of } R = \int_{-1}^{2} \left[\int_{x^2}^{x+2} dy \right] dx = \int_{-1}^{2} [y]_{x^2}^{x+2} dx$$

$$= \int_{-1}^{2} [x + 2 - x^2]\, dx = \left[\frac{x^2}{2} + 2x - \frac{x^3}{3} \right]_{-1}^{2} = \frac{9}{2}$$

Note that for *y* we write $x^2 \le y \le x + 2$, and not $x + 2 \le y \le x^2$. This is because in the range $-1 \le x \le 2,$, $x^2 \le x + 2$.

ii) The volume of the solid bound by the surface $f(x, y) = 4 + \cos x + \cos y$ and the region *R* defined by $0 \le x \le \pi;\ \ 0 \le y \le \pi$ is (Eq. A2.3)

$$V = \int_0^{\pi}\int_0^{\pi} (4 + \cos x + \cos y)\, dy\, dx \qquad \text{(i)}$$

Integrating (i) over *y* we get:

$$= \int_0^{\pi} [4y + y\cos x + \sin y]_0^{\pi}\ dx = \int_0^{\pi} [4\pi + \pi\cos x]_0^{\pi}\ dx \qquad \text{(ii)}$$

Next, integrating over *x*, we get

$$= [4\pi x + \pi \sin x]_0^{\pi} = 4\pi^2 \qquad \text{(iii)}$$

SAQ 1 - Determining area and volume using double integrals

a) Calculate the area of the region R bounded by the curves $y = x^2$ and $y = x^3$ for $x > 0;\ y > 0$.

b) Find the volume of the solid that lies below the surface of the curve $f(x,y) = x^4$ and above the region in the xy plane bounded by the curves $y = x^2$ and $y = 1$.

In physics, we also use double integrals to calculate several other quantities. We could use the double integral to determine the mass of an object like a planar lamina with a density function. We can also find the centre of mass of a laminar object or its moment of inertia about an arbitrary axis.

Before you solve an example on the applications of double integrals, let us summarize some important applications:

Recap

APPLICATIONS OF DOUBLE INTEGRALS

- Centre of mass (x_{cm}, y_{cm}) of a body with a density $\gamma(x,y)$ over a region R

$$x_{cm} = \frac{\iint\limits_R x\,\gamma(x,y)\,dx\,dy}{m};\quad y_{cm} = \frac{\iint\limits_R y\,\gamma(x,y)\,dx\,dy}{m} \tag{4.1}$$

- Mass m of a body with a density (mass/area) $\gamma(x,y)$ over a region R

$$m = \iint\limits_R \gamma(x,y)\,dx\,dy \tag{4.2}$$

- Moment of inertia of a body with a density $\gamma(x,y)$ over a region R about the x-axis, I_x and the y-axis I_y

$$I_x = \iint\limits_R y^2\,\gamma(x,y)\,dx\,dy;\quad I_y = \iint\limits_R x^2\,\gamma(x,y)\,dx\,dy \tag{4.3}$$

- The average value μ of a continuous function $f(x,y)$ over a closed region R in the xy-plane is:

$$\mu = \frac{\iint\limits_R f(x,y)\,dx\,dy}{\iint\limits_R dx\,dy};\quad \iint\limits_R dx\,dy = \text{Area of the region of integration } R \tag{4.4}$$

We study one of these applications in the following example, where we determine the mass of an object using double integrals.

EXAMPLE 4.2 : APPLICATION OF DOUBLE INTEGRAL

A rectangular plate covers the region $0 \le x \le 4;\ 0 \le y \le 3$ and has the mass density $\gamma(x,y) = x + y$. Calculate the mass of the plate.

SOLUTION ■ We use Eq. (4.2) to determine the mass of the body with the density function $\gamma(x,y) = x + y$. R is defined by the equations $0 \le x \le 4; 0 \le y \le 3,$. So the mass

$$m = \iint_R (x+y)\,dx\,dy = \int_{x=0}^{4}\int_{y=0}^{3}(x+y)\,dxdy$$

$$= \left[\int_{x=0}^{4} x dx\right]\left[\int_{y=0}^{3} dy\right] + \left[\int_{x=0}^{4} dx\right]\left[\int_{y=0}^{3} y dy\right]$$

$$= \left[\frac{x^2}{2}\right]_0^4 [y]_0^3 + [x]_0^4 \left[\frac{y^2}{2}\right]_0^3$$

$$= 42 \text{ units}$$

In the following example we study one more application of a double integral in physics.

EXAMPLE 4.3 : AVERAGE VALUE USING DOUBLE INTEGRALS

The temperature distribution at a point on a flat rectangular metal plate is $T(x,y) = 20 - 4x^2 - y^2$ °C. Calculate the average temperature on the plate, if the dimensions of the plate are described by $0 \le x \le 2; 0 \le y \le 1$.

SOLUTION ■ Using Eq. (4.4) we can write the average temperature on the plate as:

$$T_{avg} = \frac{\iint_R T(x,y)\,dx\,dy}{\iint_R dx\,dy}; \quad R: \; 0 \le x \le 2\,; 0 \le y \le 1 \qquad \text{(i)}$$

Note that $\iint_R dx\,dy = \text{Area of the rectangular plate} = 2 \text{ units}$. To evaluate the integral in the numerator of Eq. (i), we write:

$$\iint_R T(x,y)\,dx\,dy = \int_{x=0}^{2}\int_{y=0}^{1}\left(20 - 4x^2 - y^2\right) dxdy \qquad \text{(ii)}$$

$$= 20\int_{x=0}^{2}\int_{y=0}^{1} dxdy - 4\int_{x=0}^{2}\int_{y=0}^{1} x^2 dxdy - \int_{x=0}^{2}\int_{y=0}^{1} y^2\,dxdy$$

$$= 20\left[\int_{x=0}^{2}\int_{y=0}^{1} dxdy\right] - 4\left[\int_{x=0}^{2} x^2 dx\right]\left[\int_{y=0}^{1} dy\right] - \left[\int_{x=0}^{2} dx\right]\left[\int_{y=0}^{1} y^2\,dxdy\right] \text{(iii)}$$

$$= 20[2] - 4\left[\frac{x^3}{3}\right]_0^2 [y]_0^1 - [x]_0^2\left[\frac{y^3}{3}\right]_0^1 = \frac{86}{3}$$

Using Eqs. (i) and (iii), the average temperature is:

$$T_{avg} = \frac{43}{3}\,°\text{C}$$

In the next section, you will study surface integrals of vector fields. Just as line integrals are integrals along a curve, for surface integrals the region of integration is a surface. Surface integrals have several applications in physics.

4.3 SURFACE INTEGRALS

In physics, we come across many types of surface integrals. The commonest example of a surface integral is that of flux. You may recall the concept of electromagnetic induction from school physics. If we move a bar magnet *M* towards a circular coil *C* (Fig. 4.1), we know that an electromotive force is induced in the coil. This happens because the **magnetic flux** linked with the coil changes with time. The question is: How do we calculate the magnetic flux linked with the coil at a particular position?

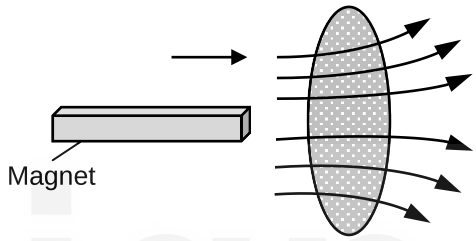

Fig. 4.1: **Magnetic flux.**

To determine the magnetic flux, we have to **integrat**e the magnetic field vector over the area enclosed by the coil. It is given by

$$\phi_B = \iint_S \vec{\mathbf{B}} . d\vec{\mathbf{S}} \tag{4.5}$$

Here $\vec{\mathbf{B}}$ is the magnetic field due to the magnet at the position of the element of area $d\vec{\mathbf{S}}$ of the coil. Here *S* is the area of the coil (the shaded region in Fig. 4.1).

This type of integral is called a **surface integral**. This involves the integral of a vector field over a surface. This is one type of surface integral. You will come across different types of surface integrals in physics as given below.

Types of Surface Integrals

Analogous to line integrals, surface integrals may appear in the following different forms:

i) $$\iint_S \phi \, d\vec{\mathbf{S}} \tag{4.6a}$$

ii) $$\iint_S \vec{\mathbf{A}} . d\vec{\mathbf{S}} \tag{4.6b}$$

iii) $$\iint_S \vec{\mathbf{A}} \times d\vec{\mathbf{S}} \tag{4.6c}$$

where ϕ is a scalar field and $\vec{\mathbf{A}}$, a vector field.

Type (ii) is the most common form of surface integrals in physics. In this unit, we focus on this type of surface integral. It is the flux of vector field $\vec{\mathbf{A}}$ through surface *S*.

4.3.1 Flux of a Vector Field

Let us consider a region of space in which we have a constant vector field $\vec{\mathbf{A}}(x, y, z) = A_0 \hat{\mathbf{i}}$. Recall what you have studied about flux in Unit 1 of BPHCT-131. You saw that the flux of rainwater can be expressed as a scalar product of the vector field representing the flow of rain and an area vector representing the top surface of the bucket.

The word *flux* is derived from the Latin word "fluxus" which means *flow.* The concept of flux is easier to understand in the context of fluid flow. You can of course determine the flux of any vector field.

Let us now learn how, in general, the flux of any vector field can be written as a surface integral. Suppose that $\vec{\mathbf{A}}$ is a vector field associated with fluid flowing through any region. Let the magnitude A_0 of the vector field be the amount of fluid that crosses unit area in unit time. Then **by definition, the flux of the field $\vec{\mathbf{A}}$ through any area is the amount of fluid that flows through that area in unit time**.

Since the loop (in *yz* plane) in Fig. 4.2a is perpendicular to fluid flow (along *x*-axis), fluid flows through it. Since the loop in Fig. 4.2b is parallel to the fluid flow, no fluid flows through it.

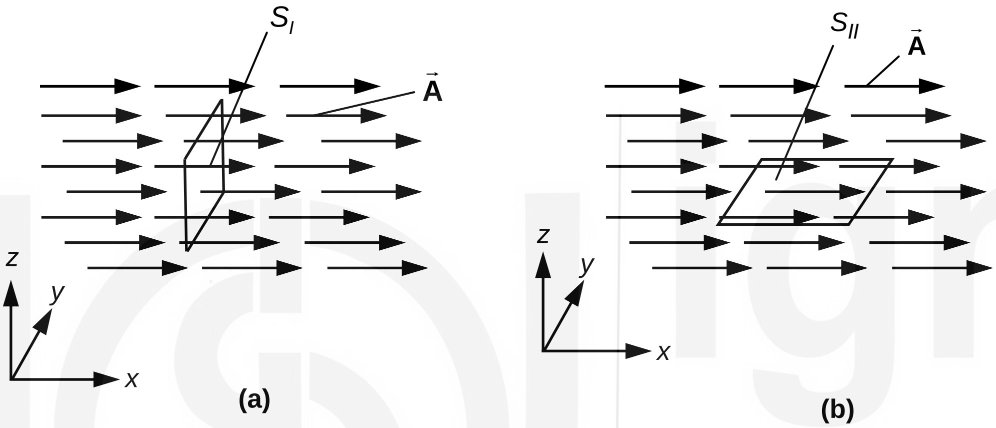

Fig. 4.2: **Flux of a vector field $\vec{\mathbf{A}}$ $(= A_x \hat{\mathbf{i}})$ through a surface a) S_I perpendicular to $\vec{\mathbf{A}}$; b) S_II parallel to it.**

Thus, the flux of $\vec{\mathbf{A}}$ through an imaginary square loop of area $(S = a^2)$ placed in the *yz* plane (Fig. 4.2a) is defined as

$$\Phi_\mathrm{I} = A_0\, a^2 \tag{4.7}$$

The flux of $\vec{\mathbf{A}}$ through the same area element placed in the *xy* plane (Fig. 4.2b) is

$$\Phi_{\mathrm{II}} = 0 \tag{4.8}$$

What happens if this imaginary loop is placed at an arbitrary angle to $\vec{\mathbf{A}}$ (Fig. 4.3a)? That is, it is neither parallel nor perpendicular to the flow.

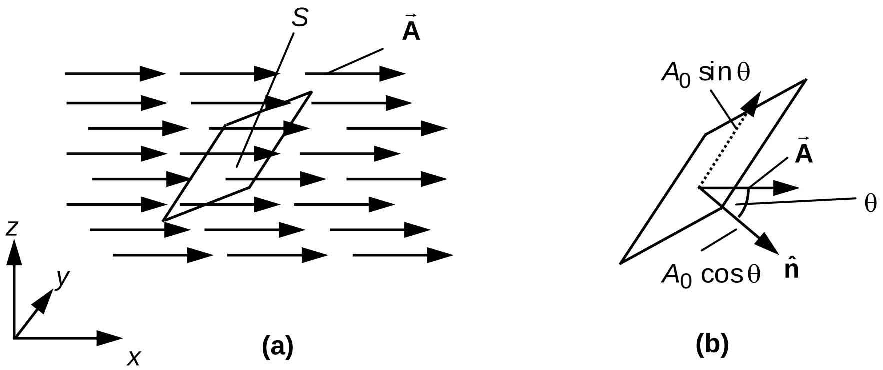

Fig. 4.3: **Flux of a vector field $\vec{\mathbf{A}}$ through a surface *S*. The normal to the surface makes an angle θ with the vector field.**

Let θ be the angle between the normal $\hat{\mathbf{n}}$ to the area element and the vector field $\vec{\mathbf{A}}$ (Fig. 4.3b). We can resolve the vector field $\vec{\mathbf{A}}$ into two components,

- one perpendicular to S: $(A_0 \cos\theta)$ and
- one parallel to it: $(A_0 \sin\theta)$.

The angle between $\hat{\mathbf{n}}$ and $\vec{\mathbf{A}}$ is θ. So $\vec{\mathbf{A}}.\hat{\mathbf{n}} = A_0 \cos\theta$, using the definition of the scalar product. Note that if we draw unit normal vectors to the surfaces S_{I} and S_{II} as well and use the expression $(\vec{\mathbf{A}}.\hat{\mathbf{n}})a^2$ we can get back Eqs. (4.7 and 4.8) because $\theta = 0$ for S_{I} and $\theta = \pi/2$ for S_{II}.

The only contribution to the flux is from the component of the field which is perpendicular to S, i.e. $A_0 \cos\theta$. So the flux is

$$\Phi_{III} = a^2 A_0 \cos\theta$$

In vector notation, we can write this flux as the following scalar product:

$$\Phi = (\vec{\mathbf{A}}.\hat{\mathbf{n}})S \tag{4.9}$$

where $\hat{\mathbf{n}}$ is the unit normal to the surface S (Fig. 4.3b).

We can write the area itself in terms of the normal vector $\hat{\mathbf{n}}$ as $\vec{\mathbf{S}} = S\hat{\mathbf{n}}$. Then, the flux Φ of the vector field $\vec{\mathbf{A}}$ is:

$$\Phi = \vec{\mathbf{A}}.\vec{\mathbf{S}} \tag{4.10}$$

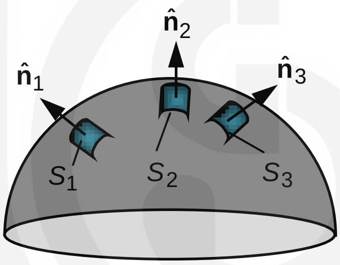

Fig. 4.4: **The unit normals to the three surface elements, S_1, S_2 and S_3 are $\hat{\mathbf{n}}_1, \hat{\mathbf{n}}_2$ and $\hat{\mathbf{n}}_3$.**

Here both the vector field $\vec{\mathbf{A}}$ and the unit normal are constant over the entire area element $(\vec{\mathbf{S}})$ over which we are defining the flux of the vector field. In general, the vector field may be a function of position (*x, y, z*). Also the surface itself may not be a plane, so the unit normal would point in different directions at different points on the surface. For example consider a part of the surface of a sphere (Fig. 4.4). In Fig. 4.4, we show the normal to this surface at different points. Note that their directions are different. How do we determine the flux in such cases?

This is where we need the concept of a surface integral.

4.3.2 Flux of a Vector Field as a Surface Integral

Let us determine the flux of a vector field $\vec{\mathbf{A}}$ (*x, y, z*) over the surface S shown in Fig. 4.5.

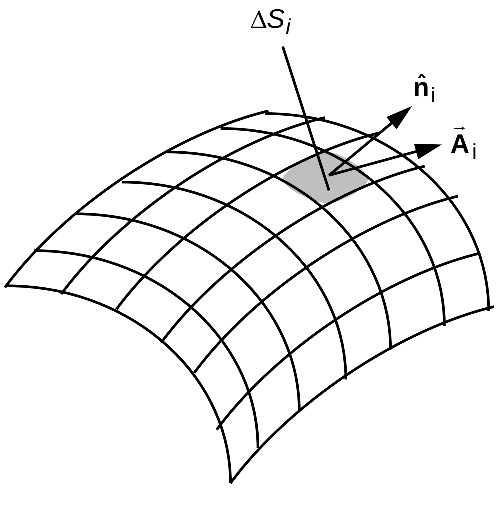

Fig. 4.5: **A surface S divided into n tiny area elements. The area of the i^{th} element is ΔS_i, it has a unit normal $\hat{\mathbf{n}}_i$ and the vector field over this area element is a constant equal to $\vec{\mathbf{A}}_i$.**

We carry out the following steps:

1. We divide the surface into n tiny elements of area. The i^{th} area element is $\Delta\vec{\mathbf{S}}_\mathbf{i} = \Delta S_i \hat{\mathbf{n}}_\mathbf{i}$ where $\hat{\mathbf{n}}_i$ is the unit normal to the surface for the area element ΔS_i (Fig 4.5).
2. Assume that the vector field over each such area element is a constant $\vec{\mathbf{A}}_i$.
3. The flux through each element of area is $\Delta\Phi_i = \vec{\mathbf{A}}_i \, . \, \Delta\vec{\mathbf{S}}_i$.
4. The flux through the entire surface is then the sum of the flux through each of these elements of area. It is

$$\Phi = \vec{\mathbf{A}}_1 . \Delta\vec{\mathbf{S}}_1 + \vec{\mathbf{A}}_2 . \Delta\vec{\mathbf{S}}_2 + ... + \vec{\mathbf{A}}_n . \Delta\vec{\mathbf{S}}_n = \sum_{i=1}^{n} \vec{\mathbf{A}}_i . \Delta\vec{\mathbf{S}}_i \tag{4.11}$$

5. In the limit as $n \to \infty$, we can write flux as an integral over the surface S:

$$\Phi = \lim_{n\to\infty} \sum_{i=1}^{n} \vec{\mathbf{A}}_i . \Delta\vec{\mathbf{S}}_i = \iint_S \vec{\mathbf{A}} . d\vec{\mathbf{S}} \tag{4.12}$$

where $d\vec{\mathbf{S}} = dS\hat{\mathbf{n}}$ is the infinitesimal element of area on this surface.

If the surface is a closed surface (like that of a sphere), we put a small circle on the sign of the integral and write flux of $\vec{\mathbf{A}}$ as

$$\Phi = \oiint_S \vec{\mathbf{A}} . d\vec{\mathbf{S}} \tag{4.13}$$

There are several physical situations in which we need to calculate the flux of a vector field. One of these is the magnetic flux through the coil given by

$$\Phi_B = \iint_S \vec{\mathbf{B}} . d\vec{\mathbf{S}}. \tag{4.14a}$$

The current i flowing through a wire is the flux of the current density ($\vec{\mathbf{J}}$) (see margin remark) vector across a cross-section of the wire, i.e.,

$$i = \iint_S \vec{\mathbf{J}} . d\vec{\mathbf{S}} \tag{4.14b}$$

where $d\vec{\mathbf{S}}$ is an area element of the cross-section of the wire.

$\vec{\mathbf{J}} = ne\vec{\mathbf{v}}$.
where n is the number of electrons per unit volume, e is the charge on an electron and $\vec{\mathbf{v}}$ is the average drift velocity of an electron.

The mass (m) of fluid flowing out of a volume V is the flux of the vector $\rho\vec{\mathbf{v}}$ across the closed surface S enclosing V. Here ρ is the density of the fluid and $\vec{\mathbf{v}}$ its **average flow velocity**.

$$m = \oiint_S \rho\vec{\mathbf{v}} . d\vec{\mathbf{S}} \tag{4.15}$$

Before we actually evaluate surface integrals, we need to know the convention used for choosing the direction of $\hat{\mathbf{n}}$. We discuss this point and define the area elements for integration in the following section.

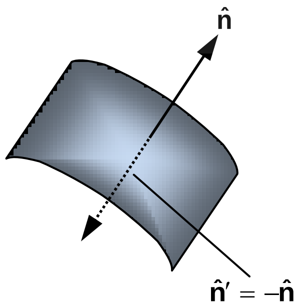

Fig. 4.6: **The unit normal to the surface of integration may point outward from the surface like $\hat{\mathbf{n}}$ or in the opposite direction as $\hat{\mathbf{n}}'$.**

4.3.3 Surface of Integration

In Fig. 4.6 you see an arbitrary surface of integration with a unit normal $\hat{\mathbf{n}}$. Note that **we could have chosen the unit normal** to be pointing downwards from the surface instead of in the upward direction, as shown by $\hat{\mathbf{n}}'$ in

Fig. 4.6. Obviously this would change the sign of the scalar product between the vector field and the unit normal in the expression for the surface integral in Eq. (4.12). How then do we decide in which direction to choose the unit normal for each area?

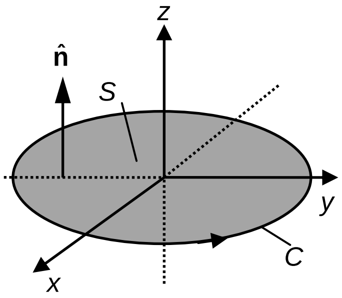

Fig. 4.7: **Direction of the normal vector for a plane surface.**

Consider a surface *S* enclosed by a closed curve *C* (Fig. 4.7) in a plane. It is an open surface lying on that plane. The direction of the normal depends on the sense in which the perimeter of this surface is traversed. If the right hand fingers are placed in the sense of travel around the perimeter, the positive normal points in the direction indicated by the thumb of the right hand (Fig. 4.7). Suppose the surface shown traversed in the sense, $+x \to +y \to -x \to -y \to +x$. The positive normal to the surface will be parallel to the positive z-axis.

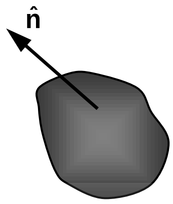

Fig. 4.8: **Outward drawn normal to a closed surface.**

If a volume is enclosed by a curved surface, it is called a closed surface (Fig. 4.8). The shell of a whole egg is an example of a closed surface. For such a surface the direction of the normal varies from point to point. However, at any point, the convention is to take the normal to the surface pointing outwards.

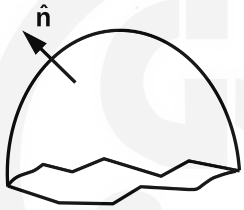

Fig. 4.9: **We choose the outer surface of the shell to be the outside and draw the outward normal.**

We may sometimes come across curved open surfaces. Examples of such surfaces are the shell of a cracked egg or a bowl (Fig. 4.9). In this case one side of the surface is chosen arbitrarily as outside and at any point the direction of the normal is outward. So we come to the general convention that:

The vector $\hat{\mathbf{n}}$ for any curved surface always points outwards from the surface.

In this unit we will study the surface integral over plane surfaces like the surface of a cube or cuboid. Surface integrals over curved surfaces are usually evaluated using non-Cartesian coordinates and this is beyond the scope of this syllabus.

Let us now describe the area element $d\vec{\mathbf{S}} = dS\,\hat{\mathbf{n}}$ for the surface of a cube or cuboid.

Area elements on the surface of a cube or cuboid

In Fig. 4.10 we show some typical area elements on the different faces of a cube/cuboid. For example, for an area element on face S_1, the outward normal is along the negative *z*-axis, so the area element is $-dxdy\hat{\mathbf{k}}$.

The integral over a closed surface like the surface of a sphere, is indicated by $\oiint_S$.

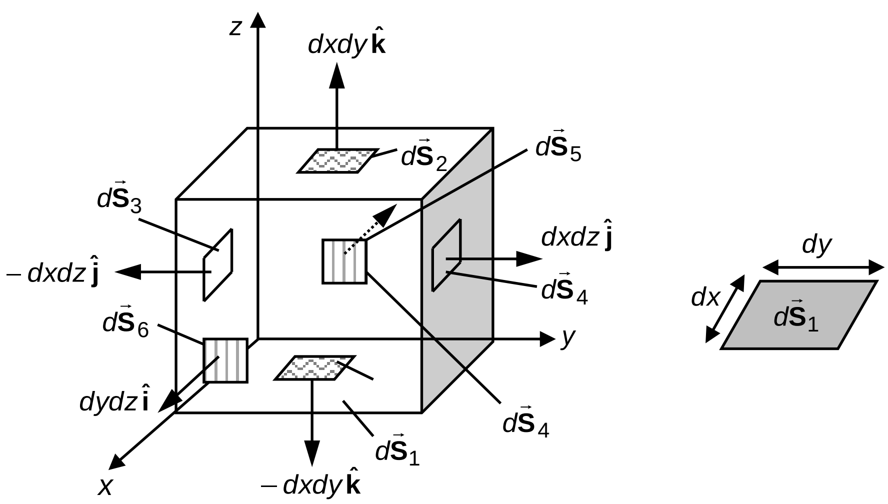

Fig. 4.10: **Surface area elements on a cube.**

You may note that the area element for the face S_6 is $dydz\,\hat{\mathbf{i}}$. You may like to write the area elements for the remaining faces. Try the following SAQ:

SAQ 2 **- Area elements on the surface of a cube/cuboid**

Write down the area element for the faces S_2, S_3, S_4 and S_5 of the cuboid in Fig. 4.10.

In many problems of surface integration, non-Cartesian coordinates are used for convenience. The choice of coordinate system is decided by the symmetry of the physical system.

4.3.4 Evaluation of Surface Integrals

A surface integral is evaluated as a double integral, over two variables. This means that we must describe both the vector field and the surface in terms of the same variables and then evaluate the double integral. In many problems on surface integrals, the choice of variables can be made by looking at the symmetry of the surface of integration.

Let us understand this by working out a few examples.

EXAMPLE 4.4: SURFACE INTEGRAL OF A VECTOR FIELD OVER A CUBE

Calculate the surface integral of a vector field $\vec{\mathbf{A}} = 2xz\hat{\mathbf{i}} + 2xz\hat{\mathbf{j}} - yz\hat{\mathbf{k}}$ over the surface of a unit cube occupying the space $0 \le x \le 1;\ 0 \le y \le 1;\ 0 \le z \le 1$.

SOLUTION ■ You have learnt about the area elements for each face of a cube in Sec. 4.3.3. Integrating $\vec{\mathbf{A}}$ over the surface of the cube means that we have to integrate over each face of the cube. So

$$\oiint_S \vec{\mathbf{A}}.d\vec{\mathbf{S}} = \iint_{S_1} \vec{\mathbf{A}}.d\vec{\mathbf{S}}_1 + \iint_{S_2} \vec{\mathbf{A}}.d\vec{\mathbf{S}}_2 + \iint_{S_3} \vec{\mathbf{A}}.d\vec{\mathbf{S}}_3 + \iint_{S_4} \vec{\mathbf{A}}.d\vec{\mathbf{S}}_4 + \iint_{S_5} \vec{\mathbf{A}}.d\vec{\mathbf{S}}_5 + \iint_{S_6} \vec{\mathbf{A}}.d\vec{\mathbf{S}}_6 \quad \text{(i)}$$

NOTE

The LHS of Eq. (i) is an integral over the entire surface whereas each integral on the RHS is on a plane, a face of the cube.

Now let us integrate $\vec{\mathbf{A}}$ over the surface S_1 which is on the plane $z = 0$ (Fig. 4.11):

$$I_1 = \iint_{S_1} \vec{\mathbf{A}}.d\vec{\mathbf{S}}_1 = \iint_{S_1} [2xz\hat{\mathbf{i}} + 2xz\hat{\mathbf{j}} - yz\hat{\mathbf{k}}].[-\,dxdy\hat{\mathbf{k}}] = \iint_{S_1} yz\,dxdy = 0\ (\because z = 0) \quad \text{(ii)}$$

We next integrate $\vec{\mathbf{A}}$ over the surface S_2 which is on the plane $z = 1$:

$$I_2 = \iint_{S_2} \vec{\mathbf{A}}.d\vec{\mathbf{S}}_2 = \iint_{S_2} \left[2xz\hat{\mathbf{i}} + 2xz\hat{\mathbf{j}} - yz\hat{\mathbf{k}}\right]\left[dxdy\hat{\mathbf{k}}\right] = -\iint_{S_2} yz\,dxdy = -\iint_{S_2} y\,dxdy$$

$$(\because z = 1) \quad \text{(iii)}$$

We can evaluate this as a double integral on a rectangular region S_2 using Eq. (4.7) with the following limits on x and y to define the region S_2:

$$0 \le x \le 1;\ 0 \le y \le 1 \quad \text{(iv)}$$

$$I_2 = -\iint_{S_2} y\,dxdy = -\left[\int_0^1 dx\right]\left[\int_0^1 y\,dy\right] = -[x]_0^1\left[\frac{y^2}{2}\right]_0^1 = -\frac{1}{2} \quad \text{(v)}$$

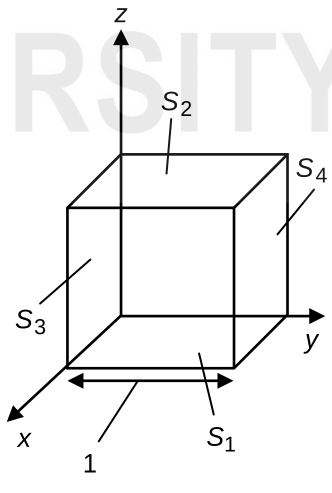

Fig. 4.11: **Unit cube of Example 10.1 with the surfaces S_1, S_2, S_3 and S_4 marked. These correspond to the planes $z = 0$, $z = 1$, $y = 0$ and $y = 1$ respectively.**

Similarly, for the surface S_3 $(0 \le x \le 1; 0 \le z \le 1)$ is on the plane y = 0, we have

$$I_3 = \iint_{S_3} \vec{\mathbf{A}}.d\vec{\mathbf{S}}_3 = \iint_{S_3} \left[2xz\hat{\mathbf{i}} \;+\; 2xz\hat{\mathbf{j}} - yz\hat{\mathbf{k}}\right]\left[-\,dxdz\hat{\mathbf{j}}\right] = -\iint_{S_3} 2xzdxdz \qquad \text{(vi)}$$

Evaluating this as a double integral we can write:

$$I_3 = -2\iint_{S_3} xzdxdz = -2\left[\int_0^1 xdx\right]\left[\int_0^1 zdz\right] = -2\left[\frac{x^2}{x}\right]_0^1\left[\frac{z^2}{2}\right]_0^1 = -\frac{1}{2} \qquad \text{(vii)}$$

You may like to work out for yourself the values of the integral of A over the faces S_4, S_5 and S_6 of the cube (SAQ 3a). You will see that

$$I_4 = \iint_{S_4} \vec{\mathbf{A}}.d\vec{\mathbf{S}}_4 = \frac{1}{2} \qquad \text{(viii)}$$

$$I_5 = \iint_{S_5} \vec{\mathbf{A}}.d\vec{\mathbf{S}}_5 = 0 \qquad \text{(ix)}$$

and

$$I_6 = \iint_{S_6} \vec{\mathbf{A}}.d\vec{\mathbf{S}}_6 = 1 \qquad \text{(x)}$$

The total flux of $\vec{\mathbf{A}}$ through the surface of the cube is found by substituting the surface integral corresponding to each surface in Eq. (i) from Eqs.(iii),(v),(vii), (viii),(ix) and (x) to get:

$$\iint_S \vec{\mathbf{A}}.\, d\vec{\mathbf{S}} = 0 - \frac{1}{2} - \frac{1}{2} + \frac{1}{2} + 0 + 1 = \frac{1}{2}$$

SAQ 3 - Surface integral

a) Calculate the integrals I_4, I_5 and I_6 from Example 4.4.

b) Calculate the surface integral $\iint_S \vec{\mathbf{r}}.d\vec{\mathbf{S}}$ where $\vec{\mathbf{r}} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$ and S is the surface of a disc of radius 2 units, lying in the plane z = 5, defined by

$$x^2 + y^2 \le 4; \quad z = 5$$

4.3.5 Solid Angle

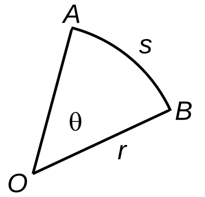

Fig. 4.12: **Angle** θ **in a plane.**

We now explain the concept of a *solid angle* which will be used in the next block. You are familiar with an angle in a plane. You know that in two-dimensions, it is the angle between two straight lines, say *AO* and *BO*, that intersect at a point *O* (Fig. 4.12). It is measured in the plane of the same lines and defined as

$$\theta = \frac{s}{r} \quad \text{(in rad)}$$

where s is the arc length of a circle of radius r passing through A and B. A solid angle is a three-dimensional analogue of the two-dimensional angle. Let us define it.

Consider an arbitrary differential area element $d\vec{\mathbf{S}}$ of a surface at a distance *r* from a point *P* to the surface element. Let $\hat{\mathbf{r}}$ denote the unit vector **from** the point *P* to the area element (Fig. 4.13). Then, by definition, the solid angle $d\Omega$ subtended by the surface area element $d\vec{\mathbf{S}}$ at the point *P* is given by

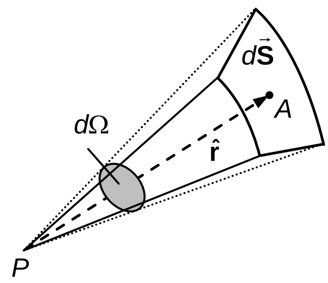

Fig. 4.13: **The solid angle $d\Omega$ subtended by an area element $d\vec{\mathbf{S}}$ at a point *O*.**

$$d\Omega = \frac{d\vec{\mathbf{S}}.\hat{\mathbf{r}}}{r^2} = \frac{dS\cos\theta}{r^2} \tag{4.16}$$

where θ is the angle between the normal to the surface element and $\hat{\mathbf{r}}$. The unit of solid angle is the *steradian* which is dimensionless.

The net solid angle subtended by the entire surface *S* is given by the surface integral:

$$\iint_S d\Omega = \iint_S \frac{\hat{\mathbf{r}}.d\vec{\mathbf{S}}}{r^2} \tag{4.17}$$

The solid angle of a closed surface is an important special case that we will use in Unit 6 of Block 2.

The net solid angle subtended by a closed surface *S* surrounding a point is given by

$$\Omega = \oiint_S \frac{\hat{\mathbf{r}}.d\vec{\mathbf{S}}}{r^2} = 4\pi \tag{4.18}$$

Note that for a closed surface, the vector $d\vec{\mathbf{S}}$ is always taken as the normal to the surface pointing *outwards*. The proof of Eq. (4.18) is beyond the scope of this course.

So, the net solid angle subtended by a closed surface of any shape, on a point enclosed by it, is 4π steradians.

You may now work out the following SAQ.

SAQ 4 **- Surface integral on the surface of a sphere**

Evaluate (i) $\oiint_S \hat{\mathbf{r}}.d\vec{\mathbf{S}}$ and (ii) $\oiint_S \frac{\hat{\mathbf{r}}.d\vec{\mathbf{S}}}{r^2}$ where *S* is a sphere of radius *R*.

Integral theorems allow you to transform one type of integral into another. We now study the Stokes theorem which allows us to transform surface integrals into line integrals, and conversely, line integrals into surface integrals.

4.4 STOKES' THEOREM

Stokes' theorem states that: **'The integral of the curl of a vector field over a surface** *S* **is equal to the line inetgral of the vector field over the closed path** *C* **bounding** *S.* It is expressed mathematically as

$$\oint_C \vec{\mathbf{A}}.d\vec{\mathbf{l}} = \iint_S \left(\vec{\nabla} \times \vec{\mathbf{A}}\right).d\,\vec{\mathbf{S}} \tag{4.19}$$

Fig. 4.14 shows some examples of surfaces bounded by closed paths.

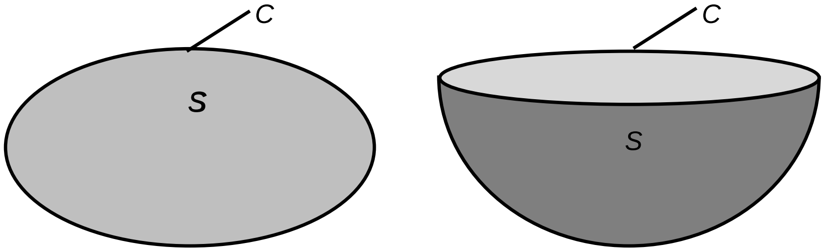

Fig. 4.14: **Surfaces bounded by closed paths.**

Let us now use Stokes' theorem to evaluate an integral.

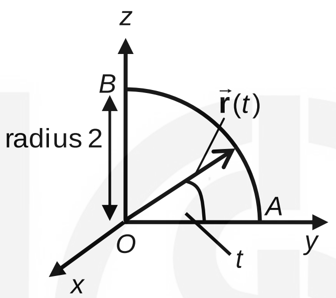

Fig. 4.15: **The contour** *C* **and surface** *S* **for Example 4.5.**

EXAMPLE 4.5 : EVALUATION OF LINE INTEGRAL USING THE STOKES' THEOREM

Verify Stokes' theorem for the vector field $\vec{\mathbf{F}} = y\hat{\mathbf{i}} + z\hat{\mathbf{j}} + x\hat{\mathbf{k}}$ over the closed contour C enclosing the plane surface S shown in the Fig. 4.15. Here AB is the arc of the circle of radius 2 with its centre at the origin.

SOLUTION ■ To verify Stokes' theorem we must show:

$$\oint_C (y\hat{\mathbf{i}} + z\hat{\mathbf{j}} + x\hat{\mathbf{k}}).d\vec{\mathbf{l}} = \iint_S \left[\vec{\nabla} \times (y\hat{\mathbf{i}} + z\hat{\mathbf{j}} + x\hat{\mathbf{k}})\right].d\vec{\mathbf{S}} \tag{i}$$

C is the closed contour OAB which encloses the quarter circle in the yz plane. The radius of the circle is 2 units. Let us first integrate the line integral on the LHS of Eq. (i). The contour C is made up of C_1, C_2 and C_3, C_1 is the straight line OA along the y-axis, C_2 is the arc AB of the circle and C_3 is the straight line BO along the z-axis. Then

$$I_1 = \oint_{C_1} (y\hat{\mathbf{i}} + z\hat{\mathbf{j}} + x\hat{\mathbf{k}}).d\vec{\mathbf{l}} + \oint_{C_2} (y\hat{\mathbf{i}} + z\hat{\mathbf{j}} + x\hat{\mathbf{k}}).d\vec{\mathbf{l}} + \oint_{C_3} (y\hat{\mathbf{i}} + z\hat{\mathbf{j}} + x\hat{\mathbf{k}}).d\vec{\mathbf{l}} \tag{ii}$$

We first evaluate the first and third integrals on the RHS of Eq. (ii). Using Eq. (3.19a), we can write

$$\int_{C_1} (y\hat{\mathbf{i}} + z\hat{\mathbf{j}} + x\hat{\mathbf{k}}).d\vec{\mathbf{l}} = \int_{C_1} (ydx + zdy + xdz) = \int_0^2 zdy \quad (\because dx = dz = 0 \text{ along } OA)$$

$$= 0 \; (\because z = 0 \text{ along } OA) \tag{iii}$$

and

$$\int_{C_3}(y\hat{\mathbf{i}}+z\hat{\mathbf{j}}+x\hat{\mathbf{k}}).d\vec{\mathbf{l}} = \int_{C_3}(ydx+zdy+xdz)=\int_2^0 xdz \quad (\because dx = dy = 0 \text{ along } BO)$$

$$= 0 \ (\because x = 0 \text{ along } BO) \qquad \text{(iv)}$$

To evaluate the line integral along C_2, we parametrize the curve AB as follows:

$$\vec{\mathbf{r}}(t) = 2\cos t\,\hat{\mathbf{j}} + 2\sin t\,\hat{\mathbf{k}};\ \ x(t) = 0,\ \ y(t) = 2\cos t,\ \ z(t) = 2\sin t;\ \ 0 \le t \le \pi/2 \qquad \text{(v)}$$

$$\therefore \vec{\mathbf{F}} = y\hat{\mathbf{i}} + z\hat{\mathbf{j}} + x\hat{\mathbf{k}} = 2\cos t\hat{\mathbf{i}} + 2\sin t\,\hat{\mathbf{j}};\ \ \frac{d\vec{\mathbf{r}}(t)}{dt} = -\,2\sin t\,\hat{\mathbf{j}} + 2\cos t\,\hat{\mathbf{k}} \qquad \text{(vi)}$$

Using Eq. (vi) in Eq. (3.28) for the line integral we get (see also margin remark):

$$\int_{C_2}(y\hat{\mathbf{i}}+z\hat{\mathbf{j}}+x\hat{\mathbf{k}}).d\vec{\mathbf{l}} = \int_{t=0}^{\pi/2}(2\cos t\,\hat{\mathbf{i}} + 2\sin t\,\hat{\mathbf{j}}).(-2\sin t\,\hat{\mathbf{j}} + 2\cos t\,\hat{\mathbf{k}})dt$$

$$= \int_{t=0}^{\pi/2}(-4\sin^2 t)\,dt = -\pi \qquad \text{(vii)}$$

$$\int_0^{\pi/2}\sin 2t\,dt = \int_0^{\pi/2}\frac{(1-\cos 2t)}{2}dt = \frac{1}{2}\int_0^{\pi/2}dt - \frac{1}{2}\int_0^{\pi/2}\cos 2t\,dt = \frac{1}{2}[t]_0^{\pi/2} - \frac{1}{2}[\sin 2t)]_0^{\pi/2} = \pi/4$$

Adding up the contributions from each segment, the line integral over OAB is found by substituting the results of Eqs. (iii), (iv) and (vii) in Eq. (ii):

$$I_1 = 0 - \pi + 0 = -\pi \qquad \text{(viii)}$$

We next evaluate the surface integral in the RHS of Eq. (i). We first calculate the curl of the vector field (see margin remark):

$$\vec{\nabla}\times\vec{\mathbf{F}} = -\,\hat{\mathbf{i}} - \hat{\mathbf{j}} - \hat{\mathbf{k}} \qquad \text{(ix)}$$

$$\vec{\nabla}\times\vec{\mathbf{F}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = -\,\hat{\mathbf{i}} - \hat{\mathbf{j}} - \hat{\mathbf{k}}$$

Note that the surface S is a plane surface on the yz plane. If we curl the fingers of our right hand around the contour in the direction of the contour, the normal to the surface is along the positive x-direction. We can consider the element of area on the yz plane to be:

$$d\vec{\mathbf{S}} = dydz\,\hat{\mathbf{i}} \qquad \text{(x)}$$

Then

$$I_2 = \iint_S \text{curl}(y\hat{\mathbf{i}}+z\hat{\mathbf{j}}+x\hat{\mathbf{k}}).d\vec{\mathbf{S}} = \iint_S (-\hat{\mathbf{i}}-\hat{\mathbf{j}}-\hat{\mathbf{k}}).(dydz\,\hat{\mathbf{i}}) = -\iint_S dydz \qquad \text{(xi)}$$

Using the area property of the double integral we can see that:

$$\iint_S dydz = \text{Area of } S = \frac{1}{4}(\text{Area of circle of radius 2}) = \pi$$

Therefore, the integral of Eq. (xi) is just

$$I_2 = -\pi \qquad \text{(xii)}$$

The line integral of Eq. (viii) and the surface integral of Eq. (xii) both give us the same result, thereby, verifying Stokes' theorem.

You may now like to work out an SAQ on solving integrals using Stokes' theorem.

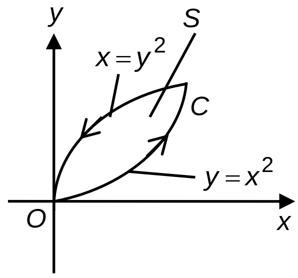

Fig. 4.16: **Figure for SAQ 5.**

SAQ 5 - Evaluation of line integral using Stokes' theorem

Using Stokes' theorem, evaluate $\oint_C \vec{\mathbf{A}}.d\vec{\mathbf{l}}$ around the closed curve C shown in Fig. 4.16 given that:

$$\vec{\mathbf{A}} = (x-y)\,\hat{\mathbf{i}} + (x+y)\,\hat{\mathbf{j}}$$

4.4.1 Applications of Stokes' Theorem

We shall now discuss an application of this theorem. The direct evaluation of $\vec{\nabla}\times\vec{\mathbf{B}}$ where $\vec{\mathbf{B}}$ is magnetic field due to a current carrying conductor is quite tedious. To obtain $\vec{\nabla}\times\vec{\mathbf{B}},$ we shall use Stokes' theorem and the circuital form of Ampere's law,

$$\oint_{C'} \vec{\mathbf{B}}.d\vec{\mathbf{l}} = \mu_0 i \tag{4.20}$$

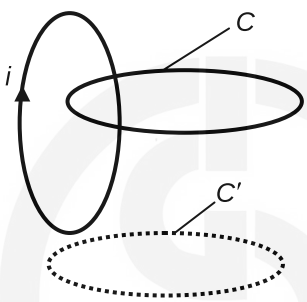

Fig. 4.17

where C is any closed path that is linked with the current i (Fig. 4.17). For a path like C', which is not linked with the current, we have

$$\oint_C \vec{\mathbf{B}}.d\vec{\mathbf{l}} = 0$$

Now, our task is to calculate $\vec{\nabla}\times\vec{\mathbf{B}}.$ From Stokes' theorem we get:

$$\oint_C \vec{\mathbf{B}}.d\vec{\mathbf{l}} = \iint_S \left(\vec{\nabla}\times\vec{\mathbf{B}}\right).d\vec{\mathbf{S}} \tag{4.21}$$

where S is enclosed by C.

Recall that in Eq. (4.14b) we have defined current in terms of the current density $\vec{\mathbf{J}}$ as:

$$I = \iint_S \vec{\mathbf{J}}.d\vec{\mathbf{S}} \tag{4.22}$$

Hence, from Eqs. (4.20), (4.21) and (4.22), we get

$$\iint_S \left(\vec{\nabla}\times\vec{\mathbf{B}}\right).d\vec{\mathbf{S}} = \iint_S \mu_0\,\vec{\mathbf{J}}.d\vec{\mathbf{S}} \tag{4.23}$$

or

$$\iint_S \left(\vec{\nabla}\times\vec{\mathbf{B}} - \mu_0\vec{\mathbf{J}}\right).d\vec{\mathbf{S}} = 0 \tag{4.24}$$

Since $d\vec{\mathbf{S}}$ is arbitrary, the integrand must be zero. Therefore,

$$\vec{\nabla}\times\vec{\mathbf{B}} = \mu_0\vec{\mathbf{J}} \tag{4.25}$$

Thus we see that $\vec{\mathbf{B}}$ has a non-vanishing curl.

You have learnt in your Mechanics Course (BPHCT-131) that the curl of a conservative force field is zero. We can prove the same result using Stokes' theorem. You can work this out yourself in the following SAQ.

SAQ 6 **- Application of Stokes' theorem**

Using Stokes' theorem, prove that curl of a conservative force field is zero everywhere.

So far you have learnt how to evaluate double integrals and surface integrals, which involve successive integrations over two variables. Next we study volume integrals (also called triple integrals) which involve successive integrations over three variables.

4.5 VOLUME INTEGRALS

Let us first define volume (also known as triple integrals) integrals, where we integrate a function of three variables, $f(x, y, z)$ over a closed volume Ω in the Cartesian coordinate system. The method we follow is similar as for defining a double integral.

4.5.1 Volume Integral of the Function $f(x, y, z)$

Like the double integral, the triple or volume integral is also defined as the limit of a sum. Let us see how this is done.

1. We first partition the three dimensional volume Ω into n parts by drawing planes parallel to the three coordinate planes. As a result, the volume Ω is filled with boxes, which we now number from 1 to n. Each box has a volume $\Delta V_i = \Delta x_i \, \Delta y_i \, \Delta z_i$.
2. We choose a point (x_i, y_i, z_i) in each of these boxes and define a sum of the form:
$$S_n = \sum_{i=1}^{n} f(x_i, y_i, z_i)\, \Delta V_i \tag{4.26}$$
3. As n increases, the volume of the boxes becomes smaller and smaller. The volume integral of the function $f(x, y, z)$ over the region Ω is defined as the limit of the sum S_n in the limit $n \to \infty$.

The volume integral of a function $f(x, y, z)$ over a closed bounded region Ω is defined by the expression:

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i, y_i, z_i)\Delta V_i = \iiint_{\Omega} f(x, y, z)\,dV = \iiint_{\Omega} f(x, y, z)\,dx\,dy\,dz \tag{4.27}$$

You have seen before that the integral of the function of a single variable with respect to that variable represents an area, and a double integral of a function of two variables represents the volume under a surface. What, then, is a volume integral? We can say that it represents a summation in a hypothetical 4th dimension.

Let us try to understand this point with an example. Imagine a balloon that is being inflated. We define the surface of the balloon with the help of an equation $z = f(x, y)$. However since the size of the balloon is changing with time, each of these variables is also a function of time t. If we integrate with respect to x and y, we get the volume of the balloon as a function of t. If we put in a value of t we will get the value of the volume of the balloon at that instant of time. However, now we can perform the integration over t to sum up the volume over the entire process of inflation that would be the volume integral.

We now write down the properties of the volume integral, which are quite similar to the properties of a double integral.

4.5.2 Properties of the Volume Integral

For any two functions $f(x,y,z)$ and $g(x,y,z)$ defined over the three dimensional region Ω, the volume integral has the following properties:

Linearity:

$$\iiint_{\Omega}[\alpha f(x,y,z)+\beta g(x,y,z)]dxdydz = \alpha\iiint_{\Omega} f(x,y,z)dxdydz + \beta\iiint_{\Omega} f(x,y,z)dxdydz \tag{4.28}$$

where α and β are constants.

Additivity:

If the region Ω can be broken up into several non-overlapping regions $\Omega_1, \Omega_2, ..., \Omega_n$, we can write:

$$\iiint_{\Omega} f(x,y,z)dxdydz$$

$$= \iiint_{\Omega_1} f(x,y,z)dxdydz + \iiint_{\Omega_2} f(x,y,z)dxdydz + ... + \iiint_{\Omega_n} f(x,y,z)dxdydz \tag{4.29}$$

Volume Property:

If the function $f(x,y,z) = 1$, then the volume integral over the region Ω gives the volume of Ω:

$$\iiint_{\Omega}[1]dxdydz = \text{Volume of the region } \Omega \tag{4.30}$$

Let us now see how a volume integral may be evaluated by iterated integration.

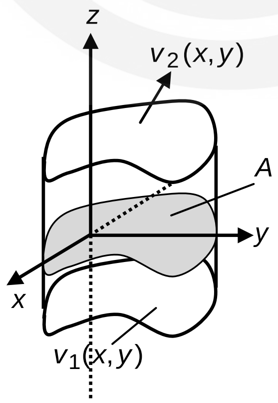

Fig. 4.18: **Limits of integration on the variable *z* in the region Ω.**

4.5.3 Evaluation of a Volume Integral

In evaluating the volume integrals we will once again perform **iterated integration**. In evaluating a double integral, where we integrate with respect to two variables, we perform a two-fold iterated integration. This, as you have seen in Sec. A2.2 of Appendix A2, can be carried out in two different ways depending on the order in which the integration over the two variables is carried out. Here we have three variables, so we carry out three-fold iterated integrations. However, in this case there are six possible ways of carrying out the repeated integral. If $f(x,y,z)$ is continuous, all the six iterated integrals are equal.

Let us consider the solid region Ω bounded below by the surface $z = v_1(x,y)$, and above by the surface $z = v_2(x,y)$, as shown in Fig. 4.18. The projection of the solid onto the *xy* plane is the region *A* (Fig. 4.18). We assume that the functions $v_1(x,y)$ and $v_2(x,y)$ are continuous in the region *A*. Then, for a function $f(x,y,z)$ continuous in the solid region Ω, we can write.

$$\iiint_{\Omega} f(x,y,z)\,dxdydz = \iint_{A}\left[\int_{z=v_1(x,y)}^{v_2(x,y)} f(x,y,z)\,dz\right]dydx \tag{4.31}$$

Once we calculate the integral within the bracket, we are left with (in general) a double integral of a function of two variables *x* and *y* to be integrated with respect to *x* and *y*. And we can use Eqs. (A.2.9) or (A.2.11) to evaluate this double integral. So if *A* is a region in the *xy* plane defined by:

$$a \le x \le b; \quad u_1(x) \le y \le u_2(x) \tag{4.32}$$

the volume integral reduces to:

$$\iiint_{\Omega} f(x,y,z)\,dxdydz = \int_{x=a}^{b}\left[\int_{y=u_1(x)}^{u_2(x)}\left[\int_{z=v_1(x,y)}^{v_2(x,y)} f(x,y,z)\,dz\right]dy\right]dx \tag{4.33}$$

As for double integrals, remember that iterated integral for the volume integral can be performed in any order of variables. Here we have chosen to integrate over *z* first, then over *y*, and finally over *x*. The choice of the order of the variables of integration is to be made according to our convenience. In Example 4.6, we integrate over *y* first, then over *z* and finally over *x*. Volume integrals are used to evaluate several quantities of interest to physicists, such as the volume and mass of an object of arbitrary shape, its centre of mass and its moment of inertia. We summarize these applications below.

APPLICATIONS OF VOLUME INTEGRALS

- Volume *V* of a region Ω:

$$V = \iiint_{\Omega} dx\,dydz \tag{4.34}$$

- Mass *m* of a body with a density $\gamma(x,y,z)$ over a region Ω:

$$m = \iiint_{\Omega} \gamma(x,y,z)\,dx\,dydz \tag{4.35}$$

- Centre of mass of a body (x_{cm}, y_{cm}, z_{cm}) with a density $\gamma(x,y,z)$ over a region Ω:

$$x_{cm} = \frac{\iiint_{\Omega} x\,\gamma(x,y,z)\,dx\,dydz}{m};\quad y_{cm} = \frac{\iiint_{\Omega} y\,\gamma(x,y,z)\,dx\,dydz}{m};\quad z_{cm} = \frac{\iiint_{\Omega} z\,\gamma(x,y,z)\,dx\,dydz}{m} \tag{4.36}$$

- Moment of inertia of a body with a density $\gamma(x,y,z)$ over a region Ω about the *x*-axis (I_x), about the *y*-axis (I_y) and about the z-axis (I_z):

$$I_x = \iiint_{\Omega} (y^2 + z^2)\gamma(x,y,z)\,dx\,dydz \tag{4.37a}$$

$$I_y = \iiint_{\Omega} (x^2 + z^2)\,\gamma(x,y,z)\,dx\,dydz; \tag{4.37b}$$

$$I_z = \iiint_{\Omega} (x^2 + y^2)\,\gamma(x,y,z)\,dx\,dydz \tag{4.37c}$$

In the following example, we determine the moment of inertia of a cube by carrying out a volume integral.

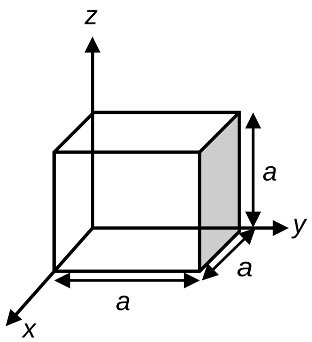

Fig. 4.19: **A cube of side *a*.**

EXAMPLE 4.6 : DETERMINING MOMENT OF INERTIA USING THE VOLUME INTEGRAL

Consider a cube with uniform density ρ and side a. The cube is placed such that its edges lie along the x, y and z axes as shown in Fig. 4.19. Determine the moment of inertia about an edge of the cube.

SOLUTION ■ To evaluate the moment of inertia about the x-axis, we use Eq. (4.37a). The limits of integration on the three variables are (Fig. 4.19):

$$0 \le x \le a; \ \ 0 \le y \le a; \ \ 0 \le z \le a$$

We write the moment of inertia as:

$$I_x = \rho \int_{x=0}^{a} \int_{y=0}^{a} \int_{z=0}^{a} \left(y^2 + z^2\right) dz\, dy\, dx$$

$$= \rho \int_{x=0}^{a} \int_{y=0}^{a} \left[y^2 z + \frac{z^3}{3} \right]_0^a dy\, dx \ \ \text{(integrating over } z \text{ first)}$$

$$= \rho \int_{x=0}^{a} \int_{y=0}^{a} \left[y^2 a + \frac{a^3}{3} \right] dy\, dx$$

$$= \rho \int_{x=0}^{a} \left[\frac{y^3 a}{3} + \frac{a^3 y}{3} \right]_0^a dx \ \ \text{(integrating over } y\text{)}$$

$$= \rho \int_0^a \left[\frac{2}{3} a^4 \right] dx \ \ = \rho \left[\frac{2}{3} a^4 x \right]_0^a = \frac{2}{3} \rho a^5 \qquad \text{(i)}$$

The mass of the cube is $M = (\text{density}) \times (\text{volume}) = \rho a^3$. Substituting for ρ in Eq. (i) we get:

$$I_x = \frac{2}{3} M a^2$$

You may now like to evaluate a few integrals by this method.

SAQ 7 - Evaluating volume integrals

a) Evaluate the volume integral of the function $f(x, y, z) = (\sin x) yz$ for $0 \le x, y, z \le \pi$.

b) Determine the mass of a unit cube of density γ (x, y, z) = x + 2y + 3z.

4.6 VOLUME INTEGRAL OF A VECTOR FIELD

So far in this unit we have discussed the volume integrals of a scalar field. Sometimes, however, you may have to evaluate the volume integral of a vector field. The volume integral of a vector field is written as:

$$\iiint_V \vec{\mathbf{A}} dV \tag{4.38}$$

where V is the volume over which the integration is to carried out. The volume element dV is a scalar and so we can write the volume integral of the vector field $\vec{\mathbf{A}} = A_1\hat{\mathbf{i}} + A_2\hat{\mathbf{j}} + A_3\hat{\mathbf{k}}$, as:

$$\iiint_V \vec{\mathbf{A}} dV = \hat{\mathbf{i}} \iiint_V A_1 dV + \hat{\mathbf{j}} \iiint_V A_2 dV + \hat{\mathbf{k}} \iiint_V A_3 dV \tag{4.39}$$

The integral of Eq. (4.39) reduces to a combination of integrals of scalar functions. The result of the integration is a vector quantity.

We now discuss another integral theorem. This theorem tells us how to convert a surface integral into a volume integral and vice versa.

4.7 THE DIVERGENCE THEOREM

The divergence theorem states that **'the integral of the divergence of a vector field over a volume *V* is equal to the surface integral of the vector over the closed surface bounding *V*.'**

The divergence theorem is sometimes also referred to as the Gauss's divergence theorem, Gauss's theorem or the divergence theorem of Gauss. It is expressed mathematically as

$$\oiint_S \vec{\mathbf{A}} \cdot d\vec{\mathbf{S}} = \iiint_V \left(\vec{\nabla} \cdot \vec{\mathbf{A}}\right) dV \tag{4.40}$$

where V is enclosed by S (Fig. 4.20).

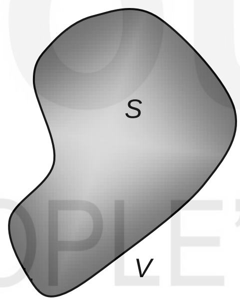

Fig. 4.20: **A closed surface *S* enclosing a volume *V*.**

Let us now work out an example to understand how to apply the divergence theorem.

EXAMPLE 4.7 : DIVERGENCE THEOREM

i) Use the divergence theorem to obtain the flux of a vector field $\vec{\mathbf{A}} = 3x\hat{\mathbf{i}} - y\hat{\mathbf{j}} + 2z\hat{\mathbf{k}}$ over a cube of side $2a$. The vertices of the cube are at $(\pm a, \pm a, \pm a)$ as shown in Fig. 4.21.

ii) Use the divergence theorem to evaluate the flux of the vector field $\vec{\mathbf{F}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ over the sphere $x^2 + y^2 + z^2 = a^2$.

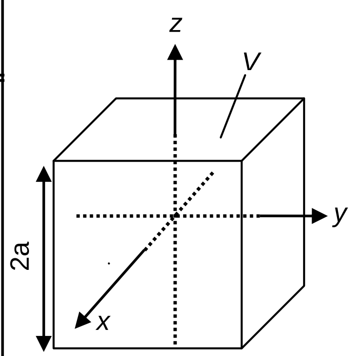

Fig. 4.21: **Cube with side $2a$. The cube is bounded by the planes** $x = \pm a$, $y = \pm a$, $z = \pm a$.

SOLUTION ■ i) Recall from Eq. (4.6b) that the flux of the vector field is defined as $\oiint_S \vec{\mathbf{A}} \cdot d\vec{\mathbf{S}}$. Here S is the surface of the cube shown in Fig. 4.21.

Using the divergence theorem, we evaluate $\iiint_V \vec{\nabla} \cdot \vec{\mathbf{A}}\, dV$, where V is the region enclosed by the surface of the cube. The region V is defined by the limits:

$$-a \le x \le a;\ -a \le y \le a;\ -a \le z \le a \qquad \text{(i)}$$

Let us first evaluate $\vec{\nabla}.\vec{\mathbf{A}}$:

$$\vec{\nabla}.\vec{\mathbf{A}} = \frac{\partial(3x)}{\partial x} + \frac{\partial(-y)}{\partial y} + \frac{\partial(2z)}{\partial z} = 4 \qquad \text{(ii)}$$

Using the result of Eq. (ii) in the divergence theorem we can write the flux of the vector field $\vec{\mathbf{A}}$ as (read the margin remark):

$$\oiint_S \vec{\mathbf{A}}.d\vec{\mathrm{S}} = \iiint_V \vec{\nabla}.\vec{\mathbf{A}}\,dV = 4\iiint_V dV = 4(2a)^3 = 32a^3$$

In writing the final result we have used the volume property of the triple integral to write $\iiint_V dV$ as the volume of the region V, which is just the volume of the cube of side $2a$ that is $8a^3$.

ii) Using the divergence theorem for the vector field $\vec{\mathbf{F}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ we can write for the flux,

$$\Phi = \oiint_S \vec{\mathbf{F}}.d\vec{\mathbf{S}} = \oiint_S \left(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}\right).d\vec{\mathbf{S}} = \iiint_V \vec{\nabla}.\left(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}\right)dV \qquad \text{(i)}$$

where V is the volume enclosed by the sphere enclosed by surface S given by $x^2 + y^2 + z^2 = a^2$. We evaluate the integral on the RHS of Eq. (i)

$$\iiint_V \vec{\nabla}.\vec{\mathbf{F}}dV = \iiint_V \left[\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}\right]dV = 3\iiint_V dV = 3\left[\frac{4}{3}\pi a^3\right] = 4\pi a^3 \qquad \text{(ii)}$$

Using the volume property of the triple integral we can see that $\iiint_V dV$ is just the volume of the sphere of radius a which is $\frac{4}{3}\pi a^3$.

You may now like to work out the following SAQ.

SAQ 8 – Evaluating surface integral using the divergence theorem

Evaluate $\oiint_S \vec{\mathbf{V}}.\hat{\mathbf{n}}\,d\vec{\mathbf{S}}$, where $\vec{\mathbf{V}} = x\cos^2 y\,\hat{\mathbf{i}} + xz\,\hat{\mathbf{j}} + z\sin^2 y\,\hat{\mathbf{k}}$ and S is the surface of a sphere with its centre at the origin and radius 3 units.

Let us now consider an application of the divergence theorem.

4.7.1 Application of the Divergence Theorem

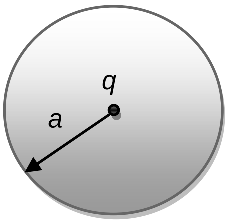

Fig. 4.22: **Electric flux due to a point charge *q* through a sphere of radius *a*.**

You have studied in your school physics courses that the electric field due to a point charge q, at a point whose position vector with respect to the location of q is $\vec{\mathbf{r}}$, is given by

$$\vec{\mathbf{E}} = \frac{kq}{r^3}\vec{\mathbf{r}} \qquad (r \ne 0) \qquad (4.41)$$

where k is a constant dependent on the nature of the medium.

Let us now determine the flux of $\vec{\mathbf{E}}$ through a sphere of radius a (Fig. 4.22) whose centre is at the position of the charge q.

The required surface integral is $\oiint_S \vec{\mathbf{E}}.d\vec{\mathbf{S}}$, where S is the surface of a sphere of radius a. Here

$$\vec{\mathbf{E}} = \frac{kq}{r^3}\vec{\mathbf{r}} = \frac{kq}{r^3} r\hat{\mathbf{r}} = \frac{kq}{r^2}\hat{\mathbf{r}} \qquad (4.42)$$

where $\hat{\mathbf{r}}$ is the unit vector along the position vector $\vec{\mathbf{r}}$ .Contribution to a surface integral comes from the surface only. So we have to know $\vec{\mathbf{E}}$ on the surface of the sphere, which is $\frac{kq}{a^2}\hat{\mathbf{r}}$. Again, we know at every point on the sphere

$d\vec{\mathbf{S}} = dS\,\hat{\mathbf{r}}$ where dS is the surface element on the surface of a sphere.

Hence, the required flux = $\oiint\limits_S \frac{kq}{a^2}\hat{\mathbf{r}}.dS\hat{\mathbf{r}} = \oiint\limits_S \frac{kq}{a^2} dS \qquad (\because\ \hat{\mathbf{r}}.\hat{\mathbf{r}} = 1)$

$$= \frac{kq}{a^2}\oiint\limits_S dS \qquad (4.43)$$

because $\oiint dS$ is the surface area of the sphere of radius a which is $4\pi a^2$, we can write

$$\oiint\limits_S \vec{\mathbf{E}}.d\vec{\mathbf{S}} = 4\pi kq \qquad (4.44)$$

where S is the surface of a sphere that encloses charge q. It can be shown that the above result is true for **any** charge distribution. Suppose that a closed surface enclosing a volume V has a continuous distribution of charge. If the charge per unit volume is ρ, then $q = \iiint\limits_V \rho dV$.

An example of such a distribution is a charged sphere. For this distribution, we have

$$\oiint\limits_S \vec{\mathbf{E}}.d\vec{\mathbf{S}} = 4\pi k \iiint\limits_V \rho\, dV \qquad (4.45)$$

But using Eq. (4.40), we have

$$\oiint\limits_S \vec{\mathbf{E}}.d\vec{\mathbf{S}} = \iiint\limits_V \vec{\nabla}.\,\vec{\mathbf{E}}\, dV \qquad (4.46)$$

From Eqs. (4.40) and (4.44),, we get

$$\iiint\limits_V \vec{\nabla}.\,\vec{\mathbf{E}}\, dV = 4\pi k \iiint\limits_V \rho dV$$

or
$$\iiint\limits_V \left(\vec{\nabla}.\,\vec{\mathbf{E}} - 4\pi k\rho\right) dV = 0 \qquad (4.47)$$

Since dV is an arbitrary infinitesimal volume element, the integrand in Eq. (4.47) must be zero:

$$\therefore \quad \vec{\nabla}.\,\vec{\mathbf{E}} - 4\pi k\rho = 0 \Rightarrow \vec{\nabla}.\,\vec{\mathbf{E}} = 4\pi k\rho \qquad (4.48)$$

Eq. (4.48) tells us that the divergence of the electric field vector due to a continuous distribution of charge is independent of the extent of distribution. It depends only on the charge per unit volume. In charge-free space, $\rho = 0$, so that

$$\vec{\nabla}.\,\vec{\mathbf{E}} = 0 \qquad (4.49)$$

The advantage of the divergence theorem is that it enables us to convert a volume integral to a surface integral and vice versa. In applications of the divergence theorem, the strategy for problem solving should be to evaluate the simpler of the two integrals.

You may now like to solve an SAQ to apply the divergence theorem.

***SAQ 9* – The divergence theorem**

a) Show that for any closed surface S the surface integral

$$\oiint_S \vec{\mathbf{r}} \cdot d\vec{\mathbf{S}} = 3V$$

where V is the volume of the region enclosed by the surface.

b) Show that for a vector $\vec{\mathbf{B}} = \vec{\nabla} \times \vec{\mathbf{A}}$

$$\oiint_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}} = 0$$

4.8 SUMMARY

Concept	Description
Applications of double integrals	■ Double integrals are used in physics to evaluate the following quantities: • **Area** A of a region R $A = \iint_R dx\,dy$ • **Mass** m **of a body** with a density $\gamma(x,y)$ over a region R $m = \iint_R \gamma(x,y)\,dx\,dy$ • **Centre of mass** (x_{cm}, y_{cm}) **of a body** with a density $\gamma(x,y)$ over a region R $x_{cm} = \dfrac{\iint_R x\,\gamma(x,y)\,dx\,dy}{m}; \quad y_{cm} = \dfrac{\iint_R y\,\gamma(x,y)\,dx\,dy}{m}$ • The **average value** μ of **a continuous function** *f(x,y)* over a closed region R in the xy plane is: $\mu = \dfrac{\iint_R f(x,y)\,dx\,dy}{\iint_R dx\,dy}; \quad \iint_R dx\,dy = \text{Area of the region of integration } R$
Surface integral	■ The surface integral of a scalar or a vector field is the generalisation of the double integral where the region of integration may be any surface. Surface integrals can occur in any of the following three forms: $\iint_S \phi\,d\vec{\mathbf{S}}, \quad \iint_S \vec{\mathbf{A}}.d\vec{\mathbf{S}} \quad \text{and} \quad \iint_S \vec{\mathbf{A}} \times d\vec{\mathbf{S}}$ The element of area is $d\vec{\mathbf{S}} = dS\hat{\mathbf{n}}$.

Flux of a vector field

- The flux of a vector field $\vec{\mathbf{A}}$ over a surface *S* is given by the surface integral

$$\Phi = \iint_S \vec{\mathbf{A}}.d\vec{\mathbf{S}}$$

Volume/triple integral of a function

- The volume/triple integral of a function *f*(*x*,*y*,*z*) over a closed bounded region Ω is written as $\iiint_\Omega f(x,y,z)dV$ or $\iiint_\Omega f(x,y,z)dxdydz$ and can be defined as the limit of a sum as follows:

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i, y_i, z_i)\Delta V_i = \iiint_\Omega f(x,y,z)dV = \iiint_\Omega f(x,y,z)dxdydz$$

Applications of volume integrals

- Volume/triple integrals are used in physics to evaluate the following quantities:
  - **Volume** *V* of a region Ω

$$V = \iiint_\Omega dx\,dydz$$

  - **Mass** *m* **of a body** with a density $\gamma(x,y,z)$ over a region Ω

$$m = \iiint_\Omega \gamma(x,y,z)\,dx\,dydz$$

  - **Centre of mass of a body** (x_{cm}, y_{cm}, z_{cm}) with a density $\gamma(x,y,z)$ over a region Ω

$$x_{cm} = \frac{\iiint_\Omega x\,\gamma(x,y,z)dx\,dydz}{m};\quad y_{cm} = \frac{\iiint_\Omega y\,\gamma(x,y,z)dx\,dydz}{m};$$

$$z_{cm} = \frac{\iiint_\Omega z\,\gamma(x,y,z)dx\,dydz}{m}$$

  - **Moment of inertia of a body** with density $\gamma(x,y,z)$ over a region Ω about the *x*-axis, I_x, about the *y*-axis I_y and about the *z*-axis, I_z.

$$I_x = \iiint_\Omega (y^2 + z^2)\gamma(x,y,z)dx\,dydz$$

$$I_y = \iiint_\Omega (x^2 + z^2)\,\gamma(x,y,z)dx\,dydz;$$

$$I_z = \iiint_\Omega (x^2 + y^2)\,\gamma(x,y,z)dx\,dydz$$

Vector integral theorems

- The **Stokes' theorem** states that the integral of the curl of a vector field over a surface *S* is equal to the line integral of the vector field over the closed path bounding *S* and is expressed mathematically as:

$$\oint_C \vec{\mathbf{A}}.d\vec{\mathbf{l}} = \iint_S \text{curl}\,\vec{\mathbf{A}}.d\vec{\mathbf{S}}$$

- The **divergence theorem** states that the integral of the divergence of a vector field over a volume *V* is equal to the surface integral of the vector field over the closed surface bounding *V* and is expressed mathematically as:

$$\oiint_S \vec{\mathbf{A}}.d\vec{\mathbf{S}} = \iiint_V \textbf{div}\,\vec{\mathbf{A}}\,dV$$

4.9 TERMINAL QUESTIONS

1. Use double integration to find the area of the region in the xy plane bounded by the curves $y = x$ and $y = x^3$ for $x > 0$.

2. Calculate the volume V of a solid which is bound above by the plane $z = 4 - y$ and below by the region R defined by the circle $x^2 + y^2 = 4$.

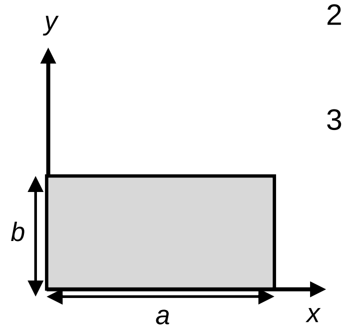

Fig. 4.23: **A rectangular lamina** $0 \le x \le a$, $0 \le y \le b$.

3. The product of inertia of a lamina in the xy plane about the x and y-axes is given by

$$I_{xy} = I_{yx} = \iint_R \sigma xy\, dx\, dy$$

where R is the region of space covered by the lamina and σ is the mass per unit area of the lamina. Determine I_{xy} for the lamina shown in Fig. 4.23.

4. A box is bounded by the planes $x = 0$; $x = 1$; $y = 0$; $y = 1$; $z = 0$ and $z = 2$. It has a density $\gamma(x, y, z) = \left(9 - z^3\right)$ kg m^{-3}. Calculate the mass of the box.

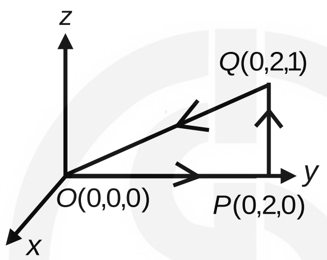

Fig. 4.24: **Path** ***OPQ*** **for TQ 6.**

5. Determine the flux of the vector field $\vec{\mathbf{F}} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} - 2z\,\hat{\mathbf{k}}$ over the surface of a sphere S defined by the equation $x^2 + y^2 + z^2 = 1$.

6. Verify Stokes' theorem for the vector field $\vec{\mathbf{A}} = z^2\hat{\mathbf{j}} + yz\hat{\mathbf{k}}$, where C is the path OPQ in the yz plane shown in Fig. 4.24.

7. Show that the line integral $\oint_C (yz\,dx + xz\,dy + xz\,dz)$ is zero along any closed contour C.

8. Using Stoke's Theorem evaluate $\oint_C \vec{\mathbf{F}}.d\vec{\mathbf{l}}$

$$\vec{\mathbf{F}} = x^2\hat{\mathbf{i}} + 2x\hat{\mathbf{j}} + z^2\hat{\mathbf{k}}$$

where C is the ellipse in the xy plane defined by

$$\frac{x^2}{16} + \frac{y^2}{64} = 1, \; z = 0$$

9. Using the divergence theorem, calculate the flux of a vector field $\vec{\mathbf{F}} = z\hat{\mathbf{i}} + 2y\hat{\mathbf{j}} - x^3\hat{\mathbf{k}}$ over a sphere of radius 2 units.

10. Evaluate the flux of the vector field $\vec{\mathbf{A}} = (2y\,\hat{\mathbf{i}} + 5y^2\,\hat{\mathbf{j}} + 4z\,\hat{\mathbf{k}})$ through the surface of a unit cube which has one corner at the origin, one corner at (1, 1, 1) and all its edges are parallel to the coordinate axes.

4.10 SOLUTIONS AND ANSWERS

Self-Assessment Questions

1. a) We have to evaluate $\iint_R dx\, dy$ where R is the region bound by $y = x^2$ and $y = x^3$. Following Example 4.1, let us first determine the points of intersection of the two curves in the region $x > 0$; $y > 0$, for this we solve the equations

$$y = x^2 \quad \text{and} \quad y = x^3$$

$$\Rightarrow \quad x^2 = x^3 \Rightarrow x^2(x-1) = 0$$

So the points of intersection are *x* = 0 and *x* = 1 and the limits on *x* and *y* are $x^3 \le y \le x^2$; $0 \le x \le 1$

Note that the limits on *y* are $x^3 \le y \le x^2$ and not $x^2 \le y \le x^3$. This is because for $0 \le x \le 1$, $x^3 < x^2$.

$$\therefore \quad A = \int_{x=0}^{1}\left[\int_{y=x^3}^{x^2} dy\,dx\right] = \left.\int_0^1 [y]_{x^3}^{x^2}dx\right| = \int_0^1 (x^2 - x^3)\,dx$$

$$= \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{1}{12}$$

b) We use Eq. (A2.3) to evaluate the integral with $f(x,y) = x^4$. The lim on *y* for the region of integration on the *xy* plane is defined by the equation:

$$x^2 \le y \le 1$$

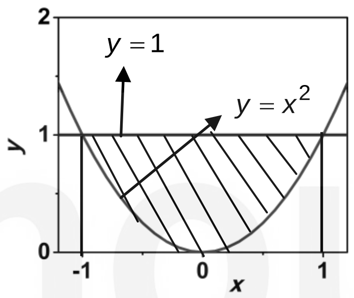

Fig. 4.25: **The region of integration for SAQ 1(b). The two curves intersect at** *x* = 1 **and**

We obtain the limits on *x* in the region of integration by determining the value of *x* at the points at which the two curves $y = x^2$ and *y = 1* intersect, as you see in Fig. 4.25. This is found by solving for *x* as follows:

$$x^2 = 1 \Rightarrow x = 1, -1$$

So the integral we have to evaluate is the following:

$$I = \int_{x=-1}^{1}\int_{y=x^2}^{1} x^4\,dy\,dx$$

Integrating over *y* first, we get:

$$I = \int_{-1}^{1}\left[y\,x^4\right]_{x^2}^{1} dx = \int_{-1}^{1}\left[x^4 - x^6\right]dx$$

Integrating over *x*, we then get:

$$I = \left[\frac{x^5}{5} - \frac{x^7}{7}\right]_{-1}^{1} = \frac{4}{35}$$

2. From Fig. 4.10 we can see that

$$d\vec{\mathbf{S}}_2 = dxdy\,\hat{\mathbf{k}}$$

$$d\vec{\mathbf{S}}_3 = -dxdz\,\hat{\mathbf{j}}$$

$$d\vec{\mathbf{S}}_4 = dxdz\,\hat{\mathbf{j}}$$

and $\quad d\vec{\mathbf{S}}_5 = -dy\,dz\,\hat{\mathbf{i}}$

3. a) Using the results of SAQ 2, we can write the surface integral over the surface $S_4(0 \le x \le 1; 0 \le z \le 1)$ on the plane $y = 1$ as

$$I_4 = \iint_{S_4} \vec{\mathbf{A}}.d\vec{\mathrm{S}}_4 = \iint_{S_4} \left[2xz\hat{\mathbf{i}} + 2xz\hat{\mathbf{j}} - yz\hat{\mathbf{k}}\right].\left[dxdz\hat{\mathbf{j}}\right] = \iint_{S_4} 2xzdxdz$$

We evaluate this double integral to get:

$$I_4 = 2\iint_{S_4} xzdxdz = 2\left[\int_0^1 xdx\right]\left[\int_0^1 zdz\right] = \frac{1}{2}$$

Similarly over the surface S_5 $(0 \le y \le 1; 0 \le z \le 1)$ on the plane $x = 0$ (Fig. 4.10) we get

$$I_5 = \iint_{S_5} \vec{\mathbf{A}}.d\vec{\mathbf{S}}_5 = \iint_{S_5} \left[2xz\hat{\mathbf{i}} + 2xz\hat{\mathbf{j}} - yz\hat{\mathbf{k}}\right]\left[-dydz\,\vec{\mathbf{i}}\right]$$

$$= -\iint_{S_5} 2xzdydz = 0 \;(\because x = 0)$$

Over the surface $S_6(0 \le y \le 1; 0 \le z \le 1)$ which is on the plane $x = 1$, we have

$$I_6 = \iint_{S_6} \vec{\mathbf{A}}.d\bar{\mathbf{S}}_6 = \iint_{S_6} \left[2xz\hat{\mathbf{i}} + 2xz\hat{\mathbf{j}} - yz\hat{\mathbf{k}}\right]\left[dydz\,\vec{\mathbf{i}}\right]$$

$$= -\iint_{S_6} 2xzdydz = \iint_{S_6} 2zdydz \;(\because x = 1)$$

We evaluate this as a double integral:

$$I_6 = 2\iint_{S_6} zdydz = 2\left[\int_0^1 dy\right]\left[\int_0^1 zdz\right] = 2[y]_0^1\left[\frac{z^2}{2}\right]_0^1 = 1$$

b) Since the disc is parallel to the xy plane, we can write as explained in Sec. 3.3,

$$d\vec{\mathbf{S}} = dxdy\,\hat{\mathbf{k}}$$

$$\therefore \iint_S \vec{\mathbf{r}}.d\vec{\mathbf{S}} = \iint_S (x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}})\;.dxdy\,\hat{\mathbf{k}}$$

$$= \iint_S z\,dx\,dy \;= 5\iint_S dx\,dy \quad \text{(since the disc lies in the plane } z = 5)$$

$$= 5\,\pi.2^2 = 20\pi \;\left(\iint_S dx\,dy \text{ is the area of the circle of radius 2 units}\right)$$

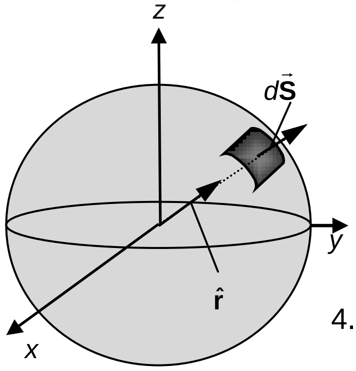

Fig. 4.26: **The unit normal for an area element on the surface of a sphere.**

4. i) Refer to Fig. 4.26. $\hat{\mathbf{r}}$ is the unit vector along the position vector $\vec{\mathbf{r}}$. Since $d\vec{\mathbf{S}}$ points along the outward drawn normal, it points along $\hat{\mathbf{r}}$ at every point on the sphere so that $d\vec{\mathbf{S}} = dS\,\hat{\mathbf{r}}$

$$\therefore \qquad \hat{\mathrm{r}}.d\vec{\mathbf{S}} = \hat{\mathbf{r}}.dS\hat{\mathbf{r}} = dS(\hat{\mathbf{r}}.\hat{\mathbf{r}}) = dS \quad (\because \hat{\mathbf{r}}\,.\,\hat{\mathbf{r}} = 1)$$

Hence $\oiint_S \hat{\mathrm{r}}.d\vec{\mathbf{S}} = \oiint_S dS = S$, which is the surface area of the sphere.

Thus $\qquad \oiint_S \hat{\mathrm{r}}.d\vec{\mathbf{S}} = 4\pi R^2$

ii) Similarly

$$\oiint_S \frac{\hat{\mathbf{r}}.d\vec{\mathbf{S}}}{r^2} = \frac{1}{R^2}\oiint_S dS = \frac{4\pi R^2}{R^2} = 4\pi$$

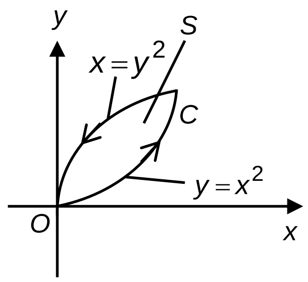

Fig. 4.27: **Figure for SAQ 5.**

5. To evaluate the line integral using Stoke's theorem as given in Eq. (4.19), we first evaluate $\vec{\nabla}\times\vec{\mathbf{A}}$ as:

$$\vec{\nabla}\times\vec{\mathrm{A}} = \begin{vmatrix} \hat{\mathrm{i}} & \hat{\mathrm{j}} & \hat{\mathrm{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x-y) & (x+y) & 0 \end{vmatrix} = \hat{\mathrm{k}}\left[\frac{\partial}{\partial x}(x+y) - \frac{\partial}{\partial y}(x-y)\right] = 2\hat{\mathrm{k}}$$

The contour C and the region R are in the xy plane as shown in Fig. 4.27 (Fig. 4.16 reproduced here), therefore $d\vec{\mathbf{S}} = dx\,dy\,\hat{\mathbf{k}}$. Substituting for $\vec{\nabla}\times\vec{\mathbf{A}}$ and $d\vec{\mathbf{S}}$ into Eq. (4.19) we can write the integral as:

$$I = \oint_C \vec{\mathbf{A}}.d\vec{\mathbf{l}} = \iint_S (\vec{\nabla}\times\vec{\mathbf{A}}).d\,\vec{\mathbf{S}} = \iint_S (2\hat{\mathbf{k}}).(dx\,dy\,\hat{\mathbf{k}}) = 2\iint_S dxdy$$

The limits on x are given by the points of intersection of the curves $x = y^2$ and $y = x^2$. By solving $\sqrt{x} = x^2$ we get the points of intersection as $x = 0$ and $x = 1$.

We define the region S (shown in Fig. 4.25) by the equations (see margin remark):

$$0 \le x \le 1;\; x^2 \le y \le \sqrt{x}$$

Then

$$I = 2\int_{x=0}^{1}\left[\int_{y=x^2}^{\sqrt{x}} dy\right]dx = 2\int_{x=0}^{1}[y]_{x^2}^{\sqrt{x}}dx = 2\int_{x=0}^{1}\left[\sqrt{x} - x^2\right]dx$$

$$= 2\left[\frac{2}{3}\left(x^{3/2}\right) - \frac{x^3}{3}\right]_0^1 = 2\left[\frac{2}{3} - \frac{1}{3}\right] = \frac{2}{3}$$

6. Refer to Fig. 4.28. You have seen that for a conservative force

$$\int_{ACB}\vec{\mathbf{F}}.d\vec{\mathbf{r}} = -\int_{-ADB}\vec{\mathbf{F}}.d\vec{\mathbf{r}}$$

or

$$\int_{ACB}\vec{\mathbf{F}}.d\vec{\mathbf{r}} + \int_{-ADB}\vec{\mathbf{F}}.d\vec{\mathbf{r}} = 0$$

i.e. $$\oint_{ACBDA}\vec{\mathbf{F}}.d\vec{\mathbf{r}} = 0$$

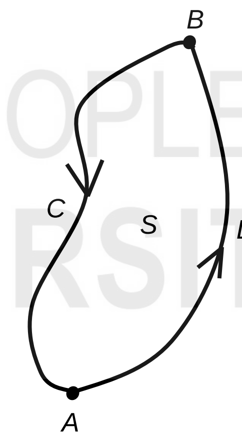

Fig. 4.28: **Diagram for the solution of SAO 6.**

From Stokes' theorem, we know that

$$\oint_{ACBDA}\vec{\mathbf{F}}.d\vec{\mathbf{r}} = \iint_S (\vec{\nabla}\times\vec{\mathbf{F}}).d\vec{\mathbf{S}}$$

So,

$$\iint_S (\vec{\nabla}\times\vec{\mathbf{F}}).d\vec{\mathbf{S}} = 0$$

But $d\vec{\mathbf{S}}$ is arbitrary. Hence the integrand is zero. Moreover, since the path *ACBDA* has been chosen anywhere in the field, we can write

$\vec{\nabla}\times\vec{\mathbf{F}}=0$ everywhere in the field.

7. a) We write the volume integral as

$$I=\iiint\limits_{\Omega}(\sin x)y\,z\,dxdydz\,,$$

where Ω is defined by the equations:

$0\le x,y,z\le\pi$

In this case, we can write the integral as:

$$I=\left[\int_0^{\pi}\sin x\,dx\right]\left[\int_0^{\pi}y\,dy\right]\left[\int_0^{\pi}z\,dz\right]=[-\cos x]_0^{\pi}\left[\frac{y^2}{2}\right]_0^{\pi}\left[\frac{z^2}{2}\right]_0^{\pi}$$

$$=\frac{\pi^4}{2}$$

b) Using Eq. (4.35) with γ (*x, y, z*) = ρ (*x, y, z*) we can write the mass of the cube as $m=\iint\limits_{\Omega}\int\rho(x,y,z)\,dx\,dy\,dz$ where Ω is the volume of the cube. For the unit cube

$$0\le x\le 1,\quad 0\le y\le 1\ \text{ and }\ 0\le z\le 1.$$

$$\therefore\qquad m=\int_0^1\int_0^1\left(\int_0^1(x+2y+3z)\,dz\right)dx\,dy$$

$$=\int_0^1\left(\int_0^1\left(xz+2yz+\frac{3z^2}{2}\right)_0^1 dy\right)dz$$

$$=\int_0^1\left(\int_0^1\left(x+2y+\frac{3}{2}\right)dy\right)dx$$

$$=\int_0^1\left(xy+y^2+\frac{3}{2}y\right)_0^1 dx$$

$$=\int_0^1\left(x+1+\frac{3}{2}\right)dx=\left(\frac{x^2}{2}+\frac{5}{2}x\right)_0^1$$

$$=3\ \text{units}$$

8. Using Eq. (2.3), we first determine the divergence of the vector field,

$$\vec{\nabla}.\vec{\mathbf{V}}=\frac{\partial}{\partial x}(x\cos^2 y)+\frac{\partial}{\partial y}(xz)+\frac{\partial}{\partial z}(z\sin^2 y)$$

$$=\cos^2 y+\sin^2 y=1 \qquad \text{(i)}$$

Using Eq. (4.39), we write using the result of Eq. (i)

$$I=\oiint\limits_{S}\vec{\mathbf{V}}.\hat{\mathbf{n}}\,d\vec{\mathbf{S}}=\iiint\limits_{\Omega}\vec{\nabla}.\vec{\mathbf{V}}\,dV=\iiint\limits_{\Omega}dV$$

where Ω is the sphere of radius 3 units with its centre at the origin.

$$\therefore \qquad I = \frac{4}{3}\pi(3^3) = 36\pi$$

9. a) Using Eq. (4.39) with $\vec{\mathbf{A}} = \vec{\mathbf{r}} = x\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ we can write

$$I = \oint\!\!\oint_S \vec{\mathbf{r}}.\, d\vec{\mathbf{S}} = \iiint_\Omega \left(\vec{\nabla}.\vec{\mathbf{r}}\right) dV \qquad \text{(i)}$$

and

$$\vec{\nabla}.\vec{\mathbf{r}} = \vec{\nabla}.\,(x\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}) = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 3$$

Replacing $\vec{\nabla}.\vec{\mathbf{r}} = 3$ in Eq. (i) we get:

$$I = \oint\!\!\oint_S \vec{\mathbf{r}}.d\vec{\mathbf{S}} = 3\left[\iiint_\Omega dV\right] \qquad \text{(ii)}$$

Using the volume property of a triple integral, the quantity in the bracket in the RHS of Eq. (ii) is just the volume of the region of integration which is V.

$$\therefore \quad I = \oint\!\!\oint_S \vec{\mathbf{r}}.\, d\vec{\mathbf{S}} = 3V$$

b) Using the divergence theorem we can write:

$$\oint\!\!\oint_S \vec{\mathbf{B}}.d\vec{\mathbf{S}} = \left[\iiint_V \vec{\nabla}.\vec{\mathbf{B}}\, dV\right] \qquad \text{(i)}$$

Given that $\vec{\mathbf{B}} = \vec{\nabla} \times \vec{\mathbf{A}}$, we can write:

$$\vec{\nabla}.\vec{\mathbf{B}} = \vec{\nabla}.\,(\vec{\nabla} \times \vec{\mathbf{A}}) = 0$$

This is because the divergence of the curl of a vector field is always zero, as you have studied in Unit 2.

$$\therefore \quad \oint\!\!\oint_S \vec{\mathbf{B}}.d\vec{\mathbf{S}} = 0$$

Terminal Questions

1. We use the area property of the double integral given in Eq. (A.2.7) to find the area. Following Example 4.1, the range of x is decided by determining the points of intersection of the curves $y = x$ and $y = x^3$ (Fig. 4.29). We solve as follows:

$$x^3 = x \Rightarrow x^2(x-1) = 0 \Rightarrow x = 0,1$$

The points of intersection are $x = 0$ and $x = 1$(Fig. 4.29). Note that in the range $0 \le x \le 1,\ x^3 \le x$. Therefore, the region of integration is:

$$0 \le x \le 1\,;\, x^3 \le y \le x$$

The area A is:

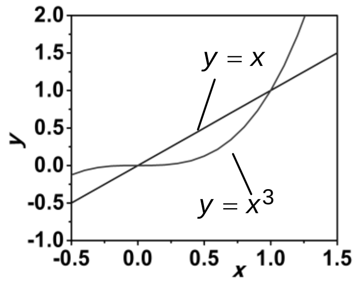

Fig. 4.29: **Region of integration is the area enclosed between the curves $y = x$ and $y = x^3$ in range** $0 \le x \le 1$.

$$A = \int_{x=0}^{1} \int_{y=x^3}^{x} (1)\, dy\, dx = \int_0^1 [y]_{x^3}^{x}\, dx = \int_0^1 \left[x - x^3\right] dx \quad \text{[Integrating over } y \text{ first]}$$

$$= \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1 = \frac{1}{4}\,\text{units}. \qquad \text{[Integrating over } x\text{]}$$

2. Using the double integral, we can define the volume under the plane $z = 4 - y$ [see Example 4.1(ii)] as:

$$V = \iint_R (4 - y)\, dx dy \qquad \text{(i)}$$

where R is the region in the xy plane which is enclosed by the circle $x^2 + y^2 = 4$.

$$\int \sqrt{a^2 - x^2}\, dx = \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right)$$

The region of integration R for Eq. (i) is

$$-2 \le x \le 2; \; -\sqrt{4 - x^2} \le y \le \sqrt{4 - x^2} \qquad \text{(ii)}$$

$$\therefore \quad V = \int_{x=-2}^{2} \left[\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4 - y)\, dy \right] dx$$

$$= \int_{-2}^{2} 8\sqrt{4 - x^2}\, dx$$

(after integrating over y).

On integrating over x this gives us (read the margin remark):

$$V = 16\left[\sin^{-1}\left(\frac{x}{2}\right)\right]_{-2}^{2} = 16\pi$$

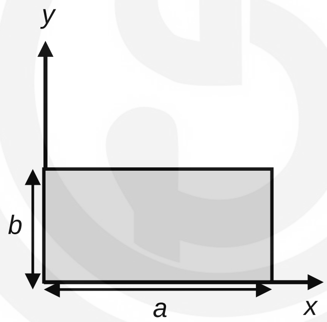

Fig. 4.30: **A rectangular lamina** $0 \le x \le a$, $0 \le y \le b$.

3. Refer to Fig. 4.30, which is Fig. 4.23 repeated here for convenience.

$$I_{xy} = \int_0^b \int_0^a \frac{m}{ab} xy\, dx\, dy$$

where m is the mass of the rectangle. Using Eq. (A2.12) we can write:

$$\therefore \quad I_{xy} = \frac{m}{ab} \left(\int_0^a x\, dx\right)\left(\int_0^b y\, dy\right)$$

Evaluating both the integrals separately we get:

$$\therefore \quad I_{xy} = \frac{m}{ab}\left[\frac{x^2}{2}\right]_0^a \left[\frac{y^2}{2}\right]_0^b = \frac{mab}{4}$$

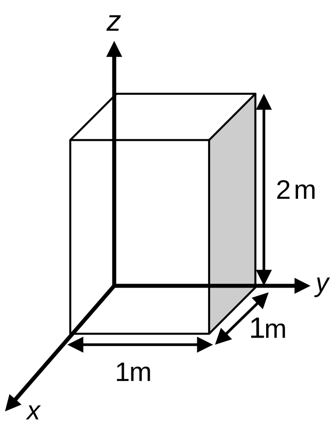

Fig. 4.31: **Diagram for solution of TQ 4.**

4. We determine the mass of the box m using Eq. (4.35) with $\gamma(x,y,z) = \left(9 - z^3\right)\text{kg m}^{-3}$ and Ω (see Fig. 4.31) as defined by the equations:

$$0 \le x \le 1;\ 0 \le y \le 1;\ 0 \le z \le 2 \qquad \text{(i)}$$

Then m is:

$$m = \iiint_{\Omega} \left(9 - z^3\right) dx\,dydz = \int_{x=0}^{1} \int_{y=0}^{1} \int_{z=0}^{2} \left(9 - z^3\right) dx\,dy\,dz \qquad \text{(ii)}$$

We can evaluate this integral as follows (read the margin remark):

$$m = 18 - [x]_0^1 [y]_0^1 \left[\frac{z^4}{4}\right]_0^2 = 14\,\text{kg}$$

Note that: $\int_{x=0}^{1} \int_{y=0}^{1} \int_{z=0}^{2} dx\,dy\,dz$ is the volume of the cube.

5. The flux of the vector field $\vec{\mathbf{F}}$ is:

$$\Phi = \oiint_S \vec{\mathbf{F}}.\,d\vec{\mathbf{S}} = \oiint_S \left(x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} - 2z\,\hat{\mathbf{k}}\right).\,d\vec{\mathbf{S}} \qquad \text{(i)}$$

We evaluate the flux using the Divergence theorem (Eq. 4.40).

$$\therefore \qquad \Phi = \oiint_S \left(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} - 2z\hat{\mathbf{k}}\right).\,d\vec{\mathbf{S}} = \iiint_V \vec{\nabla}.\left(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} - 2z\hat{\mathbf{k}}\right) dV \qquad \text{(ii)}$$

We evaluate the integral on the RHS of Eq. (ii) by first calculating:

$$\vec{\nabla}.\,(x\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} - 2z\,\hat{\mathbf{k}}) = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(-2z) = 0 \qquad \text{(iii)}$$

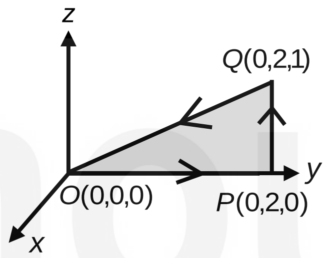

Fig. 4.32: **Path** *OPQ* **for solution of TQ 6.**

By replacing the divergence of the vector field from Eq. (iii) into Eq. (ii) we get:

$$\Phi = 0$$

6. First we shall calculate $\oint_C \vec{\mathbf{A}}.d\vec{\mathbf{l}}$ where C is shown in Fig. 4.32. Here

$$\vec{\mathbf{A}} = z^2\hat{\mathbf{j}} + yz\hat{\mathbf{k}}, d\vec{\mathbf{l}} = dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}} + dz\hat{\mathbf{k}}$$

$$\therefore \qquad \vec{\mathbf{A}}.d\vec{\mathbf{l}} = z^2 dy + yz\,dz$$

Now $\oint_{\mathbf{C}} \vec{\mathbf{A}}.d\vec{\mathbf{l}} = \int_{OP} \vec{\mathbf{A}}.d\vec{\mathbf{l}} + \int_{PQ} \vec{\mathbf{A}}.d\vec{\mathbf{l}} + \int_{QO} \vec{\mathbf{A}}.d\vec{\mathbf{l}}$

For the straight line *OP*, $x = 0$, $0 \le y \le 2$, $z = 0$. Hence $\int_{OP} \vec{\mathbf{A}}.d\vec{\mathbf{l}} = 0$.

For the straight line *PQ*, $x = 0$, $y = 2$, $0 \le z \le 1$. Hence $dy = 0$ and

$$\int_{PQ} \vec{\mathbf{A}}.d\vec{\mathbf{l}} = \int_0^1 2z\,dz = 1$$

And for the straight line *QO*, $x = 0$, $y = 2z$, $1 \le z \le 0$. Also $dy = 2dz$ (see margin remark) and

$$\int_{QO} \vec{\mathbf{A}}.d\vec{\mathbf{l}} = \int_{QO} z^2 dy + \int_{QO} yzdz = \int_{QO} 2z^2 dz + \int_{QO} (2z)\,z\,(dz) = \int_1^0 4z^2\,dz = -\frac{4}{3}$$

OQ is a straight line in the *yz* plane and its equation is $y - 2z = 0$

$$\therefore \qquad \oint_C \vec{\mathbf{A}}.d\vec{\mathbf{l}} = 0 + 1 - \frac{4}{3} = -\frac{1}{3}$$

Next we evaluate the integral using Stokes' theorem.

$$\vec{\nabla}\times\vec{\mathbf{A}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0 & z^2 & yz \end{vmatrix} = -z\hat{\mathbf{i}}$$

Since the path C is traversed anticlockwise, we have $d\vec{\mathbf{S}} = dS\,\hat{\mathbf{i}}$.

Moreover, as S lies on the yz plane, $d\vec{\mathbf{S}} = dydz\,\hat{\mathbf{i}}$

$$\therefore\ \left(\vec{\nabla}\times\vec{\mathbf{A}}\right).d\vec{\mathbf{S}} = \left(-z\hat{\mathbf{i}}\right).\left(dydz\,\hat{\mathbf{i}}\right) = -z\,dydz$$

S is defined by the equations $0 \le y \le 2; 0 \le z \le y/2$

$$\iint_S \left(\vec{\nabla}\times\vec{\mathbf{A}}\right).d\vec{\mathbf{S}} = -\iint_S z\,dy\,dz = -\int_0^2\left(\int_0^{y/2} z\,dz\right)dy$$

$$= -\int_0^2 \frac{y^2}{8}\,dy = -\frac{1}{8}\left|\frac{y^3}{3}\right|_0^2 = -\frac{1}{3}$$

$$\therefore\quad \iint_S \left(\vec{\nabla}\times\vec{\mathbf{A}}\right).d\vec{\mathbf{S}} = \int_C \vec{\mathbf{A}}.d\vec{\mathbf{l}}$$

which is Stokes' theorem.

7. Let S be the surface bounded by the closed curve C. We first note that the given line integral can be written as $\oint_C \vec{\mathbf{F}}.d\vec{\mathbf{l}}$ where

$$\vec{\mathbf{F}} = yz\hat{\mathbf{i}} + xz\hat{\mathbf{j}} + xy\hat{\mathbf{k}}$$

Applying Stokes' theorem we can write:

$$\oint_C \vec{\mathbf{F}}.d\vec{\mathbf{l}} = \iint_S \left(\vec{\nabla}\times\vec{\mathbf{F}}\right).d\vec{\mathbf{S}} = \iint_S \left[\vec{\nabla}\times\left(yz\hat{\mathbf{i}} + xz\hat{\mathbf{j}} + xy\hat{\mathbf{k}}\right)\right].d\vec{\mathbf{S}}$$

We next find $\vec{\nabla}\times(yz\hat{\mathbf{i}} + xz\hat{\mathbf{j}} + xy\hat{\mathbf{k}})$ which is:

$$\vec{\nabla}\times\vec{\mathbf{F}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix} = 0$$

Since $\vec{\nabla}\times\vec{\mathbf{F}} = \vec{\mathbf{0}}$, from Stokes' theorem we get that $\oint_C \vec{\mathbf{F}}.d\vec{\mathbf{l}} = 0$ for any closed contour C.

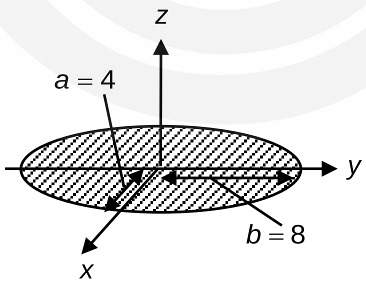

Fig. 4.33: **The shaded region is the surface of integration** *S.*

8. The surface of integration is the shaded region shown in Fig. 4.33 which is an ellipse in the xy plane defined by the equation:

$$\frac{x^2}{16} + \frac{y^2}{64} = 1; \quad z = 0$$

The parameters (semi-major and semi-minor axes) of the ellipse are a = 4 and b = 8. C is the curve enclosing the region. According to Stokes' theorem:

$$I = \oint_C \vec{\mathbf{F}}.d\vec{\mathbf{l}} = \iint_S \left(\vec{\nabla}\times\vec{\mathbf{F}}\right).d\vec{\mathbf{S}} \qquad \text{(i)}$$

We first calculate:

$$\vec{\nabla}\times\vec{\mathbf{F}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & 2x & z^2 \end{vmatrix} = 2\hat{\mathbf{k}}$$

On the xy plane $d\vec{\mathbf{S}} = dx\,dy\,\hat{\mathbf{k}}$.

$$\therefore I = \iint_S (\vec{\nabla}\times\vec{\mathbf{F}}).d\vec{\mathbf{S}} = \iint_S (2\hat{\mathbf{k}}).(dx\,dy\,\hat{\mathbf{k}}) = 2\iint_S dx\,dy$$

By the area property of the double integral, the integral is just:

$$I = 2\ (\text{Area of the Ellipse})$$

The area of the ellipse is πab so with a = 4 and b = 8 we get:

$$I = 2[\pi(4\times 8)] = 64\pi$$

9. Using the divergence theorem, the flux of the vector field $\oiint_S \vec{\mathbf{F}}.d\vec{\mathbf{S}}$ where S is the surface of the sphere of radius two units, is the volume integral $\iiint_V (\vec{\nabla}.\vec{\mathbf{F}})dV$, where V is the volume enclosed by the sphere. We first evaluate $\vec{\nabla}.\vec{\mathbf{F}}$:

$$\vec{\nabla}.\vec{\mathbf{F}} = \frac{\partial(z)}{\partial x} + \frac{\partial(2y)}{\partial y} + \frac{\partial(-x^3)}{\partial z} = 2 \qquad \text{(i)}$$

Using the result of Eq. (i) and the divergence theorem we can write the flux of the vector field $\vec{\mathbf{A}}$ as (see margin remark):

$$\oiint_S \vec{\mathbf{F}}.d\vec{\mathbf{S}} = \iiint_V (\vec{\nabla}.\vec{\mathbf{F}})dV = 2\iiint_V dV = 2\left[\frac{4}{3}\pi(2)^3\right] = \frac{64\pi}{3} \qquad \text{(ii)}$$

> We have used the volume property of the triple integral to write $\iiint_V dV$ as the volume of a sphere of radius 2 units.

10. We have to evaluate $\iint_S \vec{\mathbf{A}}.d\vec{\mathbf{S}}$, where S is the surface of the cube. Using the divergence theorem we can write

$$\iint_S \vec{\mathbf{A}}.d\vec{\mathbf{S}} = \int_0^1\int_0^1\int_0^1 (\vec{\nabla}.\mathbf{A})\,dx\,dy\,dz$$

$$= \int_0^1\int_0^1\int_0^1 10y\ dx\,dy\,dz + 4\int_0^1\int_0^1\int_0^1 dx\,dy\,dz$$

$$= 10\int_0^1 dx\int_0^1 y\,dy\int_0^1 dz + 4\int_0^1\int_0^1\int_0^1 dx\,dy\,dz$$

$$= 10\,[x]_0^1\left[\frac{y^2}{2}\right]_0^1[z]_0^1 + 4.1 = 9\ \text{units.}$$